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The L^p Boundedness of Wave Operators for Schrödinger Operators with Threshold Singuralities I. The Odd Dimensional Case

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Dedicated to Professor Barry Simon on the occasion of his 60th birthday

Abstract. Let $H = -\Delta + V(x)$ be an odd m-dimensional Schrödinger operator, $m \geq 3$, $H_0 = -\Delta$, and let $W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}$ be the wave operators for the pair (H, H_0). We say H is of generic type if 0 is not an eigenvalue nor a resonance of H and of exceptional type if otherwise. We assume that V satisfies $\mathcal{F}(\langle x\rangle^{-2\sigma}V) \in L^{m_*}$ for some $\sigma > \frac{1}{m_*}$, $m_* = \frac{m-1}{m-2}$. We show that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$ if V satisfies in addition $|V(x)| \leq C\langle x\rangle^{-m-2-\varepsilon}$ for some $\varepsilon > 0$ and if H is of generic type; and that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all p between $\frac{m}{m-2}$ and $\frac{m}{2}$ but not for p outside the closed interval $[\frac{m}{m-2}, \frac{m}{2}]$ if V satisfies $|V(x)| \leq C\langle x\rangle^{-m-3-\varepsilon}$ and if H is of exceptional type. This in particular implies that the continuous part of the propagator satisfies the L^p-L^q estimates $\|e^{-itH}P_c(H)u\|_p \leq C|t|^{\frac{1}{m}\left(\frac{1}{2}-\frac{1}{q}\right)}\|u\|_q$ for the dual exponents $\frac{1}{p}+\frac{1}{q}=1$ such that $1 \leq q \leq 2 \leq p \leq \infty$ if H is of generic type, and for $\frac{m}{m-2} < q \leq 2 \leq p < \frac{m}{2}$, $m \geq 5$, or $\frac{3}{2} < q \leq 2 \leq p < 3$, $m = 3$, if H of exceptional type.

1. Introduction

We begin with a brief review of scattering theory([17], [19], [1] and [21]). Let $H = -\Delta + V(x)$ be a Schrödinger operator on $\mathbf{R}^m$, $m \geq 1$, with real potentials $V(x)$ which satisfy

$$|V(x)| \leq C\langle x\rangle^{-\delta} \quad \text{for some} \quad \delta > 2 \tag{1.1}$$

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where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$. Then, H with domain $D(H) = H^2(\mathbf{R}^m)$, the Sobolev space of order 2, is selfadjoint in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^m)$ and $C_0^\infty(\mathbf{R}^m)$ is a core. The spectrum $\sigma(H)$ of H consists of absolutely continuous part $[0, \infty)$ and a finite number of non-positive eigenvalues $\{\lambda_j\}$ of finite multiplicities. We denote the point and the absolutely continuous subspaces for H by $\mathcal{H}_p(H)$ and $\mathcal{H}_{ac}(H)$ respectively, and the orthogonal projections onto the respective subspaces by $P_p(H)$ and $P_{ac}(H)$. The singular continuous spectrum and positive eigenvalues are absent from H. We write $H_0 = -\Delta$ for the free Schrödinger operator.

The wave operators $W_\pm = W_\pm(H, H_0)$ associated with the pair (H, H_0) are defined by the following strong limits in $\mathcal{H}$:

$$W_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}. \tag{1.2}$$

It is well known that the limits exist, $W_\pm$ are isometries and they are *asymptotically complete* in the sense that Image $W_\pm = \mathcal{H}_{ac}(H)$. It follows that

$$W_\pm^* W_\pm = I, \quad W_\pm W_\pm^* = P_{ac}(H), \tag{1.3}$$

where I is the identity operator on $\mathcal{H}$. One of the merits of the wave operators is the intertwining property: For Borel functions f on $\mathbf{R}$

$$f(H) P_{ac}(H) = W_\pm f(H_0) W_\pm^*. \tag{1.4}$$

For $\sigma \in \mathbf{R}$ we write $\mathcal{H}_\sigma$ for the weighted L^2-space $L^2(\mathbf{R}^m; \langle x \rangle^{2\sigma} dx)$. We say that H has threshold singularities or H is of exceptional type if

$$\mathcal{N} = \{u \in \mathcal{H}_{-\sigma} : (1 + (-\Delta)^{-1} V) u = 0\} \neq 0, \tag{1.5}$$

for some $\frac{1}{2} < \sigma < \delta - \frac{1}{2}$; if otherwise, we say that H is of generic type. When $m \geq 3$ and H is of generic type, we have shown that $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \leq p \leq \infty$:

$$\|W_\pm u\|_p \leq C \|u\|_p, \quad u \in L^p(\mathbf{R}^m) \cap L^2(\mathbf{R}^m) \tag{1.6}$$

under suitable smoothness and decay (at infinity) conditions on V (cf. [26], [27]). Here and hereafter we write $\|u\|_p$ for $\|u\|_{L^p}$. If $m = 2$, the same estimate (1.6) holds except at the end points $p = 1$ and $p = \infty$ under a similar generic type condition ([28]). If $m = 1$, (1.6) holds without this condition for $1 < p < \infty$ ([25], [3], [7]). The intertwining property (1.4) and the estimate (1.6) then imply that for $1 \le p, q \le \infty$ (or $1 < p, q < \infty$ if $m = 1$ or $m = 2$) and Borel functions f

$$\|f(H)P_{ac}(H)\|_{\mathbf{B}(L^p,L^q)} \le C_{pq}\|f(H_0)\|_{\mathbf{B}(L^p,L^q)}, \tag{1.7}$$

with constants C_{pq} which are independent of f. This provides a method for estimating L^p-L^q bounds of functions $f(H)P_{ac}(H)$ of continuous part of H by reducing them to those of the free Schrödinger operator $f(H_0)$ which is the Fourier multiplier by rotationally invariant function $f(\xi^2)$. Note that the point spectral part $f(H)P_p$ is a finite rank operator $\sum_{j=1}^{N} f(\lambda_j)\phi_j \otimes \phi_j$ given in terms of the eigenvalues λ_j and the corresponding normalized eigenfunctions ϕ_j of H, and its L^p continuity properties are easy to establish, thanks to the well known smoothness and decay properties of ϕ_j (see e.g. [2], [6]).

The purpose of this paper is to extend this result on wave operators to *the case when H does have threshold singularities* and, at the same time, to relax the conditions on V in the previous papers for generic case. As even dimensional cases are slightly more complex, though the main idea is similar, we exclusively deal with odd dimensional cases $m \ge 3$ in this paper, postponing the discussion on the former cases to the forthcoming paper [9].

THEOREM 1.1. *Let $m \ge 3$ be odd and $m_* = \frac{m-1}{m-2}$. Assume that*

$$\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbf{R}^m), \ \text{for some } \sigma > 1/m_*. \tag{1.8}$$

Then we have the following statements:

(1) *If V satisfies $|V(x)| \le C\langle x\rangle^{-(m+2+\varepsilon)}$ for some $\varepsilon > 0$ in addition and if H is of generic type, $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$.*

(2) *If V satisfies $|V(x)| \le C\langle x\rangle^{-(m+3+\varepsilon)}$ for some $\varepsilon > 0$ in addition and if H is of exceptional type, $W_\pm$ are bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2} < p < \frac{m}{2}$ when $m \ge 5$ and for $\frac{3}{2} < p < 3$ when $m = 3$.*

Statement (1) improves the decay and the smoothness conditions on the potential in the previous results ([26], [27]) for generic case. The condition (1.8) requires some smoothness of V and such condition is vital, in addition to certain decay condition, for the L^p boundedness of $W_\pm$ for all $1 \le p \le \infty$. This has been pointed out by Golberg and Visan ([13]), who give an example of compactly supported C^α, $\alpha < \frac{m-3}{2}$, potential V for which H is of generic type and for which $e^{-itH}P_c(H)$ violates (1.9) below for $q = 1$ and $p = \infty$. One can check that this example nearly misses the condition (1.8). We should also remark that it is long known from the local decay results of the propagator $e^{-itH}P_c(H)$ ([15], [22]) that $W_\pm$ cannot be bounded in L^p for p outside the interval $[m/(m-2), m/2]$, however, we still do not know if this is the case at the end points except for $m = 3$ (see below).

Applying estimate (1.7) for the family of functions $f(\lambda) = e^{-it\lambda}$, $-\infty < t < \infty$ we obtain the following L^p-L^q estimates for $e^{-itH}P_c(H)$, the continuous part of the propagator.

Theorem 1.2. *Let V be as in* Theorem 1.1 *and let $p \ge q$ be dual exponents of each other: $\frac{1}{p} + \frac{1}{q} = 1$. Then, the L^p-L^q estimates*

$$\|e^{-itH}P_{ac}u\|_p \le C_{pq}|t|^{m\left(\frac{1}{2}-\frac{1}{q}\right)}\|u\|_q, \quad t \ne 0 \tag{1.9}$$

is satisfied for all $2 \le p \le \infty$ if H is of generic type; and, if H is exceptional type, for $2 \le p < \frac{m}{2}$ when $m \ge 5$ and for $2 \le p < 3$ when $m = 3$.

When H is of generic type, estimate (1.9) is known for more general class of potentials in lower dimensions $1 \le m \le 3$ ([12], [11]), however, for higher dimensions $m \ge 4$, to the best knowledge of the author, Theorem 1.2 is so far the best result with respect to both decay and smoothness conditions on the potentials. When H is of exceptional type and $m = 3$, it is known that (1.9) holds if and only if $\frac{3}{2} < q \le 2 \le p < 3$ with $\frac{1}{p} + \frac{1}{q} = 1$ and at the end point

$$\|e^{-itH}P_c u\|_{L^{3,\infty}} \le C_p t^{-\frac{1}{2}}\|u\|_{L^{\frac{3}{2},1}} \tag{1.10}$$

replaces (1.9), where $L^{p,q}$ are Lorentz spaces([8], [29]). It follows that $W_\pm$ are not bounded in $L^p(\mathbf{R}^3)$ for $p = \frac{3}{2}$ and $\frac{3}{2}$ if H is of exceptional type.

We use the following notation and conventions. For $u \in \mathcal{H}_{-\gamma}$ and $v \in \mathcal{H}_\gamma$ $\langle u, v\rangle = \int_{\mathbf{R}^n} \overline{u(x)}v(x)dx$ is the standard coupling of functions; $|u\rangle\langle v| = u \otimes v$ will be interchangeably used to denote the rank 1 operator $\phi \mapsto \langle v, \phi\rangle u$. We write $R_0(z) = (H_0 - z)^{-1}$ and $R(z) = (H - z)^{-1}$ for resolvents of H_0 and H

respectively; $\Sigma = S^{m-1}$ for the unit sphere in $\mathbf{R}^m$. The upper half complex plane is denoted by $\mathbf{C}^+ = \{z \in \mathbf{C} : \Im z > 0\}$ and $\overline{\mathbf{C}}^+ = \{z \in \mathbf{C} : \Im z \geq 0\}$. We parametrize $z \in \mathbf{C} \setminus [0, \infty)$ by $z = \lambda^2$, $\lambda \in \mathbf{C}^+$, and define $G_0(\lambda) = R_0(\lambda^2)$ and $G(\lambda) = R(\lambda^2)$. For Banach spaces X and Y, $\mathbf{B}(X, Y)$ (resp. $\mathbf{B}_\infty(X, Y)$) is the Banach space of bounded (resp. compact) operators from X to Y, $\mathbf{B}(X) = \mathbf{B}(X, X)$ (resp. $\mathbf{B}_\infty(X) = \mathbf{B}_\infty(X, X)$). The identity operator is denoted by 1. The norm of L^p-spaces, $1 \leq p \leq \infty$, is denoted by $\|u\|_p = \|u\|_{L^p}$. We write $\mathcal{S}(\mathbf{R}^m)$ for the space of rapidly decreasing functions. The Fourier transform is defined by

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbf{R}^m} e^{-ix\xi} u(x) dx$$

and $\mathcal{F}^* u(\xi) = \mathcal{F}u(-\xi)$ is the conjugate Fourier transform. For functions f on the line $f^{(j)}$ is the j-th derivative of f, $j = 1, 2, \ldots$. For $a \in \mathbf{R}$, a_+ or a_- is an arbitrary number larger or smaller than a respectively; $[a]$ is the largest integer not larger than a.

The plan of this paper is as follows. Section 2 is a preparatory section and we collect results on resolvents $G_0(\lambda)$ and $G(\lambda)$ for real λ; we show, in particular, that $(1 + G_0(\lambda)V)^{-1}$ has an expansion as $\lambda \to 0$ in the form

$$(1.11) \qquad (1 + G_0(\lambda)V)^{-1} = 1 + A_0(\lambda) + \lambda^{-1} A_{-1} + \lambda^{-2} P_0 V$$

where P_0 is the projection to the 0-eigenspace of H; A_{-1} a finite rank operator if $m = 3$ or $m = 5$ and $A_{-1} = 0$ if $m \geq 7$; and $A_0(\lambda)$ is a sufficiently smooth function of λ including $\lambda = 0$ in suitable operator topologies (see Theorem 2.12). If H is of generic type, the last two terms $\lambda^{-1} A_{-1} + \lambda^{-2} P_0 V$ are absent and $A_0(\lambda)$ satisfies the same property under a weaker condition $|V(x)| \leq C\langle x \rangle^{-(m+2+\varepsilon)}$. Most results of Section 2 are well known, however, we shall sketch rather elementary proofs for readers' convenience.

We prove Theorem 1.1 in Section 3. We prove it only for W_-. The proof for W_+ is similar. As in [26] we write $W = W_-$ in terms of resolvents:

$$(1.12) \quad Wu = u - \frac{1}{\pi i} \int_0^\infty G_0(\lambda) V (1 + G_0(\lambda)V)^{-1} (G_0(\lambda) - G_0(-\lambda)) \lambda d\lambda;$$

we split W into the low and high energy parts: $W = W\Phi^2(H_0) + W\Psi^2(H_0) \equiv W_< + W_>$, by using cut-off functions $\Phi \in C_0^\infty(\mathbf{R})$ and $\Psi \in$

$C^\infty(\mathbf{R})$ such that $\Phi(\lambda)^2 + \Psi(\lambda)^2 \equiv 1$ and such that, for a suitable constant $\lambda_0 > 0$, $\Phi(\lambda^2) = 1$ for $|\lambda| \le \lambda_0/2$ and $\Phi(\lambda^2) = 0$ for $|\lambda| \ge \lambda_0$; and we study $W_<$ and $W_>$ separately. We recall $\Phi(H_0)$ and $\Phi(H)$ are integral operators with integral kernels bounded by $C_N\langle x-y\rangle^{-N}$ for $N = 1, 2, \ldots$ (see Lemma 2.2 of [27]). By virtue of the intertwining property, we have $W_> = \Psi(H)W\Psi(H_0)$ and $W_< = \Phi(H)W\Phi(H_0)$. For studying the low energy part we insert the expansion formula (1.11) for $(1 + G_0(\lambda)V)^{-1}$ into (1.12). This produces $W_< = \Phi(H)(1 - (W_{r,0} + W_r + W_{s,1} + W_{s,2}))\Phi(H_0)$ where

$$W_{r,0} = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)V(G_0(\lambda) - G_0(-\lambda))\lambda d\lambda, \tag{1.13}$$

$$W_r = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)VA_0(\lambda)(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda d\lambda, \tag{1.14}$$

$$W_{s,1} = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)VA_{-1}(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)d\lambda, \tag{1.15}$$

$$W_{s,2} = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)VP_0V(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda^{-1}d\lambda, \tag{1.16}$$

and $\tilde{\Phi}(\lambda) \in C_0^\infty(\mathbf{R})$ is such that $\tilde{\Phi}(\lambda)\Phi(\lambda^2) = \Phi(\lambda^2)$. We have shown in [26] that $\|W_{r,0}u\|_p \le C\|\mathcal{F}(\langle x\rangle^\sigma V)\|_{L^{m_*}}\|u\|_p$ for any $1 \le p \le \infty$ if $\sigma > \frac{1}{m_*}$. In subsection 3.1, we prove that $\Phi(H)W_r\Phi(H_0)$ is bounded in $L^p(\mathbf{R}^m)$ for any $1 \le p \le \infty$ by showing that its integral kernel $K(x,y)$ satisfies the condition

$$\sup_{x\in\mathbf{R}^m}\int_{\mathbf{R}^m}|K(x,y)|dy + \sup_{y\in\mathbf{R}^m}\int_{\mathbf{R}^m}|K(x,y)|dx < \infty \tag{1.17}$$

by adapting the method of [26] to this situation. We study $W_{s,1}$ and $W_{s,2}$ in subsection 3.2. We study the case $m = 3$ in paragraph 3.2.1. Thanks to the particularly simple structure of the integral kernel of $G_0(\lambda)$ in three dimension,

$$G_0(\lambda)u(x) = \int_{\mathbf{R}^3}\frac{e^{i\lambda|x-y|}}{4\pi|x-y|}u(y)dy,$$

we may write $W_{s,1}$ as a linear combination of

$$F_{jk}u(x) = \int_{\mathbf{R}^3}\frac{(V\phi_j)(y)}{4\pi|x-y|}T_k(|x-y|)dy, \quad j,k = 1,\ldots,d. \tag{1.18}$$

Here ϕ_1 is the canonical resonance and $\{\phi_2, \dots, \phi_d\}$ is the orthonormal basis of 0-eigenspace of H; $T_k(s)$ is defined by $T_k(s) = 2^{-1}(1+\tilde{\mathcal{H}})(rM_k)(s)$ by using the Hilbert transform $\tilde{\mathcal{H}}$ and the spherical average $M_k(r)$ of $(V\phi_k)*\check{u}$, $\check{u}(x) = u(-x)$:

$$M_k(r) = \int_\Sigma ((V\phi_k) * \check{u})(r\omega)d\omega. \tag{1.19}$$

It follows that, with a constant $C > 0$,

$$\|W_{s,1}u\|_p \le C\sum_{jk} \|V\phi_j\|_1 \left(\int_0^\infty r^{2-p}|T_k(r)|^p dr\right)^{\frac{1}{p}}. \tag{1.20}$$

It is well known that $|r|^a$ is an one dimensional $(A)_p$ weight if and only if $-1 < a < p-1$ and the Hilbert transform $\tilde{\mathcal{H}}$ is bounded in $L^p(\mathbf{R}, w(r)dr)$ if w is an $(A)_p$ weight ([24]). It follows that r^{2-p} is $(A)_p$ weight if and only if $3/2 < p < 3$, and for these p's the weighted inequality for $\tilde{\mathcal{H}}$ implies that

$$\begin{aligned} \|W_{s,1}u\|_p &\le C\sum_{jk} \|V\phi_j\|_1 \left(\int_{-\infty}^\infty |r|^2|M_k(r)|^p dr\right)^{\frac{1}{p}} \\ &\le C\sum_{jk} \|V\phi_j\|_1\|V\phi_k * \check{u}\|_p \le C\sum_{jk}\|V\phi_j\|_1\|V\phi_k\|_1\|u\|_p. \end{aligned} \tag{1.21}$$

For studying $W_{s,2}$, we replace λ^{-1} by the operator $|D|^{-1}$ by using that

$$\langle V\phi_j, (G_0(\lambda) - G_0(-\lambda))\lambda^{-1}u\rangle = \langle |D|^{-1}V\phi_j, (G_0(\lambda) - G_0(-\lambda))u\rangle.$$

Then, the computation used for $W_{s,1}$ implies that $W_{s,2}u$ is a linear combination of $\tilde{F}_{jk}u$, $j,k = 2, \dots, d$, which are given by the right side of (1.18) with $|D|^{-1}V\phi_k$ in place of $V\phi_k$ in the right of (1.19). We apply to $W_{s,2}u$ the estimates for $W_{s,1}u$ upto the pre-final stage of (1.21), where $V\phi_k * \check{u}$ is replaced by $(|D|^{-1}V\phi_k)*\check{u}$. Since eigenfunctions $\phi_2, \dots, \phi_d$ satisfy $\int V(x)\phi_k(x)dx = 0$,

$$|D|^{-1}V\phi_k(x) = C\int_{\mathbf{R}^3}\left(\frac{1}{|x-y|^2} - \frac{1}{|x|^2}\right)V(y)\phi_k(y)dy = \sum_{j=1}^3 \frac{C_j x_j}{|x|^4} + \rho_k(x)$$

with integrable ρ_k. Then $\||D|^{-1}(V\phi_j) * u\|_p \le C\|u\|_p$ by the Calderón-Zygmund theorem and $\|W_{s,2}u\|_p \le C\|u\|_p$. We study $W_{s,1}$ and $W_{s,2}$ in dimension $m \geq 5$ in paragraph 3.2.2. Since the kernel of $G_0(\lambda)$ becomes more complex as dimension m becomes larger, the argument becomes a bit more complicated, however, the basic idea still works. At the end of Section 3, we prove that $W_>$ is bounded in L^p for any $1 \leq p \leq \infty$ if $|V(x)| \leq C\langle x\rangle^{-(m+2+\varepsilon)}$. Because the high energy part $W_>$ is insensitive to threshold singularities, basically the same argument as in [26] applies. However, we improve some argument and substantially relax the decay and smoothness conditions on V of [26] and [27].

