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An Extension Theorem for Real Analytic Solutions to Relative Hyperbolic Systems

By Hiroshi Koshimizu

Abstract. We give a vanishing result for the cohomologies of solution complexes to microlocally relative hyperbolic systems. As an application, we prove a Bochner-type extension theorem on real analytic solutions to such systems.

Contents

1. Introduction

Let M be a real analytic manifold and N its submanifold of codimension $d \geq 1$. We denote by $Y \subset X$ a complexification of $N \subset M$. Let $\mathcal{A}_M$ (resp. $\mathcal{B}_M$) be the sheaf of real analytic functions (resp. Sato's hyperfunctions) on M and $\mathcal{D}_X$ the sheaf of holomorphic differential operators on X. Then our subject is to give a sufficient condition on a coherent $\mathcal{D}_X$-module $\mathcal{M}$ for the vanishing of the cohomologies :

$$H^j \mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq 0 \quad \text{for any} \quad j < d, \tag{1.1}$$

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where μ_N is Sato's microlocalization functor along N.

In this paper, we formulate the condition on $\mathcal{M}$ as "microlocally relative hyperbolicity" introduced in our previous paper [10] (see also Section 2.2 for the detail). For the sake of simplicity, consider a product of two real analytic manifolds $M = M' \times M'' = \{(x', x'')\}$ with its complexification $X = X' \times X''$ and a closed submanifold $N = N' \times M'' = \{x_1 = x_2 = 0\} \subset M$ of codimension $d = 2$. We set $V := X \times_{X''} T^*X''$ and $\Lambda := M \times_{M''} T^*_{M''}X''$. Choose two operators :

$$P_j = E_j \cdot Q_j + (\text{lower order terms}) \in \mathcal{D}_X,\ j = 1, 2 \tag{1.2}$$

such that

1) the coherent $\mathcal{D}_X$-module (system) $\mathcal{D}_X/\Sigma\mathcal{D}_X E_j$ is partially elliptic along V on Λ in the sense of Bony-Schapira[2].

2) for $j = 1, 2$ the operator Q_j is $1 \in \mathcal{D}_X$ or hyperbolic in the direction $p \in \dot{T}^*_N M$ and its principal symbol $\sigma(Q_j)$ is written in the form $q_j(z; \zeta')$.

3) the coherent $\mathcal{D}_X$-module $\mathcal{M} = \mathcal{D}_X/\Sigma\mathcal{D}_X P_j$ is non-microcharacteristic for Y along V on Λ.

Then we will prove the vanishing of the cohomologies (1.1) for the system $\mathcal{M} = \mathcal{D}_X/(\mathcal{D}_X P_1 + \mathcal{D}_X P_2)$.

As a result, a Bochner-type extension theorem will be deduced. Namely every real analytic solution $u \in \mathcal{A}_M$ of the system $P_1 u = P_2 u = 0$ defined on an open tuboid Ω along N automatically extends to an open neighborhood of N. The microlocal Holmgren's theorem (Bony [1]) is useful to deduce this result.

The vanishing of the cohomologies (1.1) has been considered for a long time. First Kashiwara-Kawai [4] showed the vanishing result under the additional condition of the ellipticity of $\mathcal{M}$. Recently sufficient conditions on $\mathcal{M}$ were given under the weaker condition of "microlocal hyperbolicity" (Theorem 3.9 of Takeuchi [13]) or "partial ellipticity" (Theorem 4.8 of [13]). The former is the case where M'' is reduced to a point, and the latter is the case where $Q_1 = Q_2 = 1$ in our setting (1.2). Hence we consider that our result would be a natural generalization of some results of [13] and [14]. For another generalizations of the results in [13] and [14], see also [9] and Sugiki [12] etc. It should be mentioned that the theory of bimicrolocalization and the micro-support theory developed in [7] enabled us to relax the condition on $\mathcal{M}$.

