J. Math. Sci. Univ. Tokyo
13 (2006), 1–11.

Automorphic Functions with Respect to the Fundamental Group of the Complement of the Borromean Rings

By K. Matsumoto

Abstract. We construct automorphic functions on the real 3-dimensional hyperbolic space $\mathbb{H}^3$ with respect to a subgroup B of $GL_2(\mathbb{Z}[i])$, which is isomorphic to the fundamental group of the complement of the Borromean rings. We utilize the pull-backs of theta functions on the hermitian symmetric domain $\mathbb{D}$ of type $I_{2,2}$ under an embedding from $\mathbb{H}^3$ into $\mathbb{D}$ for our construction. These automorphic functions realize the quotient space of the real 3-dimensional upper half space by B as part of an affine algebraic variety in the 6-dimensional Euclidean space.

1. Introduction

Figure 1 shows the Borromean rings L in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The complement of Borromean rings $S^3 - L$ is known to admit a hyperbolic structure: there is a group B in $GL_2(\mathbb{Z}[i])$ acting properly discontinuously on the 3-dimensional hyperbolic space $\mathbb{H}^3$, and there is a homeomorphism

$$\varphi : \mathbb{H}^3/B \xrightarrow{\cong} S^3 - L.$$

In this paper we construct automorphic functions with respect to B (analytic functions defined in $\mathbb{H}^3$ which are invariant under B), and express the homeomorphism φ in terms of these automorphic functions. We utilize the pull-backs of theta functions on the hermitian symmetric domain $\mathbb{D}$ of type $I_{2,2}$ under an embedding from $\mathbb{H}^3$ into $\mathbb{D}$ for our construction of automorphic functions.

For the Whitehead link, its complement also admits a hyperbolic structure: there exists a group W in $GL_2(\mathbb{Z}[i])$ such that $\mathbb{H}^3/W$ is homeomorphic

---

2000 *Mathematics Subject Classification*. 11F55, 14P05, 57M25.
Key words: Borromean rings, hyperbolic structures, automorphic functions, theta functions.

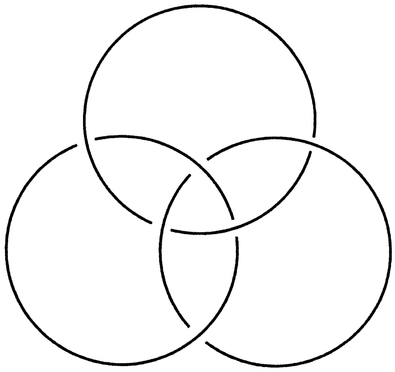

Fig. 1. The Borromean rings.

to the complement of the Whitehead link. This homeomorphism is explicitly given in [MNY]. But the embedding requires many automorphic functions (codimension of the embedding is high) and its image is rather complicated.

For the Borromean rings L, our embedding is much simpler than the case of the Whitehead link: we realize the quotient space $\mathbb{H}^3/B$ as part of an affine algebraic variety in $\mathbb{R}^6$, and write down the defining equations.

2. A Hyperbolic Structure on the Complement of the Borromean Rings

It is known that the complement of the Borromean rings admits a hyperbolic structure, i.e.,

$$S^3 - L \simeq \mathbb{H}^3/B,$$

where S^3 is the 3-dimensional sphere, L is a link called the Borromean rings, $\mathbb{H}^3$ is the upper half space $\{(z,t) \in \mathbb{C}\times\mathbb{R} \mid t > 0\}$, and B is a discrete subgroup of $GL_2(\mathbb{C})$ generated by the three elements

$$g_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2+i & 2i \\ -1 & -i \end{pmatrix},$$

and the scalar matrix iI_2 (refer to [W]). We recall the fundamental domain for B in [W] in Figure 2. Its faces are included in

$$W_1 = \{(z,t) \in \mathbb{H}^3 \mid \mathrm{Im}(z) = 0\}, \quad W_2 = \{(z,t) \in \mathbb{H}^3 \mid \mathrm{Im}(z) = 2\},$$