2. Preliminaries

The resolvents $G_0(\lambda) = R_0(\lambda^2)$ and $G(\lambda) = R(\lambda^2)$ are $\mathbf{B}(\mathcal{H})$ valued analytic or meromorphic functions of $\lambda \in \mathbf{C}^+$. The limiting absorption principle, LAP for short, says that they have continuous extension upto the boundary $\mathbf{R}$ (or $\mathbf{R} \setminus \{0\}$) when considered as, say, a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$ valued function with $\sigma, \tau > 1/2$. We denote such extensions again by $G_0(\lambda)$ and $G(\lambda)$. For studying $W_\pm$ we shall express them in terms of the boundary values of $G_0(\lambda)$ and $G(\lambda)$ on the reals and use mapping properties of these operators and their derivatives. We shall also need some information on the singularities of $G(\lambda)$ at $\lambda = 0$ when H has threshold singularities. We collect here some well known results on these matters. In what follows we assume $m = 2\nu + 1$ is odd.

2.1. Limiting absorption principle

In this subsection we assume $|V(x)| \leq C\langle x\rangle^{-\delta}$ with $\delta > 2$ unless otherwise stated. It is well known that the Fourier transform $\mathcal{F}$ is an isomorphism between $L^2_s(\mathbf{R}^m)$ and $H^s(\mathbf{R}^m)$ and $C_0^\infty(\mathbf{R}^m \setminus \{0\})$ is dense in $H^s(\mathbf{R}^m)$ if $s < \frac{m}{2}$. Since $\nu = \frac{m-1}{2}$ is an integer for odd m, it then follows by virtue of Hardy's inequality that the operator $\tilde{\Gamma}_0 : u \mapsto \lambda^\nu \hat{u}(\lambda\omega)$ is bounded from $L^2_s(\mathbf{R}^m)$ to $H^s(\mathbf{R}, L^2(\Sigma))$ for any $s \geq 0$, hence, by the Sobolev embedding theorem, to $C^{s-\frac{1}{2}}(\mathbf{R}, L^2(\Sigma))$ if $s > 1/2$. If follows that, if $s > 1/2$, the operator valued function $\Gamma(\lambda)$: $\Gamma(\lambda)u(\omega) = \lambda^{(m-1)/2}\hat{u}(\lambda\omega)$ is a $\mathbf{B}(L^2_s, L^2(\Sigma))$-valued, and therefore, $\Gamma(\lambda)^*\Gamma(\lambda)$ is a $\mathbf{B}(L^2_s, L^2_{-s})$-valued $C^{s-\frac{1}{2}}$ function of $\lambda \in \mathbf{R}$ which is bounded along with the derivatives upto the order $s - 1/2$.

By using the polar coordinates we have for $u, v \in \mathcal{S}(\mathbf{R}^m)$ that

$$(G_0(\lambda)u, v) = \int_{\mathbf{R}^m} \frac{\hat{u}(\xi)\overline{\hat{v}(\xi)}}{|\xi|^2 - \lambda^2} d\xi = \int_0^\infty \frac{\mu^{m-1}}{\mu^2 - \lambda^2} \left(\int_\Sigma \hat{u}(\mu\omega)\overline{\hat{v}(\mu\omega)} d\omega \right) d\mu$$
$$= \int_0^\infty \frac{\mu^{m-1}(\hat{u}(\mu\cdot), \hat{v}(\mu\cdot))_{L^2(\Sigma)}}{\mu^2 - \lambda^2} d\mu, \quad \lambda \in \mathbf{C}^+.$$

It follows that the free resolvent $G_0(\lambda)$, $\Im\lambda > 0$, may be written in the form

$$G_0(\lambda) = \int_0^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)}{\mu^2 - \lambda^2} d\mu = \int_0^\infty \frac{\mu^{m-1}A(\mu)}{\mu^2 - \lambda^2} d\mu \tag{2.1}$$

where $A(\lambda)$ is defined by the equation $\lambda^{2\nu} A(\lambda) = \Gamma(\lambda)^*\Gamma(\lambda)$, viz.

$$A(\lambda)u(x) = \frac{1}{(2\pi)^m} \int_\Sigma \int_{\mathbf{R}^m} e^{i\lambda\omega(x-y)} u(y) dy d\omega. \tag{2.2}$$

We define $A(\lambda)$ for $\lambda < 0$ by the right side of (2.2). It is obvious that $A(\lambda)u(x)$ is smooth in λ for $u \in \mathcal{S}(\mathbf{R}^m)$ and $A(\lambda) = A(-\lambda)$. Thus, when m is odd, $\Gamma(\lambda)^*\Gamma(\lambda)$ is also a smooth even function of λ and we may rewrite (2.1) in either of forms

$$G_0(\lambda)u = \frac{1}{2\lambda} \int_{-\infty}^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)u}{\mu - \lambda} d\mu, \quad \lambda \in \mathbf{C}^+, \tag{2.3}$$

$$G_0(\lambda)u = \frac{1}{2} \int_{-\infty}^\infty \frac{\Gamma(\mu)^*\Gamma(\mu)u}{\mu(\mu - \lambda)} d\mu. \quad \lambda \in \mathbf{C}^+. \tag{2.4}$$

It is well known that, if $f \in C_0^s(\mathbf{R})$, $0 < s < \infty$, then the Cauchy integral

$$\int_{-\infty}^\infty \frac{f(y)}{x - y} dy, \quad \Im x > 0,$$

can be extended to the closed half plane $\overline{\mathbf{C}}^+$ as a function of class C^{s-} (Privaloff's theorem). The following is elementary.

Lemma 2.1. *We have the following statements:*

(1) *Let $\sigma, \tau > 0$ be non integral and let $\sigma + \tau > 1$. Suppose that $f \in C^\sigma(\mathbf{R})$, $g \in C^\tau(\mathbf{R})$ and $f(0) = g(0) = 0$. Then, $h(x)$ defined by*

$$h(x) = \begin{cases} x^{-1} f(x) g(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is of class $C^\rho(\mathbf{R})$ for $\rho = \min(\sigma, \tau, \sigma + \tau - 1)$.

(2) *Let* $s = k + \sigma$, $t = \ell + \tau$, $k, \ell = 0, 1, \dots$ *and* $\sigma, \tau > 0$ *be non integral such that* $\sigma + \tau > 1$. *Suppose* $f \in C^s(\mathbf{R})$ *and* $g \in C^t(\mathbf{R})$ *are such that* $f^{(j)}(0) = 0$ *for* $0 \le j \le k$ *and* $g^{(j)}(0) = 0$ *for* $0 \le j \le \ell$. *Then* $f(x)g(x)/x^{k+\ell+1}$ *is of class* C^ρ, ρ *being as in* (1).

PROOF. (1) We may assume $\sigma \le \tau$. Let $0 < \sigma, \tau < 1$ first. Then $0 < \rho = \sigma + \tau - 1 < 1$ and it suffices to show that $|h(x) - h(y)| \le C|x - y|^\rho$ when $0 \le x < y < 1$. If $|x - y| \ge y/2$, we have

$$\begin{aligned} |h(x) - h(y)| &\le |h(x) - h(0)| + |h(y) - h(0)| \\ &\le C(x^\rho + y^\rho) \le C|x - y|^\rho. \end{aligned} \tag{2.5}$$

If $|x - y| < y/2$, then $y/2 < x < y$, $y^{-1} < x^{-1} < |x - y|^{-1}$ and $|h(x) - h(y)|$ is bounded by

$$\begin{aligned} &\frac{|f(x)g(x)||x - y|}{xy} + \frac{|f(x) - f(y)||g(x)|}{y} + \frac{|f(y)||g(x) - g(y)|}{y} \\ &\le \left(\frac{x^{\sigma+\tau}|x - y|}{|x|^2} + \frac{|x - y|^\sigma |x|^\tau}{y} + \frac{y^\sigma |x - y|^\tau}{y} \right) \le C|x - y|^\rho. \end{aligned} \tag{2.6}$$

Next suppose that $0 < \sigma < 1 < \tau$. Then $\rho = \sigma$ and estimates (2.5) and (2.6) with $\tau = 1$ implies $|h(x) - h(y)| \le C|x - y|^\rho$ for this case. This show that statement (1) holds if $0 < \sigma < 1$. Next let $1 < \sigma \le \tau$. Then, $\rho = \sigma$ and on the right of

$$\begin{aligned} h'(x) = (f'(x) - f'(0))\frac{g(x)}{x} &+ \frac{f(x)}{x}(g'(x) - g(0)) \\ &- \frac{f(x) - xf'(0)}{x} \cdot \frac{g(x) - xg'(0)}{x} + f'(0)g'(0), \end{aligned} \tag{2.7}$$

$(f(x) - xf'(0))/x$, $f'(x) - f'(0)$ and $f(x)/x$ are of class $C^{\sigma-1}$, and $(g(x) - xg'(0))/x$, $g'(x) - g'(0)$ and $g(x)/x$ are of class $C^{\tau-1}$. Statement (1) follows. (2) By Taylor's formula $f(x)/x^k = \int_0^1 (1 - \theta)^{k-1} f^{(k)}(\theta x) d\theta/(k - 1)!$, and $f(x)/x^k$ is of class C^σ and vanishes at $x = 0$. Likewise $g(x)/x^\ell$ is of class C^τ and vanishes at $x = 0$. Statement (2) follows from (1). $\square$

LEMMA 2.2. (1) *Let* $\sigma > 1/2$ *and* $k = 0, 1, 2$. *Then,* $\langle x \rangle^{-\sigma} G_0(\lambda) \langle x \rangle^{-\sigma}$ *is a* $\mathbf{B}_\infty(H^t, H^{t+k})$*-valued* $C^{(\sigma - \frac12)_-}$ *function of* $\lambda \in \overline{\mathbf{C}}^+ \setminus \{0\}$ *for any* $t \in \mathbf{R}$. *For* $j = 0, 1, \dots$

$$\|\langle x \rangle^{-\sigma-j} \partial_\lambda^j G_0(\lambda) \langle x \rangle^{-\sigma-j}\|_{\mathbf{B}(H^t, H^{t+k})} \le C_{jt} |\lambda|^{-1+k}, \quad |\lambda| \ge 1. \tag{2.8}$$

(2) *Let* $\sigma, \tau > 1/2$ *and* $\sigma + \tau > 2$. *Then,* $\langle x\rangle^{-\sigma} G_0(\lambda)\langle x\rangle^{-\tau}$ *is a* $\mathbf{B}_\infty(\mathcal{H})$-*valued* C^ρ *function of* $\lambda \in \overline{\mathbf{C}}^+$ *for any* ρ *such that* $\rho < \min(\tau+\sigma-2, \tau-1/2, \sigma-1/2)$. *If* $\rho = j+\kappa$, $j = [\rho]$ *and* $0 \le \kappa < 1$, *we have*

$$\sup_{\lambda\in\overline{\mathbf{C}}^+} \|\langle x\rangle^{-\sigma} G_0^{(j)}(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}} + \sup_{\lambda\neq\mu} \frac{\|\langle x\rangle^{-\sigma}(G_0^{(j)}(\lambda) - G_0^{(j)}(\mu))\langle x\rangle^{-\tau}\|_{\mathbf{B}}}{|\lambda-\mu|^\kappa} \le C.$$

Proof. Without losing generality we may assume σ and τ are non integral. Since $\langle x\rangle^{\sigma}\langle p\rangle^{-\tau}\langle x\rangle^{-\sigma}$, $p = -i\partial/\partial x$ being the momentum operator, is bounded (or has a unique bounded extension) in $\mathcal{H}$ for any $\tau \ge 0$ and $\sigma \in \mathbf{R}$ and since $-\Delta$ commutes with ∂_x, the first statement for $k=0$ follows immediately from (2.3) by using Privaloff's theorem. We have $(-\Delta)G_0(\lambda) = \lambda^2 G_0(\lambda) + 1$ and the statement for $k = 1, 2$ follows from that for $k = 0$ and interpolation. The second statement follows from (2.4) and Lemma 2.1 if we notice $\tilde\Gamma_0 L^2_\sigma(\mathbf{R}^m) \subset H^\sigma_0(\mathbf{R}\setminus\{0\})$ for $\sigma < m/2$, and, therefore, $\Gamma(0)u = 0$ for any $u \in L^2_\sigma(\mathbf{R}^m)$ if $\sigma > 1/2$. $\square$

Corollary 2.3. *Let* $1/2 < \gamma < \delta - 1/2$. *Then,* $\langle x\rangle^{-\gamma} G_0(\lambda) V \langle x\rangle^{+\gamma}$ *is a* $\mathbf{B}_\infty(\mathcal{H})$-*valued* C^ρ *function of* $\overline{\mathbf{C}}^+$ *for any* $\rho < \min(\gamma - \frac12, \delta-\gamma-\frac12, \delta-2)$. *The operator valued function* $\langle x\rangle^{+\gamma} V G_0(\lambda)\langle x\rangle^{-\gamma}$ *satisfies the same property.*

If $|V(x)| \le C\langle x\rangle^{-\delta}$ for $\delta > 2$, $H = -\Delta + V$ has no positive eigenvalues (see [17]) and the point spectral subspace $\mathcal{H}_p(H)$ for H is finite dimensional. Thus $G(\lambda) = (H-\lambda^2)^{-1}$ is a $\mathbf{B}(\mathcal{H})$-valued meromorphic function of $\lambda \in \mathbf{C}^+$ with possible poles $i\kappa_1, \dots, i\kappa_n$ on the imaginary axis such that $-\kappa_1^2, \dots, -\kappa_n^2$ are eigenvalues of H. It follows by the resolvent equation that $1 + G_0(\lambda)V$ is invertible in $\mathbf{B}(\mathcal{H})$ for $\lambda \in \mathbf{C}^+$ outside the poles and

$$G(\lambda) = (1 + G_0(\lambda)V)^{-1} G_0(\lambda). \tag{2.9}$$

Here $G_0(\lambda)V$ extends to $\lambda \in \overline{\mathbf{C}}^+$ as a $\mathbf{B}_\infty(\mathcal{H}_{-\gamma})$-valued continuous function if $1/2 < \gamma < \delta - 1/2$ by virtue of Corollary 2.3 and, for $\lambda > 0$, $-1 \in \sigma(G_0(\lambda)V)$ if and only if λ^2 is an eigenvalue of H (see [1]). Hence, the formentioned absence of positive eigenvalues implies that $G(\lambda)$ considered as a $\mathbf{B}(\mathcal{H}_\gamma, \mathcal{H}_{-\gamma})$ valued function is continuous in $\overline{\mathbf{C}}^+$ except possibly at $\lambda = 0$ and the resolvent equation (2.9) is satisfies for all $\lambda \in \mathbf{R}\setminus\{0\}$.

At $\lambda = 0$, the situation is different([14]). We set

$$\mathcal{N} = \{\phi \in \mathcal{H}_{-\gamma} : (1 + G_0(0)V)\phi = 0\}\,. \tag{2.10}$$

The space $\mathcal{N}$ is finite dimensional and is independent of $1/2 < \gamma < \delta - 1/2$; all $\phi \in \mathcal{N}$ satisfy the stationary Schrödinger equation

$$-\Delta\phi(x) + V(x)\phi(x) = 0 \tag{2.11}$$

and, conversely, any function $\phi \in \mathcal{H}_{-\frac{m}{2}}$ which satisfies (2.11) belongs to $\mathcal{N}$; the eigenspace $\mathcal{E}$ of H with eigenvalue 0 is therefore a subspace of $\mathcal{N}$.

Lemma 2.4. (1) *Sesquilinear form $-(u, Vv)$ is an inner product in $\mathcal{N}$. If $\phi_1, \ldots, \phi_n$ is an orthnormal basis of $\mathcal{N}$ with respect to this inner product, then the spectral projection Q for $G_0(0)V$ with eigenvalue -1 may be given with sufficiently small $\varepsilon > 0$ by*

$$Q = -\frac{1}{2\pi i}\int_{|z+1|=\varepsilon}(-z + G_0(0)V)^{-1}dz = -\sum_{j=1}^{n}\phi_j \otimes (V\phi_j). \tag{2.12}$$

(2) *Let $m \geq 3$ and let $\phi \in \mathcal{N}$. Then, $|\phi(x)| \leq C\langle x\rangle^{2-m}$. If $m = 3$, $\phi \in \mathcal{E}$, the eigenspace of H with zero eigenvalue if and only if $\langle V, \phi\rangle = 0$ and* $\operatorname{codim}_{\mathcal{N}}\mathcal{E} \leq 1$. *In this case $|\phi(x)| \leq C\langle x\rangle^{-2}$.*

Proof. The proof of statement (1) and that of (2) for $m = 3$ may be found in [29]. We show (2) for $m \geq 4$. By Hardy's inequality we have for $s \geq 2$,

$$\left\|\frac{u(\xi)}{|\xi|^2}\right\|_{H^{s-2}} \leq C\|u\|_{H^s}, \quad u \in C_0^\infty(\mathbf{R}^m \setminus \{0\}). \tag{2.13}$$

Since $C_0^\infty(\mathbf{R}^m \setminus \{0\})$ is dense in $H^s(\mathbf{R}^m)$ if $s < m/2$, the standard density, duality and interpolation arguments imply that (2.13) extends for all $u \in H^s(\mathbf{R}^m)$ when $2 - \frac{m}{2} < s < \frac{m}{2}$. We have $\hat{\phi}(\xi) = -\mathcal{F}(V\phi)(\xi)/|\xi|^2$ and $V\phi \in \mathcal{H}_{(\delta-1/2)_-}$ by the assumption. It follows $\phi \in \mathcal{H}_\sigma$ for any $\sigma < \min(\delta - 5/2, (m-4)/2)$. Since $\delta > 2$, the application of this argument several times yields that $\phi \in \mathcal{H}_\sigma$, which with the well known elliptic estimate (cf. Theorem 5.1 of [2]) implies $|\phi(x)| \leq C\langle x\rangle^{-\sigma}$ for any $\sigma < (m-4)/2$. Hence

$$|\phi(x)| \leq C\int_{\mathbf{R}^m}\frac{|V(y)\phi(y)|dy}{|x-y|^{m-2}} \leq C\langle x\rangle^{-\min(\sigma+\delta-2, m-2)}, \tag{2.14}$$

and, iterating (2.14), we obtain $|\phi(x)| \le C\langle x\rangle^{2-m}$. $\Box$

Following [14], we define as follows:

Definition 2.5. We say H is of generic type if $\mathcal{N} = \{0\}$ and is of exceptional type otherwise. If $m = 3$, H is of exceptional type of the first kind if $\mathcal{N} \neq \{0\}$ and $\mathcal{E} = 0$; of the second kind if $\mathcal{E} = \mathcal{N} \neq \{0\}$; and of the third kind if $\{0\} \subset \mathcal{E} \subset \mathcal{N}$ with strict inclusions. A function $\phi \in \mathcal{N} \setminus \mathcal{E}$ is called resonance of H. We denote by P_0 the orthogonal projection in $\mathcal{H}$ onto $\mathcal{E}$.

When $m = 3$ and H is of exceptional type, we shall consider only the case of the third kind as other cases may be considered as special cases. The following is Lemma 2.3 of [29].

Lemma 2.6. *For $1/2 < \sigma, \tau < \delta - \frac{1}{2}$, $\langle x\rangle^{-\sigma} G(\lambda)\langle x\rangle^{-\tau}$, as a $\mathbf{B}_\infty(\mathcal{H})$-valued function of $\lambda \in \{\lambda \in \mathbf{R} : |\lambda| > \varepsilon\}$, $\varepsilon > 0$, satisfies the same smoothness and decay properties as $\langle x\rangle^{-\sigma} G_0(\lambda)\langle x\rangle^{-\tau}$ as stated in* Lemma 2.2. *If σ, τ satisfy $\sigma + \tau > 2$ in addition and H is of generic type, this is true on the whole line $\lambda \in \mathbf{R}$.*

We write $\overline{Q} = 1 - Q$ and define $L_{00}(\lambda) = \overline{Q}(1 + G_0(\lambda)V)\overline{Q}$. We remark that, for $\gamma > \frac{1}{2}$ and $u \in \mathcal{H}_{-\delta+(\frac{1}{2})+}$, $\overline{Q}u \in \overline{Q}\mathcal{H}_{-\gamma}$ is equivalent to $u \in \mathcal{H}_{-\gamma}$ because we always have $Qu \in \mathcal{H}_{-(\frac{1}{2})+}$.