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2. Preliminary Notions and Results

2.1. A vanishing theorem for microlocally hyperbolic systems

Let M be a real analytic manifold and N its closed submanifold of codimension $d \geq 1$. We denote by $Y \subset X$ a complexification of $N \subset M$. We recall the notion of micro-hyperbolicity ([6],[7]). Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and

$$i : T^*_M X \times_M T^*M \longrightarrow T^*(T^*_M X) \simeq T_{(T^*_M X)}T^*X \tag{2.1}$$

the natural injection induced by the projection $T^*_M X \longrightarrow M$. Assume $(q,p) \in T^*_M X \times_M \dot{T}^*M$. (If there is no risk of confusion, we identify $p \in \dot{T}^*M$ with it's fiber.) Then we say that $\mathcal{M}$ is micro-hyperbolic in the direction $p \in \dot{T}^*M$ at $q \in T^*_M X$ if the characteristic variety $\mathrm{Ch}\mathcal{M}$ of $\mathcal{M}$ satisfies the condition :

$$\{i(q,p)\} \cap \mathrm{C}_{T^*_M X}(\mathrm{Ch}\mathcal{M}) = \emptyset. \tag{2.2}$$

Moreover we say that $\mathcal{M}$ is hyperbolic in the direction $p \in \dot{T}^*M$, if $\mathcal{M}$ is micro-hyperbolic in the same direction at any point (having the same base point in M as p) of $T^*_M X$. Note that by the conicness of $\mathrm{Ch}\mathcal{M}$, $\mathcal{M}$ is hyperbolic (to some direction) if and only if $\mathcal{M}$ is micro-hyperbolic at the zero-section of $T^*_M X$. Take a natural projection $\dot{\pi} : \dot{T}^*_M X \times_M T^*M \longrightarrow T^*M$.

Definition 2.1. ([13]) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say $\mathcal{M}$ is microlocally hyperbolic in the direction $p \in \dot{T}^*M$, if $\mathcal{M}$ satisfies :

$$i(\dot{\pi}^{-1}(p)) \cap \mathrm{C}_{T^*_M X}(\mathrm{Ch}\mathcal{M}) = \emptyset. \tag{2.3}$$

Note that this condition is strictly weaker than that of hyperbolicity (ellipticity). Indeed let E (resp. Q) $\in \mathcal{D}_X$ be an elliptic differential operator (resp. a hyperbolic operator in a direction $p \in \dot{T}^*_N M$) on X, and set $P = E \cdot Q +$ (lower order terms). Then the $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{D}_X P$ is microlocally hyperbolic in the direction $p \in \dot{T}^*_N M$. But it's not elliptic nor hyperbolic in general.

In this situation, the following vanishing of cohomologies was proved in [13].

Proposition 2.2. ([13]) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $p \in \dot{T}^*_N M$. If $\mathcal{M}$ is non-characteristic for Y and microlocally hyperbolic in the direction $p \in \dot{T}^*_N M$, then we have:*

$$H^j \mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq 0 \tag{2.4}$$

at $p \in \dot{T}^*_N M$ for every $j < d$.

We shall extend this result to relative cases in the next section. Note that an application of Proposition 2.2 to the extension theorem of distribution solutions (C^∞-solutions) was given in [9].

Remark 2.3. The same vanishing result was proved also for partially elliptic systems in [13]. Various edge of the wedge type extension theorems for solutions to partially elliptic systems were deduced from this result. See [13] and [14] for the details.

2.2. Microlocal relative hyperbolicity

Let $M = M' \times M''$ be a product of two real analytic manifolds and $g : M \longrightarrow M''$ the second projection. We consider a complexification $X = X' \times X''$ of $M = M' \times M''$ and denote by $g_{\mathbb{C}} : X \longrightarrow X''$ the complexification of g. Then we can define relative cotangent bundles $T^*(M/M'')$ and $T^*(X/X'')$ by the exact sequences :

$$\begin{cases} 0 \longrightarrow \Lambda = M \times_{M''} T^*_{M''} X'' \longrightarrow T^*_M X \longrightarrow T^*(M/M'') \longrightarrow 0 \\ 0 \longrightarrow V = X \times_{X''} T^* X'' \longrightarrow T^* X \longrightarrow T^*(X/X'') \longrightarrow 0. \end{cases} \tag{2.5}$$

Note that $\Sigma_g^{\mathbb{C}} := T^*(X/X'') \times_X V$ is considered as a complexification of $\Sigma_g := T^*(M/M'') \times_M \Lambda$.

