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Examples of Pseudo-Anosov Homeomorphisms with Small Dilatations

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Abstract. For a closed orientable surface Σ_g of genus $g \geq 2$, we give an upper bound for the least dilatation of pseudo-Anosov homeomorphisms of Σ_g. For this purpose, we construct a family of Birkhoff sections for a suspension Anosov flow. Birkhoff sections have been constructed by using cut and paste arguments for surfaces in 3-manifolds. But we construct them directly and then we obtain piecewise linear models of the first return mappings of the Birkhoff sections. These models enable us to investigate their dilatations explicitly, and we obtain necessary estimates.

§0. Introduction

Let Σ_g be a closed orientable surface of genus g ($g \geq 1$). An orientation preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ is called *pseudo-Anosov* if there is a pair of measured foliations $(\mathcal{F}^\sigma, \mu^\sigma)$ ($\sigma = +, -$) possibly with common prong singularities so that

(1) f preserves $\mathcal{F}^\sigma$ ($\sigma = +, -$),

(2) $\mathcal{F}^+$ and $\mathcal{F}^-$ intersect transversely outside the singularities,

(3) $f_*\mu^+ = \lambda\mu^+$, $f_*\mu^- = \lambda^{-1}\mu^-$ for some real number $\lambda > 1$,

where $f_*\mu^\sigma(\gamma) = \mu^\sigma(f^{-1}(\gamma))$ for any transverse arc γ of $\mathcal{F}^\sigma$ ($\sigma = +, -$) (see, for example, [5], [6], and [18]). Note that, in case $g = 1$, both $\mathcal{F}^+$ and $\mathcal{F}^-$ are without singularities and f is an Anosov homeomorphism. The constant λ is called the *dilatation* of f. The spectra of Σ_g are defined to be the sets

$$\mathrm{Spec}(\Sigma_g) = \{\log \lambda \mid \lambda \text{ is the dilatation of a p.A. homeomorphism of } \Sigma_g\}$$

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and

$$\mathrm{Spec}(\Sigma_g)^+ = \left\{ \log\lambda \;\middle|\; \begin{array}{l} \lambda \text{ is the dilatation of a p.A. homeomorphism of } \Sigma_g \\ \text{with orientable invariant measured foliations} \end{array} \right\}.$$

It is known that $\mathrm{Spec}(\Sigma_g)$ has no accumulation points (see [1]). We denote the least element of $\mathrm{Spec}(\Sigma_g)$ (resp. $\mathrm{Spec}(\Sigma_g)^+$) by δ_g (resp. δ_g^+). It is well known that $\delta_1 = \delta_1^+ = \log \frac{3+\sqrt{5}}{2}$ which is the logarithm of the maximal eigenvalue of a matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. In case $g = 2$, Zhirov showed that δ_2^+ is equal to the maximal modulus root of $x^4 - x^3 - x^2 - x + 1 = 0$ ([19]), but δ_2 still has not been determined. In case $g \geq 3$, we have $\frac{\log 2}{6g-6} \leq \delta_g \leq \frac{\log 6}{g}$ (see [2], [16]). In this paper, we give an upper bound of δ_g^+ which is an improvement of the last upper bound. In fact, we shall prove that

$$\delta_g^+ \leq \frac{\log(2+\sqrt{3})}{g} \qquad \text{for any } g \geq 3.$$

This paper is organized as follows. In §1, we construct following Fried [8], a family of Birkhoff sections S_n ($n \in \mathbf{N}$) for the suspension Anosov flow with monodromy $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and reconstruct by our method them without using cut and paste operations. For any $n \in \mathbf{N}$, the first return mapping of S_n gives us a pseudo-Anosov homeomorphism f_n of Σ_n. In §2, we construct a piecewise linear model of the pseudo-Anosov homeomorhism f_n found in §1. In §3, we estimate the dilatation $\lambda(f_n)$ of f_n in a geometric way and show that $\lambda(f_n) \leq \frac{\log(2+\sqrt{3})}{n}$ for any $n \in \mathbf{N}$. In §4, we calculate the characteristic polynomial of $(f_n)_* : H_1(\Sigma_n) \to H_1(\Sigma_n)$ and find that f_1 and f_2 attain δ_1 and δ_2^+ respectively. We also observe that f_2 is different from Zhirov's example mentioned above.

§1. Construction of Examples

In [8], Fried showed that every transitive Anosov flow of a closed 3-manifold has a Birkhoff section. A non-singular flow ϕ^t of a closed, connected 3-manifold M is called *Anosov* if there exists a continuous splitting

$TM = T\phi^t \oplus E^u \oplus E^s$ of the tangent bundle TM of M into $d\phi^t$-invariant one-dimensional subbundles with the following properties.

(1) $T\phi^t$ is tangent to the flow,

and moreover, given a Riemannian metric, there exist constants $C > 0$ and $0 < \lambda < 1$ such that

(2) $\|d\phi^t(v)\| \leq C\lambda^t\|v\|$ for any $v \in E^s, t > 0$, and

(3) $\|d\phi^{-t}(v)\| \leq C\lambda^t\|v\|$ for any $v \in E^u, t > 0$.

Given a flow ϕ^t on a closed connected 3-manifold M, a *Birkhoff section* for the flow is defined to be the pair of a compact connected surface S with boundary and an immersion $\iota : S \to M$ satisfying the following conditions.

(1) The restriction $\iota|Int(S)$ is an embedding transverse to the flow, where $Int(S)$ denotes the interior of S.

