J. Math. Sci. Univ. Tokyo
13 (2006), 145–204.

Studies on the Painlevé Equations, V, Third Painlevé Equations of Special Type $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$

By Yousuke Ohyama, Hiroyuki Kawamuko, Hidetaka Sakai and Kazuo Okamoto

Abstract. By means of geometrical classification ([22]) of space of initial conditions, it is natural to consider the three types, $P_{\mathrm{III}}(D_6)$, $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$, for the third Painlevé equation. The fourth article of the series of papers [17] on the Painlevé equations is concerned with $P_{\mathrm{III}}(D_6)$, generic type of the equation. The other two types, $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$ are obtained as degeneration from $P_{\mathrm{III}}(D_6)$; the present paper is devoted to investigating them in detail.

Each of $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$ is characterized through holonomic deformation of a linear differential equation and written as a Hamiltonian system. $P_{\mathrm{III}}(D_7)$ contains a parameter and admits birational canonical transformations as symmetry, isomorphic to the affine Weyl group of type $A_1^{(1)}$. Sequence of τ-functions are defined for $P_{\mathrm{III}}(D_7)$ by means of successive application of the translation of the symmetry of the equation; they satisfy the Toda equation.

The τ-functions related to algebraic solutions of $P_{\mathrm{III}}(D_7)$ are determined explicitly. The irreducibility of $P_{\mathrm{III}}(D_7)$, as well as that of $P_{\mathrm{III}}(D_8)$, is established, and there is no transcendental classical solution of these equations. A space of initial conditions is constructed for each of $P_{\mathrm{III}}(D_7)$ and $P_{\mathrm{III}}(D_8)$ by the use of successive blowing-up's of the projective plane $\mathbb{P}^2$.

Contents

2000 *Mathematics Subject Classification.* 34M55, 34M15, 34M45.

Key words: Painlevé equations, Isomonodromic deformation, Bäcklund transformation, Differential Galois theory, Space of initial conditions.

1. Introduction

Succeeding to the series of papers [17], we will study in the present article the third Painlevé equations $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{1}$$

for $\gamma = 0$ and $\alpha\delta \neq 0$, and for $\gamma = \delta = 0$.

The values of complex parameters $\alpha, \beta, \gamma, \delta$ of the third Painlevé equations can be classified into the following four cases:

(D_6) $\gamma\delta \neq 0$.

(D_7) $\gamma = 0,\ \alpha\delta \neq 0$ (or $\delta = 0, \beta\gamma \neq 0$),

(D_8) $\gamma = 0,\ \delta = 0,\ \alpha\beta \neq 0$

(Q) $\alpha = 0, \gamma = 0$ (or $\beta = 0, \delta = 0$).

In the case (Q), P_{III} is solvable by quadratures ([10], [17]). Then all of solutions of (Q) are classical in the sense of [25], so that we exclude the case (Q) from investigation on the Painlevé equations; we agree with the view point of Gromak ([5]). While the type D_6, generic case of $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$, has been studied in many articles ([17], [14]), the equations of type D_7 and D_8 are put out of main consideration of the third Painlevé equation so far. We cite [3], where the equation of type D_7 has been studied.

The significance of the equation of type D_7 and D_8 has been pointed out recently by [22]; the spaces of initial conditions for the equations of type D_6, D_7 and D_8 are different from each other.

Here, a space of initial conditions can be characterized by a pair (X, D) of a rational surface X and the anti-canonical divisor D of X. Each irreducible component of D is a rational curve and, in the case of the Painlevé equations, is called as a vertical leaf ([16]). The intersection diagram of D is given by that of the certain root lattice and in particular, we have for $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$ the three cases, $D_6^{(1)}$, $D_7^{(1)}$ and $D_8^{(1)}$; for details, see [22]. We note that it is quite natural to classify the Painlevé equations into the following eight types:

$$\begin{array}{lllll} P_{\mathrm{VI}}(D_4), & P_{\mathrm{V}}(D_5), & P_{\mathrm{III}}(D_6), & P_{\mathrm{III}}(D_7), & P_{\mathrm{III}}(D_8), \\ & & P_{\mathrm{IV}}(E_6), & P_{\mathrm{II}}(E_7), & P_{\mathrm{I}}(E_8). \end{array}$$

The following equation, which we denote by $P_{\mathrm{III'}}(\alpha, \beta, \gamma, \delta)$:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\alpha q^2}{4t^2} + \frac{\beta}{4\,t} + \frac{\gamma q^3}{4t^2} + \frac{\delta}{4q}, \tag{2}$$

is equivalent to $P_{\mathrm{III}}(\alpha, \beta, \gamma, \delta)$ through the change of variables:

$$t = x^2, \quad y = xq. \tag{3}$$

Since a realization of action of transformation group on $P_{\mathrm{III}'}$ is simpler than that on P_{III}, we consider in what follows $P_{\mathrm{III}'}$ instead of P_{III}, cf. [17]. For $P_{\mathrm{III}'}(\alpha, \beta, \gamma, \delta)$, we have thus the three cases, $P_{\mathrm{III}'}(D_6)$, $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$.

By the change of variables

$$x = \lambda x_1, \quad y = \mu y_1 \qquad (\lambda\mu \neq 0), \tag{4}$$

we can normalize values of the parameter $(\alpha, \beta, \gamma, \delta)$. The equation of type D_6 has two complex parameters and that of type D_7 has one complex parameter, while the equation of type D_8 contains no complex parameters. For $P_{\mathrm{III}'}(D_7)$, we consider the standard form:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{2q^2}{t^2} + \frac{\beta}{4t} - \frac{1}{q}, \tag{5}$$

and for $P_{\mathrm{III}'}(D_8)$, the following:

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q^2}{t^2} - \frac{1}{t}. \tag{6}$$

The aim of the present article is to study Hamiltonian structures, transformation groups, τ-functions and special solutions, with respect to the equations, $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$. We consider also holonomic deformation of a linear differential equation and show that Hamiltonian structures associated with $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$ are deduced from holonomic deformation. Moreover, we give an explicit construction of the space of initial conditions, for each of the equations. The irreducibility of the equations is also a subject of our studies.

In Section 2, we give Hamiltonian structures considered in the present article for $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$. It is known that the birational symmetries of $P_{\mathrm{III}'}(D_6)$ are given by $\widetilde{W_a}(A_1 \oplus A_1)$, where $\widetilde{W_a}(A)$ denotes the extended affine Weyl group of type A. We show that the group of birational symmetries of $P_{\mathrm{III}'}(D_7)$ is $\widetilde{W_a}(A_1)$ and that of $P_{\mathrm{III}'}(D_8)$ is $\mathbb{Z}_2$.

Section 3 is devoted to studies on holonomic deformation of a linear differential equation. We consider mainly an equation of the form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p_1(x,t)\frac{\mathrm{d}y}{\mathrm{d}x} + p_2(x,t)y = 0; \tag{7}$$

we say that (7) admits *holonomic deformation* with respect to a parameter, t, if (7) has a fundamental system of solutions whose monodromy and Stokes multipliers are not depending on t. Such deformation has been often named as monodromy preserving or as isomonodromic one. In stead of these terminology, we call it holonomic deformation. By considering the deformation related to the third Painlevé equation, we have the following degeneration scheme of the equations:

$$\begin{array}{ccccccccc}
P_{\mathrm{VI}}(D_4) & \rightarrow & P_{\mathrm{V}}(D_5) & \rightarrow & P_{\mathrm{III}}(D_6) & \rightarrow & P_{\mathrm{III}}(D_7) & \rightarrow & P_{\mathrm{III}}(D_8) \\
 & & & \searrow & & \searrow & & & \\
 & & & & P_{\mathrm{IV}}(E_6) & \rightarrow & P_{\mathrm{II}}(E_7) & \rightarrow & P_{\mathrm{I}}(E_8).
\end{array}$$

Section 4 concerns τ-functions related to $P_{\mathrm{III'}}(D_7)$ and $P_{\mathrm{III'}}(D_8)$; a τ-function is defined by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\tau = H,$$

up to a multiplicative constant, where H is a Hamiltonian function. By applying birational transformations successively, we have a sequence, $\{\tau_n\}$, of τ-functions related to $P_{\mathrm{III'}}(D_7)$, and see that it satisfies the Toda equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}t\frac{\mathrm{d}}{\mathrm{d}t}\log\tau_n = c(n)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}.$$

Bilinear forms deduced from equation $P_{\mathrm{III'}}(D_7)$ and $P_{\mathrm{III'}}(D_8)$ are investigated; for example $P_{\mathrm{III'}}(D_8)$ is equivalent to the equation:

$$\mathcal{D}^4\tau\cdot\tau + t\tau\cdot\tau = 2\mathcal{D}^2 D\tau\cdot\tau.$$

Here $\mathcal{D}$ is the Hirota derivative with respect to the derivation:

$$D = t\frac{\mathrm{d}}{\mathrm{d}t}.$$

Let y be a solution of the third Painlevé equation, and consider the differential field, $K = \mathbb{C}(t)\langle y\rangle$, $\mathbb{C}(t)$ being the field of rational functions. If the transcendental degree of K over $\mathbb{C}(t)$ is zero, then y is algebraic. In Section 5, we will show that if y is transcendental, then

$$\text{trans.deg}K/\mathbb{C}(t) = 2$$

for $P_{\text{III}'}(D_7)$ and for $P_{\text{III}'}(D_8)$. This fact establishes the irreducibility of the equations in the sense of Umemura([24]). For the other Painlevé equations, the irreducibility is established except for the determination of algebraic solutions of $P_{\text{VI}}(D_4)$; see for example [29, 30]. By virtue of the irreducibility there is no transcendental classical solution of $P_{\text{III}'}(D_7)$ and $P_{\text{III}'}(D_8)$.

The algebraic solutions of Painlevé equations are studied by many authors, and in fact, many works on this subject have been studied by Belorussian school (see [6]). An algebraic solution of the third Painlevé equations has been found by Lukashevich ([10]), and then Gromak ([4], [5]) classified all algebraic solutions of $P_{\text{III}}(D_6)$ and those of $P_{\text{III}}(D_7)$. On the other hand, Murata ([14]) has given the classification of algebraic solution of the third Painlevé equation, by using the transformation group of the equation. $P_{\text{III}}(D_8)$ has the two rational solutions, $y = \pm 1$, and then (6) has the solutions, $q = \pm\sqrt{t}$. Algebraic solutions of $P_{\text{III}'}(D_7)$ are studied in Section 6.

Section 7 is a supplement of [16], where spaces of initial conditions of the Painlevé equations are constructed. A space of initial conditions of a differential equation is, by definition, a fiber of a fiber bundle $\mathcal{P} = (E, \pi, B)$ with the following properties: There is a foliation $\mathcal{F}$ on $\mathcal{P}$ associated with the differential equation such that

a Each leaf of $\mathcal{F}$ intersects with each fiber transversally;

b Each path γ on B can be lifted to a leaf γ_p that runs through a given point $p \in \pi^{-1}(\gamma(0))$;

c $\pi|_{\gamma_p} : \gamma_p \to B$ is surjective and γ_p is a covering space of B by π.

The special cases of the third Painlevé equation, $P_{\text{III}'}(D_7)$ and $P_{\text{III}'}(D_8)$, has been settled out of consideration; they have hidden behind the generic type D_6. In [16], only equation $P_{\text{III}'}(D_6)$ is considered and we construct space of initial conditions for $P_{\text{III}'}(D_7)$ and $P_{\text{III}'}(D_8)$, in the last section.

2. Third Painlevé Equation

In this section we begin with a review of basic facts on the third Painlevé equation ([17]) and then give explicit forms of Hamiltonian functions considered in what follows. The subject of our investigation consists of the following three types of the equations:

$$P_{\mathrm{III}'}(D_6) \quad : \quad \frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\alpha q^2}{4t^2} + \frac{\beta}{4\,t} + \frac{\gamma q^3}{4t^2} + \frac{\delta}{4q} \qquad (\gamma\delta \neq 0), \tag{8}$$

$$P_{\mathrm{III}'}(D_7) \quad : \quad \frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{2q^2}{t^2} + \frac{\beta}{4t} - \frac{1}{q}, \tag{9}$$

$$P_{\mathrm{III}'}(D_8) \quad : \quad \frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - \frac{1}{t}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q^2}{t^2} - \frac{1}{t}. \tag{10}$$

2.1. Fundamental transform

We denote equation (1) by $P_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ and (2) by $P_{\mathrm{III}'}(\alpha,\beta,\gamma,\delta)$. The third Painlevé equation contains four complex parameters, and there are essentially two complex parameters; we see that by means of simple transformation. In fact we have the

Theorem 1 ([17]). (i) *By the change of variables:*

$$t = x^2, \quad y = xq, \tag{11}$$

$P_{\mathrm{III}}(\alpha,\beta,\gamma,\delta)$ and $P_{\mathrm{III}'}(\alpha,\beta,\gamma,\delta)$ are equivalent.
(ii) *By replacing q by t/q in $P_{\mathrm{III}'}(\alpha,\beta,\gamma,\delta)$, we obtain $P_{\mathrm{III}'}(-\beta,-\alpha,-\delta,-\gamma)$.*
(iii) *By the change of variables:*

$$t = t_1^2, \quad q = q_1^2, \tag{12}$$

$P_{\mathrm{III}'}(\alpha,\beta,0,0)$ is converted to $P_{\mathrm{III}'}(0,0,2\alpha,2\beta)$ with respect to (t_1,q_1).
(iv) *The change of scales:*

$$t \to \lambda t, \quad q \to \mu q \tag{13}$$

takes $P_{\mathrm{III}'}(\alpha,\beta,\gamma,\delta)$ to $P_{\mathrm{III}'}(\lambda\alpha, \mu\lambda^{-1}\beta, \lambda^2\gamma, \mu^2\lambda^{-2}\delta)$.

$P_{\mathrm{III}'}(D_8)$ is reduced to the special case of $P_{\mathrm{III}'}(D_6)$ through the quadratic transformation given by (iii). For algebraic transformation of the Painlevé equations, see [23]. By (iv), $P_{\mathrm{III}'}(\alpha,\beta,0,\delta)$ $(\alpha\delta\neq 0)$ can be normalized as (9).

2.2. Hamiltonian system

2.2.1 Hamiltonian for $P_{\mathrm{III}'}(D_7)$

The Hamiltonian associated with (9) is

$$tH = q^2p^2 + \alpha_1 qp + tp + q, \tag{14}$$

and the Hamiltonian system reads:

$$\begin{cases} t\dfrac{\mathrm{d}q}{\mathrm{d}t} = 2q^2p + \alpha_1 q + t, \\ t\dfrac{\mathrm{d}p}{\mathrm{d}t} = -2qp^2 - \alpha_1 p - 1. \end{cases} \tag{15}$$

By eliminating p from (15), we obtain in fact $P_{\mathrm{III}'}(-8, 4(1-\alpha_1), 0, -4)$, that is, the equation of the form (9). We denote (15) by $\mathcal{H}(\alpha_1)$, when considering dependence of the system on a parameter.

Defining the auxiliary Hamiltonian by:

$$h = tH + \alpha_1^2/4, \tag{16}$$

we have from (14)–(15) the expression:

$$\begin{cases} q = -\dfrac{t\,\frac{\mathrm{d}^2h}{\mathrm{d}t^2} + \alpha_1\frac{\mathrm{d}h}{\mathrm{d}t} + 1}{2\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2}, \\ p = \frac{\mathrm{d}h}{\mathrm{d}t}. \end{cases} \tag{17}$$

It follows that:

PROPOSITION 2. *The auxiliary Hamiltonian, h, satisfies the differential equation*

$$\left(t\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)^2 + 4\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\left(t\frac{\mathrm{d}h}{\mathrm{d}t} - h\right) - 2\alpha_1\frac{\mathrm{d}h}{\mathrm{d}t} - 1 = 0. \tag{18}$$

Inversely, for a solution of $h(t)$ of (18), we have a solution (q, p) of (15) by (17), provided that

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} \not\equiv 0.$$

Note that the equation (18) admits a singular solution of the form

$$h = \lambda t + \mu,$$

$$4\lambda^2 \mu + \alpha_1 \lambda + 1 = 0.$$

By means of (15) , we can show easily the

PROPOSITION 3. *There exists the one-to-one correspondence between the general solution h of* (18) *and a solution* (q, p) *of* (15).