Since X is written as a product $X' \times X''$ of two complex manifolds, we have canonical isomorphisms :

$$T^* X \simeq T^* X' \times T^* X'' \xrightarrow{\sim} \Sigma_g^{\mathbb{C}} = T^*(X/X'') \times_X V \simeq T_V(T^* X). \tag{2.6}$$

Hence we can make use of the non-degenerated 2-form Ω_X on T^*X to construct the Hamiltonian isomorphism :

$$T(T_V(T^*X)) \xrightarrow{\sim} T^*(T_V(T^*X)). \tag{2.7}$$

Since Σ_g (resp. $\Sigma_g^{\mathbb{C}}$) is locally isomorphic to T_M^*X (resp. to $T^*X = T^*X' \times T^*X''$), the isomorphism (2.7) and the injection :

$$\Sigma_g \hookrightarrow \Sigma_g^{\mathbb{C}} \simeq T_V(T^*X) \tag{2.8}$$

induce an isomorphism :

$$T_{\Sigma_g}(\Sigma_g^{\mathbb{C}}) \simeq T^*(\Sigma_g) \tag{2.9}$$

as in the same way as the definition of micro-hyperbolicity.

Finally consider the injection :

$$i_g : \Sigma_g \times_M T^*M \hookrightarrow T^*(\Sigma_g) \tag{2.10}$$

and the projection :

$$\dot{\pi}_g : \dot{\Sigma}_g \times_M T^*M \longrightarrow T^*M \quad \text{where} \quad \dot{\Sigma}_g := \dot{T}^*(M/M'') \times_M \Lambda. \tag{2.11}$$

Definition 2.4. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say that $\mathcal{M}$ is microlocally relative hyperbolic (w.r.t. g) in the direction $p \in \dot{T}^*M' \times M''$ if $\mathcal{M}$ satisfies :

$$i_g(\dot{\pi}_g^{-1}(p)) \cap \mathrm{C}_{\Sigma_g}(\mathrm{C}_V(\mathrm{Ch}\mathcal{M})) = \emptyset. \tag{2.12}$$

Assume locally $X = X' \times X'' = \mathbb{C}^l_{z'} \times \mathbb{C}^{n-l}_{z''}$ with $z' = (z_1, z_2, ..., z_l), z'' = (z_{l+1}, ..., z_n)$. Take a local coordinate system $(z; \zeta dz), z = x + iy, \zeta = \xi + i\eta$ of T^*X in which $M = \{y = 0\} \subset X$. Finally set $\hat{\xi} = (\xi_2, ..., \xi_l)$. Then the following result was proved in [10].

Lemma 2.5. (Lemma 4.3 of [10]) *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume that $\mathcal{M}$ is microlocally relative hyperbolic (w.r.t. g) in the direction*

$+dx_1 \in \dot{T}^*M' \times M''$. *Then there exists a small* $\varepsilon > 0$ *such that* $\mathrm{Ch}\mathcal{M} \cap U_\varepsilon = \emptyset$ *in an open neighborhood of* M *in* T^*X, *where we set*

$$U_\varepsilon := \{(z;\zeta) \in T^*X;\ 0 < \xi_1 < \varepsilon|\eta'|, |\hat{\xi}| < \varepsilon\xi_1, |y| \cdot |\eta'| < \varepsilon\xi_1\}. \tag{2.13}$$

Before giving an example of microlocally relative hyperbolic operators, we recall the notion of partial ellipticity. Consider the injection (2.8) and the projection $\dot{\pi}_\Lambda : \dot{\Sigma}_g \longrightarrow \Lambda$.

DEFINITION 2.6. (Bony-Schapira [2]) A coherent $\mathcal{D}_X$-module $\mathcal{M}$ is said to be partially elliptic along V at $p \in \Lambda$ if

$$\dot{\pi}_\Lambda^{-1}(p) \cap \mathrm{C}_V(\mathrm{Ch}\mathcal{M}) = \emptyset. \tag{2.14}$$

For example, an elliptic operator on X' can be considered as a differential operator on $X = X' \times X''$ partially elliptic along V on Λ. Moreover, we can easily see that the principal symbol $\sigma(E)$ of a differential operator $E \in \mathcal{D}_X$ partially elliptic along V on the whole Λ is written in the form $e(z;\zeta')$ (use the partial ellipticity on the zero-section of $\Lambda \to M$). Note also that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ partially elliptic along V on Λ is microlocally relative hyperbolic in any direction $p \in \dot{T}^*M' \times M''$.