(2) Each component of the boundary ∂S covers a periodic orbit by ι.

(3) Every orbit starting from any point of M meets S in a uniformly bounded time.

The image $\iota(S)$ is also called a Birkhoff section, and the image $\iota(\partial S)$ is called the *boundary* of $\iota(S)$.

Before explaining the main construction, we explain the method of construction of Birkhoff sections in [8]. Let ϕ^t be a flow of a closed connected 3-manifold M. For any embedded arcs J_1, J_2 in M transverse to the flow, J_1 is said to be *connected* to J_2 by ϕ^t if there exists a positive continuous function $\tau : J_1 \to \mathbf{R}$ such that for any $x \in J_1$, $\phi^{\tau(x)}(x)$ belongs to J_2 and the mapping $g : J_1 \to J_2$, defined by $g(x) = \phi^{\tau(x)}(x)$, is a homeomorphism. The minimum element among all such functions as τ above is called the *arrival time mapping* from J_1 to J_2. Then the *flow band* bounded by J_1 and J_2 is defined to be the set

$$[J_1, J_2] = \left\{\phi^t(x) \in M \mid x \in J_1,\ 0 \leq t \leq \tau(x)\right\},$$

where τ is the arrival time mapping from J_1 to J_2.

Now let ϕ^t be an Anosov flow of a closed connected 3-manifold M, and R an immersed quadrangle $XYZU$, whose interior is an embedded surface, in M transverse to ϕ^t. Suppose that the edge XY is connected to the edge ZY by ϕ^t so that $Y \in XY$ is connected to $Y \in ZY$ and also ZU to XU so that $U \in ZU$ is connected to $U \in XU$. Then the union

$$P_1 = R \cup [XY, ZY] \cup [ZU, XU]$$

is a topologically immersed surface of a pair of pants P, whose boundary consists of periodic orbits through X, Y and U. Note that the periodic orbit through Z is the same as that through X, and that some of these orbits through X, Y and U may be identical. Take a defining topological immersion $\iota_1 : P \to M$ of P_1 such that ι_1 is a covering map on each component of ∂P. Then perturb ι_1 slightly on its interior $Int(P)$ and we get an immersion $\iota : P \to M$ such that $\iota | Int(P)$ is transverse to the flow. Then we denote the image $\iota(P)$ by $P(R)$. Note that, in general, even the restriction $\iota | Int(P)$ is not an embedding. This construction plays an important role in [8]. Indeed, consider the finite union Σ' of sufficiently many such surfaces $P_1, \cdots, P_s$ as $P(R)$, for example, such that each orbit of ϕ^t goes through $Int(P_i)$ for some $1 \leq i \leq s$. If we suitably cut and paste Σ' along its self-intersection point sets, we obtatin a Birkhoff section (see, for more details, [8]). In his paper, he found such a quadrangle by using a Markov partition of an Anosov flow. But, in this paper, we will define such a quadrangle of a suspension Anosov flow explicitly as follows.

Now, let $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$, and let $\bar{A} : T^2 \to T^2$ be the hyperbolic toral automorphism defined by

$$\bar{A}(\begin{bmatrix} x \\ y \end{bmatrix}) = \left[A \begin{pmatrix} x \\ y \end{pmatrix} \right]$$

for any $\begin{bmatrix} x \\ y \end{bmatrix} \in T^2 = \mathbf{R}^2/\mathbf{Z}^2$. The mapping torus $M_{\bar{A}}$ is the 3-manifold defined by $M_{\bar{A}} = T^2 \times \mathbf{R}/ \sim$, where $\sim$ is the equivalence relation generated by $(\begin{bmatrix} x \\ y \end{bmatrix}, t+1) \sim (\bar{A} \begin{bmatrix} x \\ y \end{bmatrix}, t)$ for any $\begin{bmatrix} x \\ y \end{bmatrix} \in T^2$ and any $t \in \mathbf{R}$. Let $\pi : T^2 \times \mathbf{R} \to M_{\bar{A}}$ be the quotient map. It is well-known that the vector field $\frac{\partial}{\partial t}$ on $T^2 \times \mathbf{R}$ gives rise to an Anosov flow $\phi^t_{\bar{A}}$ of $M_{\bar{A}}$. We denote the quotient image $\pi(T^2 \times \{t\})$ by T^2_t, and we identify T^2 with T^2_0.

Let $X = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$, and $U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Let $\bar{X}\bar{Y}\bar{Z}\bar{U} \subset T_0^2 = T^2$ be the quotient image of the parallelogram $XYZU \subset \mathbf{R}^2$. Note that $\bar{X} = \bar{Z}$ and $\bar{Y} = \bar{U}$. Since $\bar{A}$ maps linearly the edge $\bar{X}\bar{Y}$ to $\bar{Z}\bar{Y}$ and $\bar{Z}\bar{U}$ to $\bar{X}\bar{U}$ and fixes $\bar{X} = \bar{Z}$ and $\bar{Y} = \bar{U}$, we obtain an immersed surface $\bar{P} = P(\bar{X}\bar{Y}\bar{Z}\bar{U})$ in $M_{\bar{A}}$. By the method of the construction of $\bar{P}$, we may assume that $\bar{P}$ intersects T_t^2 transversely for any sufficiently small $t > 0$. Choose a positive real number $0 < t_0 < 1$ such that $\bar{P}$ intersects T_t^2 for any $0 \leq t \leq t_0$, and fix it.