Lemma 2.7. *There exists $\lambda_0 > 0$ such that the following statements are satisfied for $|\lambda| < \lambda_0$:*

(1) *For $\frac{1}{2} < \gamma < \delta - \frac{1}{2}$, $L_{00}(\lambda)$ has a bounded inverse $F(\lambda) \equiv L_{00}(\lambda)^{-1}$ in $\overline{Q}\mathcal{H}_{-\gamma}$. As a $\mathbf{B}(\overline{Q}\mathcal{H}_{-\gamma})$ valued function, $F(\lambda)$ is of class C^β for any $\beta < \min(\gamma - \frac{1}{2}, \delta - \gamma - \frac{1}{2})$.*

(2) *Let $1/2 < \gamma, \tau < \delta - 1/2$ be such that $\gamma + \tau > 2$. The difference $B(\lambda) = F(\lambda) - \overline{Q}$ may be extended to a bounded operator from $\overline{Q}\mathcal{H}_{-\delta+\gamma}$ to $\overline{Q}\mathcal{H}_{-\tau}$. As a $\mathbf{B}(\overline{Q}\mathcal{H}_{-\delta+\gamma}, \overline{Q}\mathcal{H}_{-\tau})$-valued function, $B(\lambda)$ is of class $C^{\rho-}$ for $\rho < \min(\gamma - 1/2, \tau - 1/2, \tau + \gamma - 2)$. The same hold for $F(\lambda) - 1$.*

Proof. (1) Since $L_{00}(\lambda) - 1 \in \mathbf{B}_\infty(\overline{Q}\mathcal{H}_{-\gamma})$ is continuous with respect to λ and $L_{00}(0)$ is invertible by virtue of the theorem on separation of spectrum for compact operators ([18]), $L_{00}(\lambda)$ is also invertible for small $|\lambda| < \lambda_0$. The proof of Lemma 2.6 implies that $F(\lambda)$ has the same smoothness property as $G_0(\lambda)V$ (see also the proof of (2) below).
(2) If we write $B(\lambda) = -\overline{Q}G_0(\lambda)V\overline{Q}F(\lambda)$, it is easy to see from statement (1) that $B(\lambda) \in \mathbf{B}(\overline{Q}\mathcal{H}_{-\delta+\gamma}, \overline{Q}\mathcal{H}_{-\tau})$. Let $0 \le \rho < 1$ first. Since $V\phi \in \mathcal{H}_{\delta + \frac{(m-4)_-}{2}}$ and $\gamma, \tau < \delta - \frac{1}{2}$, it follows from Lemma 2.2 and statement (1) that

$$
\begin{aligned}
&\|\langle x\rangle^{-\tau}(B(\lambda) - B(\mu))\langle x\rangle^{\delta-\gamma}\|_{\mathbf{B}(\mathcal{H})} \le \|\langle x\rangle^{-\tau}F(\mu)\langle x\rangle^{\tau}\| \\
&\times \|\langle x\rangle^{-\tau}\overline{Q}(G_0(\mu) - G_0(\lambda))V\overline{Q}\langle x\rangle^{\delta-\gamma}\|\|\langle x\rangle^{-\delta+\gamma}F(\lambda)\langle x\rangle^{\delta-\gamma}\| \le C|\mu-\lambda|^{\rho}.
\end{aligned}
$$

We next let $1 \le \rho$ be an integer. We $k \le \rho$ times formally differentiate $\langle x\rangle^{-\tau}B(\lambda)\langle x\rangle^{\delta-\gamma}$ with respect to λ by using $F'(\lambda) = -F(\lambda)\overline{Q}G_0'(\lambda)V\overline{Q}F(\lambda)$. The result is a linear combination of

$$
\langle x\rangle^{-\tau}F\overline{Q}G_0^{(k_1)}V\overline{Q}F\overline{Q}G_0^{(k_2)}V\overline{Q}F\cdots F\overline{Q}G_0^{(k_\ell)}V\overline{Q}F\langle x\rangle^{\delta-\gamma} \tag{2.15}
$$

where $k_1, \dots, k_\ell \ge 1$ satisfy $k_1 + \cdots + k_\ell = k$. Take k_j', not necessarily integers, sufficiently close to $k_j + \frac{1}{2}$, $j = 1, \dots, \ell$, so that

$$
\min(\gamma, \tau) > k_j' > k_j + \frac{1}{2} \quad \text{and} \quad k_{j-1}' + k_j' < \delta, \quad j = 1, \dots, \ell,
$$

where we set $k_0' = 0$. This is possible since $\rho < \delta - 1$. Considering as

$$
\overline{Q}G_0^{(k_j)}V\overline{Q} = \langle x\rangle^{k_j'}[\langle x\rangle^{-k_j'}\overline{Q}G_0^{(k_j)}V\overline{Q}\langle x\rangle^{(\delta-k_j')}]\langle x\rangle^{-(\delta-k_j')},
$$

we write (2.15) in the form

$$
\begin{aligned}
&\langle x\rangle^{-\tau}F\langle x\rangle^{k_1'}[\langle x\rangle^{-k_1'}\overline{Q}G_0^{(k_1)}V\overline{Q}\langle x\rangle^{(\delta-k_1')}] \\
&\quad \times \langle x\rangle^{-(\delta-k_1')}F\langle x\rangle^{k_2'}[\langle x\rangle^{-k_2'}\overline{Q}G_0^{(k_2)}V\overline{Q}\langle x\rangle^{(\delta-k_2')}] \times \cdots \\
&\quad \cdots \times \langle x\rangle^{-(\delta-k_{\ell-1}')}F\langle x\rangle^{k_\ell'}[\langle x\rangle^{-k_\ell'}\overline{Q}G_0^{(k_\ell)}V\overline{Q}\langle x\rangle^{(\delta-k_\ell')}]\langle x\rangle^{-(\delta-k_\ell')}F\langle x\rangle^{\delta-\gamma}
\end{aligned}
$$

By virtue of the choice of k_j''s, all factors $\langle x\rangle^{-\tau}F\langle x\rangle^{k_1'}$, $\langle x\rangle^{-(\delta-k_\ell')}F\langle x\rangle^{\delta-\gamma}$, $\langle x\rangle^{-(\delta-k_j')}F\langle x\rangle^{k_{j+1}'}$, $j = 1, \dots, \ell-1$, and $\langle x\rangle^{-k_j'}\overline{Q}G_0^{(k_j)}V\overline{Q}\langle x\rangle^{(\delta-k_j')}$ are $\mathbf{B}(\mathcal{H})$-valued continuous functions. By taking difference quotients instead of derivatives and proceeding by the induction argument with respect to the order of differentiation, this formal computation can be easily justified and the statement (2) is satisfied for integral ρ. When $\rho = k + \varepsilon$, $0 < \varepsilon < 1$, we further apply difference operator Δ^ε to (2.15) and proceed similarly. We omit the details. The statement for $A_0(\lambda)$ is obvious. □

2.2. Threshold expansion for free Schrödinger operator

Suppose $\sigma > \frac{3}{2}$ and $k = 0, 1, \dots$ satisfies $k < \sigma - \frac{1}{2}$. Then, Lemma 2.2 implies that, as a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function, $G_0(\lambda)$ is of class $C^{(\sigma-\frac{1}{2})-}$ on $\mathbf{R}$ and

$$J_k(\lambda) = \frac{1}{\lambda^k}\left(G_0(\lambda) - \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} G_0^{(j)}(0)\right) \tag{2.16}$$

is of class $C^{(\sigma-k-1/2)-}$. Outside $\lambda = 0$, $J_k(\lambda)$ is of course $C^{(\sigma-\frac{1}{2})-}$. If $m \geq 5$, this regularity result may be improved. We define for $\lambda \in \mathbf{C}^+$,

$$H(\lambda)u(x) = \frac{i}{4(2\pi)^{m-1}} \int_\Sigma \int_{\mathbf{R}^m} e^{i\lambda|\omega(x-y)|} u(y)dyd\omega.$$

At least formally $H(\lambda)$ is an entire function of λ and it has the expansion

$$H(\lambda) = \sum_{n=0}^\infty \lambda^n H_n, H_n u(x) = \frac{i^{n+1}}{4(2\pi)^{m-1} n!} \int_{\mathbf{R}^m} \left(\int_\Sigma |\omega(x-y)|^n d\omega\right) u(y)dy.$$

Lemma 2.8. *Let $m = 2\nu + 1 \geq 3$ be odd and $\sigma > m/2$. Then, the $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function $H(\lambda)$ is analytic on the upper half plane $\mathbf{C}^+$ and it can be continued to $\overline{\mathbf{C}}^+$ as a function of class $C^{\left(\sigma-\frac{m}{2}\right)-}$. For $n = 0, 1, \dots$, $H_n \in \mathbf{B}(\mathcal{H}_{\left(\frac{m}{2}+n\right)_+}, \mathcal{H}_{-\left(\frac{m}{2}+n\right)_+})$. The integral kernel of H_n is given by $C_n|x-y|^n$. In particular, H_{2n} are of finite rank. We have*

$$G_0(\lambda)u = \sum_{j=0}^{\nu-1} \lambda^{2j}(-\Delta)^{-j-1}u + \lambda^{2\nu-1}H(\lambda)u, \quad \lambda \in \overline{\mathbf{C}}^+. \tag{2.17}$$

Proof. Schwarz inequality implies the first statement. To prove (2.17), we write $\mu^{2\nu} = (\mu^2 - \lambda^2)(\mu^{2(\nu-1)} + \lambda^2\mu^{2(\nu-2)} + \cdots + \lambda^{2(\nu-1)}) + \lambda^{2\nu}$ in (2.1). Since $A(\mu)$ is even with respect to μ, we have for $\lambda \in \mathbf{C}^+$ that

$$G_0(\lambda)u = \sum_{j=0}^{\nu-1} \lambda^{2j} \int_0^\infty \mu^{2(\nu-j-1)} A(\mu) u d\mu + \frac{\lambda^{2\nu}}{2} \int_{-\infty}^\infty \frac{A(\mu)u}{\mu^2 - \lambda^2} d\mu.$$

The first memeber on the right produces the corresponding one in (2.17). The change of order of integrations yields

$$\int_{-\infty}^\infty \frac{A(\mu)u}{\mu^2 - \lambda^2} d\mu = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \left\{ \int_\Sigma \left(\int_{-\infty}^\infty \frac{e^{i\mu\omega(x-y)}}{\mu^2 - \lambda^2} d\mu \right) d\omega \right\} u(y) dy.$$

The inner most integral is easily computed by using

$$\int_{-\infty}^\infty \frac{e^{ia\mu}}{\mu^2 - \lambda^2} d\mu = \pi i \frac{e^{i|a|\lambda}}{\lambda}, \quad \Im\lambda > 0, \ a \in \mathbf{R}.$$

The equation (2.17) follows for $\lambda \in \mathbf{C}^+$ and by continuity also for $\lambda \in \overline{\mathbf{C}}^+$. □

By virtue of (2.17), we have, for $j = 1, \dots,$

$$J_k(\lambda) = \frac{1}{\lambda^k} \left(G_0(\lambda) - \sum_{j=0}^{k-1} \lambda^j D_j \right)$$

where $D_j = 0$ for $j = 1, 3, \dots, m-4$ and

$$D_j = \begin{cases} (-\Delta)^{-1-\frac{j}{2}}, & \text{for } j = 0, 2, \dots, m-3; \\ H_{j-m+2}, & \text{for } j = m-2, \dots . \end{cases} \tag{2.18}$$

Lemma 2.9. *We have the following statements:*

(1a) *Let $0 \le k \le m-2$ be odd and $\sigma > \frac{k}{2}+1$. Then, $J_k(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in \mathbf{R}$ of class $C^{\rho-}$, $\rho = \sigma - \frac{k+2}{2}$.*

(1b) *Let $0 \le k \le m-2$ be even and $\sigma, \tau > \frac{k+1}{2}$ be such that $\sigma + \tau > k+2$. Then, $J_k(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$-valued function of $\lambda \in \mathbf{R}$ of class $C^{\rho-}$, $\rho = \min(\sigma - \frac{k+1}{2}, \tau - \frac{k+1}{2}, \sigma + \tau - k - 2)$.*

(2) *Let $k \geq m-2$, $\sigma > k - \frac{m}{2} + 2$ and $\rho = \sigma + \frac{m}{2} - (k+2)$. Then $J_k(\lambda)$ is of class $C^{\rho-}$ as a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$-valued function of $\lambda \in \mathbf{R}$.*

Proof. If $k=0$, statement (1) is contained in Lemma 2.2 (2). Let $0 < k \leq m-2$. We have from (2.3) that

$$G_0(\lambda) - \sum_{j=0}^{k-1} \lambda^j D_j = \lambda^k \int_{-\infty}^{\infty} \frac{\Gamma(\mu)^*\Gamma(\mu)}{\mu^{k+1}(\mu-\lambda)} d\mu, \quad \Im\lambda > 0.$$

If $u \in \mathcal{H}_\sigma$, we have $(\Gamma u)(\lambda)u \in H^\sigma(\mathbf{R}, \Lambda^2(\Sigma)) \cap H_0^{\min(\sigma, (\frac{m}{2})-)}(\mathbf{R}\backslash\{0\}, L^2(\Sigma))$. If $k = 2\ell - 1$ is odd, then $\ell < m/2$ and $\lambda^{-\ell}\Gamma(\lambda)u \in H^{\sigma-\ell}(\mathbf{R}, \Lambda^2(\Sigma))$ by Hardy's inequality. It follows, by the Sobolev embedding theorem, that $\lambda^{-\ell}\Gamma(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, L^2(\Sigma))$ valued, hence $\lambda^{-(k+1)}\Gamma(\lambda)^*\Gamma(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\sigma})$ valued function of class $C^{\sigma - \frac{k+2}{2}}$. If $k = 2\ell$ is even, then, $\lambda^{-\ell}\Gamma(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, L^2(\Sigma))$ valued of class $C^{\sigma-\ell-1/2}$ which vanishes at the origin. Since $\sigma + \tau - 2\ell - 1 > 1$ by the assumption, it follows by Lemma 2.1 that $\lambda^{-k-1}\Gamma(\lambda)^*\Gamma(\lambda)$ is a $\mathbf{B}(\mathcal{H}_\sigma, \mathcal{H}_{-\tau})$ valued function of class ρ. Statements (1a) and (1b) follow by Privaloff's theorem. The second statement is obvious from Lemma 2.8. □

Recall that D_j, $j = 0, 1, \ldots$ are given by (2.18) and they should not be confused with the derivatives $\partial/\partial x_j$.

Lemma 2.10. *Let $m=3$ and $\sigma > 2$ and $u \in \mathcal{H}_\sigma$ satisfy $\int u(x)dx = 0$. Then, for $1 \leq k < \sigma - \frac{1}{2}$, $D_k u \in \mathcal{H}_{-(k-1/2)_+}$ and $J_k(\lambda)u$ is an $\mathcal{H}_{-\sigma+1}$ valued function of λ of order $C^{(\sigma-k-\frac{1}{2})-}$.*

Proof. We have $\hat{u} \in H^\sigma$ and $\hat{u}(0) = 0$. It follows that $\hat{u}(\xi) = \sum_{j=1}^m \xi_j \hat{v}_j(\xi)$ with $\hat{v}_j \in H^{\sigma-1}$ and $\|\hat{v}_j\|_{H^{\sigma-1}} \leq C\|\hat{u}\|_{H^\sigma}$. This means

$$u(x) = \sum_{j=1}^m \frac{\partial v_j}{i\partial x_j}, \quad v_j \in \mathcal{H}_{\sigma-1}.$$

It follows by integration by parts that

$$\begin{aligned}\Gamma(\lambda)^*\Gamma(\lambda)u(x) &= \sum_{j=1}^m \lambda^{m-1} \int_\Sigma d\omega \int_{\mathbf{R}^m} e^{i\lambda(x-y)\omega} \frac{\partial v_j}{i\partial y_j}(y)dy \\ &= -\sum_{j=1}^m \lambda^m \int_\Sigma d\omega \int_{\mathbf{R}^m} \omega_j e^{i(x-y)\omega} v_j(y)dy = -\lambda \sum_{j=1}^m \Gamma(\lambda)^*\omega_j\Gamma(\lambda)v_j(x),\end{aligned}$$

where the last ω_j is the bounded operator on $L^2(\Sigma)$ defined by the multiplication with the coordinate variable ω_j. It follows from (2.4) that

$$\begin{aligned} G_0(\lambda)u &= -\frac{1}{2}\sum_{j=1}^{m}\int_{-\infty}^{\infty}\frac{\Gamma(\mu)^*\omega_j\Gamma(\mu)v_j}{\mu-\lambda}d\mu \\ &= -\frac{1}{2}\sum_{j=1}^{m}\left(\int_{-\infty}^{\infty}\mu^{-1}\Gamma(\mu)^*\omega_j\Gamma(\mu)v_j d\mu+\lambda\int_{-\infty}^{\infty}\frac{\mu^{-1}\Gamma(\mu)^*\omega_j\Gamma(\mu)v_j}{\mu-\lambda}d\mu\right). \end{aligned}$$

Here the first term on the right is independent of λ and is an element of $\mathcal{H}_{-\tau}$ for any $\tau > 1/2$ such that $\tau+\sigma-1>2$. In particular it is in $\mathcal{H}_{-\sigma+1}$. The second is of the same form as (2.4) except that it has an extra λ factor and that $\Gamma(\lambda)^*\Gamma(\lambda)$ is replaced by $\Gamma(\mu)^*\omega_j\Gamma(\mu)$. The latter is no worse than the former as far as the smoothness as a $\mathbf{B}(\mathcal{H}_\sigma,\mathcal{H}_{-\sigma})$-valued function is concerned. Thus, we have the same result as in Lemma 2.9 (2) with $\sigma-1$ and $k-1$ in place of σ and k, The lemma follows. □

In what follows, we use the following well known expression of the convolution kernel of $G_0(\lambda)$:

$$G_0(\lambda,x)=\sum_{j=0}^{\frac{m-3}{2}}C_j\lambda^j e^{i\lambda|x|}|x|^{2+j-m} \tag{2.19}$$

2.3. Low energy asymptotics

In this subsection, we study the singularities of the resolvent $G(\lambda)$ at $\lambda=0$ when H is of exceptional type (recall that, if $m=3$, we assume H is of exceptional type of the third kind). There is an extensive literature on the subject and apart from the condition on the potentials and smoothness properties of the remainder the following theorem is well known (cf. [14], [22] and references therein). We define

$$S(\lambda)=\begin{cases} \dfrac{P_0V}{\lambda^2}-\dfrac{P_0VD_3VP_0V}{\lambda}-\dfrac{a}{\lambda}(\varphi\otimes\varphi)V, & m=3; \\ \dfrac{P_0V}{\lambda^2}-\dfrac{c_0}{\lambda}(\varphi\otimes\varphi)V, & m=5; \\ \dfrac{P_0V}{\lambda^2}, & m\geq 7. \end{cases} \tag{2.20}$$

Here φ is the canonical resonance to be defined below and $a=4\pi i|\langle V,\varphi\rangle|^{-2}$ when $m=3$; $\varphi=P_0V$, V being considered as a function, and $c_0=i/(24\pi)^2$

when $m = 5$. The following two theorems are the main results of this subsection.

THEOREM 2.11. *Let $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 2$ and H be of generic type. Then, the following statements are satisfied:*

(1) *Let $\frac12 < \sigma < \delta - \frac12$. Then, $(1+G_0(\lambda)V)^{-1}$ is a $\mathbf{B}(\mathcal{H}_{-\sigma})$-valued function of class C^s for any $s < \min(\sigma - \frac12, \delta - \sigma - \frac12)$.*

(2) *Let $\frac12 < \sigma, \tau < \delta - \frac12$ be such that $\sigma + \tau > 2$. Then, $L(\lambda) = (I + G_0(\lambda)V)^{-1} - I$ extends to a bounded operator from $\mathcal{H}_{-\delta+\sigma}$ to $\mathcal{H}_{-\tau}$ and and $\langle x\rangle^{-\tau} L(\lambda)\langle x\rangle^{\delta-\sigma}$ is a $\mathbf{B}(\mathcal{H})$-valued function of class C^s for any $s < \min(\sigma - \frac12, \tau - \frac12, \sigma + \tau - 2)$.*

THEOREM 2.12. *Let $|V(x)| \le C\langle x\rangle^{-\delta}$ for some $\delta > 3$ and H be of exceptional type. Let $0 \le \sigma < \delta - 3$. Then, there exists $\lambda_0 > 0$ such that $(I + G_0(\lambda)V)^{-1}$ may be written as follows:*

$$(I + G_0(\lambda)V)^{-1} = I + S(\lambda) + A_0(\lambda), \quad \lambda \in (-\lambda_0, \lambda_0). \tag{2.21}$$

Here $S(\lambda)$ be given by (2.20); $A_0(\lambda)$ is a $\mathbf{B}(\mathcal{H}_{-\delta+\sigma+\frac32}, \mathcal{H}_{-\sigma-\frac32})$-valued function of $\lambda \in (-\lambda_0, \lambda_0)$ of class C^s for any $s < \sigma$ and $A_0(\lambda) - L_{00}(\lambda)^{-1}$ is of finite rank.