Example 2.7. Let $E \in \mathcal{D}_X$ be partially elliptic along V on Λ and assume that $Q \in \mathcal{D}_X$ is hyperbolic in the direction $p \in \dot{T}^*M' \times M''$ and its principal symbol $\sigma(Q)$ is written in the form $q(z;\zeta')$. Then the operator $P := E \cdot Q +$ (lower order terms) is microlocally relative hyperbolic in the direction $p \in \dot{T}^*M' \times M''$.

Example 2.8. Take two operators

$$P_j = E_j \cdot Q_j + \text{(lower order terms)},\ j = 1, 2 \tag{2.15}$$

such that the following conditions are satisfied :

1) the coherent $\mathcal{D}_X$-module (system) $\mathcal{D}_X/\Sigma\mathcal{D}_X E_j$ is partially elliptic along V on Λ.

2) for $j=1,2$ the operator Q_j is $1 \in \mathcal{D}_X$ or hyperbolic in the direction $p \in \dot{T}^*M' \times M''$ and its principal symbol $\sigma(Q_j)$ is written in the form $q_j(z;\zeta')$.

Then $\mathcal{M} = \mathcal{D}_X / \Sigma \mathcal{D}_X P_j$ is microlocally relative hyperbolic in the direction $p \in \dot{T}^*M' \times M''$.

2.3. Microlocal Holmgren's theorem

In Section 4, we make use of the microlocal version of Holmgren's theorem to sweep out microfunction solutions. Let M be a real analytic manifold and $N = \{x_1 = 0\}$ its closed submanifold of codimension one. We denote by $Y \subset X$ a complexification of $N \subset M$ and $M_+ = \{x_1 \geq 0\}$ a closed subset of M. Then the classical Holmgren's theorem for hyperfunction solutions is stated as follows :

Proposition 2.9. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module for which Y is non-characteristic. Then we have :*

$$\Gamma_{M_+}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \mid_N = 0. \tag{2.16}$$

In fact, we need a microlocal version of the above proposition. For this purpose, we consider the same situation as Section 2.2. That is $M = M' \times M''$ and consider the second projection $g : M \longrightarrow M''$. We take a closed submanifold $N = N' \times M'' \subset M$ of codimension one. Then $g \mid_N : N \longrightarrow M''$ is also surjective. In this situation, a complexification Y of N can be taken as $Y = Y' \times X''$ in $X = X' \times X''$. Now there is a natural injection :

$$T^*_Y X \times_X V \longrightarrow T^*(X/X'') \times_X V \simeq T_V(T^*X) \tag{2.17}$$

and a projection $\dot{\pi}_Y : \dot{T}^*_Y X \times_X V \longrightarrow V$.

Definition 2.10. (Bony [1]) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say $\mathcal{M}$ is non-microcharacteristic for Y along V at $p \in V$ if

$$\dot{\pi}_Y^{-1}(p) \cap \mathrm{C}_V(\mathrm{Ch}\mathcal{M}) = \emptyset. \tag{2.18}$$

Recall that $\Lambda = M \times_{M''} T^*_{M''}X'' = M \times_L T^*_L X$ where $L := g_{\mathbb{C}}^{-1}(M'') \subset X$ and set $\Lambda_N = N \times_L T^*_L X$, $\Lambda_+ = M_+ \times_L T^*_L X$.

PROPOSITION 2.11. (Bony [1]) *If $\mathcal{M}$ is non-microcharacteristic for Y along V at $p \in \Lambda_N$, then we have :*

$$\Gamma_{\Lambda_+}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_M)\mid_\Lambda)\mid_p = 0. \tag{2.19}$$

3. A Vanishing Theorem for Microlocally Relative Hyperbolic Systems

In this section, we shall give a vanishing theorem for microlocally relative hyperbolic systems. In the next section, we apply this theorem to the extension of real analytic solutions.

We consider the same situation as in Section 2.2, 2.3 and inherit the notations there. Now we take a closed submanifold $N' \subset M'$ of codimension $d \geq 1$ and set $N = N' \times M'' \subset M$. Then a complexification of N in $X = X' \times X''$ can be taken in the form $Y = Y' \times X''$. Finally we set $(g\mid_N)_{\mathbb{C}} : Y = Y' \times X'' \longrightarrow X''$.

THEOREM 3.1. *Let $p \in \dot{T}^*_N M$ and $d := codim^{\mathbb{R}}_M N \geq 1$. Assume that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ satisfies the conditions :*

(i) $\mathcal{M}$ is non-microcharacteristic for Y along V at any point of Λ_N i.e. $(\dot{T}^*_Y X \times_X \Lambda_N) \cap \mathrm{C}_V(\mathrm{Ch}\mathcal{M}) = \emptyset$.