Given a positive integer n, choose a positive real number $\epsilon > 0$ such that $n\epsilon < t_0$ and consider a fake surface $\bar{P} \cup T_\epsilon^2 \cup T_{2\epsilon}^2 \cup \cdots \cup T_{n\epsilon}^2$. Then we can cut and paste it suitably to obtain a Birkhoff section S_n. Note that the boundary ∂S_n is the union of two periodic orbits of $\phi_{\bar{A}}^t$ through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$.

In this construction, the cut and paste process prevents us from seeing the whole picture of Birkhoff section. In order to conquer the difficulties, we reconstruct S_n as follows. We first perturb $\bar{X}\bar{Y}\bar{Z}\bar{U} \cup (\bar{X}\bar{Y} \times [0, \epsilon]) \cup (\bar{Z}\bar{U} \times [0, \epsilon]) \subset M_{\bar{A}}$ slightly on its interior without moving the boundary such that the interior of the resultant surface $\bar{R}_0$ is transverse to the flow. Next, for each $1 \leq k \leq n$, we perform the same operation on $T_{k\epsilon}^2 \cup (\bar{X}\bar{Y} \times [k\epsilon, (k+1)\epsilon]) \cup (\bar{Z}\bar{U} \times [k\epsilon, (k+1)\epsilon]) \subset M_{\bar{A}}$ and we get $\bar{T}_{n,k}$. Then glue these $n+1$ surfaces along the boundaries transverse to the flow and we again obtain S_n.

Then the first return mapping of S_n induces a p.A. homemorphism $f_n : \Sigma(n) \to \Sigma(n)$ by collapsing each boundary component to a point ([6]). More precisely, the first return map of $S_n - \partial S_n$ determines a homeomorphism of the pre-immersed compact surface $\tilde{S}_n$ of S_n. Then we get a homeomorphism of a closed surface by collapsing each boundary component of $\tilde{S}_n$ to a point. It is the map f_n which is a required homeomorphism in this paper.

This reconstruction leads us to the construction of piecewise linear models of f_n and of its invariant foliations $\mathcal{F}_n^+$, $\mathcal{F}_n^-$ as in the next section, which are useful for estimating the dilatation of f_n.

§2. Piecewise Linear Models

We use the same notations as in the construction of S_n in §1. Let R_0 be the parallelogram $XYZU$ in $\mathbf{R}^2$, and n a positive integer. Put $R_n = [0,1]^2 \times \{1, 2, \cdots, n\} \cup R_0$ (disjoint union) and consider R_0 to be a subset of $[0,1]^2 \times \{0\}$. The compact surface $\tilde{S}_n$ is obtained from the rectangles $\bar{R}_0$, $\bar{T}_{n,1}$, $\cdots$, $\bar{T}_{n,n}$ by gluing together in a suitable way along their boundaries transverse to the flow (see §1). Collapsing each boundary component of $\tilde{S}_n$ to a point corresponds to collapsing each boundary component of $\bar{R}_0 \cup \bar{T}_{n,1} \cup \cdots \cup \bar{T}_{n,n}$ (disjoint union) tangent to the flow to a point. Then $\Sigma(n)$ is homeomorphic to the quotient space of R_n with respect to the following equivalence relation $\sim_n$. We define $\sim_n$ to be the equivalence relation on R_n generated by the following relations.

$$
\begin{aligned}
&(\begin{pmatrix} x \\ 0 \end{pmatrix}, k) \sim_n (\begin{pmatrix} x \\ 1 \end{pmatrix}, k) \quad (x \in [0,1],\ 1 \leq k \leq n), \\
&(\begin{pmatrix} 0 \\ y \end{pmatrix}, k) \sim_n \begin{cases} (\begin{pmatrix} 1 \\ y \end{pmatrix}, k+1) & (y \in [0, \frac{1}{2}],\ 0 \leq k \leq n-1), \\ ((1-2y)Y + 2yZ, 0) & (y \in [0, \frac{1}{2}],\ k = n), \end{cases} \\
&(\begin{pmatrix} 1 \\ y \end{pmatrix}, k) \sim_n \begin{cases} (\begin{pmatrix} 0 \\ y \end{pmatrix}, k+1) & (y \in [\frac{1}{2}, 1],\ 0 \leq k \leq n-1), \\ ((2-2y)X + (2y-1)U, 0) & (y \in [\frac{1}{2}, 1],\ k = n). \end{cases}
\end{aligned}
$$

A cell complex structure $\mathcal{C}$ of R_n is defined as follows,

0-cell : each corner point of R_n,
$\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix} \times \{k\}$, $\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} \times \{k\}$ for $0 \leq k \leq n$,
1-cell : each connected component of $\partial(R_n \setminus \{ \text{ 0-cells } \})$,
2-cell : each connected component of $Int(R_n)$.

Then we easily show the following lemma by a straightforward calculation.