Theorem 2.11 is contained in Lemma 2.7 and we prove Theorem 2.12 by a series of lemmas. The argument is more complex in lower dimensions and we focus mainly on the case $m = 3$, being sketchy for other cases. The discussion is vitually a repetition of that of section 4 of [29], however, we slightly improve some results therein. We use the following elementary lemma.

LEMMA 2.13. *Let $\mathcal{X} = \mathcal{X}_0 \dot{+} \mathcal{X}_1$ be a direct sum decomposition of a vector space $\mathcal{X}$. Suppose that a linear operator L in $\mathcal{X}$ is written in the form*

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

in this decomposition and that L_{00}^{-1} exists. Set $C = L_{11} - L_{10}L_{00}^{-1}L_{01}$. Then, L^{-1} exists if and only if C^{-1} exists. In this case

$$L^{-1} = \begin{pmatrix} L_{00}^{-1} + L_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1} & -L_{00}^{-1}L_{01}C^{-1} \\ -C^{-1}L_{10}L_{00}^{-1} & C^{-1} \end{pmatrix}. \tag{2.22}$$

We write $M(\lambda) = 1 + G_0(\lambda)V$. With respect to the direct sum decomposition $\mathcal{H}_{-\gamma} = \overline{Q}\mathcal{H}_{-\gamma} \dot{+} \mathcal{N}$, we decompose

$$M(\lambda) = \begin{pmatrix} \overline{Q}M(\lambda)\overline{Q} & \overline{Q}M(\lambda)Q \\ QM(\lambda)\overline{Q} & QM(\lambda)Q \end{pmatrix} \equiv \begin{pmatrix} L_{00}(\lambda) & L_{01}(\lambda) \\ L_{10}(\lambda) & L_{11}(\lambda) \end{pmatrix}, \tag{2.23}$$

where the right side is the definition. We already studied $L_{00}(\lambda)^{-1}$ in Lemma 2.7. In what follows in this subsection *we irrespectively denote by $E(\lambda)$ various operator valued functions of λ in the space $\mathcal{N}$ or between its subspaces which are of class $C^{(\delta-3)-}$ in a neighborhood of $\lambda = 0$.* We also irrespectively denote by $\lambda_0 > 0$ various constants which are chosen small enough to meet requirements at various stages.

LEMMA 2.14. *The operator $L_{11}(\lambda)$ is an isomorphism of $\mathcal{N}$ for any $0 < |\lambda| < \lambda_0$ and $L_{11}(\lambda)^{-1}$ is given by the following formulae, where φ, a and c_0 are as in* Theorem 2.12*:*

$$\begin{cases} \lambda^{-2}P_0V - \lambda^{-1}P_0VD_3VP_0V - a\lambda^{-1}|\varphi\rangle\langle\varphi|V + E(\lambda), & m = 3; \\ \lambda^{-2}P_0V - c_0\lambda^{-1}(\varphi \otimes V\varphi) + E(\lambda), & m = 5; \\ \lambda^{-2}P_0V + E(\lambda), & m \geq 7. \end{cases} \tag{2.24}$$

PROOF. Let $m = 3$ first. We take an orthonormal basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{N}$ with respect to the inner product $-(u, Vu)$ in such a way that $\{\phi_2, \dots, \phi_d\}$ is the basis of $\mathcal{E} = P_0\mathcal{H}$ and that $\langle \phi_1, V\rangle > 0$. The function ϕ_1 is uniquely determined by these conditions. Let $\pi_1 = -|\phi_1\rangle\langle V\phi_1|$ and $\pi_2 = -\sum_{j=2}^{d} |\phi_j\rangle\langle V\phi_j|$ and define

$$Q_0 = \overline{Q} = 1 - Q, \quad Q_1 = Q\pi_1Q, \quad Q_2 = Q\pi_2Q.$$

We have $Q = Q_1 + Q_2$, $Q_0 + Q_1 + Q_2 = I$ and $Q_jQ_k = \delta_{jk}Q_j$ $(j, k = 0, 1, 2)$ and as identities in $\mathcal{H}_{-\gamma}$:

$$(1 + D_0V)Q_1 = (1 + D_0V)Q_2 = 0. \tag{2.25}$$

$$Q_2D_1VQ_0 = 0, \quad Q_2D_1VQ_1 = 0, \quad Q_2D_1VQ_2 = 0, \tag{2.26}$$

$$Q_0D_1VQ_2 = 0, \quad Q_1D_1VQ_2 = 0. \tag{2.27}$$

In the decomposition $\mathcal{N} = Q_1\mathcal{N} \dot{+} Q_2\mathcal{N}$, we write

$$L_{11}(\lambda) = \begin{pmatrix} Q_1M(\lambda)Q_1 & Q_1M(\lambda)Q_2 \\ Q_2M(\lambda)Q_1 & Q_2M(\lambda)Q_2 \end{pmatrix} \equiv \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}. \tag{2.28}$$

Let $c(\lambda) = -\langle V\phi_1, M(\lambda)\phi_1\rangle$ so that $M_{11}(\lambda) = c(\lambda)Q_1$. Since $V\phi_1 \in \mathcal{H}_{(\delta-\frac{1}{2})_-}$, $(1 + D_0V)\phi_1 = 0$ and $D_1 = -(4\pi i)^{-1}(1 \otimes 1)$, $c(\lambda)$ is of class $C^{(\delta-1)-}$ and with a $C^{(\delta-3)-}$ function $c_1(\lambda)$

$$c(\lambda) = -\langle V\phi_1|(G_0(\lambda) - D_0)|V\phi_1\rangle = (4\pi i)^{-1}\lambda|\langle V, \phi_1\rangle|^2 + \lambda^2 c_1(\lambda).$$

Since $\langle V, \phi_1\rangle \neq 0$, it follows that $M_{11}(\lambda)$ is invertible for $0 < |\lambda| < \lambda_0$ and, with $a = 4\pi i|\langle V, \phi_1\rangle|^{-2}$,

$$M_{11}^{-1}(\lambda) = (\lambda^{-1}a + d(\lambda))Q_1, \quad d \in C^{(\delta-3)-}. \tag{2.29}$$

By virtue of Lemma 2.4 we have $V\phi_j(x) \in \mathcal{H}_{(\delta+\frac{1}{2})_-}$ and $\int V\phi_j dx = 0$ for $j = 2, \ldots, d$. Hence $Q_2 D_1 V = D_1 V Q_2 = 0$ and by virtue of Lemma 2.10,

$$M_{12}(\lambda) = \lambda^2 Q_1(D_2V + \lambda E(\lambda))Q_2, \quad M_{21}(\lambda) = \lambda^2 Q_2(D_2V + \lambda E(\lambda))Q_1.$$

Combining this with (2.29), we have

$$M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda) = \lambda^3 Q_2(aD_2VQ_1D_2V + \lambda E(\lambda))Q_2. \tag{2.30}$$

Likewise we have

$$M_{22}(\lambda) = \lambda^2 Q_2(D_2V + \lambda D_3V + \lambda^2 E(\lambda))Q_2.$$

Simple algebraic manipulations show that $Q_2D_2VQ_2$ is invertible in $\mathcal{E}$ and $(Q_2D_2VQ_2)^{-1} = P_0V$ and $P_0VQ_2 = P_0V$ (see page 499 of [29], however, we remark the discrepancy of the definitions of D_j, D_j here is i^jD_j in [29]). It follows that $M_{22}(\lambda)$ is invertible when $0 < |\lambda| < \lambda_0$ and

$$M_{22}(\lambda)^{-1} = \lambda^{-2}P_0V - \lambda^{-1}P_0VD_3VP_0V + P_0VE(\lambda)Q_2. \tag{2.31}$$

Hence, $M_{22}^{-1}M_{21}M_{11}^{-1}M_{12} = a\lambda P_0VD_2VQ_1D_2VP_0V + \lambda^2E(\lambda)$ and

$$\begin{aligned} C_{22}(\lambda) \equiv\ & M_{22}(\lambda) - M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda) \\ =\ & M_{22}(\lambda)(1 - M_{22}(\lambda)^{-1}M_{21}(\lambda)M_{11}^{-1}(\lambda)M_{12}(\lambda)) \end{aligned}$$

is invertible for $0 < |\lambda| < \lambda_0$ and

$$\begin{aligned} C_{22}^{-1}(\lambda) = \lambda^{-2}&P_0V - \lambda^{-1}P_0VD_3VP_0V \\ &+ a\lambda^{-1}P_0VD_2VQ_1D_2VP_0V + P_0VE(\lambda)P_0V. \end{aligned} \tag{2.32}$$

If we set $\tilde{\phi}_1 = P_0VD_2V\phi_1 \in P_0\mathcal{H}$, then $P_0VD_2VQ_1D_2VP_0V = -|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|V$ and the right side of (2.32) may be written in the form

$$\lambda^{-2}P_0V - \lambda^{-1}P_0VD_3VP_0V - \lambda^{-1}a|\tilde{\phi}_1\rangle\langle\tilde{\phi}_1|V + P_0VE(\lambda)P_0V. \tag{2.33}$$

Using (2.29), (2.30), (2.32) and the definition of $\tilde{\phi}_1$, we may write

$$\begin{aligned} -M_{11}^{-1}(\lambda)M_{12}(\lambda)C_{22}^{-1}(\lambda) &= -a\lambda^{-1}|\phi_1\rangle\langle\tilde{\phi}_1|V + E(\lambda). \\ -C_{22}^{-1}(\lambda)M_{21}(\lambda)M_{11}^{-1}(\lambda) &= -a\lambda^{-1}|\tilde{\phi}_1\rangle\langle\phi_1|V + E(\lambda). \\ M_{11}^{-1}(\lambda)M_{12}(\lambda)C_{22}^{-1}(\lambda)M_{21}(\lambda)M_{11}^{-1}(\lambda) &= E(\lambda). \end{aligned} \tag{2.34}$$

Combining (2.29), (2.33) and (2.34) and applying Lemma 2.13, we see that $L_{11}(\lambda)$ is invertible in $\mathcal{N}$ when $0 < |\lambda| < \lambda_0$ and $L_{11}(\lambda)^{-1}$ is the sum of

$$\begin{pmatrix} -a\lambda^{-1}|\phi_1\rangle\langle V\phi_1| & -a\lambda^{-1}|\phi_1\rangle\langle V\tilde{\phi}_1| \\ -a\lambda^{-1}|\tilde{\phi}_1\rangle\langle V\phi_1| & \lambda^{-2}P_0V - \lambda^{-1}P_0VD_3VP_0V - \lambda^{-1}a|\tilde{\phi}_1\rangle\langle V\tilde{\phi}_1| \end{pmatrix} \tag{2.35}$$

and an $E(\lambda)$. Thus, defining the canonical resonance by $\varphi = \phi_1 + \tilde{\phi}_1$ we have proven the lemma for $m = 3$.

Let $m = 5$ next. Since $(1 + D_0V)Q = 0$ and $\mathcal{N} \subset \mathcal{H}_{(\frac{1}{2})_-}$, Lemma 2.9 (2) with $m = 5$, $\sigma = \left(\delta + \frac{1}{2}\right)_-$ and $k = 4$ implies

$$L_{11}(\lambda) = Q(\lambda^2(-\Delta)^{-2} + c_0\lambda^3(1 \otimes 1) + \lambda^4E(\lambda))VQ, \tag{2.36}$$

where $c_0 = i/(24\pi^2)$ and $E(\lambda)$ is of class $C^{\rho-}$, $\rho = (\delta - 3)_-$. In the basis $\{\phi_j\}$ of $\mathcal{N}$, we have

$$Q(-\Delta)^{-2}VQ = \sum\langle D_0V\phi_j, D_0V\phi_k\rangle|\phi_j\rangle\langle V\phi_k| = \sum\langle\phi_j, \phi_k\rangle|\phi_j\rangle\langle\phi_k|V,$$

and, exactly as in the case $m = 3$, we have

$$(Q(-\Delta)^{-2}VQ)^{-1} = P_0V,\ VQP_0 = VP_0,\ P_0VQ = P_0V. \tag{2.37}$$

It follows for small $|\lambda| < \lambda_0$ that

$$\begin{aligned} L_{11}(\lambda)^{-1} &= \lambda^{-2}P_0V(1 + c_0\lambda Q(1 \otimes 1)VP_0V + \lambda^2E(\lambda)VP_0V)^{-1} \\ &= \lambda^{-2}P_0V - c_0\lambda^{-1}P_0V(1 \otimes 1)VP_0V + E(\lambda). \end{aligned}$$

Defining $\varphi = P_0V$, we obtain (2.24) for $m = 5$.

Finally let $m \geq 7$. We have $L_{11}(\lambda) = Q(\lambda^2(-\Delta)^{-2})Q + \lambda^4 QJ_4(\lambda)VQ$. Since $\mathcal{N} \subset \mathcal{H}_{\left(\frac{m-4}{2}\right)_-}$, Lemma 2.9 (1) with $\sigma = \delta + (\frac{m-4}{2})_-$ and $k = 4$ implies $QJ_4(\lambda)VQ$ is of class $C^{\rho-}$, $\rho = (\delta + \frac{m-9}{4})_- > \delta - 3$. As previously, $(Q(-\Delta)^{-2}VQ)^{-1}$ satisfies (2.37). It follows

$$L_{11}(\lambda)^{-1} = \lambda^{-2}P_0V + P_0VE(\lambda)P_0V. \tag{2.38}$$

This completes the proof of the lemma. □

Lemma 2.15. *(1) If $m = 3$, then in the decomposition $\mathcal{N} = Q_1\mathcal{N} + Q_2\mathcal{N}$,*

$$L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = \begin{pmatrix} \lambda^2 E_{11}(\lambda) & \lambda^3 E_{12}(\lambda) \\ \lambda^3 E_{21}(\lambda) & \lambda^4 E_{22}(\lambda) \end{pmatrix}, \tag{2.39}$$

where for $i, j = 1, 2$, $E_{ij}(\lambda)$ are of class $C^{(\delta-3)-}$.
(2) If $m \geq 5$, then $L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = \lambda^4 E(\lambda)$ with $E(\lambda)$ of class $C^{(\delta-3)-}$.

Proof. We let $\{\phi_1, \phi_2, \ldots, \phi_d\}$ be the basis of $\mathcal{N}$ used in the proof of Lemma 2.14. We first prove that $E_{22}(\lambda)$ is of class C^s for any $s < \delta - 3$. We assume s is an integer since extension to the non-integral case is immediate. We write $F(\lambda) = \overline{Q}L_{00}^{-1}(\lambda)\overline{Q}$ (this is consistent with the definition as $F(\lambda) = L_{00}^{-1}(\lambda)$ acts in $\overline{Q}\mathcal{H}_{-\gamma}$). We have $(1 + D_0V + \lambda D_1 V)Q_2 = 0$ and $(1 + G_0(\lambda)V)Q_2 = \lambda^2 J_2(\lambda)VQ_2$. It follows that $E_{22}(\lambda) = \lambda^4 Q_2 J_2(\lambda)VF(\lambda)J_2(\lambda)VQ_2$ and in terms of the basis

$$E_{22}(\lambda) = \sum_{i,j=2}^{d} \langle J_2(-\lambda)V\phi_j, VF(\lambda)J_2(\lambda)V\phi_i\rangle |\phi_j\rangle\langle V\phi_i|.$$

Formally differentiating $k \leq s$ times by using Leibniz' formula, we have

$$\begin{aligned}&(d/d\lambda)^k \langle J_2(-\lambda)V\phi_j, VF(\lambda)J_2(\lambda)V\phi_i\rangle \\ &\quad = \sum_{\alpha+\beta+\gamma=k} \frac{k!}{\alpha!\beta!\gamma!} \langle J_2^{(\alpha)}(-\lambda)V\phi_j, VF^{(\beta)}(\lambda)J_2^{(\gamma)}(\lambda)V\phi_i\rangle\end{aligned}$$

Choosing $\alpha' > \alpha$ and $\gamma' > \gamma$ close enough to α and γ respectively so that $\gamma' + \frac{3}{2} < \delta - (\alpha' + \frac{3}{2})$, $\delta - \alpha' - \frac{3}{2} > \beta + \frac{1}{2}$ and $\gamma' + \frac{3}{2} < \delta - \beta - \frac{1}{2}$, we write the inner product on the right side in the form

$$\langle \langle x\rangle^{-(\alpha'+\frac{3}{2})} J_2^{(\alpha)}(-\lambda)V\phi_j, \langle x\rangle^{\alpha'+\frac{3}{2}} VF^{(\beta)}(\lambda)\langle x\rangle^{\gamma'+\frac{3}{2}} \cdot \langle x\rangle^{-(\gamma'+\frac{3}{2})} J_2^{(\gamma)}(\lambda)V\phi_i\rangle.$$

By virtue of Lemma 2.10, $\langle x\rangle^{-(\alpha'+\frac{3}{2})}J_2^{(\alpha)}(\lambda)V\phi_j$ and $\langle x\rangle^{-(\gamma'+\frac{3}{2})}J_2^{(\gamma)}(\lambda)V\phi_j$ are $\mathcal{H}$-valued continuous; the $\mathbf{B}(\mathcal{H})$-valued function $\langle x\rangle^{\alpha'+\frac{3}{2}}VF^{(\beta)}(\lambda)\langle x\rangle^{\gamma'+\frac{3}{2}}$ is continuous by virtue of Lemma 2.7 (2). Thus, $E_{22}(\lambda)$ is of class C^s. To prove the same property for other E_{ij} we proceed similarly. However, we have $(1+G_0(\lambda)V)Q_1 = \lambda J_1(\lambda)VQ_1$ and we apply Lemma 2.9 (1a) instead of Lemma 2.10 for $J_1(\pm\lambda)V\phi_1$.

If $m \geq 5$, we have $L_{10}(\lambda)L_{00}(\lambda)^{-1}L_{01} = \lambda^4 QJ_2(\lambda)VF(\lambda)J_2(\lambda)VQ \equiv \lambda^4 E^*(\lambda)$. As previously we write in terms of the basis

$$E(\lambda) = \sum_{i,j=1}^{d} \langle J_2(-\lambda)V\phi_j, VF(\lambda)J_2(\lambda)V\phi_i\rangle |\phi_j\rangle\langle V\phi_i|$$

and proceed as in (1). However, for the derivatives we apply Lemma 2.9 (1b) instead of Lemma 2.10, remarking that $V\phi_j \in \mathcal{H}_{\delta+(\frac{m-4}{2})_-}$. We omit repetitious details. □

Lemma 2.16. *The operator* $C(\lambda) = L_{11}(\lambda) - L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda)$ *is invertible in* $\mathcal{N}$ *for* $0 < |\lambda| < \lambda_0$ *and* $C(\lambda)^{-1}$ *may be written in the form* (2.24).

Proof. If $m = 3$, (2.35) and (2.39) imply

$$N(\lambda) \equiv L_{11}^{-1}(\lambda)L_{10}(\lambda)L_{00}^{-1}(\lambda)L_{01}(\lambda) = \begin{pmatrix} \lambda E(\lambda) & \lambda^2 E(\lambda) \\ \lambda E(\lambda) & \lambda^2 E(\lambda) \end{pmatrix}.$$

Hence $C(\lambda) = L_{11}(\lambda)(1 - N(\lambda))$ is invertible for small $0 < |\lambda| < \lambda_0$

$$C^{-1}(\lambda) = L_{11}^{-1}(\lambda) + (1-N(\lambda))^{-1}N(\lambda)L_{11}^{-1}(\lambda) = L_{11}^{-1}(\lambda) + E(\lambda).$$

The proof for $m \geq 5$ is simpler and we may safely omit it. □

Lemma 2.17. *Let* $0 \leq \sigma < \delta - 3$. *Then, for* $|\lambda| < \lambda_0$,
(1) $F(\lambda)L_{01}(\lambda)C(\lambda)^{-1}$ *is* $\mathbf{B}(\mathcal{N}, \mathcal{H}_{-(\sigma+\frac{1}{2})_+})$*–valued function of class* C^σ.
(2) $C^{-1}(\lambda)L_{10}(\lambda)F(\lambda)$ *is* $\mathbf{B}(\mathcal{H}_{-(\delta-\sigma-\frac{1}{2})_-}, \mathcal{N})$*-valued function of class* C^σ.
(3) $F(\lambda)L_{01}(\lambda)C(\lambda)^{-1}L_{10}(\lambda)F(\lambda)$ *is* $\mathbf{B}(\mathcal{H}_{-(\delta-\sigma-\frac{1}{2})_-}, \mathcal{H}_{-(\sigma+\frac{1}{2})_+})$*-valued function of class* C^σ.

