A Semiampleness Criterion

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Abstract. We give a criterion for a real divisor to be rational and semiample.

0. Introduction

The aim of this paper is to establish the following criterion for the rationality and semiampleness of a nef divisor with real coefficients (we work over an algebraically closed field of characteristic zero):

THEOREM 0.1. *Let X be a proper nonsingular algebraic variety containing a simple normal crossings divisor $\sum_l E_l$. Let $B = \sum_l b_l E_l$ and $D = \sum_l d_l E_l$, with $d_l, b_l \in \mathbb{R}$, be $\mathbb{R}$-divisors on X satisfying the following properties:*

(1) *D is nef and $rD - (K_X + B)$ is nef and big for some $r \in \mathbb{Q}$.*

(2) *$b_l < 1$ for every l.*

(3) *For every integer $j \geq 1$, the following inclusion holds*

$$H^0(X, \sum_l \lceil jd_l - b_l \rceil E_l) \subseteq H^0(X, \sum_l \lfloor jd_l \rfloor E_l).$$

For a real number d, $\lfloor d \rfloor$ and $\lceil d \rceil$ are the largest and smallest integers, respectively, such that $\lfloor d \rfloor \leq d \leq \lceil d \rceil$.

Then there exists a positive integer m such that $md_l \in \mathbb{Z}$ for every l, and the linear system $|mD|$ is base point free.

This criterion is also valid for singular varieties, being birational in nature (see Theorem 2.1). It is a generalization of the Base Point Free Theorem (Kawamata-Shokurov [1]), and it also extends the known fact that

the existence of the Cutkosky-Kawamata-Moriwaki decomposition for a log canonical divisor of general type implies the finite generation of the log canonical ring (Kawamata [2]).

Our interest in such a criterion lies in its connection to the finite generation of Shokurov's FGA-algebras ([5], Conjecture 4.39). Shokurov has recently reduced the existence of log flips in dimension $d + 1$ to the Log Minimal Model Program in dimension d, and the finite generation of d-dimensional FGA-algebras ([5], Corollary 1.5). Thus, provided that FGA-algebras are finitely generated, and this may be established even assuming the validity of the Log Minimal Model Program in the same dimension, this would give an inductive proof of the existence of log flips. The finite generation of FGA-algebras is known in dimension one and two ([5], Main Theorem 1.7).

For the purpose of this introduction, an FGA-algebra is an $\mathbb{N}$-graded sub-algebra of a divisorial algebra

$$\mathcal{L} \subseteq \bigoplus_{i=0}^{\infty} H^0(X, iD),$$

where D is some divisor on a Fano variety X, such that $\mathcal{L}$ is *asymptotically saturated with respect to X*. More precisely, the projective normalization of any such sub-algebra admits a representation

$$\bar{\mathcal{L}} = \bigoplus_{i=0}^{\infty} H^0(X_i, M_i),$$

where each M_i is a free Cartier divisor on some resolution of singularities $\mu_i \colon X_i \to X$ such that both divisors $K_{X_i} - \mu_i^*(K_X)$ and M_i are supported by a simple normal crossings divisor on X_i. Then asymptotic saturation means that the following inclusions hold, for every $i, j \geq 1$:

$$H^0(X_i, \lceil K_{X_i} - \mu_i^*(K_X) + j\frac{M_i}{i} \rceil) \subseteq H^0(X_j, M_j).$$

FGA-algebras are not well understood at the moment, the main difficulty being in dealing with the a priori infinitely many models X_i. The successful two dimensional approach to FGA-algebras is based on the Canonical Confinement Conjecture ([5], Conjecture 6.14). Alternatively, based on the

experience from the proof of the Base Point Free Theorem, Shokurov suggested that an inductive method may be possible in a larger class of algebras: X may no longer be a Fano variety, but it should be related to $\mathcal{L}$ by some kind of adjoint-type property ([5], Remark 4.40.3). As an application of Theorem 0.1, we give two such examples of adjoint-type properties which are sufficient for finite generation:

THEOREM 0.2. *Let X be a proper nonsingular algebraic variety containing a simple normal crossings divisor $\sum_l E_l$. Let $D = \sum_l d_l E_l$ be a nef and big $\mathbb{R}$-divisor, with associated divisorial algebra*

$$\mathcal{R}_X(D) = \bigoplus_{i=0}^{\infty} H^0(X, iD).$$

Assume that there exists an $\mathbb{R}$-divisor $B = \sum_l b_l E_l$, with $b_l < 1$ for every l, such that

(1) *$\mathcal{R}_X(D)$ is asymptotically saturated with respect to (X, B).*

(2) *The $\mathbb{R}$-divisor $rD - (K_X + B)$ is nef, for some $r \in \mathbb{Q}$.*

Then $\mathcal{R}_X(D)$ is a finitely generated algebra.