LEMMA 2.1. *The quotient family $\mathcal{C}/\sim_n$ gives a cell complex structure of $R_n/\sim_n$ which has 3 vertices, $3n+2$ edges and $n+1$ faces. Then the Euler characteristic of $\Sigma(n)$ is equal to $2-2n$ which follows that $\Sigma(n)$ is a closed orientable surface of genus n.*

Since the flow lines of $\phi^t_{\bar{A}}$ are given by vertical lines in $T^2 \times \mathbf{R}$ and $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, we see that the first return mapping of S_n induces a homeomorphism f_n of $\Sigma(n)$ defined as follows. First define two maps φ_n and D_n from R_n to itself by

$$\varphi_n(\begin{pmatrix} x \\ y \end{pmatrix}, k)$$
$$= \begin{cases} (\begin{pmatrix} x \\ y \end{pmatrix}, k+1) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, 1 \leq k \leq n-1 \text{ or } \begin{pmatrix} x \\ y \end{pmatrix} \in R_0, k=0), \\ (\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, 1) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, k=n \text{ and } x+2y \leq 1), \\ (\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, k=n \text{ and } 1 < x+2y < 2), \\ (\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix}, 1) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, k=n \text{ and } 2 \leq x+2y), \end{cases}$$

and

$$D_n(\begin{pmatrix} x \\ y \end{pmatrix}, k)$$
$$= \begin{cases} (\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, n) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, k=n \text{ and } x+y \leq 1), \\ (\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}, n) & (\begin{pmatrix} x \\ y \end{pmatrix} \in [0,1]^2, k=n \text{ and } 1 < x+y), \\ (\begin{pmatrix} x \\ y \end{pmatrix}, k) & \text{otherwise.} \end{cases}$$

Since two maps are both compatible with the relation $\sim_n$, they induce homeomorphisms $\bar{\varphi}_n$, $\bar{D}_n$ of $\Sigma(n)$ respectively. Then we have $f_n = \bar{\varphi}_n \circ \bar{D}_n$.

We can also see that the invariant measured foliations $\mathcal{F}^+_n$, $\mathcal{F}^-_n$ of f_n are obtained as follows.

Let λ^+, λ^- be eigenvalues of the matrix $A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ with $\lambda^- < \lambda^+$. Let $\mathcal{F}^\sigma_A$ be the one-dimensional foliation of $\mathbf{R}^2$ whose leaves are all the lines parallel to the eigenspace of λ^σ $(\sigma = +, -)$. Copy them on $\mathbf{R}^2 \times \{k\}$ for each k and get a foliation $\mathcal{F}^\sigma_{A,n}$ of $\mathbf{R}^2 \times \{0, 1, 2, \cdots, n\}$. Then the restrictions of $\mathcal{F}^\sigma_{A,n}$ to R_n give rise to foliations $\mathcal{F}^\sigma_n$ $(\sigma = +, -)$ of $\Sigma(n)$.

Since both $\mathcal{F}_A^+$ and $\mathcal{F}_A^-$ are invariant under the associated linear transformation $A : \mathbf{R}^2 \to \mathbf{R}^2$, defined by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$, and

$$f_n(\begin{pmatrix} x \\ y \end{pmatrix}, k) = \begin{cases} (\begin{pmatrix} x \\ y \end{pmatrix}, k+1) \ \ (\text{ if } 0 \leq k \leq n-1 \text{ }), \\ (A \begin{pmatrix} x \\ y \end{pmatrix}, *) \ \ (\text{ if } \ k = n \text{ }), \end{cases}$$

the foliations $\mathcal{F}_n^{\pm}$ are preserved by f_n. Since f_n is a pseudo-Anosov homeomorphism, the foliations $\mathcal{F}_n^+$, $\mathcal{F}_n^-$ are automatically the invariant measured foliations of it ([5], [6]).

§3. An Estimate of Dilatations

Let μ_n^- be a transverse invariant measure of $\mathcal{F}_n^-$. For any segment γ in a leaf of $\mathcal{F}_n^+$, we can consider two kinds of lengths $\mu_n^-(\gamma)$ and $l_{euc}(\gamma)$ of γ, where $l_{euc}(\gamma)$ is the total sum of the Euclidean length of each connected component of the pre-image of γ in R_n.

DEFINITION 3.1. Let $r : [0,1]^2 \to \Sigma(n)$ be an immersion which is an embedding on its interior. The image $r([0,1]^2)$ is called an $\mathcal{F}_n$-rectangle if, for any $t \in [0,1]$, $r([0,1] \times \{t\})$ is a segment in a leaf of $\mathcal{F}_n^+$ and $r(\{t\} \times [0,1])$ is that in a leaf of $\mathcal{F}_n^-$. For any $t \in [0,1]$, $r([0,1] \times \{t\})$ (resp. $r(\{t\} \times [0,1])$) is called an $\mathcal{F}_n^+$-segment (resp. $\mathcal{F}_n^-$-segment) of the $\mathcal{F}_n$-rectangle $r([0,1]^2)$.

DEFINITION 3.2. Let $\mathcal{W} = \{W_1, W_2, \cdots, W_m\}$ be a family of $\mathcal{F}_n$-rectangles. A segment in a leaf of $\mathcal{F}_n^+$ (resp. $\mathcal{F}_n^-$) is called an $\mathcal{F}_n^+$-segment (resp. $\mathcal{F}_n^-$-segment) of $\mathcal{W}$, if it is an $\mathcal{F}_n^+$-segment (resp. $\mathcal{F}_n^-$-segment) of some $\mathcal{W}_i \in \mathcal{W}$.

Choose a Markov partition $\mathcal{W} = \{W_1, \cdots, W_m\}$ of $f_n : \Sigma(n) \to \Sigma(n)$ and fix it. Namely, each W_i is an $\mathcal{F}_n$-rectangle, $\cup_{i=1}^m W_i = \Sigma(n)$, and the following conditions are satisfied.

(M1) $Int(W_i) \cap Int(W_j) = \emptyset$ for $i \neq j$.

(M2) For any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$, $f_n(\gamma)$ is a finite union of $\mathcal{F}_n^+$-segments of $\mathcal{W}$.