Proof. We prove the lemma for $m = 3$ when σ is an integer. Extension to non-integral σ is immediate and the proof for $m \geq 5$ is similar. We have

$$L_{01}(\lambda) = \overline{Q}M(\lambda)Q_1 + \overline{Q}M(\lambda)Q_2 = \lambda\overline{Q}J_1(\lambda)VQ_1 + \lambda^2\overline{Q}J_2(\lambda)VQ_2$$

By virtue of Lemma 2.16, $C(\lambda)^{-1}$ is of the form $\begin{pmatrix} \lambda^{-1}E(\lambda) & \lambda^{-1}E(\lambda) \\ \lambda^{-1}E(\lambda) & \lambda^{-2}E(\lambda) \end{pmatrix}$. It follows that

$$L_{00}(\lambda)^{-1}L_{01}(\lambda)C^{-1}(\lambda) = F(\lambda)\overline{Q}J_1(\lambda)VQ_1E(\lambda) + F(\lambda)\overline{Q}J_2(\lambda)VQ_2E(\lambda).$$

We write $F(\lambda) = \overline{Q} + B(\lambda)$. As was seen previously, $J_j(\lambda)VQ_j\phi$, $\phi \in \mathcal{N}$, $j = 1, 2$, are $\mathcal{H}_{-(\sigma+\frac{3}{2})+}$–valued C^σ functions for any $0 < \sigma < \delta - 2$, hence, so are $\overline{Q}J_j(\lambda)VQ_j\phi$. And $B(\lambda)\overline{Q}J_j(\lambda)VQ_j\phi$ are $\mathcal{H}_{-(\sigma+\frac{1}{2})+}$–valued C^σ functions for any $0 < \sigma < \delta - 2$. Indeed, if $\sigma = \alpha + \beta$, then for $\alpha' > \alpha$ and $\beta' > \beta$ such that $\alpha' + \frac{1}{2} < \delta - \beta' - \frac{3}{2}$,

$$\langle x\rangle^{-(\alpha'+\frac{1}{2})}B^{(\alpha)}(\lambda)\overline{Q}\langle x\rangle^{\beta'+\frac{3}{2}}\langle x\rangle^{-(\beta'+\frac{3}{2})}J_i^{(\beta)}(\lambda)VQ_j\phi$$

is an $\mathcal{H}$-valued continuous functions by virtue of Lemma 2.7. Since $E(\lambda)$ is of class $C^{(\delta-3)-}$, this proves (1).

By a simple computation, we have

$$L_{10}F(\lambda)u = \lambda\langle J_1(-\lambda)V\phi_1, VF(\lambda)u\rangle\phi_1 + \sum_{j=2}^{d}\lambda^2\langle J_2(-\lambda)V\phi_j, VF(\lambda)u\rangle\phi_j.$$

The argument in (1) shows that, for $u \in \mathcal{H}_{-(\delta-\sigma-\frac{3}{2})-}$, $\lambda\langle J_1(-\lambda)V\phi_1, VF(\lambda)u\rangle$ and $\langle J_2(-\lambda)V\phi_j, VF(\lambda)u\rangle$ are C^σ function of λ. This proves (2). The proof for (3) is similar and we omit the details. □

Completion of the proof of Theorem 2.12. The first part of the theorem follows by combing the lemmas. By virtue of (2.22) and (2.23) and thanks to Lemma 2.16, $A_0(\lambda) - L_{00}^{-1}(\lambda)$ may be written as

$$L_{00}^{-1}L_{01}C^{-1}L_{10}L_{00}^{-1} - L_{00}^{-1}L_{01}C^{-1} - C^{-1}L_{10}L_{00}^{-1} + E,$$

and, therefore, it is of finite rank. This completes the proof of Theorem 2.12.

3. Proof of Main Theorem

We prove the theorem only for W_-. The proof for W_+ is similar. We write $W = W_-$ for simplicity. We express W_- in terms of the boundary values of resolvents in the following form where the right hand sides converge strongly in $\mathcal{H}$ ([20]):

$$\begin{aligned} W = & \ 1 - \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbf{R}} R(k+\varepsilon) V (R_0(k+i\varepsilon) - R_0(k-\varepsilon)) dk \\ = & \ 1 - \lim_{\varepsilon \downarrow 0} \frac{1}{\pi i} \int_{\varepsilon}^{\infty} G(\lambda) V (G_0(\lambda) - G_0(-\lambda)) \lambda d\lambda. \end{aligned} \tag{3.1}$$

We let $\lambda_0 > 0$ be as in Theorem 2.12 and take $\Phi \in C_0^\infty(\mathbf{R})$ and $\Psi \in C^\infty(\mathbf{R})$ as in the introduction: $\Phi(\lambda^2) = 0$ for $|\lambda| > \lambda_0$, $\Phi(\lambda^2) = 1$ for $|\lambda| < \lambda_0^2/2$ and $\Phi(\lambda)^2 + \Psi(\lambda)^2 \equiv 1$. We define $W_< = W\Phi(H_0)^2$ and $W_> = W\Psi(H_0)^2$ so that $W = W_< + W_>$. We remark that for $u, v \in \mathcal{S}(\mathbf{R}^m)$, we have

$$\langle (G_0(\lambda)u - G_0(-\lambda)) f(\lambda^2) u, v \rangle = \langle (G_0(\lambda)u - G_0(-\lambda)) f(H_0) u, v \rangle. \tag{3.2}$$

We use the following lemma which is proved in Section 2 of [26]. Recall that $m_* = \frac{m-1}{m-2}$. We define, for $n = 1, 2, \ldots$,

$$\Omega_n u = \frac{1}{\pi i} \int_0^\infty (G_0(\lambda) V)^n (G_0(\lambda) - G_0(-\lambda)) u \lambda d\lambda.$$

Notice that $\Omega_1 = W_{r,0}$.

Lemma 3.1. *Let $\sigma > 1/m_*$. Then there exists a constant $C > 0$ such that*

$$\|\Omega_1 u\|_p \le C \|\mathcal{F}\langle x \rangle^\sigma V\|_{m_*} \|u\|_p, \tag{3.3}$$

$$\|\Omega_n u\|_p \le C^n \|\mathcal{F}\langle x \rangle^{2\sigma} V\|_{m_*}^n \|u\|_p, \quad n = 2, \ldots \tag{3.4}$$

for any $1 \le p \le \infty$, $u \in L^p(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$.

3.1. Low energy estimate I, Regular Part

By virtue of the intertwining property we have $W_< = \Phi(H) W \Phi(H_0)$. We write $G(\lambda)V = G_0(\lambda)V(1 + G_0(\lambda)V)^{-1}$ in the right of (3.1) and replace $(1 + G_0(\lambda)V)^{-1}$ by the right side of (2.21) of Theorem 2.12. Then, defining

$W_{r,0}$, W_r, $W_{s,1}$ and $W_{s,2}$ as in the introduction by (1.13), (1.14), (1.15) and (1.16) respectively, we have

$$W_< = \Phi(H)(1 - (W_{r,0} + W_r + W_{s,1} + W_{s,2}))\Phi(H_0), \tag{3.5}$$

where the integrals are understood as strong limits $\lim_{\varepsilon \downarrow 0} \int_\varepsilon^\infty \cdots d\lambda$. It is well known (cf. Lemma 2.4 of [27]) that the integral kernels $\Phi_0(x,y)$ and $\Phi(x,y)$ of $\Phi(H_0)$ and $\Phi(H)$ are bounded by $C_N \langle x-y\rangle^{-N}$ for any N and $\Phi(H_0)$ and $\Phi(H)$ are bounded in L^p for all $1 \le p \le \infty$. Lemma 3.1 then implies that $\Phi(H)W_{r,0}\Phi(H_0)$ is bounded in L^p for all $1 \le p \le \infty$.

In this subsection we prove that W_r is bounded in L^p for all $1 \le p \le \infty$.

Definition 3.2. We say that the integral kernel $K(x,y)$ is admissible if it satisfies the condition (1.17):

$$\sup_x \int_{\mathbf{R}^m} |K(x,y)| dy + \sup_y \int_{\mathbf{R}^m} |K(x,y)| dx < \infty.$$

It is well known that the integral operator with an admissible integral kernel is bounded in L^p for any $1 \le p \le \infty$.

Lemma 3.3. *The integral kernel $W_r(x,y)$ of W_r is admissible.*

If δ_x denotes the Dirac mass placed at x, then $W_r(x,y)$ is given by $\langle W_r\delta_y, \delta_x\rangle$, which we may write in the form

$$\frac{1}{\pi i}\int_0^\infty \langle VA_0(\lambda)(G_0(\lambda) - G_0(-\lambda))\Phi_0(\cdot, y), G_0(-\lambda)\Phi(\cdot, x)\rangle \lambda\tilde{\Phi}(\lambda) d\lambda. \tag{3.6}$$

Recall that $\tilde{\Phi} \in C_0^\infty(\mathbf{R})$ is such that $\tilde{\Phi}(\lambda)\Phi(\lambda^2) = \Phi(\lambda^2)$, $\lambda \in \mathbf{R}$. Since $\Phi_0(z,y)$ and $\Phi(z,x)$ are bounded and rapidly decreasing in z, the integral (3.6) is absolutely convergent and defines a bounded function of (x,y).

For $j = 0, \dots, (m-3)/2$, let $G_{0,j}(\lambda)$ be the convolution operator with the kernel $e^{i\lambda|x|}|x|^{-(m-2-j)}$ so that $G_0(\lambda) = \sum_{j=0}^{(m-3)/2} C_j \lambda^j G_{0,j}(\lambda)$ by virtue of (2.19). We insert this into the right of (3.6). Then, we have

$$W_r(x,y) = \frac{1}{\pi i} \sum_{j,k=0}^{(m-3)/2} C_j C_k (-1)^k (L_{+,j,k}(x,y) - L_{-,j,k}(x,y)), \tag{3.7}$$

where, with $T_{\pm,j,k}(\lambda,x,y)=\langle VA_0(\lambda)G_{0,j}(\pm\lambda)\Phi_0(\cdot,y),G_{0,k}(-\lambda)\Phi(\cdot,x)\rangle\tilde{\Phi}(\lambda)$,

$$L_{\pm,j,k}(x,y)=\frac{(\pm 1)^j}{\pi i}\int_0^\infty T_{\pm,j,k}(\lambda,x,y)\lambda^{j+k+1}d\lambda.$$

Define, as in [26] and [27],

$$X_j(\lambda,z,x)=\int\frac{e^{i\lambda(|z-w|-|x|)}}{|z-w|^{m-2-j}}\Phi(w,x)dw=e^{-i\lambda|x|}G_{0,j}(\lambda)\Phi(\cdot,x),$$
$$Y_j(\lambda,z,y)=\int\frac{e^{i\lambda(|z-w|-|y|)}}{|z-w|^{m-2-j}}\Phi_0(w,y)dw=e^{-i\lambda|y|}G_{0,j}(\lambda)\Phi_0(\cdot,y),$$

and denote by $X_j^{(\ell)}(\lambda,x,y)$ and etc. for the ℓ-th derivatives of X_j and etc. with respect to the variable λ. We have

$$\begin{aligned}L_{\pm,j,k}(x,y)&=\frac{1}{\pi i}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\tilde{T}_{\pm,j,k}(\lambda,x,y)\lambda^{j+k+1}d\lambda,\\ \tilde{T}_{\pm,j,k}(\lambda,x,y)&=\langle VA_0(\lambda)Y_j(\pm\lambda,\cdot,y),X_k(-\lambda,\cdot,x)\rangle\tilde{\Phi}(\lambda).\end{aligned}\tag{3.8}$$

Lemma 3.4. (1) *For $\ell=0,1,2,\dots$ and $j=0,\dots,(m-3)/2$,*

$$\left|X_j^{(\ell)}(\lambda,z,x)\right|\le\frac{C_\ell\langle z\rangle^\ell}{\langle z-x\rangle^{m-2-j}},\quad\left|Y_j^{(\ell)}(\lambda,z,y)\right|\le\frac{C_\ell\langle z\rangle^\ell}{\langle z-y\rangle^{m-2-j}}.$$

(2) *Let $\nu_k=m-2-k$ if $k=0,\dots,(m-5)/2$ and $\nu_k=(m/2)_+$ if $k=(m-3)/2$. Then, for $0\le j\le k$, we have*

$$\begin{aligned}&\|\langle z\rangle^{-(\ell+\nu_k)}X_j^{(\ell)}(\lambda,z,x)\|_{L^2(\mathbf{R}^m_z)}\le C\langle x\rangle^{-(m-2-k)},\\ &\|\langle z\rangle^{-(\ell+\nu_k)}Y_j^{(\ell)}(\lambda,z,y)\|_{L^2(\mathbf{R}^m_z)}\le C\langle y\rangle^{-(m-2-k)}.\end{aligned}\tag{3.9}$$

Moreover, if $k=\frac{m-3}{2}$ and $j<k$, then (3.9) remains to hold with the exponent $m-2-k$ on the right being replaced by $\min(\nu_k,m-2-j)$.

Proof. We have $||z-w|-|x||\le|z|+|x-w|$ and $|\Phi(w,x)|\le C_N\langle x-w\rangle^{-N}$ for arbitrarily large N. It follows that

$$\left|X_j^{(\ell)}(\lambda,z,x)\right|\le\int\frac{C\langle z\rangle^\ell dw}{|z-w|^{m-2-j}\langle w-x\rangle^{N-\ell}}\le\frac{C_\ell\langle z\rangle^\ell}{\langle z-x\rangle^{m-2-j}}.$$

Then, the standard estimate implies that $X_j^{(\ell)}(\lambda, z, x)$ satisfies (3.9). Estimates for $Y_j^{(\ell)}(\lambda, z, y)$ may be proved similarly. $\square$

LEMMA 3.5. *Suppose either* $|V(x)| \le C\langle x\rangle^{-(m+2+\varepsilon)}$, $\varepsilon > 0$, *and* H *is of generic type or* $|V(x)| \le C\langle x\rangle^{-(m+3+\varepsilon)}$, $\varepsilon > 0$, *and* H *is of exceptional type. Let* $A_0(\lambda)$ *be as in* Theorem 2.12. *Then:*

$$\begin{aligned} &|\langle VA_0^{(\alpha)}(\lambda)Y_j^{(\beta)}(\pm\lambda, \cdot, y), X_k^{(\gamma)}(-\lambda, \cdot, x)\rangle| \\ &\quad \le C\langle x\rangle^{-(m-2-\kappa)}\langle y\rangle^{-(m-2-\kappa)} \end{aligned} \tag{3.10}$$

for $j, k = 0, \dots, \frac{m-3}{2}$, $\kappa = \max(j, k)$ *and* (α, β, γ) *with* $\alpha + \beta + \gamma \le \kappa + 3$. *If* $\alpha = \beta = \gamma = 0$, *we have*

$$|\langle VA_0(\lambda)Y_j(\pm\lambda, \cdot, y), X_k(-\lambda, \cdot, x)\rangle| \le C\langle x\rangle^{-(m-2-k)}\langle y\rangle^{-(m-2-j)} \tag{3.11}$$

for $j, k = 0, \dots, \frac{m-3}{2}$.

PROOF. We first prove (3.10). We may assume without loss of generality that $j \le k$ so that $\kappa = k$. Let ν_k be as in the previous Lemma 3.4. By virtue of (3.9), the left side of (3.10) is bounded by

$$C\|\langle z\rangle^{\nu_k+\gamma}VA_0^{(\alpha)}(\lambda)\langle z\rangle^{\nu_k+\beta}\|_{\mathbf{B}(\mathcal{H})}\langle x\rangle^{-(m-2-k)}\langle y\rangle^{-(m-2-k)}$$

By virtue of Theorem 2.12, $\langle z\rangle^{\nu_k+\gamma}VA_0^{(\alpha)}(\lambda)\langle z\rangle^{\nu_k+\beta}$ is a $\mathbf{B}(\mathcal{H})$-valued continuous function of λ if

$$\alpha < \min(\delta - (\nu_k + \beta) - \tfrac{3}{2},\ \delta - (\nu_k + \gamma) - \tfrac{3}{2},\ \delta - 3) \tag{3.12}$$

when H is of exceptional case, and when H is of the generic case if

$$\begin{aligned} \alpha < \min(&\delta - (\nu_k + \beta) - \tfrac{1}{2},\ \delta - (\nu_k + \gamma) - \tfrac{1}{2}, \\ &2\delta - (\beta + \gamma + 2\nu_k) - 2,\ \delta - 1). \end{aligned} \tag{3.13}$$

The conditions (3.12) and (3.13) are satisfied if $m + 3 < \delta$ and $m + 2 < \delta$ respectively and the estimate (3.10) follows. To prove (3.11), it suffices to check that $\|\langle z\rangle^{\nu_k}VA_0(\lambda)\langle z\rangle^{\nu_j}\|_{\mathbf{B}(\mathcal{H})}$ is bounded, which is, however, obvious. This completes the proof. $\square$

Lemma 3.6. *Let $j = 0, 1, \dots, (m-3)/2$. Then:*

$$\sup_y \langle y\rangle^{-(m-2-j)} \int_{\mathbf{R}^m} \langle x\rangle^{-(m-2-j)} \langle |x| \pm |y|\rangle^{-(j+3)} dx < \infty. \tag{3.14}$$

The similar estimate holds when roles of variables x and y are interchanged.

Proof. It suffices to prove the case with $\langle |x| - |y|\rangle^{-(j+3)}$. Using polar coordinates, we estimate the integral by

$$\left(\int_0^{|y|/2} + \int_{|y|/2}^{2|y|} + \int_{2|y|}^\infty\right) \frac{Cr^{m-1}dr}{(1+|r|)^{m-2-j}(1+|r-|y||)^{j+3}}$$
$$\le \int_0^{|y|/2} \frac{Cr^{m-1}dr}{(1+|r|)^{m+1}} + \frac{C|y|^{m-1}}{(1+|y|)^{m-2-j}} + \int_{2|y|}^\infty \frac{Cr^{m-1}dr}{(1+|r|)^{m+1}} \le C\langle y\rangle^{1+j}.$$

Estimate (3.14) follows because $1+j \le m-2-j$ for $j \le \frac{m-3}{2}$. □

By virtue of (3.10), we have for all $j, k = 0, \dots, (m-3)/2$

$$|L_{\pm,j,k}(x,y)| \le C\langle x\rangle^{-(m-2-\kappa)} \langle y\rangle^{-(m-2-\kappa)}, \quad \kappa = \max(j,k). \tag{3.15}$$

Since $2\kappa \le m-3$, this implies that $\chi(x,y)L_{\pm,j,k}(x,y)$ are admissible if $\chi(x,y)$ is the characteristic function of $\{(x,y) : ||x|-|y|| \le 1\}$. In what follows we thus consider $L_{\pm,j,k}(x,y)$ for $||x|-|y|| > 1$ only.

We apply integration by parts $\kappa+3$ times, $\kappa = \max(j,k)$, in the right side of

$$\begin{aligned}(3.16)\qquad &(\pm 1)^j (i(|x| \pm |y|))^{\kappa+3} L_{\pm,j,k}(x,y) \\ &\quad = \int_0^\infty \left(\frac{\partial}{\partial\lambda}\right)^{\kappa+3} e^{i(|x|\pm|y|)\lambda} \cdot \tilde{T}_{\pm,j,k}(\lambda,x,y)\lambda^{j+k+1} d\lambda.\end{aligned}$$

If $j+k+1 \ge \kappa+3$, then no boundary terms appear and

$$\begin{aligned}(3.17)\qquad &(3.16) = (-1)^{\kappa+3} \\ &\quad \times \int_0^\infty e^{i(|x|\pm|y|)\lambda} \left(\frac{\partial}{\partial\lambda}\right)^{\kappa+3} (\tilde{T}_{\pm,j,k}(\lambda,x,y)\lambda^{j+k+1}) d\lambda.\end{aligned}$$

It follows from (3.10) that

$$|L_{\pm,j,k}(x,y)| \le C\langle y\rangle^{-(m-2-\kappa)} \langle x\rangle^{-(m-2-\kappa)} \langle 1 + ||x| \pm |y||\rangle^{-(\kappa+3)} \tag{3.18}$$

and, by virtue of Lemma 3.6, $L_{\pm,j,k}(x,y)$ are admissible. If $j+k+1=\kappa+2$, then a boundary term appears:

$$(3.16) = (3.17) + (-1)^{\kappa+3}(\kappa+2)!\tilde{T}_{\pm,j,k}(0,x,y). \tag{3.19}$$

Then, (3.10) again implies (3.18) and $L_{\pm,j,k}(x,y)$ are admissible. We finally consider the case that $j+k+1=\kappa+1$, viz. $j=0$ and/or $k=0$.