(ii) $\mathcal{M}$ is microlocally relative hyperbolic (w.r.t. g) in the direction $p \in \dot{T}^*_N M$.

Then we have :

$$H^j \mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq 0 \quad \text{at } p \in \dot{T}^*_N M \quad \text{for every } j < d. \tag{3.1}$$

Note that by the assumption (i) the coherent $\mathcal{D}_X$-module $\mathcal{M}$ is non-characteristic for Y on N in the usual sense (use the non-microcharactericity of $\mathcal{M}$ on the zero-section of $\Lambda_N \to N$).

PROOF OF THEOREM 3.1. Set $L = g_{\mathbb{C}}^{-1}(M'') \subset X$, $H = (g\mid_N)_{\mathbb{C}}^{-1}(M'') \subset Y$ and $\iota : L \hookrightarrow X$. Next consider the complex $G =$

$\iota^! R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[n]$ where $n = dim^{\mathbb{C}} X$. Then the micro-support SS(G) of G is estimated by

$$\text{SS } (G) \subset \iota^{\sharp}(\text{SS}(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))) = T^*L \cap \text{C}_{T_L^*X}(\text{Ch}\mathcal{M}) \tag{3.2}$$

where $T^*L \hookrightarrow T^*T_L^*X \simeq T_{T_L^*X}(T^*X)$ is a natural injection(Corollary 6.4.4, Proposition 6.2.4 of [7]). Now we prepare the following two lemmas to prove the theorem.

Lemma 3.2. *Let $\rho_1 : T_N^*L \longrightarrow T_N^*M$ be the projection induced by $f : M \hookrightarrow L$. Then we have :*

$$\begin{aligned} &\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_p \\ &\simeq \text{R}\rho_{1!}\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \text{R}\Gamma_L(\mathcal{O}_X) \mid_L [n])_p \end{aligned} \tag{3.3}$$

Proof of Lemma 3.2. We consider the closed embedding $f : M \hookrightarrow L$. Then L.H.S. of (3.3) equals to $\mu_N(f^! G)_p$. Hence by virtue of Theorem 6.7.1 of [7], the problem is reduced to the verification of the condition :

$$f \text{ is non-characteristic for } G \text{ at } p \in T^*M. \text{ i.e. } p \notin f_\infty^\sharp(\text{SS}(G)). \tag{3.4}$$

From now on, we shall prove this by a contradiction. Assume that $p \in f_\infty^\sharp(\text{SS}(G))$, and we show that it contradicts the assumption (ii) of $\mathcal{M}$.

The problem being local, we may assume :

$$\left\{ \begin{aligned} &X = \mathbb{C}^l \times \mathbb{C}^{n-l} \supset L = \mathbb{C}^l \times \mathbb{R}^{n-l} \supset M = \mathbb{R}^l \times \mathbb{R}^{n-l} \\ &Y = (\{0\} \times \mathbb{C}^{l-d}) \times \mathbb{C}^{n-l} \supset H = \mathbb{C}^{l-d} \times \mathbb{R}^{n-l} \supset N \\ &\quad = \mathbb{R}^{l-d} \times \mathbb{R}^{n-l} \end{aligned} \right. \tag{3.5}$$

where $z = (z', z''), z' = (z_1, z_2, ..., z_l) = (z_1, \hat{z}), z'' = (z_{l+1}, ..., z_n)$ is the coordinate system of X, $X' = \mathbb{C}^l$ and $X'' = \mathbb{C}^{n-l}$ respectively, and $Y = \{z_1 = \cdots = z_d = 0\} \subset X$ $(1 \leq d \leq l)$. We take the associated coordinate system $(z; \zeta dz) = (x+iy; \xi+i\eta)$ of T^*X, and we may assume $p = (0; +dx_1) \in \dot{T}_N^*M$ without loss of generality.