(M3) For any $\mathcal{F}_n^-$-segment γ of W, $f_n^{-1}(\gamma)$ is a finite union of $\mathcal{F}_n^-$-segments of $\mathcal{W}$.

Lemma 3.3. *There exist positive constants C_1, C_2 such that, for any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$,*

$$C_1\mu_n^-(\gamma) \le l_{euc}(\gamma) \le C_2\mu_n^-(\gamma).$$

Proof. Take an immersion $r_i : [0,1]^2 \to \Sigma(n)$ defining W_i for any i. The length $l_{euc}(r_i([0,1] \times \{t\}))$ varies continuously with respect to the parameter t and is non-zero, since the foliation $\mathcal{F}_n^+$ is given by parallel Euclidean straight line segments on R_n and $W_i \cap ([0,1]^2 \times \{k\})$ ($k \in \{1, \cdots, n\}$) and $W_i \cap R_0$ are Euclidean convex polygons or finite point sets. Then there exist positive constants A_1, A_2 such that, for any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$, we have $A_1 \le l_{euc}(\gamma) \le A_2$. We also have $\mu_n^-(\gamma_1) = \mu_n^-(\gamma_2)$, for any $1 \le i \le m$ and any $\mathcal{F}_n^+$-segments γ_1, γ_2 of W_i. Then there also exist positive constants B_1, B_2 such that, for any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$, $B_1 \le \mu_n^-(\gamma) \le B_2$. So it suffices to take $C_1 = A_1/B_2$ and $C_2 = A_2/B_1$. $\square$

For any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$ and any positive integer k, $(f_n)^k(\gamma)$ is a finite union of $\mathcal{F}_n^+$-segments of $\mathcal{W}$ by the condition (M2) above. Moreover, any two different $\mathcal{F}_n^+$-segments of $\mathcal{W}$ can intersect only at their end points by the condition (M1) above. Then we have for any $\mathcal{F}_n^+$- segment of $\mathcal{W}$,

$$C_1\mu_n^-((f_n)^k(\gamma)) \le l_{euc}((f_n)^k(\gamma)) \le C_2\mu_n^-((f_n)^k(\gamma)).$$

Then the dilatation λ_n of f_n is given by

$$\lambda_n = \lim_{k\to\infty} l_{euc}((f_n)^k(\gamma))^{\frac{1}{k}}.$$

On the other hand, we can show that, for any $\mathcal{F}_n^+$-segment γ of $\mathcal{W}$,

$$\begin{aligned} l_{euc}((f_n)^n(\gamma)) &\le (2+\sqrt{3})l_{euc}(\gamma), \text{ and} \\ l_{euc}((f_n)^{n+1}(\gamma)) &\ge (2+\sqrt{3})l_{euc}(\gamma), \end{aligned}$$

because only $f_n|([0,1]^2 \times \{n\})$ expands the Euclidean length of an arc in $\mathcal{F}_n^+$ $2+\sqrt{3}$ times. Note that $\lambda^+ = 2+\sqrt{3}$. Then we have the following theorem.

THEOREM 3.4. *For any positive integer n, we have*

$$\frac{\log(2+\sqrt{3})}{n+1} \leq \lambda_n \leq \frac{\log(2+\sqrt{3})}{n}.$$

§4. Characteristic Polynomials

Since, for any $n \geq 1$, both $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$ are orientable, the dilatation λ_n of f_n is the leading eigenvalue of the homomorphism $(f_n)_* : H_1(\Sigma(n)) \to H_1(\Sigma(n))$ induced in the first homology (see, for example, [18]). The aim of this section is to prove the following theorem.

THEOREM 4.1. *Let χ_n be the characteristic polynomial of the homomorphism $(f_n)_* : H_1(\Sigma(n)) \to H_1(\Sigma(n))$. Then we have $\chi_n = \sum_{i=0}^{2n}(-1)^i x^i - 2x^n$.*

REMARKS. (1) The map $f_1 : \Sigma(1) \to \Sigma(1)$ is an orientation preserving Anosov homeomorphism of the torus $\Sigma(1)$ with only one fixed point given by $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in R_0$. We also see that it preserves transverse orientations of the invariant foliations. Then it is topologically conjugate to the Anosov diffeomorphism induced by a matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ which attains $\delta_1 = \delta_1^+$.

(2) Since $\chi_2 = x^4 - x^3 - x^2 - x + 1$, the map f_2 attains δ_2^+ (see §0). This map is not topologically conjugate to Zhirov's example, because f_2 preserves the transverse orientations of the invariant foliations but Zhirov' s not ([19]).

The rest of this section is devoted to the proof of the theorem above.

Let $a_1, b_1, \cdots, a_n, b_n$ be oriented simple closed curves in the oriented surface $\Sigma(n)$ as in Figure 4.1. For simplicity, we use the same symbols for the homology classes represented by them. Then $\{a_1, b_1, \cdots, a_n, b_n\}$ is a basis of the integral first homology group $H_1(\Sigma_n)$ of $\Sigma(n)$. For any $u, v \in H_1(\Sigma_n)$, let $u \cdot v$ denote the intersection number. This number is calculated as follows. Choose closed curves a, b which represent u, v respectively and are in general position. At each point p in the intersection $a \cap b$, define the integer $(a \cdot b)_p$ as in Figure 4.2. Then $u \cdot v = \sum_{p \in a \cap b}(a \cdot b)_p$.