LEMMA 3.7. *Let $0 \le j,k \le \frac{m-3}{2}$ and let $j=0$ and/or $k=0$. Then, $L_{+,j,k}(x,y)-L_{-,j,k}(x,y)$ is admissible.*

PROOF. We have $j+k+1=\kappa+1$ and

$$\begin{aligned}(3.16) &= (-1)^{\kappa+1}\int_0^\infty \left(\frac{\partial}{\partial\lambda}\right)^2 e^{i(|x|\pm|y|)\lambda}\cdot\left(\frac{\partial}{\partial\lambda}\right)^{\kappa+1}(\tilde{T}_{\pm,j,k}(\lambda,x,y)\lambda^{\kappa+1})d\lambda \\ &= i(-1)^{\kappa+2}(\kappa+1)!(|x|\pm|y|)\tilde{T}_{\pm,j,k}(0,x,y) + (-1)^{\kappa+1}(\kappa+2)!\tilde{T}^{(1)}_{\pm,j,k}(0,x,y) \\ &\qquad + (-1)^{\kappa+3}\int_0^\infty e^{i(|x|\pm|y|)\lambda}\left(\frac{\partial}{\partial\lambda}\right)^{\kappa+3}(\tilde{T}_{\pm,j,k}(\lambda,x,y)\lambda^{j+k+1})d\lambda,\end{aligned}$$

where $\tilde{T}^{(1)}_{\pm,j,k}(\lambda,x,y)$ is the derivative of $\tilde{T}_{\pm,j,k}(\lambda,x,y)$ with respect to λ. The argument above shows that modulo admissible terms

$$(\pm 1)^j L_{\pm,j,k}(x,y) \equiv (-1)^{\kappa+2}(\kappa+1)!\frac{\tilde{T}_{\pm,j,k}(0,x,y)}{(i(|x|\pm|y|))^{\kappa+2}}. \tag{3.20}$$

We recall $\tilde{T}_{+,j,k}(0,x,y) = \langle VA_0(0)Y_j(0,\cdot,y), VX_k(0,\cdot,x)\rangle = \tilde{T}_{-,j,k}(0,x,y)$. Then, by virtue of (3.11), modulo admissible terms, $|L_{+,j,k}(x,y) - L_{\pm,j,k}(x,y)|$ is bounded by a constant times

$$\frac{C}{\langle x\rangle^{m-2-k}\langle y\rangle^{m-2-j}}\left|\frac{1}{(|x|+|y|)^{\kappa+2}} - \frac{(-1)^j}{(|x|-|y|)^{\kappa+2}}\right| \tag{3.21}$$

Elementary computation shows that for $||x|-|y||>1$, this is bounded by a constant times

$$\sum_{a+b=j+1}\frac{\langle y\rangle^{2+j-m}\langle x\rangle^{3-m}}{\langle|x|+|y|\rangle^{j+2-a}\langle|x|-|y|\rangle^{j+2-b}}, \quad \text{if } k=0,\ 1\le j\le \tfrac{m-3}{2}; \tag{3.22}$$

$$\sum_{a+b=k+1}\frac{\langle y\rangle^{3-m}\langle x\rangle^{2+k-m}}{\langle|x|+|y|\rangle^{k+2-a}\langle|x|-|y|\rangle^{k+2-b}}, \quad \text{if } j=0,\ 1\le k\le \tfrac{m-3}{2}; \tag{3.23}$$

$$\frac{2|x||y|}{\langle x\rangle^{m-2}\langle y\rangle^{m-2}\langle|x|+|y|\rangle^2\langle|x|-|y|\rangle^2}, \qquad \text{if } j=k=0. \tag{3.24}$$

The integrals with respect to y of summands of (3.22) may be computed by using polar coordinates and we estimate them by constants times

$$
\begin{aligned}
&\int_0^\infty \frac{(1+r)^{j+2-m} r^{m-1} dr}{(1+r+|x|)^{j+2-a}(1+|r-|x||)^{j+2-b}} \\
&\le \left(\int_0^{\frac{|x|}{2}} + \int_{\frac{|x|}{2}}^{2|x|} + \int_{2|x|}^\infty\right) \frac{r^{j+1} dr}{(1+r+|x|)^{j+2-a}(1+|r-|x||)^{j+2-b}} \\
&\le \frac{C}{\langle x\rangle^{j+3}} \int_0^{\frac{|x|}{2}} r^{j+1} dr + \frac{1}{\langle x\rangle^{1-a}} \int_{\frac{|x|}{2}}^{2|x|} \frac{C dr}{(1+|r-|x||)^{j+2-b}} + \int_{2|x|}^\infty \frac{C dr}{r^2}.
\end{aligned}
$$

The first and the last terms on the right are bounded by $C\langle x\rangle^{-1}$. The second is bounded by $C\langle x\rangle^{-1}\log(1+|x|) \le C$ if $a=0$ and $b=j+1$; if otherwise by $C\langle x\rangle^{a-1} \le C\langle x\rangle^{m-3}$ since $a-1 \le j \le (m-3)/2$. Thus,

$$
\sup_{x\in \mathbf{R}^m} \int_{\mathbf{R}^m} |(3.22)| dy \le C.
$$

We likewise estimate the integrals with respect to dx by constants times

$$
\begin{aligned}
&\int_0^\infty \frac{(1+r)^{3-m} r^{m-1} dr}{(1+r+|y|)^{j+2-a}(1+|r-|y||)^{j+2-b}} \\
&\le \frac{C}{\langle y\rangle^{j+3}} \int_0^{\frac{|y|}{2}} r^2 dr + \frac{1}{\langle y\rangle^{j-a}} \int_{\frac{|y|}{2}}^{2|y|} \frac{C dr}{(1+|r-|y||)^{j+2-b}} + \int_{2|y|}^\infty \frac{C dr}{(1+r)^{j+1}}.
\end{aligned}
$$

The first and the third terms are bounded by $C\langle y\rangle^{-j} \le C\langle y\rangle^{-1}$. The second is bounded by a constant unless $a=j+1$. If $a=j+1$, then it is bounded by $C\langle y\rangle \le C\langle y\rangle^{m-2-j}$ since $j \le (m-3)/2$. Thus, we have seen that (3.22) is admissible. Since (3.23) is obtained from (3.22) by replacing the variable x and y, (3.23) is also admissible. Similar estimate shows that (3.24) is also admissible. This completes the proof. □

3.2. Low energy estimate II, Singular part

In this subsection, we show that $\Phi(H)W_{s,j}\Phi(H_0)$, $j=1,2$, are bounded in $L^p(\mathbf{R}^m)$ for $\frac{m}{m-2} < p < \frac{m}{2}$ if $m \ge 5$ and for $\frac{3}{2} < p < 3$ if $m=3$. Recall that $W_{s,j}$, $j = 1, 2$ are defined by (1.15) and (1.16) respectively:

$$W_{s,1} = \frac{1}{\pi i} \int_0^\infty G_0(\lambda) V A_{-1} (G_0(\lambda) - G_0(-\lambda)) \tilde{\Phi}(\lambda) d\lambda,$$

$$W_{s,2} = \frac{1}{\pi i} \int_0^\infty G_0(\lambda) V P_0 V (G_0(\lambda) - G_0(-\lambda)) \tilde{\Phi}(\lambda) \lambda^{-1} d\lambda,$$

and A_{-1} is given by virtue of Theorem 2.12 by

$$A_{-1} = \begin{cases} -P_0 V D_3 V P_0 V - a(\varphi \otimes \varphi) V, & m = 3; \\ -c_0 (\varphi \otimes \varphi), & m = 5; \\ 0, & m \geq 7. \end{cases}$$

We use the following two lemmas. The first is well known (cf. [24], p.218).

LEMMA 3.8. *Let $1 < p < \infty$. Then, the function $|r|^a$ on $\mathbf{R}$ is one dimensional $(A)_p$ weight if and only if $-1 < a < p - 1$. The Hilbert transform $\tilde{\mathcal{H}}$ and the Hardy-Littlewood maximal operator $\mathcal{M}$ are bounded in the weighted L^p-space $L^p(\mathbf{R}, w(r)dr)$ with $(A)_p$ weights $w(r)$.*

LEMMA 3.9. *Let $m \geq 3$ be odd. Let $\psi \in L^1(\mathbf{R}^m)$ and $u \in \mathcal{S}(\mathbf{R}^m)$. Let $c_j = |\Sigma| C_j$ where C_j are the constants in (2.19). Then*

$$\langle \psi, (G_0(\lambda) - G_0(-\lambda)) u \rangle = \sum_{j=0}^{(m-3)/2} c_j \lambda^j \int_{\mathbf{R}} e^{i\lambda r} r^{1+j} M(r, \psi * \check{u}) dr$$

where $\check{u}(x) = u(-x)$ and $M(r, f)$ is the spherical average of f:

$$M(r, f) = \frac{1}{|\Sigma|} \int_\Sigma f(r\omega) d\omega, \qquad r \in \mathbf{R}. \tag{3.25}$$

PROOF. Applying Fubini's theorem and using polar coordinates, we obtain

$$\begin{aligned} \langle \psi, G_0(\lambda) u \rangle &= \sum_{j=0}^{(m-3)/2} C_j \int_{\mathbf{R}^m} \psi(x) \left(\int_{\mathbf{R}^m} \frac{\lambda^j e^{i\lambda|y|} u(x-y)}{|y|^{m-2-j}} dy \right) dx \\ &= \sum_{j=0}^{(m-3)/2} C_j \int_{\mathbf{R}^m} \frac{\lambda^j e^{i\lambda|y|} (\psi * \check{u})(y)}{|y|^{m-2-j}} dy \\ &= \sum_{j=0}^{(m-3)/2} c_j \int_0^\infty \lambda^j e^{i\lambda r} r^{1+j} M(r, \psi * \check{u}) dr \end{aligned}$$

Note that $M(r) = M(r, \psi * \check{u})$ is smooth and $r^{m-1}M(r)$ is integrable. Since $M(r)$ is an even function of r, by changing the variable r to $-r$, we also have

$$-\langle \psi, G_0(-\lambda)u\rangle = \sum_{j=0}^{(m-3)/2} c_j \int_{-\infty}^{0} \lambda^j e^{i\lambda r} r^{1+j} M(r, \psi * \check{u})dr$$

Adding last two equations sides by sides, we obtain the lemma. □

3.2.1 The case $m = 3$

We study the case $m = 3$ separately as the argument for this case is slightly different from that for other cases.

PROPOSITION 3.10. *The operator W_{s1} is bounded in $L^p(\mathbf{R}^3)$ for $3/2 < p < 3$.*

PROOF. We write $\varphi = \phi_1$ and we let $\{\phi_2, \dots, \phi_d\}$ be an orthonormal basis of $\mathcal{E}$. Define $a_{11} = -a$, $a_{1j} = 0$ for $2 \le j \le d$ and $a_{jk} = -\langle \phi_j | VD_3V | \phi_k\rangle$ for $2 \le j, k \le d$. Then, we have $W_{s,1} = \sum_{j,k=1}^{d} F_{jk}$ where F_{jk} are defined by

$$F_{jk}u(x) = \frac{a_{jk}}{\pi i}\int_0^\infty G_0(\lambda)V\phi_j\rangle\langle V\phi_k|(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(|D|)u\rangle d\lambda. \tag{3.26}$$

It suffices to show that all F_{jk} satisfy the property of the lemma. Since $\tilde{\Phi}(|D|)$ is bounded in L^p for all $1 \le p \le \infty$, we may and do ignore it. We have $|V(x)\phi_j(x)| \le C\langle x\rangle^{-\delta-1}$ for all j. Lemma 3.9 yields that for $u \in C_0^\infty(\mathbf{R}^3)$

$$L_k(\lambda) \equiv \langle V\phi_k, (G_0(\lambda) - G_0(-\lambda))u\rangle = \int_{\mathbf{R}} e^{ir\lambda} rM(r, V\phi_k * \check{u})dr. \tag{3.27}$$

Here $M_k(r) \equiv M(r, V\phi_k * \check{u})$ is smooth, even and satisfies the decay estimates $|\partial^\ell M_k(\rho)| \le C_\ell(1+r)^{-\delta-1}$ for $\ell = 0, 2, \dots$. Hence, $L_k(\lambda)$ is a rapidly decaying $C^{\delta-}$-function. We insert (3.27) in the right of (3.26):

$$\begin{aligned} F_{jk}u(x) &= a_{jk}\int_0^\infty \left(\int_{\mathbf{R}^3} \frac{e^{-i\lambda|x-y|}(V\phi_j)(y)}{4\pi^2 i|x-y|}dy\right) L_k(\lambda)d\lambda \\ &= a_{jk}\int_{\mathbf{R}^3} \frac{V\phi_j(y)}{4\pi^2 i|x-y|}\left\{\int_0^\infty e^{-i\lambda|x-y|}L_k(\lambda)d\lambda\right\} dy. \end{aligned}$$

We denote the function inside the brace by $T_k(\rho)$, $\rho = |x-y|$ so that

$$F_{jk}u(x) = a_{jk}\int_{\mathbf{R}^3} \frac{T_k(|x-y|)(V\phi_j)(y)}{4\pi^2 i|x-y|}dy.$$

Since $L_k(\lambda)$ is the Fourier transform of $rM_k(r)$, $T_k(\rho) = \pi(1+\tilde{\mathcal{H}})[rM_k(r)](|x-y|)$ where $\tilde{\mathcal{H}}$ is the Hilbert transform. Applying Hausdorff-Young's inequality and using polar coordinates, we have

$$\|F_{jk}u\|_p \le c_p\|V\phi_j\|_1\left(\int_{\mathbf{R}}|T_k(r)|^p|r|^{2-p}dr\right)^{1/p}, \quad c_p = (4\pi)^{\frac{1}{p}}|a_{jk}|/(4\pi^2).$$

By virtue of Lemma 3.8, $|r|^{2-p}$ is an one dimensional $(A)_p$ weight if and only if $3/2 < p < 3$ and for these p's we have

$$\left(\int_{\mathbf{R}}|T_k(r)|^p|r|^{2-p}dr\right)^{1/p} \le C_p\left(\int |M_k(r)|^p r^2 dr\right)^{1/p}. \tag{3.28}$$

Since $M_k(r)^p \le |\Sigma|^{-1}\int_\Sigma |V\phi_k * \check{u}(r\omega)|^p d\omega$ by Hölder's inequality, the right side of (3.28) is bounded by

$$C_p\|V\phi_j * \check{u}\|_p \le C_p\|V\phi_j\|_1\|u\|_p. \tag{3.29}$$

This proves the proposition. $\square$

Proposition 3.11. *The operator $W_{s,2}$ is bounded in $L^p(\mathbf{R}^3)$ for $3/2 < p < 3$.*

Proof. As in the proof of previous Proposition 3.10 we ignore the cut-off function $\tilde{\Phi}(\lambda)$. It suffices to show the lemma for the summands in

$$W_{2,s}u = \frac{1}{\pi i}\sum_{j=2}^{d}\int_0^\infty G_0(\lambda)(V\phi_j)\otimes(V\phi_j)(G_0(\lambda)-G_0(-\lambda))\lambda^{-1}d\lambda\cdot u.$$

We let $|D|^{-1}$ be the Fourier multiplier by $|\xi|^{-1}$, which is the convolution with $C|x|^{-2}$. Then, via Fourier transform, we have

$$\langle V\phi_j, \lambda^{-1}(G_0(\lambda)-G_0(-\lambda))u\rangle = \langle |D|^{-1}V\phi_j, (G_0(\lambda)-G_0(-\lambda))u\rangle.$$

Write $\psi_j(x) = |D|^{-1}(V\phi_j)$. Recall that ϕ_j are eigenfunctions of H and they satisfy $|V(x)\phi_j(x)| \le C\langle x\rangle^{-\delta-2}$ and $\int V\phi_j dx = 0$. It follows that, as $|x| \to \infty$,

$$\begin{aligned}\psi_j(x) &= C\int\left(\frac{1}{|x-y|^2} - \frac{1}{|x^2|}\right)(V\phi_j)(y)dy \\ &= C\int \frac{2x\cdot y - |y|^2}{|x|^2|x-y|^2}(V\phi_j)(y)dy = \sum_{k=1}^{3}\frac{C_{jk}x_k}{|x|^4} + O(|x|^{-4}),\end{aligned}$$

and the convolution with $\psi_j(x)$ is bounded in L^p for any $1 < p < \infty$:

$$\|\psi_j * u\|_p \le C\|u\|_p, \quad 1 < p < \infty \tag{3.30}$$

([24], pp. 30-36). We then repeat the proof of Proposition 3.10 and, at the final step we substitute this estimate (3.30) for Hausdorff-Young's inequality (3.29) and conclude that $\|W_{2s}u\|_p \le C_p\|u\|_p$ for $3/2 < p < 3$. This completes the proof of Proposition 3.11. □

3.2.2 The case $m \ge 5$

By virtue of (2.20), we have $W_{s1} = 0$ if $m \ge 7$ and, if $m = 5$,

$$W_{s,1} = \frac{-c_0}{\pi i}\int_0^\infty G_0(\lambda)(V\varphi)\otimes(V\varphi)(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)d\lambda$$

with $\varphi = P_0V$, V being considered as a vector; for any $m \ge 5$

$$W_{s,2} = \frac{1}{\pi i}\int_0^\infty G_0(\lambda)VP_0V(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda^{-1}d\lambda.$$

PROPOSITION 3.12. *Let $m \ge 5$, $|V(x)| \le C\langle x\rangle^{-\delta}$ with $\delta > m+3$ and $\frac{m}{m-2} < p < \frac{m}{2}$. Then, there exists a constant C_p such that $\|W_{s,j}u\|_p \le C_p\|u\|_p$ for all $u \in C_0^\infty(\mathbf{R}^m)$, $j = 1, 2$.*

We take an orthonormal basis $\{\phi_1, \dots, \phi_d\}$ of $\mathcal{E}$ and write $W_{2,s}$ in the form $W_{2,s} = \sum_{j=1}^d (1/\pi i)Z_j$, where

$$Z_j = \int_0^\infty G_0(\lambda)(V\phi_j)\otimes(V\phi_j)(G_0(\lambda) - G_0(-\lambda))\tilde{\Phi}(\lambda)\lambda^{-1}d\lambda.$$

We shall prove that Z_j, $j = 1, \dots, d$, all satisfy the estimate of the proposition by a series of lemma. The proof will then imply the same for $W_{s,1}$ because $W_{s,1}$ is obtained from Z_j by replacing $\tilde{\Phi}$ and ϕ_j by $\lambda\tilde{\Phi}(\lambda)$ and φ respectively and because the only properties of the former functions which are used in the proof for Z_j are that $\tilde{\Phi} \in C_0^\infty(\mathbf{R})$ and that $\phi_j \in \mathcal{E}$ which are as well satisfied by the latters . In the sequel we omit suffices j from ϕ_j and Z_j for brevity.

Denoting $M(r, (V\phi) * \check{u}) = M(r)$, we define for $0 \le j, k \le (m-3)/2$:

$$K_{jk}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{-i\lambda\rho}\lambda^{k+j-1}\left(\int_{\mathbf{R}} e^{i\lambda r} r^{1+j} M(r)dr\right)\tilde{\Phi}(\lambda)d\lambda. \tag{3.31}$$

Since $|\phi(x)| \le C\langle x\rangle^{2-m}$, $M(r)$ is smooth and $|\partial^\ell M(r)| \le C_\ell\langle r\rangle^{2-m-\delta}$ for $\ell = 0, 1, \dots$. It follows that the functions inside the parenthesis are at least of oder C^δ and decay rapidly at infinity. Moreover, it satisfies $\int_{-\infty}^\infty rM(r)dr = 0$ because $M(r)$ is even. It follows that (3.31) is well defined for all j, k. Lemma 3.9 and the change of the order of integrations imply

$$Zu(x) = -\frac{2}{\pi}\sum_{j,k=0}^{(m-3)/2} c_j C_k \int_{\mathbf{R}^m} \frac{(V\phi)(y)K_{jk}(|x-y|)}{|x-y|^{m-2-k}}dy \tag{3.32}$$

where c_j and C_k are the constants in Lemma 3.9 and (2.19) respectively. In what follows we show that all integral operators on the right of (3.32),

$$W_{jk}u(x) \equiv \int_{\mathbf{R}^m} \frac{(V\phi)(y)K_{jk}(|x-y|)}{|x-y|^{m-2-k}}dy, \tag{3.33}$$

are bounded in $L^p(\mathbf{R}^m)$ when $m \ge 5$ and $m/(m-2) < p < m/2$. By interpolation, we have only to show this for $p = m/(m-2-\varepsilon)$ and for $p = m/(2+\varepsilon)$ with arbitrarily small $\varepsilon > 0$. We use the following lemma which is obvious because $|V(x)\phi(x)| \le C\langle x\rangle^{-(m-2+\delta)}$, $\delta > m+3$.