In fact suppose that $\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} \equiv 0$. We have from (16)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = p,$$

and then by the hypothesis p is constant. It follows from the second equation of (15) that q is constant; we thus arrive at contradiction with the first equation of (15).

2.2.2 Hamiltonian for $P_{\mathrm{III}'}(D_8)$

We consider the Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q} \tag{19}$$

with

$$tH = q^2 p^2 + qp - \frac{1}{2}\left(q + \frac{t}{q}\right). \tag{20}$$

We obtain from this system equation $P_{\mathrm{III}'}(-4, 4, 0, 0)$, that is, (10).

In this case we have an auxiliary Hamiltonian

$$h = tH$$

and then

$$\begin{cases} q = -\dfrac{1}{2\,\frac{\mathrm{d}h}{\mathrm{d}t}}, \\ p = t\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \frac{\mathrm{d}h}{\mathrm{d}t}. \end{cases} \tag{21}$$

We state the results without entering into details.

Proposition 4. *h satisfies the differential equation*

$$\left(t\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)^2 - \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\left(4h - 4t\frac{\mathrm{d}h}{\mathrm{d}t} + 1\right) + \frac{\mathrm{d}h}{\mathrm{d}t} = 0, \tag{22}$$

Proposition 5. *There exists the one-to-one correspondence from the general solution $h(t)$ of (22) and a solution (q,p) of (19).*

Equation (22) admits a singular solution of the form

$$\begin{aligned} &h = \lambda t + \mu, \\ &4\lambda\mu + \lambda - 1 = 0. \end{aligned}$$

2.3. Transformation group

We make the list of explicit forms of birational canonical transformation of $P_{\mathrm{III}'}(D_7)$. We have a group of such symmetries, denoted by $\mathrm{Cr}(D_7)$, and see that this is realization of the affine Weyl group, $\widetilde{W_a}(A_1)$ of type A_1. we give below the results without entering into details; verification is done by means of straightforward computation.

Theorem 6. *The birational symmetry of $P_{\mathrm{III}'}(D_7)$ is described as:*

$$\mathrm{Cr}(D_7^{(1)}) = \widetilde{W_a}(A_1) = \langle s_1, \sigma\rangle.$$

The transformations are given by the following table:

x	α_0	α_1	p	q	t	tH
$s_0(x)$	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$p - \frac{\alpha_0}{q} + \frac{t}{q^2}$	q	$-t$	$tH + \frac{t}{q} - \alpha_0$
$s_1(x)$	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$-p$	$-q - \frac{\alpha_1}{p} - \frac{1}{p^2}$	$-t$	tH
$\sigma(x)$	α_1	α_0	$-\frac{q}{t}$	tp	$-t$	$tH - qp$

(23)

where $\alpha_0 = 1 - \alpha_1$.

For example, if $(q(t), p(t))$ satisfies $\mathcal{H}(\alpha_1)$, then functions given by

$$(Q(t), P(t)) = \left(-q(-t) - \frac{\alpha_1}{p(-t)} - \frac{1}{p(-t)^2}, -p(-t)\right)$$

solve $\mathcal{H}(-\alpha_1)$. This gives the birational canonical transformation, associated with:

$$s_1 : \alpha_1 \to -\alpha_1.$$

On the other hand, for a solution $(q(t), p(t))$ of $\mathcal{H}(\alpha_1)$,

$$(Q(t), P(t)) = \left(-tp(-t), \frac{q(-t)}{t}\right)$$

satisfies $\mathcal{H}(\sigma(\alpha_1)) = \mathcal{H}(\alpha_0)$. The explicit form of the transformation $\pi = \sigma \circ s_1$ will be used below in Section 4.2; see Proposition 13.

Remark 1. By means of the table of Theorem 6, we obtain the transformation corresponding to the transformation:

$$\pi = \sigma \circ s_1 : \alpha_1 \longrightarrow \alpha_1 - 1,$$

The explicit form of the transformation is of the form

$$(q, p) \longrightarrow \left(-tp + \frac{\alpha_0 t}{q} - \frac{t^2}{q^2}, \frac{q}{t}\right). \tag{24}$$

A transformation of the form (24) has been found for the first time by Gromak [3], for equation (9) of the second order.

There is no parameter in $P_{\mathrm{III}'}(D_8)$ and we have the

Theorem 7.

$$\mathrm{Cr}(D_8^{(1)}) = \mathbb{Z}_2 = \langle \pi \rangle,$$

where

x	p	q	t
$\pi(x)$	$-\frac{q(2qp+1)}{2t}$	$\frac{t}{q}$	t

. (25)

3. Holonomic Deformation and Degeneration

Consider a linear differential equation of the form:

$$L(D_6): \quad \frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = 0,$$

$$\begin{aligned} p_1 &= \frac{\eta_0 t}{x^2} + \frac{1-\theta_0}{x} - \eta_\infty - \frac{1}{x-q}, \\ p_2 &= -\frac{tH}{x^2} + \frac{\eta_\infty(\theta_0+\theta_\infty) - 2p}{2x} + \frac{p}{x-q}, \\ H &= \frac{1}{t}\left\{q^2p^2 - (\eta_\infty q^2 + \theta_0 q - \eta_0 t)p + \frac{1}{2}\eta_\infty(\theta_0+\theta_\infty)q\right\}. \end{aligned} \tag{26}$$

It is known ([18]) that the holonomic deformation of this equation is governed by a Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}. \tag{27}$$

This system is equivalent to the third Painlevé equation $P_{\mathrm{III}'}(\alpha,\beta,\gamma,\delta)$ with $\alpha = -4\eta_\infty\theta_\infty, \beta = 4\eta_0(1+\theta_0), \gamma = 4{\eta_\infty}^2, \delta = -4{\eta_0}^2$. Here we assume $\eta_0\eta_\infty \neq 0$; the deformation given above concerns $P_{\mathrm{III}'}(D_6)$.

To characterize the other equations of the third Painlevé equations, we have to consider again the linear differential equation,

$$\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0, \tag{28}$$

for $P_{\mathrm{III}'}(D_7)$ and that for $P_{\mathrm{III}'}(D_8)$.

3.1. Holonomic deformation

In this subsection, we study two linear differential equations; $L(D_7)$ and $L(D_8)$, of the form (28). First $L(D_7)$ is, by definition, equation (28) with the following properties:

1. $L(D_7)$ has singularities at $x = 0, \infty$ and $x = q$,

2. $x = 0$ is an irregular singular point of Poincaré rank 1,

3. $x = \infty$ is an irregular singular point of Poincaré rank 1/2,

4. $x = q$ is an apparent singular point whose characteristic exponents are 0 and 2.

On the other hand, for $L(D_8)$, we demand:

1. $L(D_8)$ has singularities at $x = 0, \infty$ and $x = q$,
2. $x = 0$ and $x = \infty$ are irregular singular points of Poincaré rank $1/2$,
3. $x = q$ is an apparent singular point whose characteristic exponents are 0 and 2.

By means of suitable changes of the dependent variables, the Riemann scheme of $L(D_7)$ and $L(D_8)$ read respectively as follows:

$$L(D_7) \quad : \quad \left\{ \begin{array}{ccccc} \multicolumn{2}{c}{x = 0} & x = q & \multicolumn{2}{c}{x = \infty \ (1/2)} \\ \multicolumn{2}{c}{\overbrace{\hphantom{t \quad 1-\alpha_1}}} & & \multicolumn{2}{c}{\overbrace{\hphantom{1 \quad \alpha_1 - \frac{1}{2}}}} \\ t & 1 - \alpha_1 & 0 & 1 & \alpha_1 - \frac{1}{2} \\ 0 & 0 & 2 & -1 & \alpha_1 - \frac{1}{2} \end{array} \right\},$$

$$L(D_8) \quad : \quad \left\{ \begin{array}{ccccc} \multicolumn{2}{c}{x = 0 \ (1/2)} & x = q & \multicolumn{2}{c}{x = \infty \ (1/2)} \\ \multicolumn{2}{c}{\overbrace{\hphantom{\sqrt{2t} \quad -1/2}}} & & \multicolumn{2}{c}{\overbrace{\hphantom{\sqrt{2} \quad 1/2}}} \\ \sqrt{2t} & -1/2 & 0 & \sqrt{2} & 1/2 \\ -\sqrt{2t} & -1/2 & 2 & -\sqrt{2} & 1/2 \end{array} \right\}.$$

Hence the coefficients of $L(D_7)$ are given by:

$$\begin{aligned} p_1 &= \frac{t}{x^2} + \frac{1 + \alpha_1}{x} - \frac{1}{x - q}, \\ p_2 &= -\frac{tH}{x^2} - \frac{p - 1}{x} + \frac{p}{x - q}, \\ tH &= q^2 p^2 + \alpha_1 qp + tp + q. \end{aligned}$$

The explicit form of H is deduced from the property 4 by the use of the Frobenius method. For $L(D_8)$, we have:

$$\begin{aligned} p_1 &= \frac{2}{x} - \frac{1}{x - q}, \\ p_2 &= -\frac{t}{2x^3} - \frac{tH}{x^2} - \frac{2p + 1}{2x} + \frac{p}{x - q}, \\ tH &= q^2 p^2 + qp - \frac{1}{2}\left(q + \frac{t}{q}\right). \end{aligned}$$

Moreover by putting

$$Y = \begin{bmatrix} y \\ \dfrac{1}{x-q}\left(-py + \dfrac{dy}{dx}\right) \end{bmatrix},$$

we can rewrite these linear differential equations as the following systems:

$$L(D_7): \quad \frac{\mathrm{d}}{\mathrm{d}x} Y = A^{(1)}(x,t,q,p)Y, \tag{29}$$

$$A^{(1)}(x,t,q,p) = \begin{bmatrix} p & x-q \\ \dfrac{-qp^2 - \alpha_1 p - 1}{x^2} - \dfrac{p^2}{x} & -\dfrac{t}{x^2} - \dfrac{1+\alpha_1}{x} - p \end{bmatrix},$$

$$L(D_8): \quad \frac{\mathrm{d}}{\mathrm{d}x} Y = A^{(2)}(x,t,q,p)Y, \tag{30}$$

$$A^{(2)}(x,t,q,p) = \begin{bmatrix} p & x-q \\ -\dfrac{t}{2qx^3} - \dfrac{2qp^2 + 2p - 1}{2x^2} - \dfrac{p^2}{x} & -\dfrac{2}{x} - p \end{bmatrix}.$$

By viewing q, p as functions of t, we consider holonomic deformation of $L(D_7)$ and that of $L(D_8)$. Now we state the theorems:

Theorem 8. *The holonomic deformation of* (29) *is governed by the system of differential equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_1^{(1)} = \frac{1}{2t}[\mathcal{A}_0^{(1)}, \mathcal{A}_2^{(1)}], \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_2^{(1)} = \frac{1}{2t}\left(\mathcal{A}_2^{(1)} + [\mathcal{A}_1^{(1)}, \mathcal{A}_2^{(1)}] + [\mathcal{A}_0^{(1)}, \mathcal{A}_3^{(1)}]\right), \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_3^{(1)} = \frac{1}{2t}\left(2\mathcal{A}_3^{(1)} + [\mathcal{A}_1^{(1)}, \mathcal{A}_3^{(1)}]\right), \tag{33}$$

where

$$\mathcal{A}_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathcal{A}_1^{(1)} = \begin{bmatrix} -\alpha_1 - \frac{5}{2} & 2qp + \alpha_1 - \frac{1}{2} \\ 2qp + \alpha_1 - \frac{1}{2} & -\alpha_1 - \frac{5}{2} \end{bmatrix},$$

$$\mathcal{A}_2^{(1)} = \begin{bmatrix} 2q + 2tp & -2q + 2tp \\ 2q - 2tp & -2q - 2tp \end{bmatrix},$$

$$\mathcal{A}_3^{(1)} = \begin{bmatrix} 4t & -4t \\ -4t & 4t \end{bmatrix}.$$

THEOREM 9. *The holonomic deformation of* (30) *is governed by differential equations*

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{A}_1^{(2)} = \frac{1}{t}[\mathcal{A}_0^{(2)}, \mathcal{A}_2^{(1)}], \tag{34}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{A}_2^{(2)} = \frac{1}{t}\left(\mathcal{A}_2^{(2)} + [\mathcal{A}_1^{(2)}, \mathcal{A}_2^{(2)}]\right), \tag{35}$$

where

$$\mathcal{A}_0^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathcal{A}_1^{(2)} = \begin{bmatrix} -\frac{7}{2} & 2qp + \frac{1}{2} \\ 2qp + \frac{1}{2} & -\frac{7}{2} \end{bmatrix},$$

$$\mathcal{A}_2^{(2)} = \begin{bmatrix} -q - \frac{t}{q} & q - \frac{t}{q} \\ -q + \frac{t}{q} & q + \frac{t}{q} \end{bmatrix}.$$

We will verify these theorems at the end of this section.

It is not difficult to show that the equations (31),(32) and (33) are reduced to the Hamiltonian system:

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q}, \tag{36}$$

$$tH = q^2p^2 + \alpha_1 qp + tp + q. \tag{37}$$

Moreover, we see that the equations (34) and (35) are equivalent to Hamiltonian system:

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H}{\partial p}, \ \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial q} \tag{38}$$

$$tH = q^2p^2 + qp - \frac{1}{2}\left(q + \frac{t}{q}\right). \tag{39}$$

Therefore we have the

Theorem 10. *The holonomic deformation of* $L(D_7)$ *is governed by Hamiltonian system* (36) *with* (37) *and that of* $L(D_8)$ *is governed by* (38) *with* (39).

3.2. Degeneration

We derive the Hamiltonian systems associated with $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$ from that of $P_{\mathrm{III}'}(D_6)$ by means of successive process of degeneration.

In Hamiltonian $H = H_{\mathrm{III}'}(D_6)$, given by (26), we replace t by t/η_0 and H by $\eta_0 H$, and furthermore, put $\theta_0 = -\alpha_1, \eta_\infty = \varepsilon$ and $\theta_\infty = 1/\varepsilon$. Then the Hamiltonian H is holomorphic in ε. Letting ε to 0 in H, we arrive at Hamiltonian $H = H_{\mathrm{III}'}(D_7)$. For simplicity, we express this procedure by

$$\begin{aligned}
&(q, p, t, H) \longrightarrow \Big(q,\, p,\, \frac{t}{\eta_0},\, \eta_0 H\Big), \qquad (40)\\
&\theta_0 \longrightarrow -\alpha_1, \quad \eta_\infty \longrightarrow \varepsilon, \quad \theta_\infty \longrightarrow \frac{1}{\varepsilon}, \quad (\varepsilon \longrightarrow 0).
\end{aligned}$$

Moreover degeneration from (36) to (38) is given by:

$$\begin{aligned}
&(q, p, t, H) \longrightarrow \Big(-\frac{q}{2},\, -2p - \frac{1}{\varepsilon q},\, \frac{\varepsilon}{2}t,\, \frac{2}{\varepsilon}H + \frac{2\varepsilon - 1}{2\varepsilon^3 t}\Big), \qquad (41)\\
&\alpha_1 \longrightarrow 1 - \frac{1}{\varepsilon}, \quad (\varepsilon \longrightarrow 0).
\end{aligned}$$

We do not enter into detail of computation.

Note that (40) causes the degeneration from $L(D_6)$ to $L(D_7)$. Changing the variables y and x of $L(D_7)$ as

$$y \longrightarrow x^{(1-\alpha_1)/2} y, \quad x \longrightarrow -x/2,$$

and then applying the degeneration (41), we obtain $L(D_8)$.