By Proposition 6.2.4 of [7], we can find sequences :

$$\left\{ \begin{aligned} &(z'_n, x''_n; \zeta'_n, \xi''_n) \in \text{SS}(G) \subset T^*L \\ &(\bar{x}'_n, \bar{x}''_n) \in M \end{aligned} \right. \tag{3.6}$$

such that :

$$(3.7)\qquad \begin{cases} (z'_n, x''_n) \longrightarrow (0,0),\ (\bar{x}'_n, \bar{x}''_n) \longrightarrow (0,0) \\ ((\xi_{1n}, \hat{\xi}_n), \xi''_n) \longrightarrow ((1,0),0) \\ |(z'_n - \bar{x}'_n, x''_n - \bar{x}''_n)| \cdot |(\zeta'_n, \xi''_n)| \longrightarrow 0 \\ |(\zeta'_n, \xi''_n)| \longrightarrow +\infty. \end{cases}$$

By the above relation, we obtain :

$$(3.8)\qquad (\xi_{1n}, \hat{\xi}_n) \longrightarrow (1,0),\ |y'_n| \cdot |\eta'_n| \longrightarrow 0,\ |\eta'_n| \longrightarrow +\infty.$$

Moreover by the formula (3.2) and the well-known characterization (by series) of the normal cone $\mathrm{C}_{T_L^*X}(\mathrm{Ch}\mathcal{M})$, for every $n \in \mathbb{N}$ there exists a sequence indexed by $m \in \mathbb{N}$:

$$(3.9)\qquad \begin{cases} (z'_{nm}, z''_{nm}; \zeta'_{nm}, \zeta''_{nm}) \in \mathrm{Ch}(\mathcal{M}) \\ c_{nm} \in \mathbb{R}^+ \end{cases}$$

satisfying :

$$(3.10)\qquad \begin{cases} (y''_{nm}, \zeta'_{nm}, \xi''_{nm}) \longrightarrow (0,0,0) \\ (z'_{nm}, x''_{nm}, \eta''_{nm}) \longrightarrow (z'_n, x''_n, 0) \\ c_{nm}(y''_{nm}, \zeta'_{nm}, \xi''_{nm}) \longrightarrow (0, \zeta'_n, \xi''_n) \end{cases}$$

as $m \longrightarrow +\infty$. Hence by extracting subsequences of (3.9), we obtain the sequences :

$$(3.11)\qquad \begin{cases} (z'_j, z''_j; \zeta'_j, \zeta''_j) \in \mathrm{Ch}(\mathcal{M}) \\ c_j \in \mathbb{R}^+ \end{cases}$$

such that :

$$(3.12)\qquad \begin{cases} (c_j\xi_{1j}, c_j\hat{\xi}_j) \longrightarrow (1,0),\ |y'_j| \cdot |c_j\eta'_j| \longrightarrow 0,\ |c_j\eta'_j| \longrightarrow +\infty. \\ \eta'_j \longrightarrow 0,\ c_j y''_j \longrightarrow 0 \end{cases}$$

as $j \longrightarrow +\infty$. Then $(z'_j, z''_j; c_j\zeta'_j, c_j\zeta''_j) \in \mathrm{Ch}(\mathcal{M})$ by the conicness of $\mathrm{Ch}(\mathcal{M})$ and for any small $\varepsilon > 0$ there exists a sufficiently large $j_0 \in \mathbb{N}$ such that :

$$(3.13)\qquad \begin{cases} c_j\xi_{1j} < \varepsilon|c_j\eta'_j|,\ |c_j\hat{\xi}_j| < \varepsilon|c_j\xi_{1j}| \\ |y_j| \cdot |c_j\eta'_j| < |y'_j| \cdot |c_j\eta'_j| + |c_j y''_j| \cdot |\eta'_j| < \varepsilon|c_j\xi_{1j}| \end{cases}$$

for any $j > j_0$. According to Lemma 2.5, this contradicts the microlocal relative hyperbolicity of $\mathcal{M}$. □

LEMMA 3.3. *Let $\rho_2 : T_N^*L \longrightarrow T_N^*H$ be the projection induced by $h : H \hookrightarrow L$. Then we have :*

$$\begin{aligned}\text{(3.14)}\qquad & \mathrm{R}\rho_{2*}\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_L(\mathcal{O}_X)\mid_L [n]) \\ & \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mu_N(\mathrm{R}\Gamma_H(\mathcal{O}_Y)\mid_H)[n-d])[-d].\end{aligned}$$

PROOF OF LEMMA 3.3. Let $h : H \hookrightarrow L$ be the embedding. Then the R.H.S. of (3.14) is equal to :

$$\begin{aligned} & \mu_N(\mathrm{R}\Gamma_H(R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)[n-2d])\mid_H) \\ \text{(3.15)}\qquad & \simeq \mu_N(\mathrm{R}\Gamma_H(\mathrm{R}\Gamma_Y R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)[n])\mid_H) \\ & \simeq \mu_N(h^!G).\end{aligned}$$