$$
\begin{aligned}
W_3 &= \{(z,t) \in \mathbb{H}^3 \mid \mathrm{Re}(z) = 0\}, \quad W_4 = \{(z,t) \in \mathbb{H}^3 \mid \mathrm{Re}(z) = 4\}, \\
W_5 &= \{(z,t) \in \mathbb{H}^3 \mid |z-1|^2 + t^2 = 1, \mathrm{Im}(z) > 0\}, \\
W_6 &= \{(z,t) \in \mathbb{H}^3 \mid |z-3|^2 + t^2 = 1, \mathrm{Im}(z) > 0\}, \\
W_7 &= \{(z,t) \in \mathbb{H}^3 \mid |z-1-2i|^2 + t^2 = 1, \mathrm{Im}(z) < 2\}, \\
W_8 &= \{(z,t) \in \mathbb{H}^3 \mid |z-3-2i|^2 + t^2 = 1, \mathrm{Im}(z) < 2\}, \\
W_9 &= \{(z,t) \in \mathbb{H}^3 \mid |z-i|^2 + t^2 = 1, \mathrm{Re}(z) > 0\}, \\
W_{10} &= \{(z,t) \in \mathbb{H}^3 \mid |z-2-i|^2 + t^2 = 1, \mathrm{Re}(z) < 2\}, \\
W_{11} &= \{(z,t) \in \mathbb{H}^3 \mid |z-2-i|^2 + t^2 = 1, \mathrm{Re}(z) > 2\}, \\
W_{12} &= \{(z,t) \in \mathbb{H}^3 \mid |z-4-i|^2 + t^2 = 1, \mathrm{Re}(z) < 4\}.
\end{aligned}
$$

The set W_{2j-1} is transformed into W_{2j} by the element $g_{2j-1,2j} \in B$, where

$$
g_{1,2} = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix}, \quad g_{3,4} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \quad g_{5,6} = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix},
$$

$$
g_{7,8} = \begin{pmatrix} -3-2i & 8i \\ -1 & 1+2i \end{pmatrix}, \quad g_{9,10} = \begin{pmatrix} -2-i & 2i \\ -1 & i \end{pmatrix},
$$

$$
g_{11,12} = \begin{pmatrix} 4+i & -8-6i \\ 1 & -2-i \end{pmatrix}.
$$

The quotient space $\mathbb{H}^3/B$ has three cusps c_j; they are represented by

$$
c_1 : (z,t) = (*, \infty), \quad c_2 : (z,t) = (1+i, 0) \sim (3+i, 0),
$$

$$
c_3 : (z,t) = (0,0) \sim (2,0) \sim (4,0) \sim (2i,0) \sim (2+2i,0) \sim (4+2i,0).
$$

By considering its volume, we see that the group B is a subgroup of $\Gamma = GL_2(\mathbb{Z}[i])$ of index 48. Since the generators g_j of B belong to

$$
\Gamma_1(2) = \{g = (g_{jk}) \in \Gamma \mid g_{12}, g_{11} - g_{22} \in 2\mathbb{Z}[i]\},
$$

we have $B \subset \Gamma_1(2)$.

Proposition 1. *The group generated by the groups*

$$
\Gamma(2) = \{g = (g_{jk}) \in \Gamma \mid g_{12}, g_{21}, g_{11} - g_{22} \in 2\mathbb{Z}[i]\},
$$

and B coincides with $\Gamma_1(2)$.

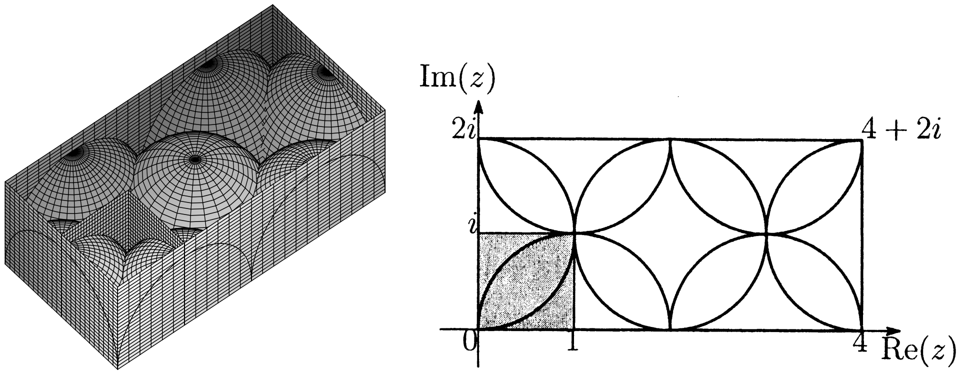

Fig. 2. Fundamental domains for B and $\Gamma_1^T(2)$.

