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Classification of Elliptic and K3 Fibrations Birational to some $\mathbb{Q}$-Fano 3-Folds

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Abstract. A complete classification is presented of elliptic and K3 fibrations birational to certain mildly singular complex Fano 3-folds. Detailed proofs are given for one example case, namely that of a general hypersurface X of degree 30 in weighted $\mathbb{P}^4$ with weights 1,4,5,6,15; but our methods apply more generally. For constructing birational maps from X to elliptic and K3 fibrations we use Kawamata blowups and Mori theory to compute anticanonical rings; to exclude other possible fibrations we make a close examination of the strictly canonical singularities of $\big(X, \frac{1}{n}\mathcal{H}\big)$, where $\mathcal{H}$ is the linear system associated to the putative birational map and n is its anticanonical degree.

1. Introduction

In [CPR] Corti, Pukhlikov and Reid proved that a general quasismooth complex variety $X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$ in one of the 'famous 95 families' of $\mathbb{Q}$-Fano 3-fold weighted hypersurfaces is *birationally rigid* — that is, if X is birational to some Mori fibre space Y/S then in fact $Y \simeq X$. A related problem is to classify elliptic and K3 fibrations birational to general hypersurfaces in these families; it was Ivan Cheltsov [Ch00] who first proved classification results of this kind for several birationally rigid smooth Fano varieties, including a general quartic 3-fold $X_4 \subset \mathbb{P}^4$ and a double cover of $\mathbb{P}^3$ branched in a general sextic surface, i.e., $X = X_6 \subset \mathbb{P}(1,1,1,1,3)$. In [Ry02] the classification of elliptic and K3 fibrations birational to general members of the remaining 93 families was addressed, but completed only for family 5, $X_7 \subset \mathbb{P}(1,1,1,2,3)$. In the present paper we aim firstly to give concise proofs of some of the more generally applicable results of [Ry02] and secondly to present a complete proof of the following theorem for family 75, which is the family referred to in the abstract. Furthermore, we state similar

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theorems for families 34, 88 and 90; these can be proved using essentially the same techniques.

1.1 THEOREM. *Let $X = X_{30} \subset \mathbb{P}(1,4,5,6,15)_{x,y,z,t,u}$ be a general member of family 75 of the 95.*

(a) Suppose $\Phi\colon X \dashrightarrow Z/T$ is a birational map from X to a K3 fibration $g\colon Z \to T$ (see 1.15 below for our assumptions on K3 fibrations, and also on elliptic fibrations and Fano 3-folds). Then there exists an isomorphism $\mathbb{P}^1 \to T$ such that the diagram below commutes, where $\pi = (x^4, y)\colon X \dashrightarrow \mathbb{P}^1$.

$$\begin{tikzcd} X \arrow[r, dashed, "\Phi"] \arrow[d, dashed, "\pi"'] & Z \arrow[d, "g"] \\ \mathbb{P}^1 \arrow[r, "\simeq"] & T \end{tikzcd}$$

(b) There does not exist an elliptic fibration birational to X.

(c) If $\Phi\colon X \dashrightarrow Z$ is a birational map from X to a Fano 3-fold Z with canonical singularities then Φ is actually an isomorphism (so in particular $Z \simeq X$ has terminal singularities).

Part (b) of this theorem was recently proved independently by Cheltsov and Park [CP05] using somewhat different methods. It is an interesting result because of its relevance to the question of whether $\mathbb{Q}$-rational points of X are potentially dense: a birational elliptic fibration is one key geometric construction used to prove potential density (see [HT00], [Ha03] and [HT01]). The proof presented here requires close examination of CS $\left(X, \frac{1}{n}\mathcal{H}\right)$ (see 1.6), where $\mathcal{H}$ is the linear system associated to a putative birational map and n is its anticanonical degree; in [CP05] more general methods are used. Our approach has the advantage that the other parts of Theorem 1.1 follow immediately from our complete classification of possible sets of strictly canonical centres.

The following results are analogous to Theorem 1.1; they can be proved with the same techniques, though we do not include all the details here. From now on we abbreviate conclusions such as that in Theorem 1.1(a) by stating that *up to a birational twist of the base, $g \circ \Phi = (x^4, y)\colon X \dashrightarrow \mathbb{P}^1$.*

The birational twist $\mathbb{P}^1 \to T$ of the base is an isomorphism in the above case because T is a smooth curve. (We assume all fibrations are morphisms of normal varieties: see 1.15.)

1.2 THEOREM. *Let $X = X_{18} \subset \mathbb{P}(1,1,2,6,9)_{x_0,x_1,y,z,t}$ be a general member of family 34 of the 95.*

(a) If $\Phi\colon X \dashrightarrow Z/T$ is a birational map from X to a K3 fibration $g\colon Z \to T$ then, up to a birational twist of the base (see above for explanation), $g \circ \Phi = (x_0, x_1)\colon X \dashrightarrow \mathbb{P}^1$.

(b) Suppose $\Phi\colon X \dashrightarrow Z/T$ is a birational map from X to an elliptic fibration $g\colon Z \to T$. Then, up to a birational twist of the base, $g \circ \Phi = (x_0, x_1, y)\colon X \dashrightarrow \mathbb{P}(1,1,2)$.

(c) If $\Phi\colon X \dashrightarrow Z$ is a birational map from X to a Fano 3-fold Z with canonical singularities then Φ is actually an isomorphism (so in particular $Z \simeq X$ has terminal singularities).

1.3 THEOREM. *Let $X = X_{42} \subset \mathbb{P}(1,1,6,14,21)_{x_0,x_1,y,z,t}$ be a general member of family 88 of the 95. Under assumptions corresponding to those in Theorem 1.2 we can conclude as follows.*

(a) Up to a birational twist of the base, $g \circ \Phi = (x_0, x_1)\colon X \dashrightarrow \mathbb{P}^1$.

(b) Up to a birational twist of the base, $g \circ \Phi = (x_0, x_1, y)\colon X \dashrightarrow \mathbb{P}(1,1,6)$.

(c) Φ is actually an isomorphism.

1.4 THEOREM. *Let $X = X_{42} \subset \mathbb{P}(1,3,4,14,21)_{x,y,z,t,u}$ be a general member of family 90 of the 95. Under assumptions corresponding to those in Theorem 1.2 we can conclude as follows.*

(a) Up to a birational twist of the base, $g \circ \Phi = (x, y)\colon X \dashrightarrow \mathbb{P}(1,3)$.

(b) Up to a birational twist of the base, $g \circ \Phi = (x, y, z)\colon X \dashrightarrow \mathbb{P}(1,3,4)$.

(c) Φ is actually an isomorphism.

Outline of paper

Our proof of Theorem 1.1 has essentially three parts; this division of the argument is closely modelled on the approach of Cheltsov to similar problems for smooth varieties — see e.g. [Ch00]. In brief, the parts are: constructing the K3 fibration birational to our X in family 75 (see §2); proving a technical result, Theorem 1.7, using exclusion arguments (§3); and deriving Theorem 1.1 from Theorem 1.7 (in §4) using an analogue of the Noether–Fano–Iskovskikh inequalities (1.10) together with an adaptation of the framework of [Ch00]. We now make some comments on each of these three.

1.5. In §2 we show that the projection $(x^4, y)\colon X \dashrightarrow \mathbb{P}^1$ is indeed a K3 fibration, after resolution of indeterminacy. The construction is Mori-theoretic: we make a Kawamata blowup of X and play out the two ray game. We also outline constructions of the elliptic fibrations in Theorems 1.2(b), 1.3(b) and 1.4(b).

1.6 **The exclusion arguments of §3.** First let us state the technical theorem mentioned above, which is proved in §3. We need the following.

Notation. Let X be a normal complex projective variety, $\mathcal{H}$ a mobile linear system on X and $\alpha \in \mathbb{Q}_{\geq 0}$. We denote by $\mathrm{CS}(X, \alpha\mathcal{H})$ the set of centres on X of valuations that are strictly canonical or worse for $K_X + \alpha\mathcal{H}$ — that is,

$$\mathrm{CS}(X, \alpha\mathcal{H}) = \{\mathrm{Centre}_X(E) \mid a(E, X, \alpha\mathcal{H}) \leq 0\}.$$

Occasionally we also use $\mathrm{LCS}(X, \alpha\mathcal{H})$, which is defined similarly as

$$\mathrm{LCS}(X, \alpha\mathcal{H}) = \{\mathrm{Centre}_X(E) \mid a(E, X, \alpha\mathcal{H}) \leq -1\}.$$

1.7 Theorem. *Let $X = X_{30} \subset \mathbb{P}(1,4,5,6,15)_{x,y,z,t,u}$ be a general member of family 75 of the 95. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ nonterminal. Then in fact $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right) = \{Q_1, Q_2\}$, where $Q_1, Q_2 \sim \frac{1}{5}(1,4,1)_{x,y,t}$ are the two singularities of X on the zu-stratum.*

For an introduction to the relationship between this result and Theorem 1.1, see 1.9 below. Our proof of 1.7 in §3 is by exclusion arguments, as mentioned above — e.g., we show in Theorem 3.7 that no smooth point can be a centre on X of a valuation strictly canonical for $K_X + \frac{1}{n}\mathcal{H}$; we refer to this as *excluding* any smooth point. As well as *absolute* exclusions such as this, there is an interesting *conditional* exclusion result for singular points, Theorem 3.25. Some of the exclusion results of §3 are extensions of arguments in [CPR], but Theorem 3.25 is an example of strikingly new behaviour, and there are substantial differences in method also for exclusion of curves (§3.1). It should be noted that, though the main aim of §3 is to prove Theorem 1.7, several of the results obtained apply to many of the 95 families other than number 75 (in particular, numbers 34, 88 and 90) — and the techniques of proof apply more generally still. [Ry02, App. A] contains a detailed list of results analogous to Theorem 1.7 for most of the 95 families, including many for which only a conjectural birational classification of elliptic and K3 fibrations is currently known. In the case of family 34, for instance, we have the following.