Lemma 3.13. *Let $p = m/(m-2-\varepsilon)$ or $p = m/(2+\varepsilon)$, $0 < \varepsilon < 1$ and q be its dual exponent. Suppose $b < m$. Then*

$$\int_{\mathbf{R}^m}\left(\int_{|x-y|<1} \frac{|V(y)\phi(y)|^p dy}{|x-y|^b}\right)^{\frac{q}{p}} dx \le C < \infty$$

In what follows we write $\mathcal{H} = (1+\tilde{\mathcal{H}})/2$, $\tilde{\mathcal{H}}$ being the Hilbert transform. By virtue of Lemma 3.8, $|r|^{m-1-p\theta}$ is a one dimensional $(A)_p$ weight if and only if $0 < \frac{m}{p} - \theta < 1$, or,

$$m - 3 - \varepsilon < \theta < m - 2 - \varepsilon \quad \text{if } p = m/(m-2-\varepsilon); \tag{3.34}$$

$$1 + \varepsilon < \theta < 2 + \varepsilon \quad \text{if } p = m/(2+\varepsilon). \tag{3.35}$$

In what follows we take $\theta = m - 3$ if $p = m/(m-2-\varepsilon)$ and $\theta = 2$ if $p = m/(2+\varepsilon)$, $0 < \varepsilon < 1$ being a small number. It is well known that for $f \in \mathcal{S}$ there exists a constant $C > 0$ such that

$$|(f * u)(t)| \le C(\mathcal{M}u)(t), \quad t \in \mathbf{R}, \tag{3.36}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator ([24], p. 57).

(i) The case $j + k = 1$ and $p = m/(m-2-\varepsilon)$. If $j + k = 1$, we have

$$K_{jk}(\rho) = \mathcal{H}((\mathcal{F}\tilde{\Phi}) * (r^{1+j}M))(\rho). \tag{3.37}$$

We split the integral (3.33) into $|x-y| < 1$ and $|x-y| \ge 1$ and estimate

$$\begin{aligned} |W_{jk}u(x)| &\le \left(\int_{|x-y|<1} + \int_{|x-y|>1} \right) \frac{|(V\phi)(y)K_{jk}(|x-y|)|}{|x-y|^{m-2-k}} dy \\ &\equiv I_1(x) + I_2(x) \end{aligned} \tag{3.38}$$

For $k = 0$ and $k = 1$ and for $|x-y| \ge 1$, we have $|x-y|^{-(m-2-k)} \le |x-y|^{-\theta}$, $\theta = m-3$, and Hausdorff-Young's inequality implies

$$\|I_2\|_p \le \|V\phi\|_1 \||x|^{-\theta} K_{jk}(|x|)\|_p = \|V\phi\|_1 \left(|\Sigma| \int_0^\infty |K_{jk}(\rho)|^p \rho^{m-1-p\theta} d\rho \right)^{\frac{1}{p}}.$$

Since $\rho^{m-1-p\theta}$ is a one dimensional $(A)_p$ weight, (3.36), (3.37) and Lemma 3.8 imply that the integral on the right is bounded by a constant times

$$\left(\int_0^\infty |r^{1+j} M(r)|^p r^{m-1-p\theta} dr \right)^{\frac{1}{p}} \le C \left(\int_{\mathbf{R}^m} |V\phi * \check{u}(x)|^p \frac{dx}{|x|^{p(\theta-1-j)}} \right)^{\frac{1}{p}}$$

where we used Hölder's inequality $|M(r,f)|^p \le |\Sigma|^{-1} \int_\Sigma |f(r\omega)|^p d\omega$ in the final step. Since $0 \le p(\theta - 1 - j) < m$ for $j = 0, 1$, the right is bounded by

$$\begin{aligned} C \left(\int_{|x|\ge 1} |V\phi * \check{u}(x)|^p dx \right)^{\frac{1}{p}} &+ C\|V\phi * \check{u}\|_\infty \left(\int_{|x|<1} \frac{dx}{|x|^{p(\theta-1-j)}} \right)^{\frac{1}{p}} \\ &\le C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p, \end{aligned}$$

where $q = m/(2+\varepsilon)$ is the dual exponent of p. Thus,

$$\||x|^{3-m} K_{jk}(|x|)\|_p \le C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p \tag{3.39}$$

and, therefore, $\|I_2\|_p \le C_{p,V}\|u\|_p$. By Hölder's inequality and (3.39)

$$\begin{aligned}|I_1(x)| &\le \left(\int_{|x-y|<1} \left(\frac{K_{jk}(|x-y|)}{|x-y|^{m-3}}\right)^p dy\right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_{|x-y|<1} \left|\frac{(V\phi)(y)}{|x-y|^{1-k}}\right|^q dy\right)^{\frac{1}{q}} \\ &\le C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p \cdot \left(\int_{|x-y|<1} \left|\frac{(V\phi)(y)}{|x-y|^{1-k}}\right|^q dy\right)^{\frac{1}{q}}\end{aligned}$$

Since $(1-k)q < m$, Lemma 3.13 implies $\|I_1\|_p \le C_{V,p}\|u\|_p$. Thus, we have proven $\|W_{jk}u\|_p \le \|u\|_p$ in this case.

(ii) The case $j+k=1$ and $p = m/(2+\varepsilon)$. We estimate $|W_{jk}u(x)| \le I_1(x) + I_2(x)$ as in (3.38). Since $m-2-k \ge \theta = 2$, we have

$$|I_2(x)| \le \int_{|x-y|\ge 1} \frac{|(V\phi)(y)K_{jk}(|x-y|)|}{|x-y|^\theta} dy$$

and $\|I_2\|_p \le \|V\phi\|_1 \||x|^{-\theta} K_{jk}(|x|)\|_p$. Again $|r|^{m-1-\theta p}$ is a one dimensional $(A)_p$ weight and $0 \le p(\theta-1-j) < m$ for $j = 0, 1$. Using (3.36) and (3.37), applying the weighted inequality, and estimating the integral in the final step by spliting it into the part on $|x| < 1$ and on $|x| \ge 1$ as previously, we obtain

$$\begin{aligned}\left\|\frac{K_{jk}(|x|)}{|x|^\theta}\right\|_p &= \left(\int_0^\infty r^{m-1-\theta p} |K_{jk}(r)|^p dr\right)^{\frac{1}{p}} \\ &\le C\left(\int_{\mathbf{R}} |r|^{m-1-(\theta-1-j)p} M(r)^p dr\right)^{\frac{1}{p}} \\ &\le C\left(\int_{\mathbf{R}^m} \frac{|V\phi * \check{u}(x)|^p dx}{|x|^{p(\theta-1-j)}}\right)^{\frac{1}{p}} \\ &\le C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p,\end{aligned} \tag{3.40}$$

where $q = m/(m-2-\varepsilon)$. Hence, $\|I_2\|_p \le C_{V,p}\|u\|_p$. For estimating $I_1(x)$ we use Hölder's inequality and then (3.40). We obtain

$$
\begin{aligned}
|I_1(x)| &\le \left(\int_{|x-y|<1} \left|\frac{(V\phi)(y)}{|x-y|^{m-4-k}}\right|^q dy\right)^{\frac{1}{q}} \\
&\quad \cdot \left(\int_{|x-y|<1} \left|\frac{K_{jk}(|x-y|)}{|x-y|^2}\right|^p dy\right)^{\frac{1}{p}} \\
&\le C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p \left(\int_{|x-y|<1} \left|\frac{(V\phi)(y)}{|x-y|^{m-4-k}}\right|^q dy\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $q(m-4-k) < m$, Lemma 3.13 implies $\|I_1\|_p \le C_{Vp}\|u\|_p$. Estimate $\|W_{jk}u\|_p \le C\|u\|_p$ holds if $j+k=1$ and $p = m/(2+\varepsilon)$.

(iii) The case $j, k \ge 1$ **and** $p = m/(m-2-\varepsilon)$. We now write

$$
K_{jk}(\rho) = \tilde{L}_{j+k} * (r^{1+j}M)(\rho), \quad \tilde{L}_{j+k}(\rho) = \frac{1}{2\pi}\int_0^\infty e^{-i\rho\lambda}\lambda^{j+k-1}\Phi(\lambda)d\lambda.
$$

Integration by parts implies $|\tilde{L}_{j+k}(\rho)| \le C\langle\rho\rangle^{-(j+k)}$. Hence,

$$
\begin{aligned}
\rho^{k-1}|K_{jk}(\rho)| &\le C\langle\rho\rangle^{k-1}\int_{\mathbf{R}} \langle r-\rho\rangle^{-(k+j)} r^{j+1}|M(r)|dr \\
&\le C\int_{\mathbf{R}} \langle r-\rho\rangle^{-1-j}\langle r\rangle^{k-1} r^{j+1}|M(r)|dr \\
&\le C\mathcal{M}(\langle r\rangle^{k-1} r^{j+1} M)(\rho).
\end{aligned} \tag{3.41}
$$

Define $Q_{jk}(\rho) = \mathcal{M}(r^{j+1}\langle r\rangle^{k-1}M)(\rho)$. Then, (3.41) yields

$$
|W_{jk}(x)| \le \int_{\mathbf{R}^m} \frac{|(V\phi)(y)|Q_{jk}(|x-y|)dy}{|x-y|^\theta}, \quad \theta = m-3,
$$

and $\|W_{jk}u\|_p \le \|V\phi\|_1 \||x|^{-\theta}Q_{jk}(|x|)\|_p$. Since $r^{m-1-p\theta}$ is a one dimensional $(A)_p$ weight, we obtain, using also Hölder's inequality, that

$$
\begin{aligned}
\left\|\frac{Q_{jk}(|x|)}{|x|^\theta}\right\|_p &\le C\left(\int_0^\infty r^{m-1-p\theta}(r^{j+1}\langle r\rangle^{k-1}M(r))^p dr\right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbf{R}^m} \frac{|(V\phi)*\check{u})(x)|^p\langle x\rangle^{p(k-1)}}{|x|^{p(m-4-j)}}dx\right)^{\frac{1}{p}}
\end{aligned} \tag{3.42}
$$

Since $0 \le p(k-1) \le p(m-4-j) < m$, we may estimate the last integral by $C(\|(V\phi) * \check{u}\|_\infty + \|(V\phi) * \check{u}\|_p) \le C\|u\|_p$ as previously, by splitting the domain of integration into $\{x : |x| < 1\}$ and $\{x : |x| \ge 1\}$. We have shown that $\|W_{jk}u\|_p \le C\|u\|_p$ for this case.

(iv) The case $j, k \ge 1$ **and** $p = m/(2+\varepsilon)$. Trading the weight in $K_{jk} = \tilde{L}_{j+k} * (r^{1+j}M)$ in the direction opposite to the one used in (3.41), we estimate

$$\begin{aligned} \langle\rho\rangle^{1-j}|K_{jk}(\rho)| &\le \int_{\mathbf{R}} \langle r-\rho\rangle^{-(k+1)}(\langle\rho\rangle\langle r-\rho\rangle)^{1-j} r^{j+1} M(r)dr \\ &\le C\int_{\mathbf{R}} \langle r-\rho\rangle^{-(k+1)}\langle r\rangle^{1-j} r^{j+1}M(r)dr \le C\mathcal{M}(r^2M)(\rho), \end{aligned} \tag{3.43}$$

or $|K_{jk}(\rho)| \le C\mathcal{M}(r^2M)(\rho)\langle\rho\rangle^{j-1}$. We estimate $|W_{jk}u(x)| \le I_1(x) + I_2(x)$ as in (3.38). Using (3.43) for $\rho > 1$ and remembering $j + k \le m-3$, we have

$$\begin{aligned} |I_2(x)| \le \ & C\int_{|x-y|>1} \frac{|(V\phi)(y)|\mathcal{M}(r^2M)(|x-y|)\langle x-y\rangle^{j-1}}{|x-y|^{m-2-k}}dy \\ \le \ & C\int_{\mathbf{R}^m} \frac{|(V\phi)(y)|\mathcal{M}(r^2M)(|x-y|)}{|x-y|^{\theta}}dy, \quad \theta = 2, \end{aligned}$$

and $\|I_2\|_p \le C\|V\phi\|_1\|\mathcal{M}(r^2M)/|x|^2\|_p$. Since r^{m-1-2p} is an $(A)_p$ weight,

$$\begin{aligned} \left\|\frac{\mathcal{M}(r^2M)(|x|)}{|x|^2}\right\|_p &= C\left(\int_0^\infty \rho^{m-1-2p}\{\mathcal{M}(r^2M)(\rho)\}^p d\rho\right)^{\frac{1}{p}} \\ &\le C\left(\int_0^\infty r^{m-1}|M(r)|^p dr\right)^{\frac{1}{p}} \le C\|V\phi * \check{u}\|_p \le C\|V\phi\|_1\|u\|_p. \end{aligned} \tag{3.44}$$

Thus, $\|I_2\|_p \le C\|u\|_p$. Estimate (3.43), Hölder's inequality and (3.44) imply

$$\begin{aligned} |I_1(x)| &\le C\int_{|x-y|<1} \frac{|V\phi(y)|\mathcal{M}(r^2M)(|x-y|)dx}{|x-y|^{m-2-k}} \\ &\le \left(\int_{|x-y|<1}\left(\frac{|(V\phi)(y)|}{|x-y|^{m-4-k}}\right)^q dy\right)^{\frac{1}{q}} \left\|\frac{\mathcal{M}(r^2M)(|x|)}{|x|^2}\right\|_p \\ &\le C\|V\phi\|_1\|u\|_p\left(\int_{|x-y|<1}\left(\frac{|(V\phi)(y)|}{|x-y|^{m-4-k}}\right)^q dy\right)^{\frac{1}{q}} \end{aligned}$$

where $q = m/(m-2+\varepsilon)$. Since $q(m-4-k) < m$, Lemma 3.13 implies $\|I_1\| \le C\|u\|_p$. We have obtained the desired $\|W_{jk}u\|_p \le C\|u\|_p$ for this case.

(v) The case $1 \le j \le (m-3)/2$, $k = 0$. We want to show

$$W_{j0} \in \mathbf{B}(L^p) \quad \text{for all } m/(m-2) < p < m/2. \tag{3.45}$$

We have shown in (i) and (ii) that (3.45) is satisfied if $j = 1$. We prove (3.45) for $j \ge 2$ by induction on j. We assume that $j \ge 2$ and that (3.45) has already been proven for smaller values of j. By integration by parts

$$\begin{aligned}
K_{j0}(\rho) &= \frac{-i}{2\pi}\int_0^\infty e^{-i\lambda\rho}\lambda^{j-1}\tilde{\Phi}(\lambda)\frac{d}{d\lambda}\left(\int_{\mathbf{R}} e^{i\lambda r}r^j M(r)dr\right)d\lambda \\
&= \frac{i}{2\pi}\int_0^\infty \frac{d}{d\lambda}\left(e^{-i\lambda\rho}\lambda^{j-1}\tilde{\Phi}(\lambda)\right)\left(\int_{\mathbf{R}} e^{i\lambda r}r^j M(r)dr\right)d\lambda \\
&= \rho K_{j-1,1}(\rho) + i(j-1)K_{j-1,0}(\rho) \\
&\quad + \frac{i}{2\pi}\int_0^\infty e^{-i\lambda\rho}\lambda^{j-2}\lambda\tilde{\Phi}'(\lambda)\left(\int_{\mathbf{R}} e^{i\lambda r}r^j M(r)dr\right)d\lambda
\end{aligned}$$

Note that the last integral is the same as $K_{j-1,0}(\rho)$ if $\lambda\tilde{\Phi}'(\lambda)$ is replaced by $\tilde{\Phi}(\lambda)$ and that the former function plays no worse role than the latter. Thus, we may ignore it by considering it being absorbed in $K_{j-1,0}(\rho)$. If we replace $K_{j0}(\rho)$ by $\rho K_{j-1,1}(\rho)$ or $K_{j-1,0}(\rho)$ in the definition (3.33) of $W_{j,0}u$, they respectively produce $W_{j-1,1}u$ or $W_{j-1,0}u$. We have already shown that $W_{j-1,1}u \in \mathbf{B}(L^p)$ for all $\frac{m}{m-2} < p < \frac{m}{2}$ in steps (iii) and (iv) above, and $W_{j-1,0}u$ does so by the induction hypothesis. Estimate (3.45) holds for all $j \le (m-3)/2$ by induction.

(vi) The case $j = 0$ and $1 \le k \le (m-3)/2$. We want to show

$$W_{0k} \in \mathbf{B}(L^p) \quad \text{for all } m/(m-2) < p < m/2. \tag{3.46}$$

We have already shown (3.46) for $k = 1$ in steps (i) and (ii) above and we prove (3.46) for $2 \le k \le (m-3)/2$ by induction on k. We assume $k \ge 2$ and that we have already proven (3.46) for smaller values of k. Integration by parts yields

$$
\begin{aligned}
K_{0k}(\rho) &= \frac{1}{2\pi i\rho}\int_0^\infty \left(-\frac{d}{d\lambda}e^{-i\lambda\rho}\right)\lambda^{k-1}\tilde{\Phi}(\lambda)\left(\int_{\mathbf{R}} e^{i\lambda r} rM(r)dr\right)d\lambda \\
&= \frac{1}{2\pi i\rho}\int_0^\infty e^{-i\lambda\rho}\left(\frac{d}{d\lambda}\right)\left\{\lambda^{k-1}\tilde{\Phi}(\lambda)\left(\int_{\mathbf{R}} e^{i\lambda r} rM(r)dr\right)\right\}d\lambda \\
&= (i\rho)^{-1}(k-1)K_{0,k-1}(\rho) + \rho^{-1}K_{1,k-1}(\rho) \\
&+\frac{1}{2\pi i\rho}\int_0^\infty e^{-i\lambda\rho}\lambda^{k-2}(\lambda\tilde{\Phi}'(\lambda))\left(\int_{\mathbf{R}} e^{ir\lambda} rM(r)dr\right)d\lambda.
\end{aligned}
$$

Again $\lambda\tilde{\Phi}'(\lambda)$ plays no worse role than $\tilde{\Phi}(\lambda)$ and we may ignore the last term on the right, considering it being absorbed in $\rho^{-1}K_{0,k-1}(\rho)$. If we replace $K_{0k}(\rho)$ by $\rho^{-1}K_{0,k-1}(\rho)$ and $\rho^{-1}K_{1,k-1}(\rho)$ in the definition (3.33) of W_{0k}, they respectively produce $W_{0,k-1}$ and $W_{1,k-1}$. We have shown $W_{1,k-1} \in \mathbf{B}(L^p)$ for all $\frac{m}{m-2} < p < \frac{m}{2}$ in (iii) and (iv) and $W_{0,k-1}$ does so by the induction hypothesis. Thus, (3.46) is satisfied for all $1 \le k \le (m-3)/2$ by induction.

(vii) The case $j = k = 0$. Since $\int_{\mathbf{R}} rM(r)dr = 0$, we have

$$
\frac{1}{i\lambda}\int_{\mathbf{R}} e^{ir\lambda} rM(r)dr = \int_{\mathbf{R}}\left(\frac{e^{ir\lambda}-1}{i\lambda}\right)rM(r)dr = \int_{\mathbf{R}}\left(\int_0^r e^{i\rho\lambda}d\rho\right)rM(r)dr
$$

We change the order of integration on the right. The result is

$$
\int_{\mathbf{R}} e^{i\lambda\rho}\left(\int_{\Delta(\rho)} rM(r)dr\right)d\rho = \int_{\mathbf{R}} e^{i\lambda\rho}\tilde{M}(\rho)d\rho
$$

where $\Delta(\rho) = \{r : \pm r > \pm\rho\}$ for $\pm\rho > 0$ and $\tilde{M}(\rho) \equiv \int_{\Delta(\rho)} rM(r)dr$. It follows that

$$
\begin{aligned}
K_{00}(\rho) &= \frac{1}{2\pi}\int_0^\infty e^{-i\lambda\rho}\tilde{\Phi}(\lambda)\left(\frac{1}{i\lambda}\int_{\mathbf{R}} e^{ir\lambda} rM(r)dr\right)d\lambda \\
&= ((\mathcal{F}\tilde{\Phi}) * (\mathcal{H}\tilde{M}))(\rho),
\end{aligned}
$$

and we have $|K_{00}(\rho)| \le C(\mathcal{M}\mathcal{H}\tilde{M})(\rho)$. We then proceed as in the case (i).