Here the first isomorphism in(3.15) is a result of [3] (We can prove it by Ex XI.1.11 of [7]). Hence by virtue of Theorem 6.7.1 of [7], it is enough to show the condition :

$$\text{(3.16)}\qquad h \text{ is non-characteristic for } G. \text{ i.e. } \mathrm{SS}(G) \cap \dot{T}_H^*L = \emptyset.$$

We can prove :

$$\text{(3.17)}\qquad \dot{T}_H^*L \cap (\mathrm{C}_{T_L^*X}(\mathrm{Ch}\mathcal{M})) \subset \dot{T}_H^*L \cap (\mathrm{C}_V(\mathrm{Ch}\mathcal{M}))$$

easily from the inclusion $T_L^*X \subset V$. Then the condition (3.16) follows from the non-microcharactericity of $\mathcal{M}$ (use $\dot{T}_H^*L \subset \dot{T}_Y^*X$) and the estimate (3.2) of $\mathrm{SS}(G)$. □

Let us continue the proof of Theorem 3.1 : The complex $\mu_N(\mathrm{R}\Gamma_H(\mathcal{O}_Y)\mid_H)[n-d]$ is concentrated in degree 0 by Kashiwara's abstract edge of the wedge theorem ([5]). Moreover it follows from the non-microcharactericity of $\mathcal{M}$ that the morphism ρ_2 is finite on supp $\mu_N R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathrm{R}\Gamma_L(\mathcal{O}_X)\mid_L [n])$. Hence the conclusion follows from Lemma 3.2 and 3.3. □

4. An Extension Theorem for Real Analytic Solutions

In this section, we shall give an extension theorem for real analytic solutions to microlocally relative hyperbolic systems under some conditions. We consider the same situation as in Section 3 with $d := \mathrm{codim}_M^{\mathbb{R}} N \geq 2$ and inherit the notations there. The problem being local, we may assume :

$$(4.1) \qquad \begin{cases} X = \mathbb{C}^l \times \mathbb{C}^{n-l} \supset L = \mathbb{C}^l \times \mathbb{R}^{n-l} \supset M = \mathbb{R}^l \times \mathbb{R}^{n-l} \\ Y = (\{0\} \times \mathbb{C}^{l-d}) \times \mathbb{C}^{n-l} \supset N \\ \quad = (\{0\} \times \mathbb{R}^{l-d}) \times \mathbb{R}^{n-l} \ (2 \leq d \leq l) \\ X' = \mathbb{C}^l, X'' = \mathbb{C}^{n-l} \text{ and } V = X \times_{X''} T^*X''. \end{cases}$$

In this section, we take a coordinate system of X as follows :

$$(4.2) \qquad z = (z_1, z_2, ..., z_d, z_{d+1}, ..., z_n) = (z_1, \bar{z}, \tilde{z}).$$

Here $Y = \{z_1 = 0, \bar{z} = 0\} \subset X$ and we also take $p = (0; +dx_1) \in \dot{T}_N^* M$.

In this situation, let us consider open tuboids $\Omega_{\delta,\varepsilon} \subset M = \mathbb{R}^n$ along N (resp. open subsets Ω_δ of $M = \mathbb{R}^n$) defined for $\varepsilon, \delta > 0$ as follows.

$$(4.3) \qquad \begin{cases} \Omega_\delta = \{x \in M = \mathbb{R}^n;\ |(x_1, \bar{x})| < \delta,\ |\tilde{x}| < \delta\} \\ \Omega_{\delta,\varepsilon} = \{x \in M = \mathbb{R}^n;\ x_1 < \varepsilon\ |\bar{x}|, |(x_1, \bar{x})| < \delta,\ |\tilde{x}| < \delta\}. \end{cases}$$

Theorem 4.1. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module satisfying the conditions :*

(i) $\mathcal{M}$ is non-microcharacteristic for Y along V.

(ii) $\mathcal{M}$ is microlocally relative hyperbolic (w.r.t.g) in the direction $p \in \dot{T}_N^* M$.

(iii) $\mathcal{M}$ is micro-hyperbolic in the $+dx_1$-direction on $\dot{T}_M^* X \backslash \dot{\Lambda}$ where $\dot{\Lambda} := M \times_L \dot{T}_L^* X$.