1.8 Theorem. *Let $X = X_{18} \subset \mathbb{P}(1,1,2,6,9)_{x_0,x_1,y,z,t}$ be a general member of family 34 of the 95. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ nonterminal. Then in fact $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and* $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ *is either $\{C, P, Q_1, Q_2, Q_3\}$ or $\{P\}$. Here $C = \{x_0 = x_1 = 0\} \cap X$ is irreducible by generality of X, and P, Q_1, Q_2 and Q_3 are the singularities of X; $P \sim \frac{1}{3}(1,1,2)_{x_0,x_1,y}$ lies on the zt-stratum and $Q_1, Q_2, Q_3 \sim \frac{1}{2}(1,1,1)_{x_0,x_1,t}$ lie on the yz-stratum.*

Like Theorem 1.2, this result is not proved in this paper, but the techniques for doing so, and for proving analogues for families 88 and 90, are essentially those we use below for Theorem 1.7.

1.9 **The derivation of Theorem 1.1 from Theorem 1.7 in §4.** In a sense the relationship between Theorems 1.7 and 1.1 is obvious: the K3 fibration given by (x^4, y) has the singularities Q_1 and Q_2 of 1.7 as centres, and no other set of centres is possible — so if, say, we try to grow an elliptic fibration birational to X, we must start with an extremal extraction of Q_1 or Q_2, but then we find we have to extract the other Q_i as well. It turns out that to make this rigorous we need some abstract machinery — in particular, results concerning the log Kodaira dimension of $(X, (\frac{1}{n} + \varepsilon)\mathcal{H})$ for small ε. It should be noted that for many of the 95 families an argument such as the above does not apply directly, because sets of centres do not distinguish objects of interest: for example, [Ry02] contains many examples of $\mathbb{Q}$-Fanos X birational to, say, an elliptic fibration and also a Fano with canonical singularities, and with the two linear systems having the same CS on X.

To describe the approach of §4 we consider a more abstract setup. Let X be a Mori Fano variety (see 1.14 below), Z a variety with canonical singularities and $\Phi\colon X \dashrightarrow Z$ a birational map. Assume furthermore that one of the following holds.

(a) $g\colon Z \to T$ is a K-trivial fibration (see 1.15) with $0 < \dim T < \dim Z$. Let $\mathcal{H}_Z = g^*|H|$ be the pullback of a very ample complete linear system of Cartier divisors on T and $\mathcal{H}$ its birational transform on X.

(b) Z is a Fano variety with canonical singularities (see 1.14) and Φ is not an isomorphism. Let $\mathcal{H}_Z = |H|$ be a very ample complete linear system of Cartier divisors on Z and $\mathcal{H}$ its birational transform on X.

In either case (a) or (b), define $n \in \mathbb{Q}$ by $K_X + \frac{1}{n}\mathcal{H} \sim_{\mathbb{Q}} 0$.

1.10 **NFI-type inequality** ([Ch00]). In either of the above situations, $K_X + \frac{1}{n}\mathcal{H}$ is nonterminal, that is, strictly canonical or worse.

This is a standard result used in [Ch00] and [Is01]; but we give a proof of it in §4, under the assumptions (a), because in this situation it follows from results we need anyway. Clearly 1.10 is motivation enough to address Theorem 1.7 on the way towards proving Theorem 1.1, but more work is needed to complete the proof of the latter. §4 contains the necessary arguments; two of the propositions are results of Cheltsov (though we give a proof of one of them), but in order to conclude we need a rather delicate argument that traces a log Kodaira dimension through a two ray game diagram.

What is special about families 34, 75, 88 and 90?

It is natural to ask why we are able to prove Theorems 1.1, 1.2, 1.3 and 1.4 for families 75, 34, 88 and 90, but are not able to prove similar theorems for other families out of the 95 with the same methods. There is no one answer to this, but the following are major factors.

- General members of families 34, 75, 88 and 90 are *superrigid*, i.e., there are no nonautomorphic birational selfmaps. For nonsuperrigid families (the majority of the 95) one obtains much bigger anticanonical rings after Kawamata blowups of the singularities that are centres of involutions: see [CPR]. This makes impossible any direct generalisation of arguments such as that in our final proof of Theorem 1.1(b).

- As already mentioned, it frequently occurs for other families out of the 95 that sets of centres on X do not distinguish between different birational maps to elliptic or K3 fibrations or to Fanos with canonical singularities. [Ry02, Ch. 4] discusses this phenomenon in some detail using family 5, $X_7 \subset \mathbb{P}(1,1,1,2,3)$, as an extended example. Whilst the analogue of Theorem 1.1 is eventually proved for this family in [Ry02], the proof requires complicated exclusion arguments on blown up models of X, and there is no obvious way of avoiding these. See [CM04] for a situation that is in some ways analogous.

The recent work of Cheltsov and Park [CP05] proves a number of results that complement those here: they show, for example, that a general member of any of the 95 families is birational to a K3 fibration, but do not classify the K3 fibrations so obtained; they also prove that a general member of family N is not birational to an elliptic fibration if and only if $N \in \{3, 60, 75, 84, 87, 93\}$ and, for some 23 values of N not in this set, they show that up to a birational twist of the base there is a unique birational elliptic fibration: it is obtained by projection (x_0, x_1, x_2) onto the first three coordinates [CP05, 4.13]. They do not, however, prove full analogues of our Theorem 1.7 for these families; this is one reason why they cannot classify birational K3 fibrations.

More recently (since the preprint of the present paper appeared), Cheltsov has succeeded in classifying elliptic fibrations birational to a general member of any of the 95 families [Ch05]. Again the methods used differ somewhat from those of this paper and do not permit analogues of our Theorem 1.7 to be proved, or birational K3 fibrations to be classified. Our methods do make possible the classification of K3 fibrations, but only in cases where whenever we blow up canonical centres not excluded by general results we obtain small, manageable anticanonical rings. With our current technology this restricts us to families such as 34, 75, 88 and 90, where there are few birational maps to elliptic and K3 fibrations and no nontrivial birational maps to Fanos with canonical singularities.

Conventions and assumptions

Our notations and terminology are mostly as in, for example, [KM], but we list here some conventions that are nonstandard, together with assumptions that will hold throughout.

1.11. All varieties considered are complex, and they are projective and normal unless otherwise stated.

1.12. For details on the famous 95 families, see [Fl00] and [CPR]. In brief, $X_d \subset \mathbb{P}(1, a_1, \ldots, a_4)$ belongs to one of the families if (1) X is *quasismooth*, i.e., its singularities are all quotient singularities forced by the weighted $\mathbb{P}^4$; (2) the singularities are *terminal* — 3-fold terminal quotient singularities are necessarily of the form $\frac{1}{r}(1, a, r-a)$ with $r \geq 1$ and $(a, r) = 1$; and (3) $a_1 + \cdots + a_4 = d$, so by the adjunction formula $-K_X = \mathcal{O}_X(1)$ is very ample. Whenever X is a member of one of the 95 families, we let $A = -K_X = \mathcal{O}_X(1)$ denote the positive generator of the class group; moreover, if $f\colon Y \to X$ is a birational morphism then B denotes $-K_Y$.

1.13. As in [CPR] and [Ry02], we refer to the weighted blowup with weights $\frac{1}{r}(1, a, r-a)$ of a 3-fold terminal quotient singularity $\frac{1}{r}(1, a, r-a)$ with $r \geq 2$ and $(a, r) = 1$ as the *Kawamata blowup* — see [Ka96], the main theorem of which is reproduced here as Theorem 3.14.

1.14 Definitions. A *Mori fibre space* $f\colon X \to S$ is a Mori extremal contraction of fibre type, that is, $\dim S < \dim X$. This means that X and S are projective varieties, X has $\mathbb{Q}$-factorial, terminal singularities, $f_*\mathcal{O}_X = \mathcal{O}_S$, $\rho(X/S) = 1$ and $-K_X$ is f-ample. If $S = \{*\}$ is a point then X is a *Mori Fano variety.* We use the term *Fano variety* more generally to refer to any normal, projective variety X with $-K_X$ ample and $\rho(X) = 1$.