3.3. Verification of Theorem 8 and Theorem 9

In this subsection, we verify Theorem 8 and Theorem 9. For $L(D_7)$, we show the

LEMMA 11. (i) *By the change of variables*

$$Y = P\,\mathcal{Y}, \quad x = -\frac{\xi^2}{4}, \qquad \left(P = \begin{bmatrix} -\xi^3/4 & -\xi^3/4 \\ -p\xi - 2 & -p\xi + 2 \end{bmatrix}\right) \tag{42}$$

system (29) *is converted to the system:*

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{Y} = \mathcal{A}^{(1)}(\xi, t, q, p)\mathcal{Y}, \tag{43}$$

$$\begin{aligned} \mathcal{A}^{(1)}(\xi, t, q, p) &= P^{-1}\left(-\frac{\xi}{2}A^{(1)}(-\xi^2/4, t, q, p)P - \frac{d}{d\xi}P\right) \\ &= \mathcal{A}_0^{(1)} + \frac{\mathcal{A}_1^{(1)}}{\xi} + \frac{\mathcal{A}_2^{(1)}}{\xi^2} + \frac{\mathcal{A}_3^{(1)}}{\xi^3}. \end{aligned}$$

(ii) *System* (43) *has formal solutions,* $Y^{(\infty)}(\xi, t)$ *and* $Y^{(0)}(\xi, t)$*, of the form:*

$$Y^{(\infty)}(\xi, t) = \hat{Y}^{(\infty)}(\xi, t)\, e^{T^{(\infty)}(\xi, t)},$$

$$\begin{aligned} T^{(\infty)}(\xi, t) &= \begin{bmatrix} \xi + (\alpha_1 + \frac{5}{2})\log(1/\xi) & 0 \\ 0 & -\xi + (\alpha_1 + \frac{5}{2})\log(1/\xi) \end{bmatrix}, \\ \hat{Y}^{(\infty)}(\xi, t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \omega_1 & \omega_2 \\ -\omega_2 & -\omega_1 \end{bmatrix} \xi^{-1} + \cdots, \end{aligned} \tag{44}$$

$$\begin{aligned} \omega_1 &= -2q^2p^2 - (2\alpha_1 - 1)qp - 2q - 2tp - \frac{(2\alpha_1 - 1)^2}{8}, \\ \omega_2 &= -qp - \frac{2\alpha_1 - 1}{4}, \end{aligned}$$

$$Y^{(0)}(\xi, t) = G\,\hat{Y}^{(0)}(\xi, t)\, e^{T^{(0)}(\xi, t)},$$

$$\begin{aligned} G &= \begin{bmatrix} f & g \\ f & -g \end{bmatrix}, \quad T^{(0)}(\xi, t) = \begin{bmatrix} -3\log\xi & 0 \\ 0 & -\frac{4t}{\xi^2} - 2(\alpha_1 + 1)\log\xi \end{bmatrix}, \\ \hat{Y}^{(0)}(\xi, t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \frac{g}{f}\frac{q}{2t} \\ -\frac{f}{g}\frac{p}{2} & 0 \end{bmatrix} \xi + \begin{bmatrix} \omega_3 & 0 \\ 0 & -\omega_3 - \frac{qp}{4t} \end{bmatrix} \xi^2 + \cdots, \end{aligned} \tag{45}$$

$$\omega_3 = -\frac{1}{4t}(q^2p^2 + \alpha_1 qp + q + tp),$$

f *and* g *being functions of* t.

(iii) *The holonomic deformation of* (43) *is governed by* (31)–(33).

PROOF. The assertions (i) and (ii) can be established by a straightforward calculation. We verify the assertion (iii). Let $\Phi_k^{(\infty)}(t), \Phi_k^{(0)}(t)$ be matrices defined by

$$\hat{Y}^{(\infty)} \frac{\partial T^{(\infty)}}{\partial t} \hat{Y}^{(\infty)}{}^{-1} = \sum_{k=0}^{\infty} \Phi_k^{(\infty)}(t)\xi^{-k},$$
$$\hat{Y}^{(0)} \frac{\partial T^{(0)}}{\partial t} \hat{Y}^{(0)}{}^{-1} = \sum_{k=-3}^{\infty} \Phi_k^{(0)}(t)\xi^{k}.$$

By using (44) and (45), we obtain the explicit forms of $\Phi_{-3}^{(0)}(t), \Phi_{-2}^{(0)}(t)$, $\Phi_{-1}^{(0)}(t), \Phi_0^{(0)}(t)$ and $\Phi_k^{(\infty)}(t)$, as follows:

$$\Phi_{-3}^{(0)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \Phi_{-2}^{(0)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}, \Phi_{-1}^{(0)}(t) = \begin{bmatrix} 0 & -\frac{2gq}{tf} \\ -\frac{2fp}{g} & 0 \end{bmatrix},$$
$$\Phi_0^{(0)}(t) = \begin{bmatrix} -\frac{qp}{t} & 0 \\ 0 & \frac{qp}{t} \end{bmatrix}, \Phi_k^{(\infty)}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (k \in \mathbb{Z}).$$

Assume that (43) has a fundamental matrix solution $\mathcal{Y}$ whose monodromy groups and Stokes multipliers are independent of t. Then there exist a matrix, $\Omega'^{(1)}(\xi,t)$, depending rationally on ξ, and matrix $\Theta(t)$, such that

$$\frac{\partial}{\partial t}\mathcal{Y} = \Omega'^{(1)}(\xi,t)\,\mathcal{Y}, \quad \frac{\partial}{\partial t}G = \Theta(t)\,G.$$

Since the eigenvalues of $\mathcal{A}_0^{(1)}$ and $\mathcal{A}_3^{(1)}$ are mutually distinct, $\Omega'^{(1)}(\xi,t)$ and $\Theta(t)$ can be written in the form:

$$\begin{aligned}\Omega'^{(1)}(\xi,t) =& \Big\{ G\,(\Phi_{-3}^{(0)}(t)\xi^{-3} + \Phi_{-2}^{(0)}(t)\xi^{-2} + \Phi_{-1}^{(0)}(t)\xi^{-1})\,G^{-1} \\ &\qquad + \Phi_0^{(\infty)}(t)\xi \Big\} \Big|_{q=q(t),p=p(t)} \\ =& -\frac{1}{t\xi}\begin{bmatrix} q(t)+tp(t) & -q(t)+tp(t) \\ q(t)-tp(t) & -q(t)-tp(t) \end{bmatrix} \\ =& -\frac{1}{2t\xi^2}\left(\mathcal{A}_2^{(1)}\xi + \mathcal{A}_3^{(1)}\right),\end{aligned} \tag{46}$$

$$\begin{aligned}\Theta(t) &= -G\,\Phi_0^{(0)}(t)\,G^{-1}\Big|_{q=q(t),p=p(t)}\\ &= \begin{bmatrix} 0 & -\frac{q(t)}{tp(t)} \\ -\frac{q(t)}{tp(t)} & 0 \end{bmatrix}. \end{aligned} \tag{47}$$

Substituting (46), (47) into the integrability condition of

$$\frac{\partial}{\partial \xi}\mathcal{Y} = \mathcal{A}^{(1)}\mathcal{Y}, \tag{48}$$

$$\frac{\partial}{\partial t}\mathcal{Y} = \Omega'^{(1)}\mathcal{Y}, \tag{49}$$

we obtain (31)–(33). Conversely, we assume (31)–(33). Let $\mathcal{Y}$ be a matrix solution of (48)–(49), then the monodromy groups and Stokes multipliers of $\mathcal{Y}$ are independent of t because the entries of $\Omega'^{(1)}$ are rational function of ξ, see [7]. We have thus established the lemma. □

Proof of Theorem 8. It is sufficient to show that the holonomic deformation of (43) is equivalent to that of (29). We assume that (43) has a solution whose monodromy groups and Stokes multipliers are independent of t. Then there exists a function $\Omega = \Omega(x,t)$, rational in x, such that the system

$$\frac{\partial}{\partial x}Y = A^{(1)}Y, \quad \frac{\partial}{\partial t}Y = \Omega Y \tag{50}$$

is completely integrable. By (42), equation (50) is converted to the system

$$\frac{\partial}{\partial \xi}\mathcal{Y} = \mathcal{A}^{(1)}\mathcal{Y}, \quad \frac{\partial}{\partial t}\mathcal{Y} = \widetilde{\Omega}\mathcal{Y},$$

where $\widetilde{\Omega} = P^{-1}\Big(\Omega(-\xi^2/4,t)P - \frac{\partial P}{\partial t}\Big)$. Since the entries of $\widetilde{\Omega}$ are rational in ξ, it follows that (43) admits a holonomic deformation. Conversely, we assume that (43) admits a holonomic deformation. Then the system (48)-(49) is completely integrable. Changing the variables $Z \to Y, \xi \to x$, we have

$$\frac{\partial}{\partial x}Y = A^{(1)}Y, \quad \frac{\partial}{\partial t}Y = \Omega Y,$$

where

$$\Omega = \Omega(x,t) = \left\{ \frac{\partial}{\partial t} P + P \cdot \Omega'^{(1)}(\xi, t) \right\} \cdot P^{-1} \Big|_{\xi^2 = -4x}$$

$$= \begin{bmatrix} \dfrac{q(t)\,p(t)}{t} & -\dfrac{1}{4}\dfrac{q(t)\,x}{t} \\ 4\dfrac{q(t)\,p(t)^2 + t\frac{dp(t)}{dt}}{tx} & -\dfrac{q(t)\,p(t) + 4t}{tx} \end{bmatrix}.$$

Since entries of Ω are rational function of x, (29) admits a holonomic deformation. □

Next we consider $L(D_8)$.

Proof of Theorem 9. By the change of variables

$$Y = \begin{bmatrix} -\xi^3/2 & \xi^3/2 \\ -p\xi + 1 & -p\xi - 1 \end{bmatrix} \mathcal{Y}, \quad x = \frac{\xi}{2},$$

the equation (30) is written

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{Y} = \mathcal{A}^{(2)}\mathcal{Y}, \tag{51}$$

$$\mathcal{A}^{(2)} = \mathcal{A}_0^{(2)} + \frac{\mathcal{A}_1^{(2)}}{\xi} + \frac{\mathcal{A}_2^{(2)}}{\xi^2}.$$

In a way similar to the proof in Lemma 11 and Theorem 8, we can show that the holonomic deformation of (51) is equivalent to that of (30), and equation (51) admits a monodromy preserving deformation if and only if the system

$$\frac{\partial}{\partial \xi}\mathcal{Y} = \mathcal{A}^{(2)}\mathcal{Y}, \quad \frac{\partial}{\partial t}\mathcal{Y} = \Omega'^{(2)}\mathcal{Y},$$

$$\Omega'^{(2)}(\xi, t) = -\frac{1}{2t\xi}\mathcal{A}_2^{(2)}$$

is completely integrable. By using these fact, we establish Theorem 9. □

4. τ-Function

For any solution (q,p) of the Hamiltonian system,

$$\frac{\mathrm{d}q}{\mathrm{d}t}=\frac{\partial H}{\partial p},\quad \frac{\mathrm{d}p}{\mathrm{d}t}=-\frac{\partial H}{\partial q}, \tag{52}$$

a τ-function $\tau(t)$ is defined by

$$\frac{d}{dt}\log\tau(t)=H(t,q,p), \tag{53}$$

up to multiplicative constant.

4.1. Global behavior of τ-function

By means of the Painlevé property of the equation, $H=H(t,q(t),p(t))$ has only poles as movable singularities. As is well-known for the other Painlevé equations we have the following result also for $P_{\mathrm{III}'}(D_7)$, $P_{\mathrm{III}'}(D_8)$.

Theorem 12. *The τ-function $\tau(t)$ is holomorphic on the universal covering of $\mathbb{C}-\{0\}$ and has simple zeros.*

Since the equations we consider are derived from monodromy preserving deformations, we can verify the theorem by the use of the result obtained by Miwa ([13],[11]). Here we give direct proof of Theorem 12 for $P_{\mathrm{III}'}(D_7)$. In fact, let h be the auxiliary Hamiltonian given by (16). If h has a pole at $t=t_0$ ($t_0\neq 0$), then we deduce from (18):

$$h\sim\frac{t_0}{t-t_0}+O((t-t_0)^0),$$

where $O((t-t_0)^0)$ denote the Landau's symbol. It follows that:

$$H=\frac{1}{t}\left(h-\frac{\alpha_1^2}{4}\right)\sim\frac{1}{t-t_0}+O((t-t_0)^0).$$

By definition, $\tau(t)$ has a simple zero at $t=t_0$. We can verify the theorem for $P_{\mathrm{III}'}(D_8)$ in a similar way.

It will useful to give local expression of a solution around a pole, $t = t_0$. Let $T = t - t_0$ be a local coordinate; then we have for $P_{\mathrm{III}'}(D_7)$:

$$\begin{aligned} q &= T[1 + \frac{\alpha_1}{2t_0}T + cT^2 + \cdots], \quad p \ : \ \text{holomorphic}, \\ q &= T[1 + \frac{2-\alpha_1}{2t_0}T + cT^2 + \cdots], \quad p = \frac{1}{T^2}[-t_0 + c'T^2 + \cdots], \\ q &= -\frac{t_0^2}{T^2}[1 + \frac{1}{t_0}T + cT^2 + \cdots], \quad p = \frac{T}{t_0}[1 + \frac{\alpha_1 - 1}{2t_0}T + c'T^2 + \cdots]. \end{aligned} \tag{54}$$

Here c denotes an arbitrary constant and c' is determined in terms of t_0, α_1 and c. In the case of (54) H has a pole at $T = 0$, while H is holomorphic for the other cases. For $P_{\mathrm{III}'}(D_8)$, we obtain the following expansion:

$$\begin{aligned} q &= T^2[\frac{1}{2t_0} + cT^2 + \cdots], \qquad pq = \frac{t_0}{T}[1 + \frac{1}{2t_0}T + cT^2 + \cdots], \\ q &= \frac{2t_0^2}{T^2}[1 + \frac{1}{t_0}T + cT^2 + \cdots], \quad pq = -\frac{t_0}{T}[1 + \frac{1}{t_0}T + cT^2 + \cdots]. \end{aligned}$$

A pole of H appears from the former.

4.2. Toda equation

In the present subsection, we concern τ-functions of $P_{\mathrm{III}'}(D_7)$. Let $\mathcal{H}(\alpha_1)$ be Hamiltonian system (52) with (14), (q, p) a solution of $\mathcal{H}(\alpha_1)$ and $h = h(t, q, p, \alpha_1)$ an auxiliary function given by (16). We define a new auxiliary function $\bar{h}$ by

$$\bar{h} = h - qp + \frac{-2\alpha_1 + 1}{4}. \tag{55}$$

Then, by using the differential equations, we can show

$$\begin{cases} q = t\frac{\mathrm{d}\bar{h}}{\mathrm{d}t}, \\ p = \dfrac{-1 + \frac{\mathrm{d}\bar{h}}{\mathrm{d}t} - \alpha_1 \frac{\mathrm{d}\bar{h}}{\mathrm{d}t} + t\,\frac{\mathrm{d}^2\bar{h}}{\mathrm{d}t^2}}{2\,t\left(\frac{\mathrm{d}\bar{h}}{\mathrm{d}t}\right)^2}. \end{cases} \tag{56}$$

Let us consider (Q, P), given by:

$$(Q, P) = \left(-\frac{t^2}{q^2} + \frac{\alpha_0 t}{q} - tp, \frac{q}{t}\right). \tag{57}$$

We have the

PROPOSITION 13. *(Q, P) is a solution of (52) with Hamiltonian $H(t, Q, P, \alpha_1 - 1)$.*

This fact follows from Theorem 6; transformation (57) is corresponding to $\pi = \sigma \circ s_1$. We give another verification of Proposition, by showing that $\bar{h}(t, q, p, \alpha_1)$ coincides with the auxiliary function, $h(t, Q, P, \alpha_1 - 1)$. In fact, we can verify by computation

$$H(t, Q, P, \alpha_1 - 1) = H(t, q, p, \alpha_1) - \frac{qp}{t}, \tag{58}$$

and it is easy to see the transformation given by (57) and (58) is canonical. Moreover,

$$\begin{aligned} h(t, Q, P, \alpha_1 - 1) &= tH(t, Q, P, \alpha_1 - 1) + \frac{(\alpha_1 - 1)^2}{4} \\ &= tH(t, q, p, \alpha_1) - qp + \frac{(\alpha_1 - 1)^2}{4} \\ &= h(t, q, p, \alpha_1) - qp + \frac{1 - 2\alpha_1}{4} = \bar{h}(t, q, p, a_1). \end{aligned}$$

REMARK 2. Put $X = qp$. We obtain from (17)

$$X = -\frac{t \frac{\mathrm{d}^2 h}{\mathrm{d}t^2} + \alpha_1 \frac{\mathrm{d}h}{\mathrm{d}t} + 1}{2 \frac{\mathrm{d}h}{\mathrm{d}t}}, \tag{59}$$

and on the other hand (56) results

$$X = \frac{t \frac{\mathrm{d}^2 \bar{h}}{\mathrm{d}t^2} + (1 - \alpha_1) \frac{\mathrm{d}\bar{h}}{\mathrm{d}t} - 1}{2 \frac{\mathrm{d}\bar{h}}{\mathrm{d}t}}. \tag{60}$$

Starting from a solution (q, p) of $\mathcal{H}(\alpha_1)$, arbitrary fixed, we consider now the sequence

$$(q_n, p_n) = (\pi^n(q), \pi^n(p)),$$

of pairs of functions, such that $(q_0, p_0) = (q, p)$.