PROOF. It is clear that $\Gamma_1(2)$ contains the group $\langle\Gamma(2), B\rangle$ generated by $\Gamma(2)$ and B. Note that the group $\Gamma(2)$ is normal in $\Gamma_1(2)$ and that the quotient group $\Gamma_1(2)/\Gamma(2)$ is isomorphic to $(\mathbb{Z}_2)^2$. This quotient group is generated by the representatives

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = g_1 \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = i g_3 \begin{pmatrix} -1-2i & -2 \\ 2 & 1-2i \end{pmatrix}.$$

Thus $\Gamma_1(2) \subset \langle\Gamma(2), B\rangle$. $\square$

Let T be the involution

$$T : (z, t) \mapsto (\bar{z}, t).$$

For a subgroup $G \in GL_2(\mathbb{C})$, the group generated by G and T with relations $gT = T\bar{g}$ for any $g \in G$ is denoted G^T.

The group $\Gamma_1^T(2)$ is generated by the six reflections

$$\gamma_1 = T, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} T, \quad \gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T,$$

$$\gamma_4 = \begin{pmatrix} 1 & 2i \\ 0 & 1 \end{pmatrix} T, \quad \gamma_5 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} T, \quad \gamma_6 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} T,$$

with mirrors

$$\mathrm{Im}(z) = 0, \quad |z - i|^2 + t^2 = 1, \quad \mathrm{Re}(z) = 0,$$

$$\mathrm{Im}(z) = 1, \quad \mathrm{Re}(z) = 1, \quad |z-1|^2 + t^2 = 1,$$

respectively. The mirrors of reflections γ_j and γ_{j+3} are tangent at the cusp c_j, and their product $\gamma_j\gamma_{j+3}$ belongs to B.

We have the following inclusion relations:

$$\begin{array}{ccccc}
\Gamma_1^T(2) & & & & \\
& \mathbb{Z}_2 \searrow & & & \\
(\mathbb{Z}_2)^2 \,\big| & & \Gamma_1(2) & & \\
\Gamma^T(2) & & (\mathbb{Z}_2)^2 \,\big| & \searrow & \\
& \mathbb{Z}_2 \searrow & & & \\
& & \Gamma(2) & & B.
\end{array}$$

3. Automorphic Functions with Respect to $\Gamma_1^T(2)$

The upper half space $\mathbb{H}^3$ can be embedded into the hermitian symmetric domain $\mathbb{D} = \{\tau \in M_{2,2}(\mathbb{C}) \mid (\tau - \tau^*)/2i \text{ is positive definite}\}$ of type $I_{2,2}$ by

$$\jmath : \mathbb{H}^3 \ni (z,t) \mapsto \frac{i}{t}\begin{pmatrix} t^2 + |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in \mathbb{D}.$$

Through this embedding, $GL_2(\mathbb{C})$ and T act on $\mathbb{D}$ as

$$\jmath(g \cdot (z,t)) = \frac{1}{|\det(g)|} g\, \jmath(z,t)\, g^*, \quad \jmath(T \cdot (z,t)) = {}^t\jmath(z,t).$$

Theta functions $\Theta\binom{a}{b}$ on $\mathbb{D}$ are defined as

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(\tau) = \sum_{n \in \mathbb{Z}[i]^2} \mathbf{e}[(n+a)\tau(n+a)^* + 2\mathrm{Re}(nb^*)],$$

where $\tau \in \mathbb{D}$, $a, b \in \mathbb{Q}[i]^2$. The pull back of $\Theta\binom{a}{b}(\tau)$ by $\jmath$ is denoted $\Theta\binom{a}{b}(z,t)$. For $a, b \in (\frac{\mathbb{Z}[i]}{2})^2$, we use the following convention:

$$\Theta\begin{pmatrix} a \\ b \end{pmatrix}(z,t) = \Theta\begin{bmatrix} 2a \\ 2b \end{bmatrix}(z,t) = \Theta\begin{bmatrix} 2a \\ 2b \end{bmatrix}.$$

Set

$$x_0 = \Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}, \quad x_1 = \Theta\begin{bmatrix} 1+i, 1+i \\ 1+i, 1+i \end{bmatrix}, \quad x_2 = \Theta\begin{bmatrix} 1+i, 0 \\ 0, 1+i \end{bmatrix}, \quad x_3 = \Theta\begin{bmatrix} 0, 1+i \\ 1+i, 0 \end{bmatrix}.$$

By the definition, we can see that x_0 is invariant under the action of Γ^T. By Lemmas 2.1.1 and 2.1.2 in [M1], we have the following.