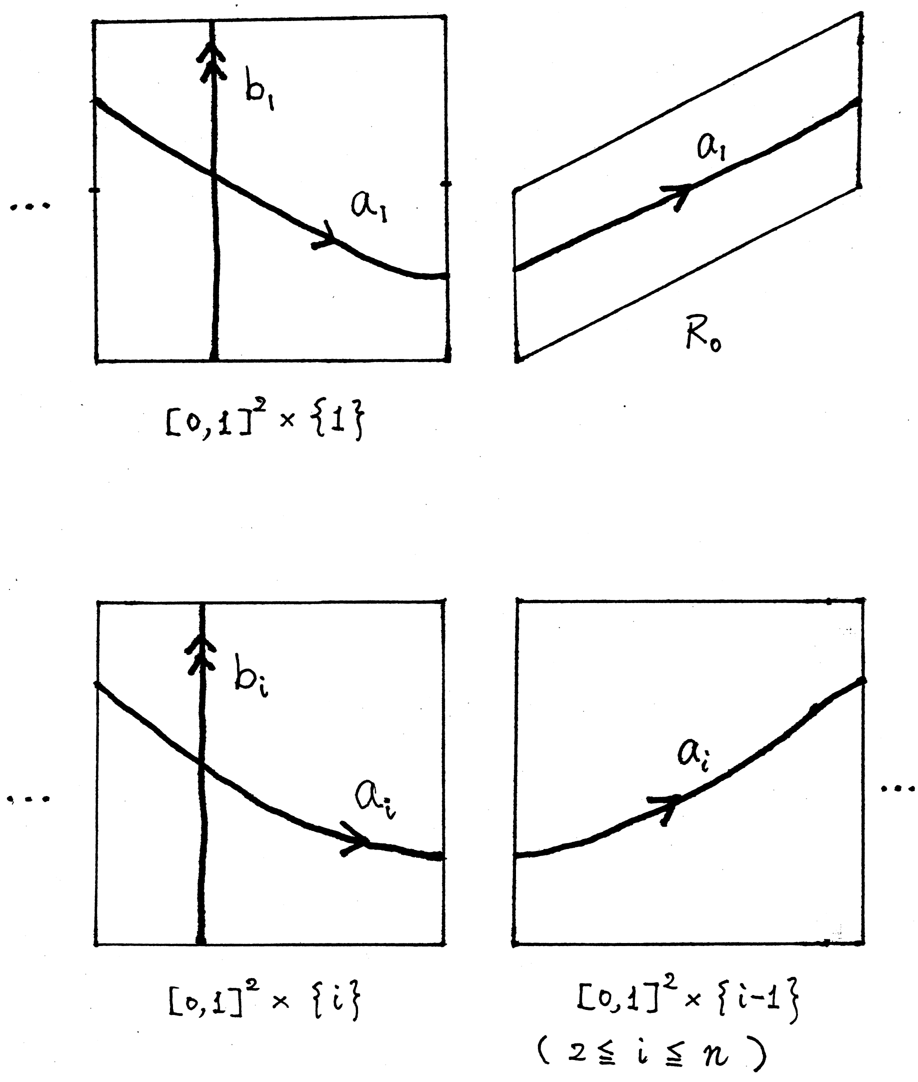

Figure 4.1.

Now let

$\bar{a}_1 = a_1,$
$\bar{b}_i = b_i$ for any $1 \leq i \leq n,$ and
$\bar{a}_i = \sum_{k=1}^{i}(-1)^{i-k}a_k + \sum_{k=1}^{i-1}(-1)^{i-k}b_k$ for any $2 \leq i \leq n.$

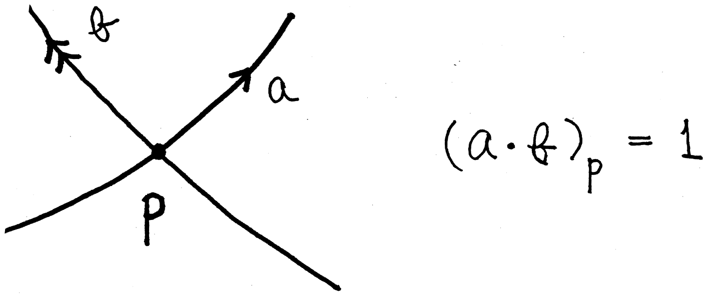

Figure 4.2.

Then we see the following lemma by a straightforward calculation.

LEMMA 4.2. *The collection* $\{\bar{a}_1, \bar{b}_1, \cdots, \bar{a}_n, \bar{b}_n\}$ *is a symplectic basis of* $H_1(\Sigma_n)$. *That is to say, for any* $i, j \in \{1, \cdots, n\}$, *we have that* $\bar{a}_i \cdot \bar{a}_i = 0$, $\bar{b}_i \cdot \bar{b}_i = 0$, *and* $\bar{a}_i \cdot \bar{b}_j = \delta_{ij}$, *where* δ_{ij} *denotes Kronecker's delta.*

By the definitions of $\bar{\varphi}_n$ and $\bar{D}_n$, we easily check the following.

$$\begin{aligned}
&(\bar{\varphi}_n)_*(a_i) = a_{i+1} \text{ for any } 1 \leq i \leq n-1,\\
&(\bar{\varphi}_n)_*(b_i) = b_{i+1} \text{ for any } 1 \leq i \leq n-1,\\
&(\bar{\varphi}_n)_*(b_n) = a_1 + b_1,\\
&(\bar{D}_n)_*(a_i) = a_i \text{ for any } 1 \leq i \leq n-1,\\
&(\bar{D}_n)_*(a_n) = a_n + b_n,\\
&(\bar{D}_n)_*(b_i) = b_i \text{ for any } 1 \leq i \leq n.
\end{aligned}$$