Then (q_n, p_n) solves $\mathcal{H}(\alpha_1 - n)$. We have a sequence $\{\tau_n\}$ of τ-functions, defined by:

$$\frac{d}{dt}\log\tau_n = H(t, q_n, p_n, \alpha_1 - n).$$

Theorem 14. *$\{\tau_n\}$ satisfies the Toda equation:*

$$\frac{d}{dt}t\frac{d}{dt}\log\tau_n = c(n)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}, \tag{61}$$

$c(n)$ being non zero constants.

Proof. Put $X_n = q_n p_n$. From (58) we have

$$H(t, q_{n+1}, p_{n+1}, \alpha_1 - n - 1) = H(t, q_n, p_n, \alpha_1 - n) - \frac{X_n}{t}.$$

It follows that:

$$X_n = t\frac{d}{dt}\log\frac{\tau_n}{\tau_{n+1}}. \tag{62}$$

Let h_n be the auxiliary Hamiltonian for (q_n, p_n). From (59), we have

$$X_n = -\frac{t\,\frac{\mathrm{d}^2 h_n}{\mathrm{d}t^2} + (\alpha_1 - n)\,\frac{\mathrm{d}h_n}{\mathrm{d}t} + 1}{2\frac{\mathrm{d}h_n}{\mathrm{d}t}}.$$

On the other hand, by defining $\bar{h}_n$ in a way similar to (55), we deduce from Lemma 13

$$\bar{h}_{n-1} = h_n,$$

and then from (60)

$$X_{n-1} = \frac{t\,\frac{\mathrm{d}^2 h_n}{\mathrm{d}t^2} + (n - \alpha_1)\,\frac{\mathrm{d}h_n}{\mathrm{d}t} - 1}{2\,\frac{\mathrm{d}h_n}{\mathrm{d}t}}.$$

Therefore we have

$$X_{n-1} - X_n = \frac{t\,\frac{\mathrm{d}^2 h_n}{\mathrm{d}t^2}}{\frac{\mathrm{d}h_n}{\mathrm{d}t}} = t\frac{d}{dt}\log\frac{\mathrm{d}h_n}{\mathrm{d}t}. \tag{63}$$

Finally we obtain from (62) and (63)

$$\frac{\mathrm{d}h_n}{\mathrm{d}t} = c(n)\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

which show the theorem. □

Since τ-functions are determined up to multiplicative constants, we can normalize the functions, for example, as $c(n) = 1$. For algebraic solutions, it is convenient to put $c(n) = -3$; see Section 6.1.

4.3. Bilinear forms

In this subsection, we rewrite equations $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$ in terms of bilinear forms, by using the method of [19], where a bilinear form of the second Painlevé equation has been considered. The Hirota derivatives considered in what follows are, by definition, given by the following expressions:

$$\mathcal{D}g \cdot f = (Dg)f - g(Df), \qquad \mathcal{D}^2 g \cdot f = (D^2 g)f - 2(Dg)(Df) + g(D^2 f),$$
$$\mathcal{D}^3 g \cdot f = (D^3 g)f - 3(D^2 g)(Df) + 3(Dg)(D^2 f) - g(D^3 f), \quad \dots,$$

where $D = t\frac{\mathrm{d}}{\mathrm{d}t}$. We show the

THEOREM 15. *$P_{\mathrm{III}'}(D_7)$ is equivalent to the bilinear form:*

$$\mathcal{D}^2\tau_1 \cdot \tau_0 - \alpha_1 \mathcal{D}\tau_1 \cdot \tau_0 = \tau_1 \cdot D\tau_0, \tag{64}$$
$$\mathcal{D}^3\tau_1 \cdot \tau_0 - \alpha_1 \mathcal{D}^2\tau_1 \cdot \tau_0 = \mathcal{D}\tau_1 \cdot D\tau_0 + 2t\tau_1 \cdot \tau_0, \tag{65}$$

with respect to two τ-functions, τ_0 and τ_1.

PROOF. In general, consider the three functions of t, H, H_1 and X, such that

$$X = H_1 - H,$$

and let f and g be functions defined by

$$H_1 = D\log g, \qquad H = D\log f,$$

respectively; we have

$$X = D\log\frac{g}{f} = \frac{\mathcal{D}g \cdot f}{g \cdot f}.$$

It is easy to verify the following fundamental formulae of differentiation:

$$\begin{aligned} DH_1 + DH &= \frac{\mathcal{D}^2 g\cdot f}{g\cdot f} - \left(\frac{\mathcal{D}g\cdot f}{g\cdot f}\right)^2, \\ D^2H_1 - D^2H &= \frac{\mathcal{D}^3 g\cdot f}{g\cdot f} - 3\frac{\mathcal{D}^2 g\cdot f}{g\cdot f}\frac{\mathcal{D}g\cdot f}{g\cdot f} + 2\left(\frac{\mathcal{D}g\cdot f}{g\cdot f}\right)^3. \end{aligned}$$

Now consider $P_{\mathrm{III}'}(D_7)$ and put

$$H = tH(t,q,p,\alpha_1) = t\frac{\mathrm{d}}{\mathrm{dt}}\log\tau_0, \quad H_1 = tH(t,Q,P,\alpha_1 - 1) = t\frac{\mathrm{d}}{\mathrm{dt}}\log\tau_1,$$

τ_0, τ_1 being τ-functions. In this case we have

$$X = -qp, \quad DH = tp, \quad DH_1 = tP = q.$$

It follows that:

$$DH + DH_1 = tp + q = H - q^2p^2 - \alpha_1 qp = H - X^2 + \alpha_1 X,$$

from which we obtain the first bilinear equation, (64). Moreover, since

$$D^2H_1 - D^2H = D(q - tp) = (q + tp)(2qp + \alpha_1) - tp + 2t,$$

we have

$$D^2H_1 - D^2H = (DH_1 + DH)(\alpha_1 - 2X) - DH + 2t,$$

from which we deduce (65).

Note that bilinear forms (64)–(65) are corresponding to the transformation $\alpha_1 \mapsto \alpha_1 - 1$. It is easy to verify that (64)–(65) are equivalent to $P_{\mathrm{III}'}(D_7)$. $\square$

In the case of $P_{\mathrm{III}'}(D_8)$, which contains no parameter, we have no sequence of τ-functions. We obtain instead a bilinear equation of the other form, by using differential equation (22) satisfied by the auxiliary function. In fact consider (22) and by using

$$\begin{aligned} &h = D\tau/\tau, \qquad Dh = \mathcal{D}^2\tau\cdot\tau/2\tau^2, \\ &D^2h + 2hDh = \mathcal{D}^2 D\tau\cdot\tau/\tau^2, \quad D^3h + 6(Dh)^2 = \mathcal{D}^4\tau\cdot\tau/2\tau^2, \end{aligned}$$

we obtain the bilinear equation:

$$\mathcal{D}^4\tau\cdot\tau+t\tau\cdot\tau=4\mathcal{D}^2D\tau\cdot\tau. \tag{66}$$

The same method can be applied also to $P_{\mathrm{III}'}(D_7)$, and we have the bilinear equation:

$$\mathcal{D}^4\tau\cdot\tau+(1-\alpha_1^2)\mathcal{D}^2\tau\cdot\tau-2\alpha_1t\tau\cdot\tau=4\mathcal{D}^2D\tau\cdot\tau. \tag{67}$$

5. Irreducibility Theorem

In this section we will establish irreducibility of $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$, that is, we will show the

THEOREM 16. *None of $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$ does have transcendental classical solutions.*

By virtue of the theorem, $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$ may have only algebraic solutions as classical solutions. We will prove the Theorem 16 by using the method of Umemura and Watanabe [26], and then determine the algebraic solutions of each equation in the next section.

Let K be a differential extention of $\mathbb{C}(t)$ with respect to the derivation: $t\frac{\mathrm{d}}{\mathrm{d}t}$ and $K[p,q]$ the polynimial ring over K. For $P_{\mathrm{III}'}(D_7)$ we consider on $K[p,q]$ the Hamiltonian vector field:

$$X_{D_7}(\alpha_1)=t\frac{\partial}{\partial t}+\left(2q^2p+\alpha_1q+t\right)\frac{\partial}{\partial q}-\left(2qp^2+\alpha_1p+1\right)\frac{\partial}{\partial p}. \tag{68}$$

To establish Theorem 16 for $P_{\mathrm{III}'}(D_7)$, we introduce the following condition, (J):

(J) For any differential field extension $K/\mathbb{C}(t)$, there exists no principal ideal I of $K[p,q]$ such that $0\subsetneq I\subsetneq K[p,q]$ and $X(\alpha_1)I\subset I$.

By means of the theory of irreducibility given by [24], Theorem 16 follows from Proposition 17 given below. In fact, if $X=X_{D_7}(\alpha_1)$ enjoys condition (J), a transcendental solution of $P_{\mathrm{III}'}(D_7)$ is non classical; see [26].

PROPOSITION 17. *The derivation $X=X_{D_7}(\alpha_1)$ satisfies the condition (J).*

For $P_{\mathrm{III}'}(D_8)$, Theorem 16 is an immediate consequence of irreducibility of $P_{\mathrm{III}'}(D_6)$ by the use of the transformation given above in Theorem 1 (iii), cf. [27]. On the other hand, we will give below another proof by considering the vector field:

$$X_{D_8} = t\frac{\partial}{\partial t} + \left(2q^2p + q\right)\frac{\partial}{\partial q} - \left(2qp^2 + p - \frac{1}{2}\left(1 - \frac{t}{q^2}\right)\right)\frac{\partial}{\partial p}, \tag{69}$$

on $K[q,p]$. Since

$$X_{D_8}(q) = (2qp+1)q,$$

we have a principal ideal $I = (q)$ such that $X(I) \subset I$. To establish Theorem 16 for $P_{\mathrm{III}'}(D_8)$ in a way similar to the case of $P_{\mathrm{III}'}(D_7)$, we have to introduce instead of condition (J) the following condition :

$(J)'$ For any differential field extension $K/\mathbb{C}(t)$, there exists no principal ideal I of $K[p,q,q^{-1}]$ such that $0 \subsetneq I \subsetneq K[p,q,q^{-1}]$ and $XI \subset I$.

In this section we will verify the

Proposition 18. *The derivation $X = X_{D_8}$ satisfies the condition $(J)'$.*

The irreducibility of $P_{\mathrm{III}'}(D_8)$ follows from Proposition 18; in fact, we have the

Proposition 19. *Let $L \subset K \subset \mathbb{C}(t)$ be a sequence of extensions of differential fields. If $q \in L$ be a solution of $P_{\mathrm{III}'}(D_8)$ and* $\mathrm{trans.deg.}_K L \le 1$, *then q is algebraic over K.*

Proof. Put $M = K(q,p)$, where $p = \frac{1}{2q^2}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{1}{2q}$, then M is a subfield of L and $\mathrm{trans.deg.}_K M \le 1$. We assume $\mathrm{trans.deg.}_K M = 1$. Since q is transcendental, p is algebraic over $K(q)$. Let $F(q,P) \in K[q,P]$ be the minimal polynomial of p over $K(q)$.

Differentiating $F(q,p) = 0$ with respect to t, we obtain

$$X_{D_8}(F)(q,p) = 0.$$

Therefore $X_{D_8}(F)(q,P)$ is divisible by $F(q,P)$; and then it follows from condition $(J)'$ that $F \in K \cdot q^l$ for certain l. F being the minimal polynomial of p, we arrive at contradiction; hence $\mathrm{trans.deg.}_K M = 0$, which shows q is algebraic over K. $\square$

5.1. Proof of Proposition 17

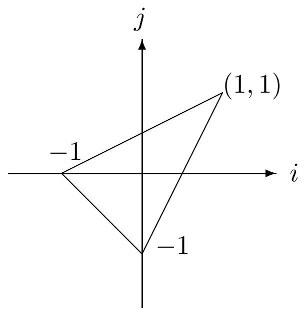
We prove Proposition 17 by reductio ad absurdum; assume that there would exist a principal ideal I of $K[p,q]$, invariant under the action of $X(\alpha_1)$. Let $F \in K[p,q]$ be a generator of I. Then we have

$$X(\alpha_1)F = GF, \tag{70}$$

for some $G \in K[p,q]$. Such a polynomial, F, will be called as an *invariant divisor* in the following of this section.

5.1.1 Weights on $K[p,q]$

We can associate a Newton polygon with derivation on $K[p,q]$; the Newton polygon of $X(\alpha_1)$ is of the form:



Taking the Newton polygon into consideration, we introduce two kinds of weights on $K[p,q]$. First one, ω_1, is defined by:

$$\omega_1(q) = -1, \quad \omega_1(p) = 2;$$

the weight of aq^ip^j is $2j-i$ for any $a \in K$, $a \neq 0$. Let R_d be the K-linear subspace of $K[p,q]$ generated by all the monomials of weight d; we have

$$R_{-d} = K[q^2p]q^d, R_{2d} = K[q^2p]p^d, R_{2d-1} = K[q^2p]qp^d$$

for a non-negative integer d. Then we obtain the decomposition:

$$K[p,q] = \bigoplus_{d\in\mathbb{Z}} R_d, \quad R_d \cdot R_{d'} = R_{d+d'}.$$

Consider three homogeneous derivations, X_{-2}, X_0, X_1, given by:

$$X_1 = \left(2q^2p + t\right)\frac{\partial}{\partial q} - 2qp^2\frac{\partial}{\partial p},$$

$$X_0 = t\frac{\partial}{\partial t} + \alpha_1 q\frac{\partial}{\partial q} - \alpha_1 p\frac{\partial}{\partial p},$$
$$X_{-2} = -\frac{1}{2}\frac{\partial}{\partial p}.$$

Note that each X_i maps R_d to R_{d+i} and $X(\alpha_1) = X_1 + X_0 + X_{-2}$.

On the other hand, we introduce the second weight ω_2, defined by:

$$\omega_2(q) = 2, \quad \omega_2(p) = -1.$$

Let S_d be the K-linear subspace of $K[p,q]$ generated over K by all monomials of weight d. We have, for any non negative integer d,

$$S_{-d} = K[qp^2]p^d, \quad S_{2d} = K[qp^2]q^d, \quad S_{2d-1} = K[qp^2]q^d p,$$

and

$$K[p,q] = \bigoplus_{d\in\mathbb{Z}} S_d, \quad S_d \cdot S_{d'} = S_{d+d'}.$$

The three homogeneous derivations Y_1, Y_0, Y_{-2} by

$$\begin{aligned}
Y_1 &= 2q^2p\frac{\partial}{\partial q} - \left(1+2qp^2\right)\frac{\partial}{\partial p},\\
Y_0 &= t\frac{\partial}{\partial t} + \alpha_1 q\frac{\partial}{\partial q} - \alpha_1 p\frac{\partial}{\partial p},\\
Y_{-2} &= t\frac{\partial}{\partial q},\\
X(\alpha_1) &= Y_1 + Y_0 + Y_{-2},
\end{aligned}$$

and Y_i maps S_d to S_{d+i}.