LEMMA 1. *By the actions of g_1, g_2 and g_3, the functions x_1, x_2 and x_3 are transformed as follows:*

$$\begin{aligned}(x_1,x_2,x_3)\cdot g_1 &= (x_1,x_2,x_3)\begin{pmatrix} & & 1\\ & 1 & \\ 1 & & \end{pmatrix},\\ (x_1,x_2,x_3)\cdot g_2 &= (x_1,x_2,x_3),\\ (x_1,x_2,x_3)\cdot g_3 &= (x_1,x_2,x_3)\begin{pmatrix} & & -1\\ & 1 & \\ -1 & & \end{pmatrix}.\end{aligned}$$

The functions $x_1 + x_3$ and $x_1 - x_3$ are invariant modulo sign under the action of $\Gamma_1^T(2)$: they are invariant under the action of T, and change as

$$(x_1+x_3)\cdot g = \mathbf{e}[\mathrm{Im}(r)](x_1+x_3), \quad (x_1-x_3)\cdot g = \mathbf{e}[\mathrm{Re}(r)](x_1-x_3)$$

by the action of $g = I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix} \in \Gamma_1(2)$. Especially, by the actions of g_1, g_2 and g_3, their signs change as

	g_1	g_2	g_3
$x_1 + x_3$	$+$	$+$	$-$
$x_1 - x_3$	$-$	$+$	$+$

This lemma together with Theorem 3.2 in [MY] implies the following.

PROPOSITION 2. *The functions x_0, x_2, x_1x_3, $x_1^2 + x_3^2$ are invariant under the action of $\Gamma_1^T(2)$. The map*

$$\varphi_0 : \mathbb{H}^3 \ni (z,t) \mapsto \frac{1}{x_0^2}(x_0x_2, x_1x_3, x_1^2 + x_3^2) \in \mathbb{R}^3$$

induces an isomorphism between $\mathbb{H}^3/\Gamma_1^T(2)$ and the image $\varphi_0(\mathbb{H}^3)$.

4. Automorphic Functions with Respect to B

Set

$$w_1 = \Theta\begin{bmatrix}1,0\\0,1\end{bmatrix}, \quad w_2 = \Theta\begin{bmatrix}i,0\\0,1\end{bmatrix}, \quad w_3 = \Theta\begin{bmatrix}1,1+i\\1+i,1\end{bmatrix}, \quad w_4 = \Theta\begin{bmatrix}i,1+i\\1+i,1\end{bmatrix}.$$

By Lemmas 2.1.1 and 2.1.2 in [M1], we have the following.

Lemma 2. *The functions $w_1, \ldots, w_4$ are invariant modulo sign under the action of $\Gamma_1^T(2)$. By the actions of $g = I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix} \in \Gamma_1(2)$ and T, their signs change as*

	g $(r \in (1+i)\mathbb{Z}[i])$	g $(r \notin (1+i)\mathbb{Z}[i])$	T
w_1	$\mathbf{e}[\mathrm{Re}(q)]$	$\mathbf{e}[\mathrm{Re}(q)]$	$+$
w_2	$\mathbf{e}[\mathrm{Im}(q)]$	$\mathbf{e}[\mathrm{Im}(q)]$	$+$
w_3	$\mathbf{e}[\mathrm{Re}(p+q+r+s) + \mathrm{Im}(p+s)]$	$-\mathbf{e}[\mathrm{Re}(p+s) + \mathrm{Im}(p+q+r+s)]$	$+$
w_4	$\mathbf{e}[\mathrm{Re}(p+r+s) + \mathrm{Im}(p+q+s)]$	$\mathbf{e}[\mathrm{Re}(p+q+s) + \mathrm{Im}(p+r+s)]$	$-$

Especially, by the actions of g_1, g_2 and g_3, their signs change as

	g_1	g_2	g_3
w_1	$+$	$+$	$-$
w_2	$+$	$-$	$+$
w_3	$-$	$+$	$+$
w_4	$+$	$-$	$+$

Lemmas 1 and 2 imply the following Proposition.