1.15 Definitions. Let Z be a normal projective variety with canonical singularities. A *fibration* is a morphism $g\colon Z \to T$ to another normal projective variety T such that $\dim T < \dim Z$ and $g_*\mathcal{O}_Z = \mathcal{O}_T$. We say the fibration is *K-trivial* if and only if $K_Z C = 0$ for every contracted curve C.

g is an *elliptic fibration*, resp. a *K3 fibration*, if and only if its general fibre is an elliptic curve, resp. a K3 surface.

1.16. Usually when we write an equation explicitly or semi-explicitly in terms of coordinates we omit scalar coefficients of monomials; this is the 'coefficient convention'.

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2. Constructions

2.1. K3 fibrations

The following observation does not apply to family 75, our main object of study, but it does apply to families 34 and 88; in any case, it needs to be noted because it describes all 'easy' K3 fibrations birational to members of the famous 95 — cf. Lemma 2.3 for the 'hard' case.

2.1 Proposition. *Let $X = X_d \subset \mathbb{P}(1,1,a_2,a_3,a_4)$ be general in one of the families with $a_1 = 1$ and $a_2 > 1$. Then a general fibre S of $\pi = (x_0, x_1)\colon X \dashrightarrow \mathbb{P}^1$ is a quasismooth Du Val K3 surface and, setting $\mathcal{P}$ to be the pencil $\langle x_0, x_1\rangle$, we have*

$$\mathrm{CS}(X,\mathcal{P}) = \{C, P_1, \dots, P_r\},$$

where C is the curve $\{x_0 = x_1 = 0\} \cap X$, which is irreducible by generality of X, and $P_1, \dots, P_r$ are all the singularities of X.

Proof. Because S is a general element of $|\mathcal{O}_X(1)|$, it is certainly quasismooth. The adjunction formula for $S_d \subset \mathbb{P}(1,a_2,a_3,a_4) =: \mathbb{P}$ gives $K_S = 0$ and the cohomology long exact sequence from

$$0 \to \mathcal{I}_{S,\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_S \to 0,$$

with the standard cohomology results for weighted projective space, gives $h^1(S, \mathcal{O}_S) = 0$. Therefore S is a quasismooth Du Val K3 surface.

Let $f\colon Y \to X$ be the blowup of the ideal sheaf $\mathcal{I}_{C,X}$ of C and $E \subset Y$ the unique exceptional divisor of f which dominates C. Then clearly $m_E(\mathcal{P}) = a_E(K_X) = 1$, so $C \in \mathrm{CS}(X, \mathcal{P})$. The fact that $P_1, \dots, P_r \in \mathrm{CS}(X, \mathcal{P})$ is a consequence of Corollary 3.17 of Kawamata's Lemma (below). Therefore

$$\mathrm{CS}(X, \mathcal{P}) \supset \{C, P_1, \dots, P_r\};$$

the reverse inclusion follows from Theorem 3.7. □

2.2 REMARK. If $X = X_d \subset \mathbb{P}(1, 1, 1, a_3, a_4)$ is general in a family with $a_1 = a_2 = 1$ then clearly a general element S of any pencil $\mathcal{P} \subset |\mathcal{O}_X(1)|$ is a Du Val K3 surface, provided one can prove it is quasismooth. This can be fiddly, the problem being that while X is general and $S \in \mathcal{P}$ is general, $\mathcal{P}$ must be able to be *any* pencil inside $|\mathcal{O}_X(1)|$.

In contrast to the situation considered above, for families $X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$ with $a_1 > 1$ it is not immediately clear whether there exist K3 fibrations birational to X: to construct them, or at least to make sense of the construction, we need Mori theory. Here we consider only family 75, but the technique applies to many other families — see [Ry02] — and, in particular, to family 90, the subject of our Theorem 1.4.

2.3 LEMMA. *Let $X = X_{30} \subset \mathbb{P}(1, 4, 5, 6, 15)_{x,y,z,t,u}$ be a general member of family 75 of the 95; for the notations P, Q_1, Q_2, R_1, R_2 for the singularities of X, see Theorem 1.7. Let $Q_i \in X$ be either Q_1 or Q_2 and $f\colon Y \to X$ the Kawamata blowup of Q_i.*

(1) *Let $R \subset \overline{\mathrm{NE}}\, Y$ be the ray with $\mathrm{cont}_R = f$. Then the other ray $Q \subset \overline{\mathrm{NE}}\, Y$ is contractible and its contraction $g = \mathrm{cont}_Q\colon Y \to Z$ is antiflipping.*

(2) *The antiflip $Y \dashrightarrow Y'$ of g exists and Y' has canonical singularities.*

(3) *Let $Q' \subset \overline{\mathrm{NE}}\, Y'$ be the ray whose contraction is $g'\colon Y' \to Z$ and $R' \subset \overline{\mathrm{NE}}\, Y'$ the other ray. Then R' is contractible and its contraction $f'\colon Y' \to \mathbb{P}^1$, which is in fact the anticanonical morphism*

$\varphi_{|-4K_{Y'}|} = \varphi_{|4B'|}$, *is a K3 fibration — that is, a general fibre T' of f' has Du Val singularities, $K_{T'} = 0$ and $h^1(T', \mathcal{O}_{T'}) = 0$.*

It follows that the total composite $X \dashrightarrow \mathbb{P}^1$ of the two ray game we have played, illustrated below, is $\pi = (x, y)\colon X \dashrightarrow \mathbb{P}(1,4) = \mathbb{P}^1$.

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[dr, "g"] & & Y' \arrow[dl, "g'"'] \arrow[dr, "f'"] & \\
X & & Z & & \mathbb{P}^1
\end{tikzcd}$$

Therefore $R(Y, B) = R(Y', B') = k[x, y]$.

Proof of 2.3. (1) The first part of the following argument — showing that the curve C defined below generates $Q \subset \overline{\mathrm{NE}}\, Y$ — is one case of [CPR, 5.4.3]; but in [CPR] this point Q_i is excluded as a maximal centre by the test class method, so there is no need for the two ray game to be played out.

By generality of X, the curve $\{x = y = 0\} \cap X$ is irreducible. $f\colon Y \to X$ is the $\frac{1}{5}(1,4,1)_{x,y,t}$ weighted blowup of the $\frac{1}{5}(1,4,1)_{x,y,t}$ point Q_i. Let $S \in |B|$ be the unique effective surface and $T \in |4B|$ a general element, where as always $B = -K_Y$. One can check explicitly, by looking at the three affine pieces of Y locally over Q_i, that $C := S \cap T$ is an irreducible curve inside Y; this uses the generality of X. We also know that

$$B^3 = A^3 - \frac{1}{ra(r-a)} = \frac{1}{60} - \frac{1}{20} < 0,$$

so in particular $BC = 4B^3 < 0$. It follows that the ray $Q \subset \overline{\mathrm{NE}}\, Y$ is generated by C — indeed, suppose this is not the case; then C is in the interior of the 2-dimensional cone $\overline{\mathrm{NE}}\, Y$, so we can pick an effective 1-cycle $\sum_{i=0}^{p} \alpha_i C_i$ that lies strictly between Q and the half-line generated by C. This 1-cycle is B-negative, because R is B-positive (i.e., K-negative) and $BC < 0$; but $\mathrm{Bs}\,|4B|$ is supported on C and therefore one of the C_i, say C_0, is in fact C. The geometry of the cone now implies that, after we subtract off $\alpha_0 C_0$, $\sum_{i=1}^{p} \alpha_i C_i$ is again strictly between Q and the half-line generated by C; so we can repeat the argument to deduce that some other C_i is C — and of course this contradicts our initial (implicit) assumption that the C_i

were distinct. This argument has also shown that C is the only irreducible curve in the ray Q.

We show Q is contractible with a general Mori-theoretic trick — afterwards, clearly, $g = \mathrm{cont}_Q$ is antiflipping, because C is the only contracted curve and $K_Y C = -BC > 0$. Firstly note that S has canonical singularities and so in particular is klt. We apply Shokurov's inversion of adjunction (see [KM, 5.50]) to deduce that the pair (Y, S) is plt. (In fact $K_Y + S$ is Cartier so the log discrepancy of any valuation for (Y, S) is an integer, and therefore (Y, S) is canonical.) But plt is an open condition so for $\varepsilon \in \mathbb{Q}_{>0}$, $\varepsilon \ll 1$, the pair $(Y, S + \varepsilon T)$ is plt as well. Now

$$(K_Y + S + \varepsilon T)C = \varepsilon TC = \varepsilon B^3 < 0,$$

so Q is contractible by the (log) contraction theorem for $(Y, S + \varepsilon T)$.

(2) This follows immediately from Mori's result [FA, 20.11]: S and T are effective divisors, $T \sim 4S$ and $T \cap S = C$ is precisely the exceptional set of g. Since $S \sim B = -K_Y$ the antiflip of g is precisely its 'opposite with respect to S', to use the language of [FA]. This can be constructed as the normalisation of the closure of the image of

$$g \times \pi_Y \colon Y \dashrightarrow Z \times \mathbb{P}^1,$$

where $\pi_Y = \pi \circ f \colon Y \dashrightarrow \mathbb{P}(1,4) = \mathbb{P}^1$ corresponds to the pencil $\langle 4S, T\rangle = |4B|$.