Since the weight of $X(\alpha)$ is one with respect to both of the weights, G can be written as:

$$G = \lambda qp + \mu$$

for some $\lambda, \mu \in K$.

5.1.2 Highest terms of F

We consider the following decompositions of an invariant divisor, F:

$$\begin{aligned}
F &= F_m + F_{m-1} + \cdots + F_{m-m_0},\\
&\quad F_k \in R_k, \quad F_m \neq 0, \quad F_{m-m_0} \neq 0,
\end{aligned} \tag{71}$$

$$F = f_n + f_{n-1} + \cdots + f_{n-n_0}, \qquad f_k \in S_k, \quad f_n \neq 0, \quad f_{n-n_0} \neq 0, \tag{72}$$

corresponding to the weights, ω_1 and ω_2. The homogeneous part of equation (70) reads:

$$X_1 F_{k-1} + X_0 F_k + X_{-2} F_{k+2} = \lambda pq F_{k-1} + \mu F_k, \tag{73}$$

$$Y_1 f_{k-1} + Y_0 f_k + Y_{-2} f_{k+2} = \lambda pq f_{k-1} + \mu f_k, \tag{74}$$

respectively, where we agree to put $F_k = 0$ for $k < m - m_0$ or $k > m$ and $f_k = 0$ for $k < n - n_0$ or $k > n$. In particular, we have

$$X_1 F_m = \lambda qp F_m, \tag{75}$$

$$Y_1 f_n = \lambda qp f_n. \tag{76}$$

We show the

LEMMA 20. (i) *If $X_1 F = \lambda qp F$ for $F \in K[q,p]$, then F is not divisible by q.*
(ii) *If $Y_1 f = \lambda qp f$ for $f \in K[q,p]$, then f is not divisible by p.*

PROOF. (i) Assume that

$$X_1 F = \lambda qp F \tag{77}$$

and that $F = q^k F'$, where $k \geq 0$, $F' \in K[q,p]$: We obtain from (77):

$$q(X_1(F') + 2kqpF') + ktF' = \lambda qpF',$$

and then $k = 0$. The second assertion of the lemma can be verified in a similar way. □

Therefore m, n are non-negative, even integers; we put

$$m = 2r, \quad n = 2s,$$

r, s being non-negative integers.

REMARK 3. Put $L = q^2 p + t$, $M = q^2 p + 1$; we have:

$$X_1(L) = 2qpL, \quad L \in R_0,$$
$$Y_1(M) = -2qpM, \quad M \in S_0.$$

We can write F_m $(m = 2r)$ as

$$F_{2r} = p^r \sum_{j \geq 0} b_j L^j, \quad b_j \in K.$$

and since

$$X_1(F_{2r}) = qp^{r+1} \sum_{j \geq 0} 2(j-r) b_j L^j.$$

we deduce from (75) that $\lambda = 2(j_0 - r)$ for a non-negative integer j_0 and

$$F_{2r} = bp^r L^{r+\lambda/2}, \quad b \in K^\times. \tag{78}$$

Moreover we obtain from (76)

$$f_{2s} = cq^s M^{s-\lambda/2}, \quad c \in K^\times. \tag{79}$$

The highest term of F_{2r} with respect to ω_2 is:

$$q^{2r+\lambda/2} p^{2r+\lambda/2},$$

and we obtain

$$2r + \frac{3}{2}\lambda \leq 2s.$$

On the other hand, by considering the highest term of f_{2s} with respect to ω_1, we have

$$2s - \frac{3}{2}\lambda \leq 2r.$$

It follows that

$$2r + \frac{3}{2}\lambda = 2s, \tag{80}$$

and moreover, by comparing the coefficient; we have

$$b = c.$$

If F is an invariant divisor, then so is $b^{-1}F$ for any $b \in K^\times$. Here, without loss of generality, we assume $b = c = 1$ in (78)–(79).

5.1.3 Determination of F_{2r-1} and f_{2s-1}

By means of (73), F_{2r} and F_{2r-1} satisfy the equations

$$X_1(F_{2r-1}) + X_0(F_{2r}) = \lambda qpF_{2r-1} + \mu F_{2r}, \tag{81}$$

By writing F_{2r-1} as:

$$F_{2r-1} = p^r q \sum_{j=0}^{k_1} d_j L^j, \quad d_j \in K,$$

we deduce from (81) that

$$\begin{aligned} &X_1(F_{2r-1}) - \lambda qpF_{2r-1} \\ &\quad = p^r \sum_{j=0}^{k_1} d_j \left[(2-\lambda-2r+2j)L^{j+1} + (\lambda - 1 + 2r - 2j)tL^j\right]. \end{aligned} \tag{82}$$

On the other hand we have

$$\begin{aligned} \mu F_{2r} - X_0(F_{2r}) = &\left(\mu - \frac{\alpha_1\lambda}{2}\right) L^{r+\frac{\lambda}{2}} p^r \\ &+ (\alpha_1 - 1)t\left(r + \frac{\lambda}{2}\right) L^{r+\frac{\lambda}{2}-1} p^r. \end{aligned} \tag{83}$$

Comparing (83) and (82), we obtain $k_1 = r + \lambda/2 - 1$, and then

$$\mu - \frac{\alpha_1\lambda}{2} = 0. \tag{84}$$

Moreover we have

$$d_0 = 0,\ d_1 = 0, ..., d_{k-1} = 0, d_k = (\alpha_1 - 1)\left(r + \frac{\lambda}{2}\right).$$

It follows that:

$$F_{2r-1} = \frac{1}{2}(\alpha_1 - 1)(2r + \lambda)\, p^r q L^{r+\lambda/2-1}.$$

We can compute f_{2s-2} in a way similar to F_{2r-1}; we obtain in fact from (79):

$$f_{2s-1} = \frac{1}{2}\alpha_1 (2s - \lambda)\, q^s p M^{s-\lambda/2-1}.$$

F_{2r-1} contains the monomial:

$$p^{2r+\lambda/2-1}q^{2r+\lambda-1},$$

and f_{2s-1} contains the monomial

$$q^{2s-\lambda/2-1}p^{2s-\lambda-1}.$$

These two terms coincide each other by (80) and then we have

$$\frac{1}{2}(\alpha_1-1)(2r+\lambda)=\frac{\alpha_1}{2}(2s+\lambda).$$

It follows again from (80) that:

$$\lambda=\frac{4r}{\alpha_1-2}. \tag{85}$$

5.1.4 Condition (J)

By taking into consideration the birational canonical transformations given above in Section 2.3, we can assume, without loss of generality,

$$0\le\Re\,\alpha_1<1.$$

Since r, s are non-negative integers , $\frac{3}{4}\lambda$ is an integer by means of (80). Then by (85) α_1 is rational and $0\le\alpha_1<1$; hence λ is a non-negative integer. We put, for a non-negative integer l,

$$\lambda=-4l.$$

It follows from (80) and (85) that:

$$r-3l=s,$$
$$l=\frac{r}{2-\alpha_1},$$

respectively. We obtain:

$$\frac{r}{3}\ge l,\qquad \frac{r}{2}\le l<r,$$

which shows

$$r=l=0,$$

and then $\lambda=s=\mu=0$. Definitively, if F satisfies (70), then $F\in K$. We have thus arrived at contradiction.

5.2. Verification of Proposition 18

In this subsection, we establish condition $(J)'$ for $P_{\mathrm{III}'}(D_8)$, in a way similar to the case of condition (J) for $P_{\mathrm{III}'}(D_7)$. Let X be the Hamiltonian vector field given by (69); we suppose that there exist $F, G \in K[p, q, q^{-1}]$ such that

$$XF = GF. \tag{86}$$

F is called an invariant divisor, also in this case. To prove Proposition 18, it is sufficient to show that, if F is an invariant divisor, then $F \in K[q, q^{-1}]$. Note that

$$X(q) = (2q + 1)q.$$

By putting

$$x = q, \qquad z = 2qp + 1,$$

we have from (69)

$$X = zx\frac{\partial}{\partial x} + \left(x - \frac{t}{x}\right)\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}. \tag{87}$$

If F is an invariant divisor, then we can assume $F \in K[x, z]$ without a loss of generality; in fact, if F satisfies (86), then we have for $F' = x^{-n}F$

$$XF' = (G + nx)F'.$$

We introduce two kind of weights, ω_1 and ω_2, as follows:

$$\begin{aligned} \omega_1(z) &= 1, \qquad & \omega_1(x) &= -2, \\ \omega_2(z) &= 1, \qquad & \omega_2(x) &= 2. \end{aligned}$$

Let R_d (S_d resp.) be the K-linear subspace of $K[x, z]$ generated by monomials of weight d with respect to ω_1 (ω_2 resp.); we have the decompositions:

$$\begin{aligned} K[x, z] &= \oplus_{d \in \mathbb{Z}} R_d \\ &= \oplus_{d \geq 0} S_d. \end{aligned}$$

We have homogeneous derivations X_j (Y_j resp.) with respect to ω_1 (ω_2 resp.) given by:

$$\begin{aligned} &X_1 = zx\frac{\partial}{\partial x} = \frac{t}{x}\frac{\partial}{\partial z}, \quad Y_1 = zx\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \\ &X_0 = Y_0 = t\frac{\partial}{\partial t}, \qquad X_{-3} = x\frac{\partial}{\partial z}, \qquad Y_{-3} = -\frac{t}{x}\frac{\partial}{\partial z}, \end{aligned} \tag{88}$$

such that

$$\begin{aligned} X &= X_1 + X_0 + X_{-3} \\ &= Y_1 + Y_0 + Y_{-3}. \end{aligned}$$

Then X_j maps R_d to R_{d+j} and Y_j maps to S_d to S_{d+j}.

5.2.1 Decomposition of an invariant divisor

We see firstly that G in (86) can be written in the form:

$$G = \lambda z + \mu, \qquad \lambda, \mu \in K;$$

in fact, the highest degree of terms of G is at most 1 with respect to both weights.

We rewrite F as a sum of homogeneous polynomials:

$$F = F_m + F_{m-1} + \cdots, \tag{89}$$

with respect to the weight, ω_1. By considering terms of the highest degree in (86), we have the equation:

$$X_1 F_m = \lambda z F_m, \tag{90}$$

where X_1 is given by (88).

It is easy to see that F_m is not divisible by z; hence m is a non-positive even integer. If we put $m = -2n$, n being a non-negative integer, then $F_m = F_{-2n}$ is written in the form:

$$F_{-2n} = x^n F_0, \qquad F_0 \in R_0.$$

We obtain successively

$$\begin{aligned} F_{-2n-1} &= x^n z F_0', \qquad F_0' \in R_0, \\ F_{-2n-2} &= x^{n+1} F_0'', \qquad F_0'' \in R_0, \end{aligned}$$

and so on. Then F can be decomposed as

$$F = x^n F', \qquad F' \in K[x, z];$$

we have another invariant divisor F'. Therefore we can assume in (89) $m = 0$, and put

$$F_0 = \sum a_k L^k, \qquad a_k \in K,$$

where

$$L = xz^2 - 2t;$$

note that $L \in R_0$ and $X_1 L = zL$. It follows from (90) that λ is a non-negative integer and that:

$$F_0 = aL^\lambda, \qquad a \in K. \tag{91}$$

We can assume without a loss of generality $a = 1$.

On the other hand, let

$$F = f_m + f_{m-1} + \cdots$$

be the weighted homogeneous decomposition of F with respect to ω_2. We have the equation:

$$Y_1 f_m = \lambda z f_m. \tag{92}$$

It is easy to show that f_m is not divisible by z, and then m is even. Since f_m is a homogeneous polynomial in x and z^2, we can put

$$f_m = \sum_{2(\alpha+\beta)=m} b_{\alpha,\beta} x^\alpha M^\beta, \qquad b_{\alpha,\beta} \in K,$$

where

$$M = 2x - z^2, \qquad M \in S_2.$$

By taking $Y_1 M = 0$ into consideration, we deduce from (92) that f_m can be written in the form:

$$f_m = bx^\lambda M^\nu, \qquad b \in K, \quad m = 2(\lambda + \nu). \tag{93}$$

Now we claim that $\lambda = \nu$. In fact, with respect to ω_2, the highest degree of monomials contained in L^λ is 4λ. Hence,

$$4\lambda \leq 2(\lambda + \nu). \tag{94}$$

On the other hand, the highest degree of terms of f_m is $-2\lambda + 2\nu$, with respect to ω_1, so that,

$$-2\lambda + 2\nu \leq 0. \tag{95}$$

We deduce from (94)–(95) that $\lambda = \nu$, and then $m = 4\lambda$.

Since both F_0 and f_m contain $x^\lambda z^{2\lambda}$, we can put $a = b = 1$. Therefore we have

$$\begin{aligned} F_0 &= L^\lambda, && (96)\\ f_m &= x^\lambda M^\lambda, \qquad m = 4\lambda, && (97) \end{aligned}$$

for a non-negative integer λ, where

$$L = xz^2 - 2t, \qquad M = 2x - z^2.$$

5.2.2 Condition $(J)'$

To finish the verification of Proposition 18, we compute terms, F_{-1} and f_{m-1}, following to F_0 and f_m, respectively.

We begin with the term, F_{-1}, which is determined by the equation:

$$XF_{-1} + X_0F_0 = \lambda z F_{-1} + \mu F_0.$$

It follows from (96) that

$$XF_{-1} - \lambda z F_{-1} = \mu L^\lambda + 2t\lambda L^{\lambda-1}. \tag{98}$$

By putting

$$F_{-1} = xz \sum_{k=0}^{k_1} b_k L^\lambda, \quad b_k \in K, \quad b_{k_1} \neq 0,$$

and then by computing the left hand side of (98), we obtain

$$\sum_{k=0}^{k_1} (1 + k - \lambda) b_k L^{k+1} + t \sum_{k=0}^{k_1} (1 + 2k - 2\lambda) b_k L^k = \mu L^\lambda + 2t\lambda L^{\lambda-1}. \tag{99}$$

We claim that:

$$\mu = 0, \qquad F_{-1} = -2\lambda xzL^{\lambda-1}.$$

In fact, we have from (99)

$$k_1 + 1 = \lambda, \qquad (1 + k_1 - \lambda)b_{k_1} = \mu,$$

which shows $\mu = 0$, and then we see

$$b_0 = b_1 = \cdots = b_{k_1-1} = 0, \qquad b_{k_1} = b_{\lambda-1} = -2\lambda.$$

F_{-1} contains the term, $x^\lambda z^{2\lambda-1}$, whose degree is

$$4\lambda - 1 = m - 1,$$

with respect to ω_2. So we compute f_{m-1} by means of the equation:

$$Y_1 f_{m-1} + Y_0 f_m = \lambda z f_{m-1} + \mu f_m, \qquad m = 4\lambda.$$

Since $Y_0 f_m = 0$, $\mu = 0$, we have

$$Y_1 f_{m-1} = \lambda z f_{m-1}.$$

Since a solution of the equation (92) is given by (93), we have $f_{m-1} = 0$, and then $\lambda = 0$. It follows that, if $XF = GF$, then $G = 0$ and $F \in K$. We have thus established the proposition.

6. Algebraic Solutions

In the present section, we consider algebraic solutions of $P_{\mathrm{III}'}(D_7)$ and $P_{\mathrm{III}'}(D_8)$. We have shown in the previous section that these equations do not admit a transcendental classical solutions, and so all classical solutions of them are algebraic.