Proposition 3. *The functions $f_1 = w_2 w_4$, $f_2 = (x_1 + x_3)w_1$ and $f_3 = (x_1 - x_3)w_3$ are invariant under the action of B. By the action of $g = I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix} \in \Gamma_1(2)$, their signs change as*

	g $(r \in (1+i)\mathbb{Z}[i])$	g $(r \notin (1+i)\mathbb{Z}[i])$
f_1	$\mathbf{e}[\mathrm{Re}(p+r+s) + \mathrm{Im}(p+s)]$	$\mathbf{e}[\mathrm{Re}(p+q+s) + \mathrm{Im}(p+q+r+s)]$
f_2	$\mathbf{e}[\mathrm{Re}(q) + \mathrm{Im}(r)]$	$\mathbf{e}[\mathrm{Re}(q) + \mathrm{Im}(r)]$
f_3	$\mathbf{e}[\mathrm{Re}(p+q+s) + \mathrm{Im}(p+s)]$	$\mathbf{e}[\mathrm{Re}(p+s) + \mathrm{Im}(p+q+s)]$

Especially, we have

	γ_1	γ_2	γ_3
f_1	$-$	$+$	$+$
f_2	$+$	$-$	$+$
f_3	$+$	$+$	$-$

where

$$\gamma_1 = T, \quad \gamma_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} T, \quad \gamma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} T.$$

Let Iso_j be the subgroup of $\Gamma_1^T(2)$ consisting of elements keeping f_j invariant for $j = 1, 2, 3$, and Iso_0 the subgroup of $\Gamma_1^T(2)$ consisting of elements keeping $f_1 f_2 f_3$ invariant.

PROPOSITION 4. *We have*

$$\Gamma_1(2) = \mathrm{Iso}_0, \quad B = \mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3.$$

The group B is normal in $\Gamma_1^T(2)$; the quotient group $\Gamma_1^T(2)/B$ is isomorphic to $(\mathbb{Z}_2)^3$.

PROOF. The group $\Gamma_1^T(2)$ is generated by the group B and the reflections γ_1, γ_2 and γ_3. Since the index $[\Gamma_1^T(2) : B]$ is eight, we have $B = \mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3$ and $\Gamma_1^T(2)/B \simeq (\mathbb{Z}_2)^3$ by Proposition 3. Proposition 3 also shows that the function $f_1 f_2 f_3$ is invariant under the action of $\Gamma_1(2)$, and that it changes its sign by T. $\square$

REMARK 1. The quotient group $\Gamma_1^T(2)/B \simeq (\mathbb{Z}_2)^3$ corresponds to some symmetries of the Borromean rings L. We draw three congruent ellipses E_j with center at the origin in the plane $t_j = 0$ $(j = 1, 2, 3)$ so that the minor axis of E_j is in $t_{j-1} = 0$, where we regard t_0 as t_3. Then E_j form the Borromean rings L. The three reflections with mirrors $t_j = 0$ act on the complement of the Borromean rings L and form the group isomorphic to $(\mathbb{Z}_2)^3$; see Figure 3.

We can assume that any element $g \in \Gamma_1(2)$ takes the form $I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix}$, otherwise multiply i to g. For example, $ig_3 = I_2 + \begin{pmatrix} -2+2i & -2 \\ -i & 0 \end{pmatrix}$.