Alternatively, since we have already observed that the ray Q is $(K_Y + S + \varepsilon T)$-negative, (2) follows from Shokurov's general result that log flips of lc pairs exist in dimension 3.

(3) From the construction of the antiflip, the transforms S' and T' of S and T on Y' are disjoint, so $\mathrm{Bs}\,|-4K_{Y'}| = \mathrm{Bs}\,|4B'| = \emptyset$. Therefore $f' = \mathrm{cont}_{R'}$ exists and is (the Stein factorisation of) $\varphi_{|4B'|}$; T' is a general fibre of f'. In the diagram below, $\nu \colon U \to g(T)$ is the normalisation of $g(T)$.

$$\begin{tikzcd}
Y & \supset & T \arrow[d, "f"'] \arrow[dr, "h"] \arrow[ddr, bend right, "g"'] & & T' \arrow[dl, "h'"'] \arrow[dr, "f'"] \arrow[ddl, bend left, "g'"] & \subset & Y' \\
X & \supset & T_X & U \arrow[d, "\nu"] & & \{*\} & \subset & \mathbb{P}^1 \\
& & & g(T) & & & &
\end{tikzcd}$$

Now $K_T = (K_Y + T)|_T = 3B|_T = 3C$, so $K_U = h_*(3C) = 0$. Furthermore, the Leray spectral sequence for $f\colon T \to T_X$,

$$0 \to 0 = H^1\left(T_X, \mathcal{O}_{T_X}\right) \to H^1\left(T, \mathcal{O}_T\right) \to H^0\left(T_X, R^1 f_* \mathcal{O}_T\right) = 0,$$

shows that $h^1(T, \mathcal{O}_T) = 0$ (here $h^0(T_X, R^1 f_* \mathcal{O}_T) = 0$ because the singularity $\frac{1}{5}(1,1)$ of T_X at Q_i is rational); and now the Leray spectral sequence for $h\colon T \to U$,

$$0 \to H^1\left(U, \mathcal{O}_U\right) \to H^1\left(T, \mathcal{O}_T\right) = 0,$$

shows that $h^1(U, \mathcal{O}_U) = 0$. The only thing left to do is to show that U has Du Val singularities — it is then clear that T' is a Du Val K3 surface because

$$K_{T'} = \left(K_{Y'} + T'\right)|_{T'} = 3B'|_{T'} = 0,$$

so the minimal resolution $\widetilde{U} \to U$ of U factors through $h'\colon T' \to U$.

To show U has Du Val singularities we observe first that T has two singularities: a $\frac{1}{5}(1,1)_{x,t}$ point over $Q_j \in T_X$, where $\{i,j\} = \{1,2\}$, and a $\frac{1}{3}(1,2)_{x,z}$ point over $P \in T_X$. The curve C passes through both and is locally defined by $x = 0$ in a neighbourhood of each. Let $\xi\colon \widetilde{T} \to T$ be the minimal resolution of T, with exceptional curves E_1, E_2 lying over P and E_3 lying over Q_j, where $E_1^2 = E_2^2 = -2$, $E_1 E_2 = 1$, $E_1 \widetilde{C} = 0$, $E_2 \widetilde{C} = 1$, $E_3^2 = -5$ and $E_3 \widetilde{C} = 1$ (here $\widetilde{C}$ is the birational transform of C on $\widetilde{T}$). We can calculate that $(\widetilde{C})^2 < 0$ and $K_{\widetilde{T}} \widetilde{C} < 0$, from which it follows that $\widetilde{C}$ is a -1-curve. (To do this, first show that $C_T^2 < 0$ and $K_T C < 0$; then express $\widetilde{C}$ as the pullback $\xi^* C$ minus nonnegative multiples of E_1, E_2, E_3; then use the fact that $(E_i E_j)_{i,j=1}^2$ is negative definite, $K_{\widetilde{T}}(\xi^* C) = K_T C < 0$ and $K_{\widetilde{T}}$ is ξ-nef by minimality of ξ.)

Now the minimal resolution $\widetilde{U}$ of U is obtained from $\widetilde{T}$ by running a minimal model program over U — so we start by contracting $\widetilde{C}$, which is exceptional over U, and then E_2, which has become a -1-curve, and finally E_1. E_3 is left as a -2-curve. This MMP can be summarised by

$$(2,2,1,5) \to (2,1,4) \to (1,3) \to (2).$$

It follows that U has a single A_1 Du Val singularity, as required. $\square$

2.2. Elliptic fibrations

Our main aim is to prove Theorem 1.1, part of which is the statement that there is no elliptic fibration birational to a general X in family 75. In this subsection, however, we digress briefly to discuss the construction of elliptic fibrations birational to hypersurfaces in other families out of the 95. In particular we are concerned with families 34, 88 and 90, the subjects of Theorems 1.2, 1.3 and 1.4.

2.4 *Example.* Let $X = X_{18} \subset \mathbb{P}(1,1,2,6,9)_{x_0,x_1,y,z,t}$ be a general member of family 34 of the 95. We claim that

(a) the projection $\pi = (x_0, x_1, y)\colon X \dashrightarrow \mathbb{P}(1,1,2)$ gives an elliptic fibration birational to X after resolution of indeterminacy; and

(b) the indeterminacy may be resolved as shown below, where $f\colon Y \to X$ is the Kawamata blowup of the unique singularity $P \sim \frac{1}{3}(1,1,2)_{x_0,x_1,y}$ on the zt-stratum, and $\pi_Y := \pi \circ f$ is the anticanonical morphism $\varphi_{|2B|}$, $B = -K_Y$.

$$\begin{tikzcd} Y \arrow[d, "f"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[r, dashed, "\pi"] & \mathbb{P}(1,1,2) \end{tikzcd}$$

Proof. (a) This is easy to see: a general fibre of the rational map π is a curve $E_{18} \subset \mathbb{P}(1,6,9)_{x,z,t}$ and, when we write down the Newton polygon of the defining equation of such a curve, there is a unique internal monomial (namely x^3zt, the vertices being t^2, x^{18} and z^3). Consequently E_{18} is birational to an elliptic curve, by standard toric geometry.

(b) The linear system $\mathcal{L}$ defining π is $|2A| = \langle x_0^2, x_1^2, y\rangle$ and one can calculate directly that its birational transform $\mathcal{L}_Y$ on Y is free. Furthermore

$$\mathcal{L}_Y = f^*\mathcal{L} - \tfrac{2}{3}E \sim 2B$$

and $\mathcal{L}_Y$ is clearly a complete linear system. $\square$

As well as exhibiting one elliptic fibration birational to a general hypersurface in family 34 (actually, according to Theorem 1.2, the only such), this example shows us one way to look for elliptic fibrations when we consider other families: namely, find a singular point P with $B^3 = 0$ (for $B = -K_Y$, $f\colon Y \to X$ the Kawamata blowup of P), take the anticanonical morphism on Y (if B is eventually free) and see if it maps to a surface. It turns out that this method works for families 88 and 90, as well as 34, so as far as the present paper is concerned, we are done. There are, however, other ways in which elliptic fibrations occur birational to hypersurfaces in the 95 families. Here is a brief list; see [Ry02] for more details.

- Sometimes elliptic fibrations have more than one singular point of X in $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$; they can be constructed by blowing up all these points and taking the anticanonical morphism.
- It can also occur that $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ consists of only one singular point P, but $\mathrm{CS}\left(Y, \frac{1}{n}\mathcal{H}_Y\right) \neq \emptyset$, where $Y = \mathrm{B}_P\, X$ is the Kawamata blowup; in this case further blowups of Y are necessary before taking the anticanonical morphism.
- Finally, there are examples of elliptic fibrations with a curve in $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ — but only for families 1 and 2; see [Ry02].

3. Exclusions, Absolute and Conditional

For an initial introduction to the contents of this section, see 1.6. We divide the material for proving Theorem 1.7 into three subsections: in §3.1 we show that all curves are excluded absolutely and in §3.2 we prove the corresponding result for smooth points; finally in §3.3 we deal with singular points. Much of the material in this section applies more widely than to family 75: for example, out of all the singular points on members of the 95 families, we show that those satisfying a certain condition turn out to be excluded absolutely, while those satisfying a different (but closely related) condition are excluded *conditionally* — that is, we prove that if they belong to $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ for some $\mathcal{H}$ then other centres must also exist in $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$.

3.1. Curves

First we state the main curve exclusion theorem proved in [Ry02].

3.1 Theorem ([Ry02, Curves Theorem A]). *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a general hypersurface in one of the 95 families and*

$C \subset X$ a reduced, irreducible curve. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and $C \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then there exist two linearly independent forms ℓ, ℓ' of degree 1 in $(x_0, \ldots, x_4)$ such that

$$C \subset \{\ell = \ell' = 0\} \cap X. \tag{1}$$

We do not reproduce the proof in full here, but instead restrict ourselves to the case we need, namely $X_{30} \subset \mathbb{P}(1,4,5,6,15)$. This follows immediately from the following lemmas, the first of which is standard.