We begin with equation $P_{\mathrm{III}'}(D_8)$, which reduces to an equation of the type $P_{\mathrm{III}'}(D_6)$ through the quadratic transformation given above in Theorem 1 (iii). The classification of algebraic solutions of $P_{\mathrm{III}'}(D_6)$ is known; see [5] and [14]. By means of the classification, we have the

PROPOSITION 21. *$P_{\mathrm{III}'}(D_8)$ has only two rational solutions.*

In fact, it is easy to see that $P_{\mathrm{III}}(\alpha, \beta, 0, 0)$ has the constant solutions:

$$y = \pm\sqrt{-\beta/\alpha},$$

and then $P_{\mathrm{III}'}(-4, 4, 0, 0)$ has algebraic solutions:

$$q = \pm\sqrt{t}.$$

6.1. Algebraic solutions of $P_{\mathrm{III}'}(D_7)$

Let $\mathcal{H}(\alpha_1)$ be the Hamiltonian system with the Hamiltonian

$$H = \frac{1}{t}[q^2p^2 + \alpha_1 pq + tp + q].$$

We study algebraic solutions of $P_{\mathrm{III}'}(D_7)$; it is known that

Proposition 22 ([10, 4]). *If $\alpha_1 = 1$, $\mathcal{H}(1)$ has the algebraic solution*

$$q(t) = -\frac{1}{2}(2t)^{2/3}, \qquad p(t) = \frac{1}{3(2t)^{2/3}} - \frac{1}{(2t)^{1/3}}.$$

$\mathcal{H}(\alpha_1)$ has one and only one algebraic solution if and only if α_1 is an integer.

The rational solution given above has been found for the first time by Lukashevich [10] for $P_{\mathrm{III}'}(D_7)$ and we have rewritten it in terms of the Hamiltonian structure.

We give below a few of algebraic solutions; when $\alpha_1 = 0$, we have

$$(q, p) = \left(-\frac{1}{6}(2t)^{1/3} - \frac{1}{2}(2t)^{2/3}, -\frac{1}{(2t)^{1/3}}\right),$$

and when $\alpha_1 = -1$,

$$(q, p) = \left(-\frac{5(2t)^{\frac{1}{3}} + 24t + 9(2t)^{\frac{4}{3}}}{2\left(1 + 3(2t)^{\frac{1}{3}}\right)^2}, -\frac{1 + 3(2t)^{\frac{1}{3}}}{3(2t)^{\frac{2}{3}}}\right).$$

Starting from the algebraic solution of $\mathcal{H}(0)$, we can obtain that of $\mathcal{H}(-n)$ by means of the transformation, π^n; see Section 4.2.

The aim of this subsection is to determine τ-functions related to algebraic solutions. When $\alpha_1 = 1$, the Hamiltonian function associated with the algebraic solutions given by

$$tH = -\frac{3}{4}(2t)^{\frac{2}{3}} + \frac{1}{2}(2t)^{\frac{1}{3}} - \frac{5}{36}.$$

Then the τ-function is determined up to a multiplicative constant, c, as:

$$\tau = c \exp\left(-\frac{9}{8}(2t)^{\frac{2}{3}} + \frac{2}{3}(2t)^{\frac{1}{3}}\right) t^{-\frac{5}{36}}.$$

Moreover, when $\alpha_1 = 0$, we obtain

$$tH = -\frac{3}{4}(2t)^{\frac{2}{3}} + \frac{1}{36}$$

and then

$$\tau = c' \exp\left(-\frac{9}{8}(2t)^{\frac{2}{3}}\right) t^{\frac{1}{36}},$$

c' being a non zero constant. We put

$$s = 3(2t)^{\frac{1}{3}},$$

and let τ_n be the τ-function related to the algebraic solution of $\mathcal{H}(-n)$. The τ-function satisfy Toda equation (61); in what follows, we consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} t \frac{\mathrm{d}}{\mathrm{d}t} \log \tau_n = -3 \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}s} s \frac{\mathrm{d}}{\mathrm{d}s} \log \tau_n = -\frac{1}{2} s^2 \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2}. \tag{100}$$

Moreover we put

$$\begin{aligned} \tau_{-1} &= \exp\left(-\frac{1}{8}s^2 + \frac{1}{2}s\right) s^{-\frac{5}{12}}, \\ \tau_0 &= \exp\left(-\frac{1}{8}s^2\right) s^{\frac{1}{12}}. \end{aligned} \tag{101}$$

Remark 4. Let (q_0, p_0) be the algebraic solution of $\mathcal{H}(0)$ and H_0 the Hamiltonian function associated to it. By means of the canonical transformation given by (57), the Hamiltonian function at $\alpha_1 = -1$ is given by

$$tH = tH_0 - q_0 p_0 = -\frac{3}{4}(2t)^{\frac{2}{3}} - \frac{1}{2}(2t)^{\frac{1}{3}} - \frac{5}{36}.$$

Hence we obtain

$$\tau_1 = c'' \exp\left(-\frac{9}{8}(2t)^{\frac{2}{3}} - \frac{3}{2}(2t)^{\frac{1}{3}}\right) t^{-\frac{5}{36}},$$

c'' being a non zero constant. On the other hand, it follows from (100) for $n = 0$ and (101) that:

$$\tau_1 = \exp\left(-\frac{1}{8}s^2 - \frac{1}{2}s\right) s^{-\frac{5}{12}}.$$

The τ-function, $\tau_n = \tau_n(s)$, can be determined by means of (100). In fact we have the

Theorem 23. *The τ-functions related to the algebraic solutions are of the form:*

$$\tau_n = \exp\left(-\frac{1}{8}s^2 - \frac{1}{2}ns\right) s^{-d_n/12} S_n(s),$$

where $S_n(s)$ are monic polynomials in s with integral coefficients, such that $S_n(0) \neq 0$. Here

$$d_n = \begin{cases} 9n^2 - 1 & n \text{ is even}, \\ 9n^2 - 4 & n \text{ is odd}. \end{cases} \tag{102}$$

Remark 5. $S_n(s)$ satisfy the equation:

$$(s+n)S_n(s)^2 - 2sS_n(s)S_n(s)'' + 2sS_n'(s)^2 - 2S_n(s)S_n'(s) = \begin{cases} sS_{n-1}(s)S_{n+1}(s) & \text{n is even}, \\ S_{n-1}(s)S_{n+1}(s) & \text{n is odd}. \end{cases} \tag{103}$$

by virtue of Toda equation (100).

We give below the list of $S_n(s)$ for $n = 0, 1, 2, 3, 4, 5$:

$$S_{-3} = s^2 - 4s + 5,$$

$$S_{-2} = s - 1,$$

$$S_{-1} = 1,$$

$$S_0(s) = 1,$$

$$S_1(s) = 1,$$

$$S_2(s) = s + 1,$$

$$S_3(s) = s^2 + 4s + 5,$$

$$S_4(s) = s^4 + 10s^3 + 40s^2 + 70s + 35,$$

$$S_5(s) = s^6 + 20s^5 + 175s^4 + 840s^3 + 2275s^2 + 3220s + 1925.$$

Note that, for $n \geq 2$, $S_n(s)$ admit only simple zeros, by means of Theorem 12. By comparing $S_n(s)$ with Yablonskii-Vorob'ev polynomials, which appear in the case of the second Painlevé equations, we might expect that $S_n(s)$ could be written in terms of the Schur functions; cf. [9]. This problem remains unsettled.

6.2. Proof of Theorem 23

For any integer n, $S_n(s)$ is determined in a unique way by means of (103) and the initial condition:

$$S_0(s) = S_1(s) = 1.$$

Moreover it is easy to see:

Lemma 24. *Equation* (103) *admits a symmetry of the form:*

$$S_{-n}(s) = (-1)^{e_n} S_n(-s),$$

where

$$e_n = \begin{cases} \frac{1}{4}n^2 & n \text{ is even}, \\ \frac{1}{4}(n^2 - 1) & n \text{ is odd}. \end{cases} \tag{104}$$

Note that $d_{-n} = d_n$, $e_{-n} = e_n$. By agreeing that $S_n(s)$ are polynomials, we can show

$$e_n = \deg S_n(s).$$

By virtue of Lemma 24 it is sufficient for verification of the theorem to consider only the case when n is non-negative. Assume that

$$S_k(s) \qquad (0 \leq k \leq n)$$

satisfy the following condition, as a function of s:

$(P)_k$ $S_k(s)$ is a monic polynomial with integral coefficients and $S_k(0)$ is an odd integer.

We see that $S_k(s)$ $(0 \leq k \leq 5)$ satisfy $(P)_k$ and we establish the theorem by showing that $S_{n+1}(s)$ also fulfills $(P)_{n+1}$.

We begin with considering the case when n is an odd integer. Let us denote by $T_n(s)$ the left hand side of (103). By virtue of the assumption of induction, $T_n(s)$ is a monic polynomial with integral coefficients, so is

$$T_n(0) = nS_n(0)^2 - 2S_n(0)S_n'(0);$$

in particular, $T_n(0) \neq 0$. We determine $S_{n+1}(s)$ by

$$T_n(s) = S_{n-1}(s)S_{n+1}(s); \tag{105}$$

$S_{n+1}(s)$ is holomorphic at $s = 0$ and may be rational in s. On the other hand, a τ-function has to be holomorphic at any point $s = s_0$, such that $s_0 \neq 0$. It follows that $S_{n+1}(s)$ is a polynomial; $T_n(s)$ can be divided by $S_{n-1}(s)$. Since $S_{n-1}(s)$ and $T_n(s)$ are monic polynomials with integral coefficients, so is $S_{n+1}(s)$. Moreover, both $T_n(0)$ and $S_{n-1}(0)$ are odd integers, and then so is $S_{n+1}(0)$. We have thus arrived at $(P)_{n+1}$.

We proceed to study the case when n is even. Recurrence formula (103) reads as follows:

$$\begin{aligned} S_n(s)^2 - 2S_n(s)S_n(s)'' + 2S_n'(s)^2 - 2S_n(s)\frac{nS_n(s) - 2S_n'(s)}{s} \\ = S_{n-1}(s)S_{n+1}(s). \end{aligned} \tag{106}$$

Let us denote by $U_n(s)$ the left hand side of (106), and we can show that $U_n(s)$ is a monic polynomial with integral coefficients. In fact, we have the

LEMMA 25. *If n is even, then*

$$nS_n(0) - 2S_n'(0) = 0.$$

PROOF. Consider the auxiliary function:

$$h = tH_n + \frac{n^2}{4}, \qquad H_n = \frac{\mathrm{d}}{\mathrm{dt}} \log \tau_n, \tag{107}$$

which satisfies the differential equation:

$$\left(t\frac{\mathrm{d}^2h}{\mathrm{d}t^2}\right)^2 + 4\left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2\left(t\frac{\mathrm{d}h}{\mathrm{d}t} - h\right) + 2n\frac{\mathrm{d}h}{\mathrm{d}t} - 1 = 0;$$

see Section 4.2. This equation can be written as:

$$\left(s\frac{\mathrm{d}^2h}{\mathrm{d}s^2} - 2\frac{\mathrm{d}h}{\mathrm{d}s}\right)^2 + 12\left(\frac{\mathrm{d}h}{\mathrm{d}s}\right)^2\left(s\frac{\mathrm{d}h}{\mathrm{d}s} - 3h\right) + ns^2\frac{\mathrm{d}h}{\mathrm{d}s} - \frac{1}{36}s^4 = 0, \tag{108}$$

with respect to $s = 3(2t)^{1/3}$. On the other hand, we deduce from (107) with

$$\tau_n = \exp\left(-\frac{1}{8}s^2 - \frac{1}{2}ns\right) s^{-\frac{1}{12}(9n^2-1)} S_n(s),$$

the following expression:

$$h = \frac{1}{36} + Bs + O(s^2),$$

where $O(\cdot)$ is the Landau Symbol and

$$B = \frac{2S_n'(0) - nS_n(0)}{6S_n(0)}.$$

Then, by putting $s = 0$ in (108), we obtain $B = 0$, which establishes the lemma. $\square$

Therefore we can deduce from

$$U_n(s) = S_{n-1}(s)S_{n+1}(s) \tag{109}$$

that $S_{n+1}(s)$ is a monic polynomial with integral coefficients, in a way similar to the case studied just above. Since n is even and $S_n(0)$ is odd,

$$U_n(0) = S_n(0)^2 - 2S_n(0)S_n''(0) + 2S_n'(0)^2 + (nS_n'(0) - 2S_n''(0))S_n(0)$$

is an odd integer; in particular, $U_n(0) \neq 0$. It follows that $S_{n+1}(0)$ is an odd integer. We have thus finished up the verification of the theorem.

7. Space of Initial Conditions

As is mentioned in Introduction, it is quite natural to distinguish the three types of the third Painlevé equations from a point of view of geometrical studies on the equations; cf [22]. By virtue of the Painlevé property, to construct a space of initial conditions, we have only to determine a compact space $\bar{X}$ and a set $\{D_j\}$ of subvarieties of $\bar{X}$, satisfying the following properties:

(i) $D_j \cong \mathbb{P}^1$ and D_j contains a leaf of the foliation entirely;

(ii) $D_j \cup D_k (j \neq k)$ defines a singularity of the first class of the foliation associated with the equation;

(iii) $D_j \cap D_k \cap D_l = \emptyset, \qquad (j,\ k,\ l$ are distinct).

Given such a space $\bar{X}$, then $X = \bar{X} - \cup_j D_j$ is a space of initial conditions. In fact, by constructing $\bar{X}$ and $\{D_j\}$, we obtain the fiber space $\mathcal{P}$ and the foliation $\mathcal{F}$, with properties, a, b and c, stated in Introduction. And we have

$$X \cong \pi^{-1}(t_0),$$

for any point $t_0 \in B$; cf [16]. We call each D_j a vertical leaf. A singular point of the first class is defined as a singular point which does not belong to the closure of any leaf except vertical one. Provided that the Painlevé property would be established, a singular point $b = \{(y, z) = (0, 0)\}$ of the following equation is of the first class:

$$zy' = \lambda + f(t, y, z), \qquad yz' = -\mu + g(t, y, z). \tag{110}$$

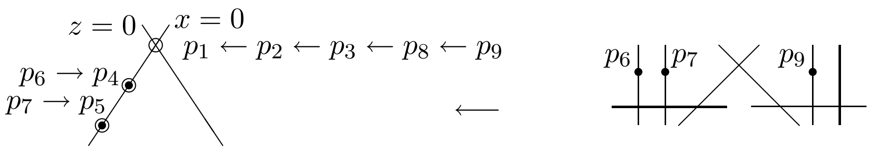
Here we assume that λ, $\mu > 0$ and f, g are holomorphic near b with $f(t,0,0) \equiv 0$, $g(t,0,0) \equiv 0$.

7.1. $D_7^{(1)}$-surface and $D_8^{(1)}$-surface

Such a compact space $\bar{X}$ for the third Painlevé equations of type $D_i^{(1)}$ is called $D_i^{(1)}$-surface in [22] ($i = 6, 7, 8$), in fact this surface contains vertical leaves $\{D_j\}$, whose intersection form is expressed by the Cartan matrix of $D_i^{(1)}$ type. A $D_6^{(1)}$-surface has been constructed from the Hirzeburch surface, $\Sigma_{(\epsilon)}^{(2)}$, by 8 points blowing-ups in [16], and now we begin with $\mathbb{P}^2$. Note that, since $\Sigma_{(\epsilon)}^{(2)}$ and $\mathbb{P}^2$ are birational each other, $D_i^{(1)}$-surface is determined by this way.