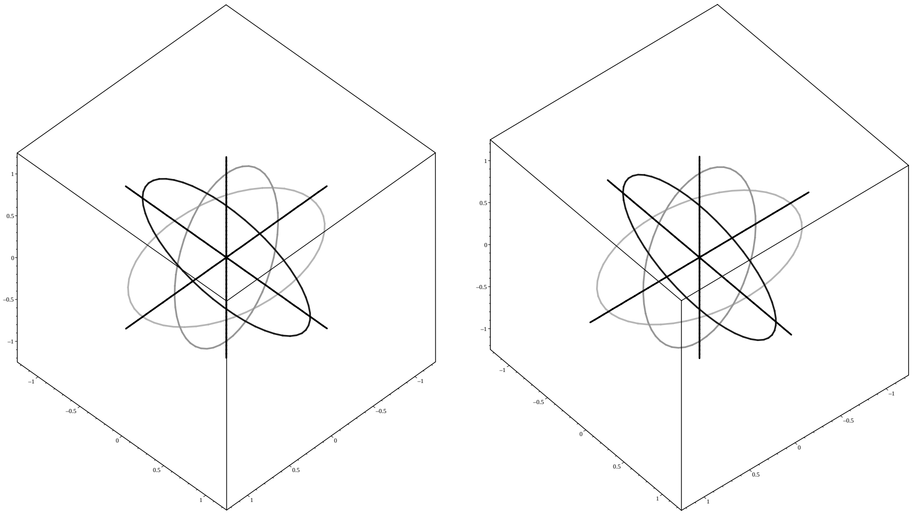

Fig. 3. Stereographic figures of the Borromean rings.

Theorem 1. *The element* $g = I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix} \in \Gamma_1(2)$ *belongs to* B *if and only if*

$$\mathrm{Re}(q) + \mathrm{Im}(r) \equiv 0,$$

$$\frac{1 + (-1)^{\mathrm{Re}(r)+\mathrm{Im}(r)}}{2}\mathrm{Re}(q) + \frac{1 - (-1)^{\mathrm{Re}(r)+\mathrm{Im}(r)}}{2}\mathrm{Im}(q) \equiv \mathrm{Re}(p+s) + \mathrm{Im}(p+s),$$

modulo 2.

Proof. We have only to write down the conditions for $I_2 + \begin{pmatrix} 2p & 2q \\ r & 2s \end{pmatrix}$ to belong to $\mathrm{Iso}_1 \cap \mathrm{Iso}_2 \cap \mathrm{Iso}_3$. $\square$

Proposition 5. *We have*

$$4w_1^2 = 4\Theta\begin{bmatrix} 1,0 \\ 0,1 \end{bmatrix}^2$$

$$= 2\Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 1+i,0 \\ 0,1+i \end{bmatrix} + 2\Theta\begin{bmatrix} 1+i,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 0,0 \\ 0,1+i \end{bmatrix} - 2\Theta\begin{bmatrix} 1+i,1+i \\ 1+i,1+i \end{bmatrix}\Theta\begin{bmatrix} 0,1+i \\ 1+i,0 \end{bmatrix}$$

$$= \quad (x_0+x_1+x_2+x_3)(x_0-x_1+x_2-x_3),$$

$$4w_2^2 = 4\Theta\begin{bmatrix} i,0 \\ 0,1 \end{bmatrix}^2$$
$$= \quad 2\Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 1{+}i,0 \\ 0,1{+}i \end{bmatrix} + 2\Theta\begin{bmatrix} 1{+}i,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 0,0 \\ 0,1{+}i \end{bmatrix} + 2\Theta\begin{bmatrix} 1{+}i,1{+}i \\ 1{+}i,1{+}i \end{bmatrix}\Theta\begin{bmatrix} 0,1{+}i \\ 1{+}i,0 \end{bmatrix}$$
$$= \quad (x_0+x_1+x_2-x_3)(x_0-x_1+x_2+x_3),$$

$$4w_3^2 = 4\Theta\begin{bmatrix} 1,1{+}i \\ 1{+}i,1 \end{bmatrix}^2$$
$$= \quad -2\Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 1{+}i,0 \\ 0,1{+}i \end{bmatrix} + 2\Theta\begin{bmatrix} 1{+}i,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 0,0 \\ 0,1{+}i \end{bmatrix} + 2\Theta\begin{bmatrix} 1{+}i,1{+}i \\ 1{+}i,1{+}i \end{bmatrix}\Theta\begin{bmatrix} 0,1{+}i \\ 1{+}i,0 \end{bmatrix}$$
$$= \quad (x_0+x_1-x_2-x_3)(x_0-x_1-x_2+x_3),$$

$$4w_4^2 = 4\Theta\begin{bmatrix} i,1{+}i \\ 1{+}i,1 \end{bmatrix}^2$$
$$= \quad -2\Theta\begin{bmatrix} 0,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 1{+}i,0 \\ 0,1{+}i \end{bmatrix} + 2\Theta\begin{bmatrix} 1{+}i,0 \\ 0,0 \end{bmatrix}\Theta\begin{bmatrix} 0,0 \\ 0,1{+}i \end{bmatrix} - 2\Theta\begin{bmatrix} 1{+}i,1{+}i \\ 1{+}i,1{+}i \end{bmatrix}\Theta\begin{bmatrix} 0,1{+}i \\ 1{+}i,0 \end{bmatrix}$$
$$= \quad (x_0+x_1-x_2+x_3)(x_0-x_1-x_2-x_3).$$