3.2 Lemma. *Let X be any hypersurface in one of the 95 families and $C \subset X$ a curve, reduced but possibly reducible. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical and each irreducible component of C belongs to $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then $\deg C = AC \leq A^3$.*

3.3 Lemma. *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, a_2, a_3, a_4)$ be a hypersurface in one of the families with $a_1 > 1$ and suppose that either*

(a) $d < a_1 a_4$ or

(b) $d < a_2 a_4$ and the curve $\{x = y = 0\} \cap X$ is irreducible (which holds for general X in a family with $a_1 > 1$ by Bertini's theorem).

Then any curve $C \subset X$ that is not contracted by $\pi_4 \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ has $\deg C > A^3$. Consequently C is excluded absolutely by Lemma 3.2.

For the proofs of Lemmas 3.2 and 3.3, see below. It is straightforward to check that they imply the following.

3.4 Corollary. *Let $X = X_{30} \subset \mathbb{P}(1,4,5,6,15)_{x,y,z,t,u}$ be any (quasi-smooth) member of family 75. Then Theorem 3.1 holds for X, that is, no reduced, irreducible curve $C \subset X$ can belong to $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ for any mobile system $\mathcal{H}$ of degree n with $K_X + \frac{1}{n}\mathcal{H}$ strictly canonical.*

Proof of Lemma 3.2. Let s be a natural number such that sA is Cartier and very ample, and pick general members $H, H' \in \mathcal{H}$. Now by assumption

$$\mathrm{mult}_{C_i}(H) = \mathrm{mult}_{C_i}(H') = n$$

for each irreducible component C_i of C, so for a general $S \in |sA|$

$$A^3 sn^2 = SHH' \geq sn^2 AC = sn^2 \deg C,$$

which proves $\deg C \leq A^3$. $\square$

Proof of Lemma 3.3. We prove this lemma under the additional assumption that $(a_1, a_2) = 1$ — which is the case for family 75 ($a_1 = 4$, $a_2 = 5$); if $(a_1, a_2) > 1$ then a little trick, described in [Ry02], is needed.

Suppose that $C \subset X$ has $\deg C \leq A^3$ and is not contracted by π_4; let $C' \subset \mathbb{P}(1, a_1, a_2, a_3)$ be the set-theoretic image $\pi_4(C)$. Note that $\deg C' \leq \deg C$ — indeed, if H denotes the hyperplane section of $\mathbb{P}(1, a_1, \ldots, a_4)$ and H' that of $\mathbb{P}(1, a_1, a_2, a_3)$, we pick $s \geq 1$ such that $|sH|$ and $|sH'|$ are very ample, and calculate that

$$\begin{aligned} s \deg C &= (sH)C = \pi_4^*(sH')C \\ &= sH'(\pi_4)_* C = srH'C' = sr \deg C' \geq s \deg C', \end{aligned}$$

where $r \geq 1$ is the degree of the induced morphism $\pi_4|_C \colon C \to C'$. So in fact $\deg C$ is a multiple of $\deg C'$ by the positive integer r. (The point of $|sH|$ being very ample is that we can move it away from P_4, where π_4 is undefined, and apply the projection formula to the morphism $\pi_4|_{\mathbb{P}(1,a_1,\ldots,a_4) \smallsetminus \{P_4\}}$.)

Now form the diagram below.

$$\begin{array}{ccc} C & \subset & \mathbb{P}(1, a_1, a_2, a_3, a_4) \\ \downarrow & & \Big\downarrow{\scriptstyle \pi_4} \\ C' & \subset & \mathbb{P}(1, a_1, a_2, a_3) \\ \downarrow & & \Big\downarrow{\scriptstyle \pi_3} \\ \{*\} & \subset & \mathbb{P}(1, a_1, a_2) \end{array}$$

C' is contracted by π_3 — indeed, if its image were a curve C'' we would have

$$\deg C'' \leq \deg C' \leq \deg C \leq A^3,$$

but $A^3 = d/(a_1 a_2 a_3 a_4) < 1/(a_1 a_2)$, since $d < a_3 a_4$ in either case (a) or (b), and on the other hand $1/(a_1 a_2) \leq \deg C''$ simply because $C'' \subset \mathbb{P}(1, a_1, a_2)$ — contradiction.

Now by our extra assumption $(a_1, a_2) = 1$, the point $\{*\} \subset \mathbb{P}(1, a_1, a_2)$ is, up to coordinate change, one of

$$\{y = z = 0\}, \quad \{y^{a_2} + z^{a_1} = x = 0\}, \quad \{x = z = 0\} \quad \text{and} \quad \{x = y = 0\},$$

using the coefficient convention in $y^{a_2} + z^{a_1} = 0$. It follows that the curve $C' \subset \mathbb{P}(1, a_1, a_2, a_3)$ is defined by the same equations. In the first case, this means that $\deg C' = 1/a_3 > d/(a_1a_2a_3a_4) = A^3$, contradiction. In the second case $\deg C' = 1/a_3$ again, because $C' \simeq \{y^{a_2} + z^{a_1} = 0\} \subset \mathbb{P}(a_1, a_2, a_3)$ passes only through the singularity $(0, 0, 1)$, using $(a_1, a_2) = 1$ — so we obtain a contradiction as in the first case. In the case $C' = \{x = z = 0\}$, we have $\deg C' = 1/(a_1a_3)$ and we easily obtain a contradiction from $a_2a_4 > d$. In the final case, $C' = \{x = y = 0\}$, if the assumptions in part (a) of the statement hold then we have

$$\deg C' = 1/(a_2a_3) > d/(a_1a_2a_3a_4) = A^3,$$

contradiction; while if the assumptions in part (b) hold then

$$C = \{x = y = 0\} \cap X$$

(because the right hand side is irreducible), but

$$\deg (\{x = y = 0\} \cap X) = a_1A^3 > A^3,$$

since we also assumed $a_1 > 1$ — contradiction. $\square$

Note that for the case $a_1 = 1, a_2 > 1$ there is the following analogue of Lemma 3.3 — see [Ry02, 3.3] for the proof, which is similar to but much shorter than the one we have just seen.

3.5 Lemma. *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1, 1, a_2, a_3, a_4)$ be a hypersurface in one of the families with $a_1 = 1$ and $a_2 > 1$; suppose that $d < a_2a_4$. Then any curve $C \subset X$ that is not contracted by π_4 and that satisfies $\deg C \leq A^3$ is contained in $\{x_0 = x_1 = 0\} \cap X$.*

When $a_1 = a_2 = 1$, however, the situation is different and more work is required to prove sufficiently strong results.

We note also for completeness that in the case of family 75 we have $P_4 = P_u \notin X$ so the question of whether C is contracted by

$\pi_4 \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ never arises. For families that do contain curves contracted by π_4, [Ry02, 3.5] shows that in almost all cases these curves are of degree greater than A^3, so they are excluded by Lemma 3.2. The remaining few families to which this result does not apply are also dealt with in [Ry02].

3.6 **Curves for families 34, 88 and 90.** We have dealt with curves inside a general X in family 75. The arguments presented above, or variants of them, deal also with curves in general members of families 88 and 90, given the following observation: family 88 has $a_1 = 1$ and $a_2 > 1$, and consequently there is a birational K3 fibration obtained by the projection $(x_0, x_1) \colon X \dashrightarrow \mathbb{P}^1$. The corresponding pencil $\mathcal{P} = \langle x_0, x_1 \rangle$ has $C \in \mathrm{CS}(X, \mathcal{P})$, where C is the curve $\{x_0 = x_1 = 0\} \cap X$. The important point is that, by taking X general in its family, C is *irreducible*. If it were not, and we had

$$\{x_0 = x_1 = 0\} \cap X = C_0 \cup \cdots \cup C_r,$$

we would have to prove a conditional exclusion result of the form 'if $C_0 \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ then all $C_i \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$'. This can be done — see [Ry02, 3.12] — but we avoid it using our generality assumption on X.

The case of family 34 is more problematic because Lemma 3.5 above does not apply. We omit the argument for curve exclusion for this family; it can be found in [Ry02, §3.4].

3.2. Smooth points

In this subsection we present a proof of the following theorem.

3.7 Smooth Points Theorem. *Let $X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, \ldots, a_4)$ be a general hypersurface in one of families $3, 4, \ldots, 95$ and $P \in X$ a smooth point. For any $n \in \mathbb{Z}_{\geq 1}$ and any mobile linear system $\mathcal{H}$ of degree n on X we have $P \notin \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$.*

The proof closely follows the argument used in [CPR] to exclude smooth points as maximal centres of a pair $\left(X, \frac{1}{n}\mathcal{H}\right)$. First we need to quote some theoretical results.

3.8 Theorem (Shokurov's inversion of adjunction). *Let $P \in X$ be the germ of a smooth 3-fold and $\mathcal{H}$ a mobile linear system on X. Assume*

$n \in \mathbb{Z}_{\geq 1}$ is such that $P \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then for any normal irreducible divisor S containing P such that $\mathcal{H}|_S$ is mobile we have $P \in \mathrm{LCS}\left(S, \frac{1}{n}\mathcal{H}|_S\right)$.