Then there exists a sufficiently small $\varepsilon > 0$ such that :

$$(4.4) \qquad \varinjlim_{\delta>0} \Gamma(\Omega_\delta; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)) \xrightarrow{\sim} \varinjlim_{\delta>0} \Gamma(\Omega_{\delta,\varepsilon}; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M)).$$

Namely every real analytic solution to $\mathcal{M}$ on an open tuboid $\Omega_{\delta,\varepsilon}$ for $0 < \delta << 1$ extends to an open neighborhood of N as a real analytic solution.

PROOF. We have by Theorem 3.1 :

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)_0 \xrightarrow{\sim} \varinjlim_{\delta,\varepsilon>0} \Gamma(\Omega_{\delta,\varepsilon}; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M)). \tag{4.5}$$

This means that for every real analytic solution $u \in \Gamma(\Omega_{\delta,\varepsilon}; \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M))$ with sufficiently small $\varepsilon, \delta > 0$:

$$\begin{cases} \text{there exist } 0 < \varepsilon' < \varepsilon \text{ and } 0 < \delta' < \delta \text{ such that} \\ \text{the restriction } u\,|_{\Omega_{\delta',\varepsilon'}} \text{ of } u \text{ to } \Omega_{\delta',\varepsilon'} \\ \text{extends to an open neighborhood of } N \\ \text{as a hyperfunction solution } \tilde{u} \text{ of } \mathcal{M}. \end{cases} \tag{4.6}$$

By the analytic continuation theorem it is sufficient to show that $\tilde{u}$ is analytic. We shall make use of a microlocal viewpoint to prove it.

First note that :

$$\tilde{u} \equiv 0 \quad \text{on} \quad \dot{T}^*_M X \cap \{x_1 < 0\} \tag{4.7}$$

as a microfunction solution to $\mathcal{M}$. Next set $N \subset N_0 := \{x_1 = 0\} \subset M$ and consider its complexification $Y_0 := \{z_1 = 0\} \subset X$. We also set $M_+ = \{x_1 \geq 0\}$, $\dot{\Lambda}_+ = M_+ \times_L \dot{T}^*_L X$ and $\dot{\Lambda}_{N_0} = N_0 \times_L \dot{T}^*_L X$. Then by (4.7) and the microlocal Holmgren's theorem (Proposition 2.11) we obtain :

$$\tilde{u} \equiv 0 \quad \text{on an open neighborhood of } \dot{\Lambda}_+ \text{ in } \dot{T}^*_M X. \tag{4.8}$$

It remains to show :

$$\tilde{u} \equiv 0 \quad \text{on} \quad (M_+ \times_M \dot{T}^*_M X)\backslash\dot{\Lambda}_+ = M_+ \times_M (\dot{T}^*_M X \backslash \dot{\Lambda}). \tag{4.9}$$

But this follows from the micro-hyperbolicity of $\mathcal{M}$ on $\dot{T}^*_M X\backslash\dot{\Lambda}$. This completes the proof. □

Example 4.2. Assume that $d = \operatorname{codim}^{\mathbb{R}}_M N = 2$ and take two differential operators $P_1, P_2 \in \mathcal{D}_X$ so that the conditions of Theorem 4.1 are satisfied for $\mathcal{M} := \mathcal{D}_X/\Sigma\mathcal{D}_X P_j$. For example we can take $P_j = E_j \cdot Q_j (j = 1, 2)$ by setting $E_1 = D_1 + \sqrt{-1}D_3$, $E_2 = D_2 + \sqrt{-1}D_4$, $Q_j = D_1 + (-1)^j a(x_5, ..., x_n)D_2$. Here $a(x_5, ..., x_n)$ is a non-zero real-valued analytic function of $x_5, ..., x_n$. Then there exists a sufficiently small $\varepsilon > 0$ such that every real analytic solution :

$$u \in \mathcal{A}_M \quad \text{satisfying} \quad P_1 u = P_2 u = 0 \tag{4.10}$$

on an open tuboid $\Omega_{\delta,\varepsilon}$ (for $0 < \delta << 1$) extends to an open neighborhood of N as a real analytic solution.

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Geographical Survey Institute
Kitasato 1, Tsukuba
Ibaraki 305-0811, Japan
E-mail: koshimi@gsi.go.jp