PROOF. Use Theorem 1 in [M2] and Lemma 3.2 in [MY]. □

THEOREM 2. *The map*

$$\varphi : \mathbb{H}^3 \ni (z,t) \mapsto \frac{1}{x_0^2}(x_0x_2, x_1x_3, x_1^2+x_3^2, f_1, f_2, f_3) \in \mathbb{R}^6$$

induces an isomorphism between $\mathbb{H}^3/B$ *and the image* $\varphi(\mathbb{H}^3)$*. The squares of* f_j *can be expressed in terms of* $\Gamma_1^T(2)$*-invariant functions:*

$$\begin{aligned}
16f_1^2 &= (x_0^2{-}x_2^2)^2 - 2(x_0^2{+}x_2^2)(x_1^2{+}x_3^2) + (x_1^2{+}x_3^2)^2 - 4(x_1x_3)^2 \\
&\quad - 8(x_0x_2)(x_1x_3), \\
4f_2^2 &= (x_1^2+x_3^2+2x_1x_3)((x_0+x_2)^2 - (x_1^2+x_3^2) - 2x_1x_3), \\
4f_3^2 &= (x_1^2+x_3^2-2x_1x_3)((x_0-x_2)^2 - (x_1^2+x_3^2) + 2x_1x_3).
\end{aligned}$$

These relations together with the image of the map φ_0 *determine the image of the map* φ*.*

Proof. By Proposition 5, f_j vanishes only on the mirror of the reflection γ_j for $j = 1, 2, 3$. Note that the space $\mathbb{H}^3/B$ is the eight fold covering of $\mathbb{H}^3/\Gamma_1^T(2)$ branching along the union of the mirrors of γ_1, γ_2 and γ_3, which corresponds to the zero locus of $f_1 f_2 f_3$. Thus the map φ realize this covering. Use Proposition 5 to express f_j^2 in terms of $x_0, \dots, x_3$. $\square$

References

[F] Freitag, E., Modulformen zweiten Grades zum rationalen und Gaußschen Zahlkörper, Sitzungsber. Heidelb. Akad. Wiss. **1** (1967), 1–49.

[M1] Matsumoto, K., Theta functions on the bounded symmetric domain of type $I_{2,2}$ and the period map of 4-parameter family of K3 surfaces, Math. Ann. **295** (1993), 383–408.

[M2] Matsumoto, K., Algebraic relations among some theta functions on the bounded symmetric domain of type $I_{r,r}$, to appear in Kyushu J. Math.

[MNY] Matsumoto, K., Nishi, H. and M. Yoshida, Automorphic functions for the Whitehead-link-complement group, preprint 2005.

[MSY] Matsumoto, K., Sasaki, T. and M. Yoshida, The monodromy of the period map of a 4-parameter family of K3 surfaces and the Aomoto-Gel'fand hypergeometric function of type (3,6), Internat. J. of Math. **3** (1992), 1–164.

[MY] Matsumoto, K. and M. Yoshida, Invariants for some real hyperbolic groups, Internat. J. of Math. **13** (2002), 415–443.

[T] Thurston, W., *Geometry and Topology of 3-manifolds*, Lecture Notes, Princeton Univ., 1977/78.

[W] Wielenberg, N., The structure of certain subgroups of the Picard group, Math. Proc. Cambridge Philos. Soc. **84** (1978), no. 3, 427–436.

[Y] Yoshida, M., *Hypergeometric Functions, My Love*, Aspects of Mathematics, E32, Friedr Vieweg & Sohn, Braunschweig, 1997.

(Received August 4, 2005)

Keiji Matsumoto
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: matsu@math.sci.hokudai.ac.jp