For a readable account of the proof see [KM, 5.50]. Note that under the given assumptions, but without assuming $\mathcal{H}|_S$ is mobile, [KM, 5.50] says $K_S + \frac{1}{n}\mathcal{H}|_S$ is not klt near P, which does not preclude the centre on S of the relevant valuation being a *curve* containing P rather than P itself. This curve would of course be in $\mathrm{Bs}(\mathcal{H})$, so the problem is eliminated by assuming $\mathcal{H}|_S$ is mobile — and it will be clear we may assume this in our application. With the extra assumption $K_S + \frac{1}{n}\mathcal{H}|_S$ is not plt near P, as required.

3.9 Theorem (Corti). *Let $(P \in \Delta_1 + \Delta_2 \subset S) \simeq (0 \in \{xy = 0\} \subset \mathbb{C}^2)$ be the analytic germ of a normal crossing curve on a smooth surface; let $\mathcal{L}$ be a mobile linear system on S and $L_1, L_2 \in \mathcal{L}$ general members. Suppose there exist $n \in \mathbb{Z}_{\geq 1}$ and $a_1, a_2 \in \mathbb{Q}_{\geq 0}$ such that*

$$P \in \mathrm{LCS}\left(S, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + \tfrac{1}{n}\mathcal{L}\right).$$

Then

$$(L_1 \cdot L_2)_P \geq \begin{cases} 4a_1a_2n^2 & \text{if } a_1 \leq 1 \quad \text{or} \quad a_2 \leq 1; \\ 4(a_1 + a_2 - 1)n^2 & \text{if both } a_1, a_2 > 1. \end{cases}$$

This is proved as in the original [Co00], but replacing 'log canonical' by 'purely log terminal' and strict inequalities by $\leq$ or $\geq$ as appropriate. Now combining Theorems 3.8 and 3.9 we obtain the following.

3.10 Corollary. *Let $P \in X$ be the germ of a smooth 3-fold and $\mathcal{H}$ a mobile linear system on X. Assuming as in Theorem 3.8 that $P \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ for some $n \in \mathbb{Z}_{\geq 1}$ we have*

$$\mathrm{mult}_P(H \cdot H') \geq 4n^2$$

where $H, H' \in \mathcal{H}$ are general and $H \cdot H'$ is their intersection cycle.

Now we need to borrow an additional result from [CPR]. First we recall the following definition.

3.11 Definition (cf. [CPR, 5.2.4]). Let L be a Weil divisor class in a 3-fold X and $\Gamma \subset X$ an irreducible curve or a closed point. We say that L *isolates* Γ, or is a Γ-*isolating class*, if and only if there exists $s \in \mathbb{Z}_{\geq 1}$ such that the linear system $\mathcal{L}^s_\Gamma := |\mathcal{I}^s_\Gamma(sL)|$ satisfies

- $\Gamma \in \operatorname{Bs} \mathcal{L}^s_\Gamma$ is an isolated component (i.e., in some neighbourhood of Γ the subscheme $\operatorname{Bs} \mathcal{L}^s_\Gamma$ is supported on Γ); and
- if Γ is a curve, the generic point of Γ appears with multiplicity 1 in $\operatorname{Bs} \mathcal{L}^s_\Gamma$.

3.12 Theorem ([CPR, 5.3.1]). *Let $X = X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$ be a general hypersurface in one of families $3, 4, \dots, 95$ and $P \in X$ a smooth point. Then for some positive integer $l < 4/A^3$ the class lA is P-isolating.*

3.13 Remark. [CPR, 5.3.1] says $l \leq 4/A^3$, but the statement there is for all the families except number 2 — that is, including number 1 — and a trivial check shows that in fact $l = 4/A^3$ only for number 1.

Proof of Theorem 3.7. We know that $P \in \operatorname{Bs} |\mathcal{I}^s_P(slA)|$ is an isolated component for some $l, s \in \mathbb{Z}_{\geq 1}$ with $l < 4/A^3$. Take a general surface $S \in |\mathcal{I}^s_P(slA)|$ and general elements $H, H' \in \mathcal{H}$. If we assume that P belongs to $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ then Corollary 3.10 tells us that $\operatorname{mult}_P(H \cdot H') \geq 4n^2$, so

$$S \cdot H \cdot H' \geq \left(S \cdot H \cdot H'\right)_P \geq 4sn^2.$$

But we know

$$S \cdot H \cdot H' = sln^2 A^3 < \frac{4}{A^3} sn^2 A^3 = 4sn^2,$$

contradiction. □

3.3. Singular points

The following two results are fundamental.

3.14 Theorem (Kawamata, [Ka96]). *Let $P \in X \simeq \frac{1}{r}(1, a, r-a)$, with $r \geq 2$ and $(a, r) = 1$, be the germ of a 3-fold terminal quotient singularity, and*

$$f\colon (E \subset Y) \to (\Gamma \subset X)$$

a divisorial contraction such that $P \in \Gamma$ (so Y has terminal singularities, $\operatorname{Exc} f = E$ is an irreducible divisor and $-K_Y$ is f-ample). Then $\Gamma = P$ and f is isomorphic over X to the $(1, a, r-a)$ weighted blowup of $P \in X$.

3.15 LEMMA (Kawamata, [Ka96]). *Let $P \in X \simeq \frac{1}{r}(1, a, r-a)$ be as in Theorem 3.14 and $f\colon (E \subset Y) \to (P \in X)$ the $(1, a, r-a)$ weighted blowup; let $g\colon \widetilde{X} \to X$ be a resolution of singularities with exceptional divisors $\{E_i\}$. Fix an effective Weil divisor H on X and define $a_i = a_{E_i}(K_X)$ and $m_i = m_{E_i}(H)$ in the usual way via*

$$\begin{aligned} K_{\widetilde{X}} &= g^* K_X + \textstyle\sum a_i E_i, \\ g_*^{-1} H &= g^* H - \textstyle\sum m_i E_i; \end{aligned}$$

define a_E and m_E similarly using f. Then $m_i/a_i \le m_E/a_E$ for all i.

In [Ka96] Lemma 3.15 is used to prove Theorem 3.14, but it is an interesting result in its own right; in particular, it has two corollaries that are of great importance for our problem.

3.16 COROLLARY. *Let $P \in X \simeq \frac{1}{r}(1, a, r-a)$ and $f\colon (E \subset Y) \to (P \in X)$ the Kawamata blowup as in Lemma 3.15. Suppose $\mathcal{H}$ is a mobile linear system on X and $n \in \mathbb{Z}_{\ge 1}$ is such that $P \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then the valuation v_E of E is strictly canonical or worse for $\left(X, \frac{1}{n}\mathcal{H}\right)$.*

PROOF. Let $g\colon \widetilde{X} \to X$ be any resolution of singularities with exceptional divisors $\{E_i\}$ and $H \in \mathcal{H}$ a general element. The assumption $P \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ means that $n \le m_i/a_i$ for some i, so by Lemma 3.15 $n \le m_E/a_E$ as well. $\square$

This Corollary 3.16 tells us that we can exclude a singular point from any $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ simply by excluding the valuation v_E; it has other uses as well.

3.17 COROLLARY. *Let $P \in X \simeq \frac{1}{r}(1, a, r-a)$ and $f\colon Y \to X$ be the Kawamata blowup of P as in Lemma 3.15. Suppose $C \subset X$ is a curve containing P, $\mathcal{H}$ is a mobile linear system on X and $n \in \mathbb{Z}_{\ge 1}$ is such that $C \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then $P \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ also.*

PROOF. Let $g\colon \widetilde{X} \to X$ be a resolution of singularities with exceptional divisors $\{E_i\}$ at least one of which has centre C on X and is strictly canonical or worse for $\bigl(X, \frac{1}{n}\mathcal{H}\bigr)$. The rest of the proof is the same as that of Corollary 3.16. □

3.18 **Absolute exclusion of singular points with $B^3 < 0$ and $T \sim bB + cE$, $b, c > 0$.** Suppose P is a singular point of a hypersurface X in one of the 95 families and $P \in X$ is locally isomorphic to $\frac{1}{r}(1, a, r-a)$. Let $f\colon (E \subset Y) \to (P \in X)$ be the Kawamata blowup and suppose $B^3 < 0$ (where as always $B = -K_Y$). We denote by S the surface $f_*^{-1}\{x_0 = 0\}$, which is an element of $|B|$ and is irreducible, assuming X is general.

3.19 LEMMA (see [CPR, 5.4.3]). *If $B^3 < 0$ then there exist integers b, c with $b > 0$ and $b/r \geq c \geq 0$ and a surface $T \in |bB + cE|$ such that*

(a) the scheme theoretic complete intersection $\Gamma = S \cap T$ is a reduced, irreducible curve and

(b) $T\Gamma \leq 0$.

3.20 THEOREM. *Suppose $B^3 < 0$ and the integer c provided by Lemma 3.19 is strictly positive. Then P is excluded absolutely, that is, there does not exist a mobile linear system $\mathcal{H}$ of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is canonical and $P \in \mathrm{CS}\bigl(X, \frac{1}{n}\mathcal{H}\bigr)$.*

For the proof we need only the following two lemmas.