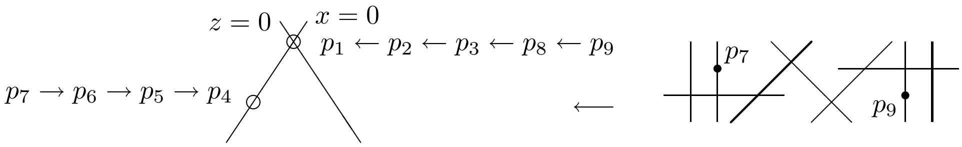
If we regard $\bar{X}$ as a blowing-up surface of $\mathbb{P}^2$ centered at 9 points, we have only to write down the positions of the 9 points, including infinitely near points, in $\mathbb{P}^2$ to describe $\bar{X}$. We can express $D_7^{(1)}$-surface and $D_8^{(1)}$-surface as follows:

$D_7^{(1)}$-surface:



$$
\begin{aligned}
p_1 :\ (0:1:0) \quad &\leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_3 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,0) \leftarrow \\
&\leftarrow p_8 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,-2t) \leftarrow \\
&\leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^2 z + 2tx^3)}{x^4}\right) = (0, 4\alpha_0 t), \\
p_4 :\ (0:0:1) \quad &\leftarrow p_6 : \left(\frac{x}{z}, \frac{y}{x}\right) = (0,0), \\
p_5 :\ (0:1:1) \quad &\leftarrow p_7 : \left(\frac{x}{z}, \frac{y-z}{x}\right) = (0, 2\alpha_1).
\end{aligned}
$$

$D_8^{(1)}$-surface:



$$
\begin{aligned}
p_1 :\ (0:1:0) \quad & \leftarrow p_2 : \left(\frac{x}{y}, \frac{z}{x}\right) = (0,0) \leftarrow p_3 : \left(\frac{x}{y}, \frac{yz}{x^2}\right) = (0,0) \leftarrow \\
& \leftarrow p_8 : \left(\frac{x}{y}, \frac{y^2 z}{x^3}\right) = (0,4t) \leftarrow \\
& \leftarrow p_9 : \left(\frac{x}{y}, \frac{y(y^2 z - s x^3)}{x^4}\right) = (0,8t), \\
p_4 :\ (0:0:1) \quad & \leftarrow p_5 : \left(\frac{y}{z}, \frac{x}{y}\right) = (0,0) \leftarrow p_6 : \left(\frac{y}{z}, \frac{zx}{y^2}\right) = (0,1) \leftarrow \\
& \leftarrow p_7 : \left(\frac{y}{z}, \frac{z(zx - y^2)}{y^3}\right) = (0,0).
\end{aligned}
$$

Here $(x:y:z)$ denotes a homogeneous coordinate of $\mathbb{P}^2$ and p_k $(k = 1, \ldots 9)$ are the 9 points in $\mathbb{P}^2$; $p_l \leftarrow p_k$ signifies that p_k is infinitely near p_l.

Each vertical leaf is represented by the positive divisor D_j, which represents a divisor class $\mathcal{D}_j \in \mathrm{Pic}(\bar{X})$; here we give the table of $\mathcal{D}_j$:

$D_7^{(1)}$-surface:

$$
\begin{aligned}
&\mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_3 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_3, \\
&\mathcal{D}_4 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_4 - \mathcal{E}_6, \quad \mathcal{D}_7 = \mathcal{E}_5 - \mathcal{E}_7, \\
&\mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9,
\end{aligned}
$$

$D_8^{(1)}$-surface:

$$
\begin{aligned}
&\mathcal{D}_1 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3, \quad \mathcal{D}_2 = \mathcal{E}_3 - \mathcal{E}_8, \quad \mathcal{D}_3 = \mathcal{E}_2 - \mathcal{E}_3, \\
&\mathcal{D}_4 = \mathcal{E}_1 - \mathcal{E}_2, \quad \mathcal{D}_5 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_4 - \mathcal{E}_5, \quad \mathcal{D}_6 = \mathcal{E}_5 - \mathcal{E}_6, \quad \mathcal{D}_7 = \mathcal{E}_4 - \mathcal{E}_5, \\
&\mathcal{D}_8 = \mathcal{E}_6 - \mathcal{E}_7, \quad \mathcal{D}_0 = \mathcal{E}_8 - \mathcal{E}_9.
\end{aligned}
$$

Here $\mathcal{E}_0$ is the class of total transform of a line in $\mathbb{P}^2$ and $\mathcal{E}_k$ is that of the closed point p_k $(k = 1, \ldots, 9)$. These surfaces and vertical leaves satisfy property (iii) given above.

7.2. Foliation associated with $P_{\mathrm{III}'}(D_7)$

We define a foliation induced from the each Painlevé equation on this surface and show that properties (i) and (ii) are established. We begin with the Painlevé equation of type $D_7^{(1)}$.

For canonical variables, q, p of the Hamiltonian system, we take a homogeneous coordinates as $(x : y : z) = \left(\frac{p}{2} : -qp^2 : 1\right)$. We have then the following differential equations

$$tx_1' = y_1 - \alpha_1 x_1 - \frac{1}{2}, \qquad ty_1' = -\frac{ty_1(y_1 - 1)}{2x_1} + tx_1^2 + \alpha_1 y_1, \tag{111}$$

$$tx_2' = -\frac{t}{y_2} - \frac{1}{2}x_2 y_2, \qquad ty_2' = y_2\left(\frac{1}{2}y_2 - x_2 + \alpha_1\right), \tag{112}$$

$$tx_3' = \frac{x_3(x_3 - 1)}{y_3} + ty_3^2 + \alpha_1 x_3, \qquad ty_3' = \frac{ty_3^3}{x_3} + \frac{1}{2}x_3. \tag{113}$$

with respect to the coordinates $(x_1, y_1) = \left(\frac{x}{z}, \frac{y}{z}\right)$, $(x_2, y_2) = \left(\frac{y}{x}, \frac{z}{x}\right)$, $(x_3, y_3) = \left(\frac{z}{y}, \frac{x}{y}\right)$.

In $\mathbb{P}^2$ there are vertical leaves, $D_1 = \{z = 0\}$ and $D_5 = \{x = 0\}$, and singular points:

$$p_1 = \{(x_3, y_3) = (0,0)\}, p_4 = \{(x_1, y_1) = (0,0)\},$$
$$p_5 = \{(x_1, y_1) = (0,1)\} = \{(x_3, y_3) = (1,0)\}.$$

By blowing up successively p_1, p_4 and p_5, we will obtain resolution of these singularities. Since properties, (i) and (ii), are fulfilled apart from these singular points, we have only to study the foliation around an exceptional divisor, obtained by a blowing-up process at a singular point.

Put

$$(f, g) = (x_1, y_1) = \left(\frac{x}{z}, \frac{y}{z}\right), \qquad G = \frac{y_1}{x_1}, \quad F = \frac{x_1}{y_1},$$

then we have the system:

$$tf' = fG - \alpha_1 f - \frac{1}{2}, \qquad tG' = -\frac{G}{2f} - tf, \tag{114}$$

$$tF' = \frac{1}{2g} + tgF^3, \qquad tg' = \frac{g-1}{2F} + \alpha_1 g - tg^2F^2. \tag{115}$$

This system appears as result of a blowing-up at the point p_4 and we see easily that the total transform of closed point p_4 consists of the vertical leaf, D_6, and the singular points:

$$p_6 = \{(f,G) = (0,0)\}, \qquad b_{5,6} = \{(F,g) = (0,0)\}.$$

The latter is of the first class; the system can be reduced to a system of the form (110) by a suitable change of variables.

Blowing up the point p_6, we obtain

$$tf' = f^2G - \alpha_1 f - \frac{1}{2}, \qquad tG' = -fG^2 + \alpha_1 G - t, \tag{116}$$

$$tF' = F\left(g + tF - \alpha_1\right), \qquad tg' = \frac{1}{2F} - tgF, \tag{117}$$

where

$$(f,g) = \left(\frac{x}{z}, \frac{y}{x}\right), \quad G = \frac{yz}{x^2}, \quad F = \frac{x^2}{yz}.$$

The foliation defined by the system (116) is transversal to the fiber, and the total transform of p_6 does never give a vertical leaf. The resolution of singularity, p_6, is thus finished up.

We proceed to the blowing-up at singular point p_5; we have

$$tf' = fG - \alpha_1 f + \frac{1}{2}, \qquad tG' = \frac{G - 2\alpha_1}{2f} - tf, \tag{118}$$

$$tF' = \frac{2\alpha_1 F - 1}{2g} + tgF^3, \qquad tg' = \frac{g+1}{F} - \alpha_1(g+1) + tg^2F^2, \tag{119}$$

where

$$(f,g) = (x_1, y_1 - 1) = \left(\frac{x}{z}, \frac{y-z}{z}\right),$$

$$G = \frac{y_1 - 1}{x_1} = \frac{y-z}{x}, \quad F = \frac{x_1}{y_1 - 1} = \frac{x}{y-z}.$$

The total transform consists of vertical leaf D_7 and singular points,

$$p_7 = \{(f,G) = (0, 2\alpha_1)\} = \{(F,g) = (1/2\alpha_1, 0)\},$$
$$b_{5,7} = \{(F,g) = (0,0)\}.$$

The latter is of the first class.

To resolve singular point p_7, we put

$$(f,g)=\left(\frac{x}{z},\frac{y-z-2\alpha_1 x}{x}\right),$$
$$G=\frac{z(y-z-2\alpha_1 x)}{x^2},\quad F=\frac{x^2}{z(y-z-2\alpha_1 x)}.$$

It follows that

$$tf'=f^2G+\alpha_1 f+\frac{1}{2},\qquad tG'=-fG^2-\alpha_1 G-2, \tag{120}$$
$$tF'=F\left(g+tF+\alpha_1\right),\qquad tg'=\frac{1}{2F}-tgF. \tag{121}$$

We next blow up point p_1; we have

$$tf'=\frac{f-1}{G}+\alpha_1 x+tf^2G^2,\qquad tG'=\frac{1}{f}-\alpha_1 G-\frac{1}{2}, \tag{122}$$
$$tF'=-\frac{F}{f}+\frac{1}{2}F\left(F+2\alpha_1\right),\qquad tg'=\frac{g^2}{F}+\frac{1}{2}gF, \tag{123}$$

with

$$(f,g)=(x_3,y_3)=\left(\frac{z}{y},\frac{x}{y}\right),\quad G=\frac{x}{z},\quad F=\frac{z}{x}.$$

The total transform of p_1 defines vertical leaf D_4, and singular points $p_2=\{(F,g)=(0,0)\}$ and $b_{4,5}=\{(f,G)=(0,0)\}$.

Since $b_{4,5}$ is of the first class, we blow up p_2; putting

$$(f,g)=\left(\frac{z}{x},\frac{x}{y}\right),\quad G=\frac{x^2}{yz},\quad F=\frac{yz}{x^2}.$$

we obtain

$$tf'=-\frac{1}{G}+f\left(\frac{1}{2}f+\alpha_1\right),\qquad tG'=\frac{1}{f}+G(tG-\alpha_1), \tag{124}$$
$$tF'=-\frac{F}{g}+\frac{\alpha_1 t}{2}F-t,\qquad tg'=\frac{2g}{F}+\frac{1}{2}g^2F, \tag{125}$$

We have again vertical leaf D_3 and two singular points, $p_3=\{(F,g)=(0,0)\}$ and $b_{3,4}=\{(f,G)=(0,0)\}$.

By blowing up p_3, we obtain the following system:

$$tf' = -\frac{1}{G} + \alpha_1 f - t, \qquad tG' = \frac{1+2tG}{f} + G\left(\frac{1}{2}f^2 - \alpha_1\right), \tag{126}$$

$$tF' = -\frac{F+2t}{f} - \left(\frac{1}{2}g^2 F - \alpha_1\right), \qquad tg' = \frac{t}{F} + \frac{1}{2}g^2F^2, \tag{127}$$

where

$$(f,g) = \left(\frac{yz}{x^2}, \frac{x}{y}\right), \quad G = \frac{x^3}{y^2 z}, \quad F = \frac{y^2 z}{x^3}.$$

The total transform consists of vertical leaf D_2 and singular points $p_8 = \{(F,g) = (-2t, 0)\} = \{(f,G) = (0, -1/2t)\}$, $b_{2,3} = \{(f,G) = (0,0)\}$ and $b_{1,2} = \{(F,g) = (0,0)\}$. Singular points $b_{2,3}$ and $b_{1,2}$ as well as $b_{3,4}$ are of the first class.

When we blow up p_8, we arrive at

$$\begin{aligned} tf' &= -\frac{1}{G} - 2t^2 f^2 G^2 + 2tf^3G^2 + 2t\alpha_0 - \frac{1}{2}f^4G^2 + \alpha_1 f, \\ tG' &= \frac{1-4\alpha_0 tG}{2f} + \frac{1}{2(f-2t)} \\ &\quad + G\left(2t^2 fG^2 - 3tf^2G^2 + f^3G^2 - \alpha_1\right), \end{aligned} \tag{128}$$

$$\begin{aligned} tF' &= -\frac{F-4\alpha_0 t}{2g} - \frac{F^2}{2(gF-2t)} \\ &\quad - g^2\left(2t^2 g - 3tg^2F + g^3F^2 - \alpha_1 F\right), \\ tg' &= \frac{2t}{gF-2t} - g^3\left(\frac{1}{2}yF - t\right), \end{aligned} \tag{129}$$

with

$$(f,g) = \left(\frac{y^2z + 2tx^3}{x^3}, \frac{x}{y}\right), \quad G = \frac{x^4}{y(y^2z+2tx^3)}, \quad F = \frac{y(y^2z+2tx^3)}{x^4}.$$

The total transform yields vertical leaf D_0, a singular point, $p_9 = \{(F,g) = (4\alpha_0 t, 0)\} = \{(f,G) = (0, -1/4\alpha_0 t)\}$, and another one, $b_{0,2} = \{(f,G) = (0,0)\}$, of the first class.

We can resolve singular point p_9 as follows:

$$\begin{aligned}
tf' &= -\frac{1}{2G} - \frac{(f+4\alpha_0 t)^2}{2(f^2G+4\alpha_0 tfG-2t)} - \\
&\qquad - \frac{1}{2}f^5G^3 - 4\alpha_0 tf^4G^3 - 8\alpha_0^2t^2f^3G^3 + \\
&\qquad + \frac{3}{2}tf^3G^2 + 6t^2\alpha_0 f^2G^2 + \frac{1}{2}\alpha_1 f + 2\alpha_0^2 t, \\
tG' &= \frac{G(4\alpha_0^2 tfG + 16\alpha_0^3t^2G + f + 6\alpha_0 t)}{2(f^2G+4\alpha_0 tfG-2t)} + \frac{3}{2}f^4G^4 + 10\alpha_0 tf^3G^4 + \\
&\quad - 16\alpha_0^2t^2f^2G^4 - 4tf^2G^3 - 12\alpha_0^2t^2fG^3 + 2t^2G^2 - \alpha_1 G,
\end{aligned} \tag{130}$$

$$\begin{aligned}
tF' &= -\frac{gF^2 + 4\alpha_0^2 tgF + 6t\alpha_0 F + 16\alpha_0^3t^2}{2(g^2F - 4\alpha_0 tg - 2t)} + \\
&\qquad - \frac{3}{2}g^3F^4 - 10t\alpha_0 g^3F^3 - 16\alpha_0^2t^2g^3F^2 + 4tg^2F^2 + \\
&\qquad + 12\alpha_0 t^2g^2F - 4t^2g + \alpha_1, \\
tg' &= \frac{2t}{g^2F - 4\alpha_0 tg - 2t} - g^3\left(\frac{1}{2}g^2F - 2\alpha_0 tg - t\right),
\end{aligned} \tag{131}$$

where

$$\begin{aligned}
(f,g) &= \left(\frac{y(y^2z+2tx^3) - 4\alpha_0 tx^4}{x^4}, \frac{x}{y}\right), \\
G &= \frac{x^5}{y(y(y^2z+2tx^3) - 4\alpha_0 tx^4)}, \quad F = \frac{y(y(y^2z+2tx^3) - 4\alpha_0 tx^4)}{x^5}.
\end{aligned}$$

In conclusion, we obtained the foliation satisfying (i), (ii) and (iii), defined by the Painlevé equation of $D_7^{(1)}$ type. The vertical leaves are $D_0, \ldots, D_7$ with singular points of first class $b_{j,k} = D_j \cap D_k$ $((jk) = (02), (12), (23), (34), (45), (56), (57))$ and all of $b_{j,k}$.