First let $p = m/(m-2-\varepsilon)$ and write $\theta = m-3$. We estimate $|W_{00}u(x)|$ as in (3.38). Since $|x-y|^{-(m-2)} \le |x-y|^{-\theta}$ for $|x-y| \ge 1$, we have by applying Hausdorff-Young's inequality that $\|I_2\|_p \le C\|V\phi\|_1 \||x|^{-\theta}K_{00}(|x|)\|_p$. Since $r^{m-1-p\theta}$ is an $(A)_p$ weight and $m - p\theta > 0$, Lemma 3.8 and then Hardy's inequality ([23], p. 274) imply

$$\begin{aligned}\||x|^{-\theta}K_{00}(|x|)\|_p &\le C\left(\int_0^\infty \rho^{m-1-p\theta}|(\mathcal{M}\mathcal{H}\tilde{M})(\rho)|^p d\rho\right)^{\frac{1}{p}} \\ &\le C\left(\int_{\mathbf{R}} \rho^{m-1-p\theta}|\tilde{M}(\rho)|^p d\rho\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \rho^{m-1-p\theta}\rho^{2p}M(\rho)^p d\rho\right)^{\frac{1}{p}}.\end{aligned}$$

Since $0 \le p(\theta-2) = p(m-5) < m$, the right hand side is as previously bounded by $C(\|V\phi\|_1 + \|V\phi\|_q)\|u\|_p$, $q = \frac{m}{2+\varepsilon}$, and

$$\|I_2\|_p \le C\||x|^{-(m-3)}K_{00}(|x|)\|_p \le C\|u\|_p. \tag{3.47}$$

By virtue of Hölder's inequality and (3.47) we have

$$\begin{aligned}|I_1(x)| &\le \left(\int_{|x-y|<1}\left|\frac{(V\phi)(y)}{|x-y|}\right|^q dy\right)^{\frac{1}{q}} \cdot \left(\int_{|x-y|<1}\left|\frac{K_{00}(|x-y|)}{|x-y|^{m-3}}\right|^p dy\right)^{1/p} \\ &\le C\|u\|_p\left(\int_{|x-y|<1}\left|\frac{(V\phi)(y)}{|x-y|}\right|^q dy\right)^{\frac{1}{q}},\end{aligned}$$

where $q = m/(2+\varepsilon) < m$. Lemma 3.13 implies $\|I_1\|_p \le C\|u\|_p$. This proves $\|W_{00}u\|_p \le C\|u\|_p$ when $p = m/(m-2-\varepsilon)$, $0 < \varepsilon < 1$.

Next we let $p = m/(2+\varepsilon)$. We first estimate $|W_{00}(x)|$ by the right of (3.38). Since $m-2 \ge 2$,

$$|I_2(x)| \le \int_{|x-y|>1} \frac{|(V\phi)(y)K_{00}(|x-y|)|dy}{|x-y|^2}$$

and $\|I_2\|_p \le \|V\phi\|_1\||x|^{-2}K_{00}\|_p$. Since r^{m-1-2p} is an $(A)_p$ weight and $m-$

$2p > 0$, we may apply Lemma 3.8 and then Hardy's inequality to obtain

$$\||x|^{-2}K_{00}(|x|)\|_p \le C\left(\int_0^\infty \rho^{m-1-2p}|(\mathcal{M}\mathcal{H}\tilde{M})(\rho)|^p d\rho\right)^{\frac{1}{p}}$$

$$\le C\left(\int_0^\infty \rho^{m-1-2p}|\tilde{M}(\rho)|^p d\rho\right)^{\frac{1}{p}} \le C\left(\int_0^\infty \rho^{m-1}M(\rho)^p d\rho\right)^{\frac{1}{p}} \le C\|u\|_p$$

Thus, $\|I_2\|_p \le C\|V\phi\|_1\|u\|_p$. If $q = m/(m-2-\varepsilon)$ is the dual exponent of p,

$$|I_1 u(x)| \le \left(\int_{|x-y|<1}\left|\frac{(V\phi)(y)}{|x-y|^{m-4}}\right|^q dy\right)^{\frac{1}{q}} \cdot \left(\int_{|x-y|<1}\left|\frac{K_{00}(|x-y|)}{|x-y|^2}\right|^p dy\right)^{1/p}.$$

The first factor is an L^p function of $x \in \mathbf{R}^m$ by virtue of Lemma 3.13 and the second is bounded by $\||x|^{-2}K_{00}(|x|)\|_p \le C\|u\|_p$. Thus $\|I_1\|_p \le C\|u\|_p$. This proves $\|W_{00}u\|_p \le C\|u\|_p$ for the case $p = m/(2+\varepsilon)$, $0 < \varepsilon < 1$. This completes the proof of Proposition 3.12.

3.3. High energy estimate

We prove in this subsection that $W_>$ is bounded in $L^p(\mathbf{R}^m)$ for all $1 \le p \le \infty$ if V satisfies $|V(x)| \le C\langle x\rangle^{-\delta}$, $\delta > m+2$, in addition to (1.8), viz. $\mathcal{F}(\langle x\rangle^{2\sigma}V) \in L^{m_*}(\mathbf{R}^m)$ for $\sigma > \frac{1}{m_*}$. Since threshold singularities are irrelevant to the high energy part $W_> = W\Psi(H_0)^2$, we can (and do) follow basically the same idea as in [26], however, we shall substantially relax the regularity and the decay conditions on the potentials by improving some of the arguments there.

We want to show that the integral kernel of $W_>$ is admissible by the method used for the low energy case. However, in this way we shall encounter two difficulties: (1) The λ integral extends to non-compact interval and the integral corresponding to (3.6) does not converge absolutely; (2) the local singularities of the convolution kernel $C_j e^{i\lambda|x|}|x|^{-(m-2-j)}$ of $G_{0,j}(\lambda)$ are strong and they are not locally in L^2 except the case $j = (m-3)/2$. To overcome the first difficulty we shall exploit Lemma 2.2 that the resolvent decays as $|\lambda| \to \infty$: $\|\langle x\rangle^{-\sigma}G_0(\lambda)\langle x\rangle^{-\tau}\|_{\mathbf{B}(\mathcal{H})} \le C\lambda^{-1}$ for $\sigma, \tau > 1/2$. In order to do this, we expand $(1+G_0(\lambda)V)^{-1}$ and write $W_>$ in the following form:

$$\begin{aligned}
&\Psi(H_0)^2 - \frac{1}{\pi}\int_0^\infty G_0(\lambda)V(1+G_0(\lambda)V)^{-1}(G_0(\lambda)-G_0(-\lambda))\Psi(H_0)^2\lambda d\lambda \\
&\quad = \Psi(H_0)^2 + \sum_{j=1}^{2n}\frac{(-1)^j}{i\pi}\int_0^\infty (G_0(\lambda)V)^j(G_0(\lambda)-G_0(-\lambda))\lambda d\lambda \Psi(H_0)^2 \\
&\quad -\frac{1}{i\pi}\int_0^\infty (G_0(\lambda)V)^n G(\lambda)V(G_0(\lambda)V)^n(G_0(\lambda)-G_0(-\lambda))\lambda\Psi(\lambda^2)^2 d\lambda \\
&\quad = \Psi(H_0)^2 + \sum_{j=1}^{2n}(-1)^j\Omega_j\Psi(H_0)^2 - \tilde{\Omega}_{2n+1}.
\end{aligned}$$

The operator $\Psi(H_0)$ is bounded in L^p for any $1 \le p \le \infty$ and, by virtue of Lemma 3.1, $\|\Omega_n u\|_p \le C^n\|\mathcal{F}(\langle x\rangle^{2\sigma}V)\|_{m_*}^n\|u\|_p$ for $\sigma > 1/m_*$.

In what follows we show that, if n is large enough, the integral kernel

$$\tilde{\Omega}_{2n+1}(x,y) = \int_0^\infty \langle \tilde{K}_n(\lambda)(G_0(\lambda)-G_0(-\lambda))\delta_y, G_0(-\lambda)\delta_x\rangle\lambda\Psi^2(\lambda^2)d\lambda,$$

of $\tilde{\Omega}_{2n+1}$, where $\tilde{K}_n(\lambda) = (VG_0(\lambda))^{n-1}VG(\lambda)V(G_0(\lambda)V)^n$, is admissible by applying the argument similar to the one used for studying $W_r(x,y)$. However, we need modify it because of the difficulty (2) mentioned above. We write the integral kernel of $G_0(\lambda)$ in the form

$$G_0(\lambda,z,x) = e^{i\lambda|x|}\sum_{j=0}^{\frac{m-3}{2}} C_j\lambda^j\frac{e^{i\lambda(|x-z|-|x|)}}{|x-z|^{m-2-j}} \equiv e^{i\lambda|x|}\tilde{G}_0(\lambda,z,x).$$

Note that we have now included the factor λ^j into $\tilde{G}_0(\lambda,z,x)$. For $|\lambda| \ge 1$, we have

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^j\tilde{G}_0(\lambda,z,x)\right| \le C\lambda^{\frac{m-3}{2}}\left(\frac{\langle z\rangle^j}{|x-z|^{m-2}} + \frac{\langle z\rangle^j}{|x-z|^{\frac{m-1}{2}}}\right) \tag{3.48}$$

Define $T_\pm(\lambda,x,y) = \langle\tilde{K}_n(\lambda)\tilde{G}_0(\pm\lambda,\cdot,y), \tilde{G}_0(-\lambda,\cdot,x)\rangle$ so that

$$\begin{aligned}
&\tilde{\Omega}_{2n+1}(x,y) \\
&\quad = \frac{1}{\pi i}\int_0^\infty \left(e^{i\lambda(|x|+|y|)}T_+(\lambda,x,y) - e^{i\lambda(|x|-|y|)}T_-(\lambda,x,y)\right)\tilde{\Psi}(\lambda)d\lambda,
\end{aligned}$$

where $\tilde{\Psi}(\lambda) \equiv \Psi(\lambda^2)^2$. Note that the smoothing factors $\Phi_0(x,y)$ and $\Phi(x,y)$ for the low energy case are absent here. For proceeding further we take a partition of unity $\chi(z) + \tilde{\chi}(z) \equiv 1$ such that $\chi \in C_0^\infty(\mathbf{R}^m)$, $\chi(z) = 1$ for $|z| \le 1$ and $\chi(z) = 0$ for $|z| \ge 2$ and split $G_0(\lambda,x,y)$ and $\tilde{G}_0(\lambda,x,y)$ into sums of compactly supported part and smooth part:

$$\begin{aligned} G_0(\lambda,x,y) &= \chi(x-y)G_0(\lambda,x,y) + \tilde{\chi}(x-y)G_0(\lambda,x,y), \\ \tilde{G}_0(\lambda,x,y) &= \chi(x-y)\tilde{G}_0(\lambda,x,y) + \tilde{\chi}(x-y)\tilde{G}_0(\lambda,x,y). \end{aligned}$$

We set $G_{0,<}(\lambda,x,y) = \chi(x-y)G_0(\lambda,x,y)$, $G_{0,>}(\lambda,x,y) = \tilde{\chi}(x-y)G_0(\lambda,x,y)$ and let $G_{0,<}(\lambda)$ and $G_{0,>}(\lambda)$ be the operators with the respectivel integral kernels. We use similar notation for $\tilde{G}_0(\lambda,x,y)$.

LEMMA 3.14. *Let $0 \le s \le \frac{m+3}{2}$. For sufficiently large n, we have*

$$\left|\left(\frac{\partial}{\partial\lambda}\right)^s T_\pm(\lambda,x,y)\right| \le C_{ns}\lambda^{-2}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}. \tag{3.49}$$

PROOF. By using Leibniz' formula we write $\left(\frac{\partial}{\partial\lambda}\right)^s T_\pm(\lambda,x,y)$ as a linear combination of

$$\begin{aligned} &\langle VG_0^{(s_n)}(\lambda)V\cdots G_0^{(s_1)}(\lambda)V\tilde{G}_0^{(s_0)}(\pm\lambda,\cdot,y), \\ &\qquad\qquad G^{(t_n)}(-\lambda)VG_0^{(t_{n-1})}(-\lambda)V\cdots G_0^{(t_1)}(-\lambda)V\tilde{G}_0^{(t_0)}(-\lambda,\cdot,x)\rangle \end{aligned}$$

where $s_0 + \cdots + s_n + t_0 + \cdots + t_n = s$. If $\frac{1}{2} < \gamma$ is sufficiently close to $\frac{1}{2}$ so that $\delta - (s_0 + s_1 + \gamma) > \frac{m}{2}$, then we have for the smooth part $\tilde{G}_{0,>}^{(s_0)}(\lambda,\cdot,y)$ that

$$\|\langle\cdot\rangle^{s_1+\gamma}V\tilde{G}_{0,>}^{(s_0)}(\lambda,\cdot,y)\|_2 \le C\lambda^{\frac{m-3}{2}}\langle y\rangle^{-\left(\frac{m-1}{2}\right)}. \tag{3.50}$$

We then obtain that for such $\gamma > \frac{1}{2}$ and $s_{n+1} = t_n$:

$$\|\langle\cdot\rangle^{s_{n+1}+\gamma}VG_0^{(s_n)}V\cdots G_0^{(s_1)}V\tilde{G}_{0,>}^{(s_0)}(\lambda,\cdot,y)\|_2 \le C\lambda^{\left(-n+\frac{m-3}{2}\right)}\langle y\rangle^{-\frac{m-1}{2}}$$

by considering the quantity inside the norm in the form

$$\left(\prod_{i=1}^n V\langle z\rangle^{s_i+s_{i+1}+2\gamma}\langle z\rangle^{-s_i-\gamma}G_0^{(s_i)}(\lambda)\langle z\rangle^{-s_i-\gamma}\right)\langle z\rangle^{s_1+\gamma}V\tilde{G}_{0,>}^{(s_0)}(\lambda,\cdot,y)$$

and by applying Lemma 2.2. For the function produced by the singular part $\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)$, we further split

$$G_0^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)=G_{0,>}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)+G_{0,<}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y).$$

We may estimate $S(\lambda,x,y)\equiv\langle x\rangle^{-s_1-\gamma}\{G_{0,>}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\}(x)$ by

$$\int_{\mathbf{R}^m}\frac{C\lambda^{m-3}|V(z)||\chi(z-y)|\langle z\rangle^{s_0}dz}{\langle x\rangle^{s_1+\gamma}\langle x-z\rangle^{\frac{m-1}{2}-s_1}|z-y|^{m-2}}\le\frac{C\lambda^{(m-3)}}{\langle x\rangle^{s_1+\gamma}\langle x-y\rangle^{\frac{m-1}{2}-s_1}\langle y\rangle^{\delta-s_0}}.$$

Since $\gamma>\frac{1}{2}$ and $\delta-s_0>(m-1)/2$, Hölder's inequality implies

$$\|S(\lambda,\cdot,y)\|_2\le C\lambda^{m-3}\langle y\rangle^{-(\delta-s_0)}\le C\lambda^{m-3}\langle y\rangle^{-\frac{m-1}{2}}.$$

Thus, by the same argument as above we obtain

$$\|\langle\cdot\rangle^{s_{n+1}+\gamma}VG_0^{(s_n)}V\cdots G_0^{(s_2)}VG_{0,>}^{(s_1)}V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\|\le C\lambda^{m-n-2}\langle y\rangle^{-\frac{m-1}{2}}.$$

Since $|G_{0,<}^{(s_1)}(\lambda,z,x)|\le C\lambda^{\left(\frac{m-3}{2}\right)}|x-z|^{-(m-2)}\chi(x-z)$, we have

$$\left|\int G_{0,<}^{(s_1)}(\lambda,x-z)V(z)\tilde{G}^{(s_0)}_{0,<}(\pm\lambda,z,y)dz\right|\le\frac{C\lambda^{m-3}\langle y\rangle^{-\delta+s_0}}{|x-y|^{m-4}}\chi\left(\frac{x-y}{4}\right).$$

Notice that the singularity at $x=y$ is tamed by the factor 2 at the cost of increasing the factor $\lambda^{\frac{m-3}{2}}$ with respect to λ. If $m\le 7$, we then have

$$\|G_{0,<}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\|_2\le C\lambda^{m-3}\langle y\rangle^{-\frac{m-1}{2}},$$

hence, $\|\langle\cdot\rangle^{s_{n+1}+\gamma}VG_0^{(s_n)}V\cdots G_0^{(s_2)}VG_{0,<}^{(s_1)}V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\|\le C\lambda^{m-n-2}\times\langle y\rangle^{-\frac{m-1}{2}}$. Then, by summing up the estimates obtained so far, we conclude that

$$\|\langle\cdot\rangle^{s_{n+1}+\gamma}VG_0^{(s_n)}V\cdots G_0^{(s_1)}V\tilde{G}^{(s_0)}_0(\lambda,\cdot,y)\|\le C\lambda^{m-n-2}\langle y\rangle^{-\frac{m-1}{2}},\quad|\lambda|>1.$$

If $m-4\ge m/2$, viz. $m\ge 8$, however, $|x-y|^{4-m}\chi((x-y)/4)$ is still not in L^2. We then further split

$$\begin{aligned}&G_0^{(s_2)}(\lambda)VG_{0,<}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\\&=G_{0,<}^{(s_2)}(\lambda)VG_{0,<}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)+G_{0,>}^{(s_2)}(\lambda)VG_{0,<}^{(s_1)}(\lambda)V\tilde{G}^{(s_0)}_{0,<}(\lambda,\cdot,y)\end{aligned}$$

and repeat the argument above. Then, after doing this $\ell = (m-3)/2$ times, the function produced by using singular kernels only also becomes L^2 function and

$$\|G_{0,<}^{(s_\ell)}(\lambda)V\cdots VG_{0,<}^{(s_1)}(\lambda)V\tilde{G}_{0,<}^{(s_0)}(\lambda,\cdot,y)\|_{L^2} \le C\lambda^{(\ell+1)(m-3)/2}\langle y\rangle^{-\frac{m-1}{2}}.$$

We also have for any $\gamma > \frac{1}{2}$ that

$$\begin{aligned}&\|\langle\cdot\rangle^{-(s_\ell+\gamma)}G_{0,>}^{(s_\ell)}(\lambda)V\cdots VG_{0,<}^{(s_1)}(\lambda)V\tilde{G}_{0,<}^{(s_0)}(\lambda,\cdot,y)\|_{L^2}\\ &\quad \le C\lambda^{(\ell+1)(m-3)/2}\langle y\rangle^{-\frac{m-1}{2}}.\end{aligned}$$

Summing up the estimates thus obtained, we conclude by taking n sufficiently large that

$$\|\langle\cdot\rangle^{s_{n+1}+\gamma}\prod_{i=1}^{n}(VG_0^{(s_i)}(\lambda))V\tilde{G}_0^{(s_0)}(\lambda,\cdot,y)\| \le C\lambda^{-1}\langle y\rangle^{-\frac{m-1}{2}}.$$

Entirely similar argument implies

$$\|\langle\cdot\rangle^{s_{n+1}+\gamma}\prod_{i=1}^{n}(VG_0^{(s_i)}(\lambda))V\tilde{G}_0^{(s_0)}(-\lambda,\cdot,y)\| \le C\lambda^{-1}\langle y\rangle^{-\frac{m-1}{2}}$$

and also that

$$\|\langle x\rangle^{-t_n-\gamma}G^{(t_n)}(-\lambda)V\prod_{i=1}^{n-1}(G_0^{(t_i)}(-\lambda)V)\tilde{G}_0^{(t_0)}(-\lambda,\cdot,x)\| \le C\lambda^{-1}\langle x\rangle^{-\frac{m-1}{2}}.$$

Last two estimates imply the lemma. □

We now apply integration by parts $(m+3)/2$ times to obtain

$$\begin{aligned}&\int_0^\infty e^{i\lambda(|x|\pm|y|)}T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)d\lambda\\ &= \frac{1}{(|x|\pm|y|)^{\frac{m+3}{2}}}\int_0^\infty e^{i\lambda(|x|\pm|y|)}\left(i\frac{\partial}{\partial\lambda}\right)^{\frac{m+3}{2}}\Big(T_\pm(\lambda,x,y)\tilde{\Psi}(\lambda)\Big)d\lambda\end{aligned}$$

and estimate the right hand side by using (3.49). We obtain

$$|\tilde{\Omega}_{n+1}(x,y)| \le C\langle|x|-|y|\rangle^{-\frac{m+3}{2}}\langle x\rangle^{-\frac{m-1}{2}}\langle y\rangle^{-\frac{m-1}{2}}$$

and $\tilde{\Omega}_{n+1}(x, y)$ is admissible. This completes the proof of Theorem 1.1.

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