3.21 LEMMA. *Let X be a Fano 3-fold hypersurface in one of the 95 families and $\mathcal{H}$ a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is strictly canonical; suppose $\Gamma \subset X$ is an irreducible curve or a closed point satisfying $\Gamma \in \mathrm{CS}\bigl(X, \frac{1}{n}\mathcal{H}\bigr)$, and furthermore that there is a Mori extremal divisorial contraction*

$$f\colon (E \subset Y) \to (\Gamma \subset X), \quad \mathrm{Centre}_X\, E = \Gamma,$$

such that $E \in \mathrm{V}_0\bigl(X, \frac{1}{n}\mathcal{H}\bigr)$. Then $B^2 \in \overline{\mathrm{NE}}\, Y$.

PROOF. We know that

$$K_Y + \tfrac{1}{n}\mathcal{H}_Y \sim_{\mathbb{Q}} f^*\bigl(K_X + \tfrac{1}{n}\mathcal{H}\bigr) \sim_{\mathbb{Q}} 0.$$

It follows that $B \sim_{\mathbb{Q}} \frac{1}{n}\mathcal{H}_Y$, and therefore $B^2 \in \overline{\mathrm{NE}}\, Y$, because $\mathcal{H}_Y$ is mobile. $\square$

3.22 LEMMA (see [CPR, 5.4.6]). *If $B^3 < 0$, let T and $\Gamma = S \cap T$ be as in the conclusion of Lemma 3.19. Write R for the extremal ray of $\overline{\mathrm{NE}}\, Y$ contracted by $f\colon Y \to X$ and let $Q \subset \overline{\mathrm{NE}}\, Y$ be the other ray. Then $Q = \mathbb{R}_{\geq 0}[\Gamma]$.*

PROOF OF THEOREM 3.20. Suppose $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is canonical and $P \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Corollary 3.16 of Kawamata's Lemma tells us that the Kawamata blowup $f\colon Y \to X$ of P extracts a valuation v_E (where $E = \mathrm{Exc}\, f$) which is strictly canonical for $\left(X, \frac{1}{n}\mathcal{H}\right)$. The test class Lemma 3.21 now implies that $B^2 \in \overline{\mathrm{NE}}\, Y$.

But the ray $Q \subset \overline{\mathrm{NE}}\, Y$ is generated by $(bB + cE)B$ for some $b, c > 0$, and certainly $EB \in \overline{\mathrm{NE}}\, Y$, so if $B^2 \in \overline{\mathrm{NE}}\, Y$ we have both $B^2, EB \in Q$ (by definition of 'extremal'). It follows that EB is numerically equivalent to αB^2 for some positive $\alpha \in \mathbb{Q}$; but

$$EB \cdot B = E\Big(A - \frac{1}{r}E\Big)^2 = \frac{1}{r^2}E^3 = \frac{1}{a(r-a)} > 0,$$

while $B^2 \cdot B = B^3 < 0$ by assumption — contradiction. $\square$

3.23 COROLLARY. *Let $X = X_{30} \subset \mathbb{P}(1,4,5,6,15)_{x,y,z,t,u}$ be a general member of family 75 of the 95 and $\mathcal{H}$ a mobile linear system of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ strictly canonical. Then no singular point of X other than the two $\frac{1}{5}(1,4,1)$ points in the zu-stratum can belong to* $\mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$.

PROOF. Here is the complete list of singular points of X, together with the sign of B^3 and $bB + cE \sim T$ (see Lemma 3.19) for each of them.

$P_y \sim \frac{1}{4}(1,1,3)_{x,z,u}$	$B^3 < 0$	$10B + E$
$P_tP_u \cap X = P \sim \frac{1}{3}(1,1,2)_{x,y,z}$	$B^3 < 0$	$5B + E$
$P_zP_u \cap X = Q_1, Q_2 \sim \frac{1}{5}(1,4,1)_{x,y,t}$	$B^3 < 0$	$4B$
$P_yP_t \cap X = R_1, R_2 \sim \frac{1}{2}(1,1,1)_{x,z,u}$	$B^3 < 0$	$5B + 2E$

Clearly all these apart from Q_1 and Q_2 satisfy the hypotheses of Theorem 3.20, and consequently are excluded absolutely. $\square$

3.24 **Singular points with $B^3 < 0$ and $T \sim bB$, $b > 0$.** Assume $P \in X$ is a singular point of a hypersurface in one of the 95 families which is locally isomorphic to $\frac{1}{r}(1, a, r-a)$ with $B^3 < 0$. We keep the notations $T \sim bB + cE$, $S = f_*^{-1}\{x_0 = 0\}$ and $\Gamma = S \cap T$ of Lemma 3.19; in the following paragraphs we consider the case where the integer c provided by Lemma 3.19 is zero. Out of all such singular points, the vast majority live in a family with $b = a_1 < a_2$, where a_1, a_2 are the weights of x_1, x_2. For such points we have the following result.

3.25 Theorem. *Let $P \in X$ be a singular point satisfying $B^3 < 0$, $T \sim bB$ and $b = a_1 < a_2$. Assume that the curve $C = \{x_0 = x_1 = 0\} \cap X$ is irreducible. Then $R(Y, B) = k[x_0, x_1]$. It follows that if $\mathcal{H}$ is a mobile linear system of degree n on X such that $K_X + \frac{1}{n}\mathcal{H}$ is canonical and $P \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ then in fact $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right) = \operatorname{CS}\left(X, \frac{1}{b}f_*\,|bB|\right)$, where $f\colon Y \to X$ is the Kawamata blowup of P.*

Proof for family 75. We do not prove this theorem here for every case. As explained in [Ry02], the first statement, namely $R(Y, B) = k[x_0, x_1]$, follows from two ray game calculations such as that in the proof of our Lemma 2.3; we have already shown this for family 75. The second part of the theorem follows easily from the first: let $\mathcal{H}$ be mobile of degree n on X with $K_X + \frac{1}{n}\mathcal{H}$ canonical and $P \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$. Then $\mathcal{H}_Y \subset |nB| = \left\langle x_0^n, x_0^{n-b}x_1, \ldots, x_0^r x_1^q \right\rangle$, where $n = qb + r$, $0 \le r < b$ — so $\mathcal{H} \subset f_*|nB| = \langle x_0^n, \ldots, x_0^r x_1^q \rangle$, while of course $f_*|bB| = \left\langle x_0^b, x_1 \right\rangle$, and therefore $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right) = \operatorname{CS}\left(X, \frac{1}{b}f_*|bB|\right)$. $\square$

We are now in a position to put together all the exclusion results obtained so far to prove Theorem 1.7 for family 75.

Proof of Theorem 1.7. First, since X is superrigid by [CPR, §6], $K_X + \frac{1}{n}\mathcal{H}$ nonterminal implies $K_X + \frac{1}{n}\mathcal{H}$ strictly canonical; therefore $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ is nonempty. Corollary 3.4 tells us that no curve belongs to $\operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$, Theorem 3.7 says that the same is true for smooth points and Corollary 3.23 shows the same for all singular points other than $Q_1, Q_2 \sim \frac{1}{5}(1, 4, 1)_{x,y,t}$. Therefore at least one $Q_i \in \operatorname{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$; without loss of generality we may assume this holds for Q_1.

Now by Theorem 3.25

$$\operatorname{CS}\left(X, \tfrac{1}{n}\mathcal{H}\right) = \operatorname{CS}\left(X, \tfrac{1}{b}f_*|bB|\right)$$

where $b = 4$ because $T \sim 4B$ (in the notation above) and $f\colon Y \to X$ is the Kawamata blowup of Q_1. But $R(Y, B) = k[x_0, x_1] = k[x, y]$, so $f_*|bB|$ is just $\langle x^4, y\rangle$; and both x^4 and y have local vanishing order 4/5 at Q_2, so $Q_2 \in \mathrm{CS}\left(X, \frac{1}{n}\mathcal{H}\right)$ also, as required. □

4. Birational Classification of Elliptic and K3 Fibrations

We have now proved Theorem 1.7 for family 75; all that remains is to deduce the main Theorem 1.1 from it. For this we need the following theorem–definition and propositions.

4.1 Theorem–Definition. *Let X be a variety, normal and projective over $\mathbb{C}$ as always, and $\mathcal{H}$ a mobile linear system on X. Fix $\alpha \in \mathbb{Q}_{\geq 0}$ and let $f\colon Y \to X$ be a birational morphism such that $K_Y + \alpha\mathcal{H}_Y$ is canonical. We define the* log Kodaira dimension $\kappa(X, \alpha\mathcal{H})$ *to be the D-dimension of $K_Y + \alpha\mathcal{H}_Y$, that is,*

$$\kappa(X, \alpha\mathcal{H}) = D(K_Y + \alpha\mathcal{H}_Y) = \max\{\dim(\operatorname{im}\varphi_{|m(K_Y+\alpha\mathcal{H}_Y)|})\},$$

taking the max over all $m \geq 1$ such that $m(K_Y + \alpha\mathcal{H}_Y)$ is integral; if all the linear systems $|m(K_Y + \alpha\mathcal{H}_Y)|$ are empty, by definition

$$\kappa(X, \alpha\mathcal{H}) = D(K_Y + \alpha\mathcal{H}_Y) = -\infty.$$

Then $\kappa(X, \alpha\mathcal{H})$ is independent of the choice of Y and is attained for a particular Y using any sufficiently large m such that $m(K_Y + \alpha\mathcal{H}_Y)$ is integral.