LEMMA 4.3.

$$(\bar{\varphi}_n)_*(a_n) = \begin{cases} \sum_{i=1}^n (-1)^{i+1} a_i & (\textit{ n is odd }),\\ \sum_{i=1}^n (-1)^i a_i + 2\sum_{i=1}^n (-1)^i b^k & (\textit{n is even }).\end{cases}$$

PROOF. Since the case $n = 1$ is easy, we show the case $n \geq 2$. By using the picture of $\bar{\varphi}_n(a_n)$ in Figure 4.3, we easily check the following.

$$\begin{aligned}
&\bar{a}_1 \cdot (\bar{\varphi}_n)_*(a_n) = a_1 \cdot (\bar{\varphi}_n)_*(a_n) = -1,\\
&a_i \cdot (\bar{\varphi}_n)_*(a_n) = 0 \qquad (\ 2 \leq i \leq n-1,\ n \geq 3),\\
&a_n \cdot (\bar{\varphi}_n)_*(a_n) = 1\\
&\bar{b}_i \cdot (\bar{\varphi}_n)_*(a_n) = b_1 \cdot (\bar{\varphi}_n)_*(a_n) = 0\ (\ 1 \leq i \leq n-1\ ),\\
&\bar{b}_n \cdot (\bar{\varphi}_n)_*(a_n) = b_n \cdot (\bar{\varphi}_n)_*(a_n) = -1
\end{aligned}$$

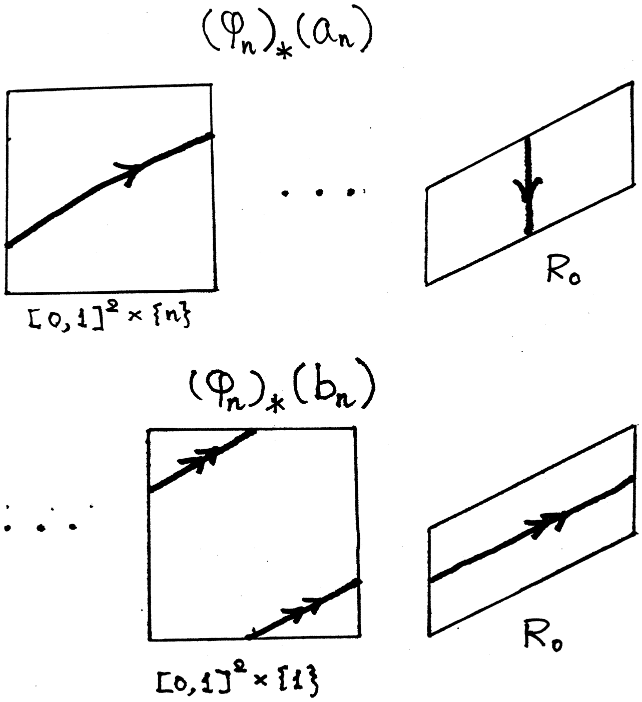

Figure 4.3.

Then we have

$$\bar{a}_i \cdot (\bar{\varphi}_n)_*(a_n) = (-1)^{i-1} a_1 \cdot (\bar{\varphi}_n)_*(a_n) = (-1)^i \ (2 \leq i \leq n-1,\ n \geq 3),$$
$$\bar{a}_n \cdot (\bar{\varphi}_n)_*(a_n) = a_n \cdot (\bar{\varphi}_n)_*(a_n) + (-1)^{n-1} a_1 \cdot (\bar{\varphi}_n)_*(a_n) = 1 + (-1)^n.$$

Since $\{\bar{a}_1, \bar{b}_1, \cdots, \bar{a}_n, \bar{b}_n\}$ is a symplectic basis, by these data, we have

$$\begin{aligned}
&(\bar{\varphi}_n)_*(a_n) \\
= \ & \bar{a}_n + \textstyle\sum_{i=1}^{n-1} (-1)^i \bar{b}_i + (1 + (-1)^n) \bar{b}_n \\
= \ & \textstyle\sum_{i=1}^{n} (-1)^{n-i} a_i + \sum_{i=1}^{n-1} (-1)^{n-i} b_i + \sum_{i=1}^{n-1} (-1)^i b_i + (1 + (-1)^n) b_n \\
= \ & \textstyle\sum_{i=1}^{n} (-1)^{n-i} a_i + \sum_{i=1}^{n} (-1)^i (1 + (-1)^n) b_i. \ \square
\end{aligned}$$

By Lemma 4.3 and the data just before it, we obtain the matrices representing $(\bar{\varphi}_n)_*$ and $(\bar{D}_n)_*$ with respect to the basis $\{a_1, b_1, \cdots, a_n, b_n\}$ of $H_1(\Sigma_n)$ as follows:

$(\bar{\varphi}_n)_*$:

$$\left(\begin{array}{c|cc} O_{2,2n-2} & 1 & 1 \\ & 0 & 1 \\ \hline & -1 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & 0 & 0 \\ I_{2n-2} & \vdots & \vdots \\ & -1 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & 0 & 0 \end{array}\right), \quad \left(\begin{array}{c|cc} O_{2,2n-2} & -1 & 1 \\ & -2 & 1 \\ \hline & 1 & 0 \\ & 2 & 0 \\ & -1 & 0 \\ & -2 & 0 \\ I_{2n-2} & \vdots & \vdots \\ & -1 & 0 \\ & -2 & 0 \\ & 1 & 0 \\ & 2 & 0 \end{array}\right)$$

n : odd $\qquad$ n : even

$(\bar{D}_n)_*$:

$$\left(\begin{array}{ccc|cc} & & & 0 & 0 \\ & I_{2n-2} & & \vdots & \vdots \\ & & & 0 & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{array}\right)$$

Here, we denote the $k \times l$ zero matrix by $O_{k,l}$ and the $m \times m$ identity matrix by I_m respectively. In order to prove Theorem 4.1, it suffices to show the following lemma.