7.3. Space of initial conditions for $P_{\mathrm{III}'}(D_8)$

We determine foliation on $D_8^{(1)}$-surface in a way similar to the case of $D_7^{(1)}$-surface. If we take the homogeneous coordinates:

$$(x:y:z) = (1 : 2qp+1 : 2q),$$

the differential equations read as follows:

$$tx_1' = -y_1, \qquad ty_1' = -\frac{y_1^2}{x_1} - t2x_1^2 + \frac{1}{2}, \tag{132}$$

$$tx_2' = -\frac{2t}{y_2} + \frac{1}{2}y_2, \qquad ty_2' = x_2y_2, \tag{133}$$

$$tx_3' = \frac{x_3}{y_3} + 2ty_3^2 - \frac{1}{2}x_3^2, \qquad ty_3' = \frac{2ty_3^3}{x_3} - \frac{1}{2}x_3y_3, \tag{134}$$

where $(x_1, y_1) = \left(\frac{x}{z}, \frac{y}{z}\right)$, $(x_2, y_2) = \left(\frac{y}{x}, \frac{z}{x}\right)$, $(x_3, y_3) = \left(\frac{z}{y}, \frac{x}{y}\right)$.

In $\mathbb{P}^2$ we have vertical leaves defined by $D_1 = \{z = 0\}$ and $D_5 = \{x = 0\}$ and singular points, $p_1 = \{(x_3, y_3) = (0, 0)\}$ and $p_4 = \{(x_1, y_1) = (0, 0)\}$. By blowing up p_1 and p_4, we arrive at resolution of these singularities.

We can pursue the process of blowing-up's, obtaining vertical leaves and singular points. In what follows, we give only results of computations.

By bowing-up at p_4, we have:

$$tf' = -fG, \qquad tG' = \frac{1}{2f} - 2tf, \tag{135}$$

$$tF' = -\frac{F}{2g} + 2tgF^3, \qquad tg' = -\frac{g}{2F} - tg^2F^2 + \frac{1}{2}, \tag{136}$$

with

$$(f, g) = (x_1, y_1) = \left(\frac{x}{z}, \frac{y}{z}\right), \quad G = \frac{y_1}{x_1} = \frac{y}{x}, \quad F = \frac{x_1}{y_1} = \frac{x}{y},$$

$$\begin{aligned} \text{vertical leaf} \quad &: \quad D_7 = \{g = 0\}, \\ \text{singular point} \quad &: \quad p_5 = \{(F, g) = (0, 0)\}. \end{aligned}$$

By blowing-up at p_5, we have:

$$tf' = -\frac{1}{2G} + 2tf^2G^2, \qquad tG' = \frac{1 - G}{f} - 4tf^3G^2, \tag{137}$$

$$tF' = \frac{1 - F}{g} + 4tg^3F^3, \qquad tg' = -\frac{1}{2F} - 2tg^4F^2 + \frac{1}{2}, \tag{138}$$

with

$$(f, g) = \left(\frac{x}{y}, \frac{y}{z}\right), \quad G = \frac{y^2}{zx}, \quad F = \frac{zx}{y^2},$$

$$\begin{aligned} \text{vertical leaf} \quad &: \quad D_6 = \{g = 0\}, \\ \text{singular points} \quad &: \quad p_6 = \{(F, g) = (1, 0)\} = \{(f, G) = (0, 1)\}, \\ & \quad\;\; b_{5,6} = \{(F, g) = (0, 0)\}, \quad b_{6,7} = \{(G, f) = (0, 0)\}. \end{aligned}$$

By blowing-up at p_6, we have:

$$\begin{aligned} tf' &= -\frac{1}{G} + 4tf^3(f+1)^3G^3, \\ tG' &= \frac{1}{2f} + \frac{1}{f+1} - 2tf^2(3f^3 + 8f^2 + 7f + 2)G^4, \end{aligned} \tag{139}$$

$$\begin{aligned} tF' &= -\frac{F}{2g} - \frac{F^2}{gF+1} + 2tg^2(3g^3F^3 + 8g^2F^2 + 7gF + 2), \\ tg' &= -\frac{1}{gF+1} - 2tg^6F^2 - 4tg^5F - 2tg^4 + \frac{1}{2}, \end{aligned} \tag{140}$$

with

$$(f, g) = \left(\frac{zx - y^2}{y^2}, \frac{y}{z}\right), \quad G = \frac{y^3}{z(zx - y^2)}, \quad F = \frac{z(zx - y^2)}{y^3},$$

$$\begin{aligned} \text{vertical leaf} \quad &: \quad D_8 = \{g = 0\}, \\ \text{singular points} \quad &: \quad p_7 = \{(F, g) = (0, 0)\}, \quad b_{6,8} = \{(f, G) = (0, 0)\}. \end{aligned}$$

Definitively by resolution of singularity, p_7, we arrive at:

$$\begin{aligned} tf' &= -\frac{1}{2G} - \frac{f^2}{f^2G+1} + 2tf^2G^2(3f^6G^3 + 8f^4G^2 + 7f^2G + 2), \\ tG' &= \frac{2fG}{f^2G+1} - 4tfG^3(2f^6G^3 + 5f^4G^2 + 4f^2G + 1), \end{aligned} \tag{141}$$

$$\begin{aligned} tF' &= -\frac{gF^2}{g^2F+1} + 4tg(2g^6F^3 + 5g^4F^2 + 4g^2F + 1), \\ tg' &= -\frac{1}{g^2F+1} - 2tg^8F^2 - 4tg^6F - 2tg^4 + \frac{1}{2}, \end{aligned} \tag{142}$$

with

$$(f, g) = \left(\frac{z(zx - y^2)}{y^3}, \frac{y}{z}\right), \quad G = \frac{y^4}{z^2(zx - y^2)}, \quad F = \frac{z^2(zx - y^2)}{y^4}.$$

We have thus separated all leaves passing through singular point p_4. Singular points, $b_{5,6}$, $b_{6,7}$, $b_{6,8}$ are of the first class. We proceed now to what concerns singular point p_1.

By blowing-up at p_1, we have:

$$tf' = -\frac{1}{G} + 2tf^2G^2 - \frac{t}{2}f^2, \qquad tG' = \frac{1}{f}, \tag{143}$$

$$tF' = \frac{F}{g}, \qquad tg' = \frac{2tg^2}{F} - \frac{1}{2}g^2F, \tag{144}$$

with

$$(f,g) = (x_3, y_3) = \left(\frac{z}{y}, \frac{x}{y}\right), \quad G = \frac{y_3}{x_3} = \frac{x}{z}, \quad F = \frac{x_3}{y_3} = \frac{z}{x},$$

$$\begin{aligned} \text{vertical leaf} \quad &: \quad D_4 = \{g = 0\}, \\ \text{singular points} \quad &: \quad p_2 = \{(F,g) = (0,0)\}, \quad b_{4,5} = \{(f,G) = (0,0)\}. \end{aligned}$$

By blowing-up at p_2, we have:

$$tf' = \frac{1}{G}, \qquad tG' = -\frac{1}{f} + 2tG^2 - \frac{1}{2}f^2G^2, \tag{145}$$

$$tF' = \frac{F}{g} + \frac{1}{2}g^2F^2 - 2t, \qquad tg' = \frac{2tg}{F} - \frac{1}{2}g^3F, \tag{146}$$

with

$$(f,g) = \left(\frac{z}{x}, \frac{x}{y}\right), \quad G = \frac{x^2}{yz}, \quad F = \frac{yz}{x^2},$$

$$\begin{aligned} \text{vertical leaf} \quad &: \quad D_3 = \{g = 0\}, \\ \text{singular points} \quad &: \quad p_3 = \{(F,g) = (0,0)\}, \quad b_{3,4} = \{(f,G) = (0,0)\}. \end{aligned}$$

By blowing-up at p_3, we have:

$$tf' = \frac{1}{G} + \frac{1}{2}f^4G^2 - 2t, \qquad tG' = \frac{4tG - 1}{f} - f^3G^3, \tag{147}$$

$$tF' = \frac{F - 4t}{g} - g^3F^3, \qquad tg' = \frac{2t}{F} - \frac{1}{2}g^4F, \tag{148}$$

with

$$(f,g)=\left(\frac{yz}{x^2},\frac{x}{y}\right),\quad G=\frac{x^3}{y^2z},\quad F=\frac{y^2z}{x^3},$$

$$\begin{aligned}\text{vertical leaf} &: \quad D_2=\{g=0\},\\ \text{singular points} &: \quad p_8=\{(F,g)=(4t,0)\}=\{(f,G)=(0,1/4t)\},\\ &\quad\ \ b_{2,3}=\{(f,G)=(0,0)\},\quad b_{1,2}=\{(F,g)=(0,0)\}.\end{aligned}$$

By blowing-up at p_8, we have:

$$\begin{aligned}tf' &= \frac{1}{G}+16t^2f^3G^3+8tf^4G^3+f^5G^3-4t,\\ tG' &= \frac{8tG-1}{2f}-\frac{1}{2(f+4t)}-f^2G^4\left(\frac{3}{2}f^2+10tf+16t^2\right),\end{aligned}\tag{149}$$

$$\begin{aligned}tF' &= \frac{F-8t}{2g}+\frac{F^2}{2(gF+4t)}+g^2\left(\frac{3}{2}g^2F^2+10tgF+16t^2\right),\\ tg' &= \frac{2t}{gF+4t}-g^4\left(\frac{1}{2}yF+2t\right),\end{aligned}\tag{150}$$

with

$$(f,g)=\left(\frac{y^2z-4tx^3}{x^3},\frac{x}{y}\right),\quad G=\frac{x^4}{y(y^2z-4tx^3)},\quad F=\frac{y(y^2z-4tx^3)}{x^4},$$

$$\begin{aligned}\text{vertical leaf} &: \quad D_0=\{g=0\},\\ \text{singular points} &: \quad p_9=\{(F,g)=(8t,0)\},\quad b_{0,2}=\{(f,G)=(0,0)\}.\end{aligned}$$

Resolution of singularity, p_9, gives the following:

$$\begin{aligned}tf' &= \frac{1}{2G}+\frac{(f+8t)^2}{2(f^2G+8tfG+4t)}+96t^2f^4G^4+24tf^5G^4+\\ &\qquad +\frac{3}{2}f^6G^4+80t^2f^3G^3+16t^2f^2G^2+10tf^4G^3-8t,\\ tG' &= \frac{G(64t^2G+8tfG-12t-f)}{8tfG+4t+f^2G}+\\ &\qquad -fG^3\Big(24t^2f^2G^2+20t^2fG+4t^2+\\ &\qquad\qquad +7tf^3G^2+3tf^2G+\frac{1}{2}f^4G^2\Big),\end{aligned}\tag{151}$$

$$tF' = \frac{-64t^2 - 8tgF + 12tF + gF^2}{8tg + 4t + g^2F}$$
$$+ g\left(24t^2g^2 + 20t^2g + 4t^2 + 7tg^3F + 3tg^2F + \frac{1}{2}g^4F^2\right), \quad (152)$$
$$tg' = \frac{2t}{8tg + 4t + g^2F} - g^4\left(4tg + 2t + \frac{1}{2}g^2F\right),$$

with

$$(f, g) = \left(\frac{y(y^2z - 4tx^3) - 8tx^4}{x^4}, \frac{x}{y}\right),$$
$$G = \frac{x^5}{y(y(y^2z - 4tx^3) - 8tx^4)}, \quad F = \frac{y(y(y^2z - 4tx^3) - 8tx^4)}{x^5}.$$

Finally we have vertical leaves,$D_0, \ldots, D_8$ and singular points $b_{j,k} = D_j \cap D_k$ $((jk) = (02), (12), (23), (34), (45), (56), (67), (68))$, of the first class.

Acknowledgments. This work was partially supported by Grant-in-Aid no.14204012 and no.15740099, the Japan Society for the Promotion of Science and was also partially supported by Grant for Basic Science Research Projects from Sumitomo Foundation.

References

[1] Fuchs, R., Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen, Math. Ann. **63** (1907), 301–321.

[2] Gambier, B., Sur les équations diffé
retielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes, Acta. Math. **33** (1910), 1–55.

[3] Gromak, V. I., The solutions of Painlevé's third equation, Differential Equations **9** (1973), 1599–1600.

[4] Gromak, V. I., Algebraic solutions of the third Painlevé equation, Dokl. Akad. Nauk BSSR **23** (1979), 499–502.

[5] Gromak, V. I., Reducibility of the Painlevé equations, Differential Equations **20** (1984), 1191–11198.

[6] Gromak, V. I. and N. A. Lukashevich, Special classes of solutions of Painlevé equations, Differential Equations **18** (1980), 317–326.

[7] Jimbo, M., Miwa, T. and K. Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I, Physica **2D** (1981), 306–352.

[8] Jimbo, M. and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II, Physica **2D** (1981), 407–448.

[9] Kajiwara, K. and Y. Ohta, Determinant structure of the rational solutions for the Painlevé II equation, J. Math. Phys. **37** (1996), 4693–4704.

[10] Lukashevich, N. A., Elementary solutions of certain Painlevé equations, Differential Equations **1** (1965), 561–564.

[11] Malgrange, B., Sur les Deformations Isomonodromiques I, Singularités Régulières, Seminaire de l'Ecole Normale Superieure 1979–1982, *Progress in Mathematics* **37**, Birkhäuser, Boston (1983), 401–426.

[12] Malmquist, J., Sur les équations différentielles du secondordre dont l'intégrale générale a ses points critiques fixes, Ark. Mat. Astr. Fys. **17** (1922), 1–89.

[13] Miwa, T., Painlevé Property of Monodromy Preserving Deformation Equations and the Analyticity of τ-function, Publ. RIMS, Kyoto Univ. **17** (1981), 703–721.

[14] Murata, Y., Classical solutions of the third Painlevé equation, Nagoya Math. J. **139** (1995), 37–65.

[15] Noumi, M. and Y. Yamada, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys. **199** (1998), no. 2, 281–295.

[16] Okamoto, K., Sur les feuilletages associés aux équation du second ordre à points crtiques fixes de P. Painlevé, Japan. J. Math. **5** (1979), 1–79.

[17] Okamoto, K., Studies on the Painlevé equations I. Annali di Matematica pura ed applicata **CXLVI** (1987), 337–381; II. Jap. J. Math. **13** (1987), 47–76; III. Math. Ann. **275** (1986), 221–255; IV. Funkcial. Ekvac. Ser. Int. **30** (1987), 305–332.

[18] Okamoto, K., Isomonodromic deformation and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 575–618.

[19] Okamoto, K., The Hamiltonians associated with the Painlevé equations, *The Painlevé property: One Century Later*, Springer (1999), 735–787.

[20] Painlevé, P., Sur les équations differetielles du second ordre dont l'intégrale générale et uniforme, Oeuvre t. **III**, 187–271.

[21] Painlevé, P., Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris **127** (1898), 945–948. Oeuvres t. III 35–38.

[22] Sakai, H., Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. **220** (2001), 165–229.

[23] Tsuda, T., Okamoto, K. and H. Sakai, Folding transformation of the Painlevé equations, Math. Annalen **331** (2005), 713–738.

[24] Umemura, H., On the irreducibility of the first differential equation of Painlevé, *Algebraic geometry and commutative algebra in Honor of Masayoshi NAGATA*, Vol. II, Kinokuniya, Tokyo (1988), 771–789.

[25] Umemura, H., Special polynomials associated with the Painlevé equations. I, preprint.

[26] Umemura, H. and H. Watanabe, Solutions of the second and fourth Painlevé equations, I, Nagoya Math. J. **148** (1997), 151–198.

[27] Umemura, H. and H. Watanabe, Solutions of the third Painlevé equation. I, Nagoya Math. J. **151** (1998), 1–24.

[28] Vorob'ev, A. P., On the rational solutions of the second Painlevé equation, Differential Equations **1** (1965), 58–59.

[29] Watanabe, H., Solutions of the fifth Painlevé equation I, Hokkaido Math. J. **24** (1995), 231–267.

[30] Watanabe, H., Birational canonical transformations and classical solutions of the sixth Painlevé equation, Ann. Scuola. Norm. Sup. Pisa Cl. Sci. (4)**27** (1998), 379–425.

(Received November 7, 2005)

Yousuke OHYAMA
Graduate School of Information
Sciences and Technology
Osaka University
Japan

Hiroyuki KAWAMUKO
Faculty of Education
Mie University
Japan

Hidetaka SAKAI and Kazuo OKAMOTO
Graduate School of Mathematical Sciences,
The Univrsity of Tokyo
Japan