Proof. This result is standard and is used in, e.g., [Ch00] and [Is01]. The methods employed in [Sh96] to show uniqueness of a log canonical model can be used to prove it; alternatively see [FA, Ch. 2], particularly Theorem 2.22 and the Negativity Lemma 2.19, which is an essential ingredient. □

4.2. Now let X be a Mori Fano variety and $\mathcal{H}$ a mobile linear system of degree n on X, that is, $n \in \mathbb{Q}$ is such that $K_X + \frac{1}{n}\mathcal{H} \sim_{\mathbb{Q}} 0$ — of course $n \in \mathbb{Z}_{\geq 1}$ if X is a hypersurface in one of the 95 families.

4.3 Proposition. *Assume that $K_X + \frac{1}{n}\mathcal{H}$ is canonical.*

(a) Let $\varepsilon \in \mathbb{Q}$. Then

$$\kappa\left(X, \left(\tfrac{1}{n} + \varepsilon\right)\mathcal{H}\right) = \begin{cases} -\infty & \text{if } \varepsilon < 0 \\ 0 & \text{if } \varepsilon = 0 \\ d \geq 1 & \text{if } \varepsilon > 0 \end{cases}$$

(b) If $1 \leq \kappa(X, (\frac{1}{n}+\varepsilon)\mathcal{H}) \leq \dim X - 1$ for some $\varepsilon \in \mathbb{Q}_{>0}$ then $K_X + \frac{1}{n}\mathcal{H}$ is nonterminal, i.e., strictly canonical (so $K_X + (\frac{1}{n} + \varepsilon)\mathcal{H}$ is noncanonical).

Proof. See the survey [Is01, III.2.3–2.4]. □

Recall that the NFI-type inequality 1.10 was stated under two alternative sets of assumptions: either

(a) X is a Mori Fano and $\Phi\colon X \dashrightarrow Z/T$ a birational map to the total space Z of a K-trivial fibration $g\colon Z \to T$; or

(b) X is a Mori Fano and $\Phi\colon X \dashrightarrow Z$ a birational map to a Fano variety with canonical singularities.

4.4 Proposition. *In situation (a) above, assume that $K_X + \frac{1}{n}\mathcal{H}$ is canonical. Then for any $\varepsilon \in \mathbb{Q}_{>0}$, $\kappa\left(X, \left(\frac{1}{n} + \varepsilon\right)\mathcal{H}\right) = \dim T$.*

Proof. Fix $\varepsilon \in \mathbb{Q}_{>0}$. $K_X + \frac{1}{n}\mathcal{H}$ is canonical so $\kappa(X, \frac{1}{n}\mathcal{H}) = 0$, and therefore $\kappa(Z, \frac{1}{n}\mathcal{H}_Z) = 0$ by the birational invariance of log Kodaira dimension (4.1). But in fact $K_Z + \frac{1}{n}\mathcal{H}_Z$ is canonical (as is $K_Z + (\frac{1}{n}+\varepsilon)\mathcal{H}_Z$, because K_Z is canonical and $\mathcal{H}_Z$ is free), so $\kappa(Z, \frac{1}{n}\mathcal{H}_Z)$ (and $\kappa(Z, (\frac{1}{n}+\varepsilon)\mathcal{H}_Z)$) can be computed on Z as the ordinary D-dimension. Consequently for $m \gg 0$ and such that $m(K_Z + \frac{1}{n}\mathcal{H}_Z)$ is integral, it is in fact effective and fixed. Fix such an m with the additional property that $m\varepsilon \in \mathbb{N}$, so that $m(K_Z + (\frac{1}{n}+\varepsilon)\mathcal{H}_Z)$ is integral as well; let $F \in \left|m(K_Z + \frac{1}{n}\mathcal{H}_Z)\right|$ be the unique element. Now for any curve C contracted by g, $FC = 0$, because by assumption $K_Z C = 0$ and $\mathcal{H}_Z = g^*|H|$. But F is effective, so it must be a pullback g^*F_T of some effective F_T on T. Furthermore, for any $m' \in \mathbb{N}$,

$$H^0(T, m'F_T) = H^0(Z, g^*(m'F_T)) = H^0(Z, m'F),$$

so F_T is fixed, $D(F_T)=0$ and it is easy to see that $D(F_T+m\varepsilon H)=\dim T$, because H is ample and F_T is effective. Now $g^*(F_T+m\varepsilon H)=F+m\varepsilon\mathcal{H}_Z$, so

$$\begin{aligned}\kappa\left(X,\left(\tfrac{1}{n}+\varepsilon\right)\mathcal{H}\right) &= D\left(m\left(K_Z+\left(\tfrac{1}{n}+\varepsilon\right)\mathcal{H}_Z\right)\right)\\ &= D(F+m\varepsilon\mathcal{H}_Z)=\dim T\end{aligned}$$

as required. □

PROOF OF NFI-TYPE INEQUALITY 1.10 IN SITUATION (a). We note that under the assumptions (a), 1.10 is an immediate consequence of Propositions 4.3 and 4.4. Under assumptions (b), the proof of [Co95, 4.2] can be easily adapted to give an argument. Like Theorem–Definition 4.1, this is a standard result used in [Ch00] and [Is01], so we omit the details. □

All that remains is to prove the main theorem for family 75. Arguments similar to the following also prove Theorems 1.2, 1.3 and 1.4.

PROOF OF THEOREM 1.1. It is simplest to prove part (b) first; we then indicate how the argument can be easily adapted to demonstrate (a) and (c) also.

(b) Suppose that $\Phi\colon X \dashrightarrow Z/T$ is a birational map from X to an elliptic fibration $g\colon Z\to T$. By the NFI-type inequality 1.10, $K_X+\frac{1}{n}\mathcal{H}$ is nonterminal, where as usual the system $\mathcal{H}=\Phi_*^{-1}\mathcal{H}_Z=\Phi_*^{-1}g^*|H|$ is the transform of a very ample complete system of Cartier divisors on T. By Theorem 1.7, $K_X+\frac{1}{n}\mathcal{H}$ is strictly canonical and $\mathrm{CS}\left(X,\frac{1}{n}\mathcal{H}\right)=\{Q_1,Q_2\}$.

Let Q be either Q_1 or Q_2. As in Lemma 2.3 we blow up Q and play out the two ray game; in the notation of the lemma,

$$R(Y,B)=R(Y',B')=k[x,y],$$

f' is the anticanonical morphism of Y' and the composite $\pi\colon X\dashrightarrow\mathbb{P}^1$ is (x^4,y). Now because $Q\in\mathrm{CS}\left(X,\frac{1}{n}\mathcal{H}\right)$, we have that

$$\mathcal{H}_Y\subset|-nK_Y|=|nB|=k[x,y]_n=k\left[x^n,x^{n-4}y,\dots,x^{n\bmod 4}y^{\lfloor n/4\rfloor}\right];$$

the same is true of Y' (since Y and Y' are isomorphic in codimension one) and therefore $\mathcal{H}_{Y'}=(f')^*\mathcal{H}_{\mathbb{P}^1}$ is the pullback of a mobile system on $\mathbb{P}^1$.

But any mobile system on $\mathbb{P}^1$ is free, so we can deduce (as in the proof of Proposition 4.4) that for any $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon \ll 1$,

$$\kappa\left(X, \left(\tfrac{1}{n} + \varepsilon\right)\mathcal{H}\right) = \kappa\left(Y', \left(\tfrac{1}{n} + \varepsilon\right)\mathcal{H}_{Y'}\right) = D(F + m\varepsilon\mathcal{H}_{\mathbb{P}^1}) \leq 1 \tag{2}$$

where F is a fixed effective divisor on $\mathbb{P}^1$ (so in fact $F = 0$) and $m \in \mathbb{Z}_{>0}$.

But by Proposition 4.4 applied to $\Phi\colon X \dashrightarrow Z/T$ we have $\kappa\left(X, \left(\frac{1}{n} + \varepsilon\right)\mathcal{H}\right) = \dim T = 2$, which contradicts (2). This proves (b).

(a) We can follow the proof of (b) but in the end, rather than a contradiction, we deduce that the system $\mathcal{H} = \Phi_*^{-1}g^*|H|$ is actually a pullback $\pi_*^{-1}\mathcal{H}_{\mathbb{P}^1}$ via the map $\pi = (x^4, y)\colon X \dashrightarrow \mathbb{P}^1$. This induces an isomorphism $\mathbb{P}^1 \to T$ such that the specified diagram commutes.

(c) If we assume Φ is not an isomorphism, we can follow the argument for (b) to deduce that $\kappa\left(X, \left(\frac{1}{n} + \varepsilon\right)\mathcal{H}\right) = 1$, which is obviously a contradiction since $\mathcal{H}_Z = |H|$ is very ample. (Note that the NFI-type inequality 1.10 requires us to assume Φ is not an isomorphism in the Fano case; for the elliptic and K3 cases this is of course not necessary.)

This completes the proof. □

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