LEMMA 4.4. *The characteristic polynomial of the matrix*

$$\left(\begin{array}{c|cc} O_{2,2n-2} & c_1 & d_1 \\ & c_2 & d_2 \\ \hline & c_3 & d_3 \\ I_{2n-2} & \vdots & \vdots \\ & c_{2n} & d_{2n} \end{array}\right)$$

is equal to

$$\begin{vmatrix} c_1 + c_3 x + \cdots + c_{2n-1} x^{n-1} - x^n & d_1 + d_3 x + \cdots + d_{2n-1} x^{n-1} \\ c_2 + c_4 x + \cdots + c_{2n} x^{n-1} & d_2 + d_4 x + \cdots + d_{2n} x^{n-1} - x^n \end{vmatrix}.$$

PROOF. Let n be a positive interger and P the $2n \times 2n$ matrix in the statement of this lemma. Define $2n \times 2n$ matrices $P(0), P(1), \cdots, P(2n-2)$ inductively as follows:

(1) Let $P(0) = P - xI_{2n}$.

(2) Let $P(k+1)$ be the matrix obtained from $P(k)$ by adding the $(2n-k)$-th row of $P(k)$ multiplied by x to the $(2n-k-2)$-th row for each $0 \leq k \leq 2n-3$.

Since the matrices $P(0), \cdots, P(2n-2)$ have the same determinants, the characteristic polynomial $\det(P - xI_{2n})$ of P is equal to the determinant of $P(2n-2)$. Since the matrix $P(2n-2)$ has the form $\left(\begin{array}{c|c} O_{2,2n-2} & Q \\ \hline I_{2n-2} & * \end{array}\right)$, where the 2×2 matrix Q has the form

$$\begin{pmatrix} c_1 + c_3 x + \cdots + c_{2n-1}x^{n-1} - x^n & d_1 + d_3 x + \cdots + d_{2n-1}x^{n-1} \\ c_2 + c_4 x + \cdots + c_{2n}x^{n-1} & d_2 + d_4 x + \cdots + d_{2n}x^{n-1} - x^n \end{pmatrix},$$

we have

$$\begin{aligned} &\det P(2n-2) \\ &= \det Q \\ &= \begin{vmatrix} c_1 + c_3 x + \cdots + c_{2n-1}x^{n-1} - x^n & d_1 + d_3 x + \cdots + d_{2n-1}x^{n-1} \\ c_2 + c_4 x + \cdots + c_{2n}x^{n-1} & d_2 + d_4 x + \cdots + d_{2n}x^{n-1} - x^n \end{vmatrix}. \quad \square \end{aligned}$$

Now, it is time to complete the proof of Theorem 4.1. For any positive integer n, the matrix M_n of $(f_n)_* = (\bar{\varphi}_n)_*(\bar{D}_n)_*$ relative to the basis $\{a_1, b_1, \cdots, a_{2n}, b_{2n}\}$ above has the form

$$\left(\begin{array}{c|cc} O_{2,2n-2} & 2 & 1 \\ & 1 & 1 \\ \hline & -1 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & 0 & 0 \\ I_{2n-2} & \vdots & \vdots \\ & -1 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & 0 & 0 \end{array}\right),$$

if n is odd, and

$$\left(\begin{array}{c|cc} O_{2,2n-2} & 0 & 1 \\ & -1 & 1 \\ \hline & 1 & 0 \\ & 2 & 0 \\ & -1 & 0 \\ & -2 & 0 \\ I_{2n-2} & \vdots & \vdots \\ & -1 & 0 \\ & -2 & 0 \\ & 1 & 0 \\ & 2 & 0 \end{array}\right)$$

if n is even. Then, by Lemma 4.4, the determinant of the matrix $M_n - xI_{2n}$ is equal to

$$\begin{vmatrix} 2 - x + \cdots + (-1)^i x^i + \cdots + x^{n-1} - x^n & 1 \\ 1 & 1 - x^n \end{vmatrix} = \sum_{i=0}^{2n} (-1)^i x^i - 2x^n$$

if n is odd, and

$$\begin{vmatrix} x - x^2 + \cdots + (-1)^{i-1} x^i + \cdots + x^{n-1} - x^n & 1 \\ -1 + 2x - \cdots + (-1)^{i-1} 2x^i + \cdots + 2x^{n-1} & 1 - x^n \end{vmatrix} = \sum_{i=0}^{2n} (-1)^i x^i - 2x^n$$

if n is even. This completes the proof of Theorem 4.1. □

REMARKS. (1)For any integer $n \geq 1$, Hironaka and Kin found a homeomorphism h_n of Σ_n whose dilatation is equal to that of f_n in this paper (see [11]). They found them in the study of a family of pseudo-Anosov braids. It seems that studying the relations between f_n and h_n is an interesting problem.

(2) Leininger found an example of pseudo-Anosov homeomorphism of Σ_5 whose dilatation is smaller than that of f_5 in this paper (see [12]).

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