THEOREM 0.3. *Let X be a proper nonsingular algebraic variety containing a simple normal crossings divisor $\sum_l E_l$. Let $M_i = \sum_l m_{i,l} E_l$ be divisors on X, for $i \geq 0$, satisfying the following properties:*

(i) *The linear system $|M_i|$ is base point free, for every i.*

(ii) *$M_0 = 0$, $M_i + M_j \leq M_{i+j}$ for every i, j.*

(iii) *The limit $d_l = \lim_{i \to \infty} \frac{1}{i} m_{i,l}$ exists, for every l. Set $D = \sum_l d_l E_l$.*

The sequence $(M_i)_{i \geq 0}$ defines an algebra

$$\mathcal{L} = \bigoplus_{i=0}^{\infty} H^0(X, M_i) \subseteq \mathcal{R}_X(D).$$

Assume moreover that there exists an $\mathbb{R}$-divisor $B = \sum_l b_l E_l$, with $b_l < 1$ for every l, such that

(1) $\mathcal{L}$ *is asymptotically saturated with respect to* (X, B). *Equivalently,*
$H^0(X, \sum_l \lceil j\frac{m_{i,l}}{i} - b_l \rceil E_l) \subseteq H^0(X, M_j)$ *for every* $i, j \geq 1$.

(2) *The* $\mathbb{R}$-*divisor* $rD - (K_X + B)$ *is nef and big, for some* $r \in \mathbb{Q}$.

Then $\mathcal{L}$ *is a finitely generated algebra.*

The paper is organized as follows. In Section 1, we collect some basic definitions and properties on log pairs, b-divisors and functional algebras. In Section 2, we establish the semiampleness criterion. The proof is quite similar to that of the Base Point Free Theorem, and we use a nonvanishing result for real divisors due to Kawamata ([2], Theorem 3). In Section 3, we apply this criterion to the finite generation of some functional algebras.

1.   Preliminary

A *variety* is a reduced and irreducible separated scheme of finite type, defined over an algebraically closed field k of characteristic zero. A *contraction* is a proper morphism of normal varieties $f \colon X \to Y$ with $\mathcal{O}_Y = f_* \mathcal{O}_X$.

1.A.   Divisors

Let X a normal variety. A Weil divisor on X is an element of the free abelian group $\mathrm{WDiv}(X)$ generated by codimension one cycles on X. A non-zero rational function $a \in k(X)^\times$ defines the principal Weil divisor

$$(a) = \sum_{\mathrm{codim}(P,X)=1} \mathrm{mult}_P(a) P.$$

Let $L \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. An L-Weil divisor is an element of

$$\mathrm{WDiv}(X)_L := \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} L.$$

We say that $D_1, D_2 \in \mathrm{WDiv}(X)_{\mathbb{R}}$ are L-*linearly equivalent*, denoted $D_1 \sim_L D_2$, if there exist finitely many $q_i \in L$ and $a_i \in k(X)^\times$ such that $D_1 - D_2 = \sum_i q_i(a_i)$. We say that $D \in \mathrm{WDiv}(X)_L$ is L-*Cartier* if $D \sim_L 0$ in a neighborhood of each point of X. We denote by

$$\mathrm{CDiv}(X)_L \subset \mathrm{WDiv}(X)_L$$

the subgroup of L-Cartier L-Weil divisors. To each $D \in \mathrm{WDiv}(X)_{\mathbb{R}}$ we can associate a coherent $\mathcal{O}_X$-sub-module $\mathcal{O}_X(D)$ of the constant sheaf of

rational functions $k(X)$ on X, whose sections on open subsets U of X are given by

$$H^0(U, \mathcal{O}_X(D)) = \{a \in k(X)^\times; ((a) + D)|_U \geq 0\} \cup \{0\}.$$

For $D \in \mathrm{WDiv}(X)_\mathbb{R}$, its *round down* $\lfloor D \rfloor$, *round up* $\lceil D \rceil$, *fractional part* $\{D\}$ are defined componentwise (for $d \in \mathbb{R}$, $\lfloor d \rfloor$ and $\lceil d \rceil$ are the unique integers in the half-open intervals $(d-1, d]$ and $[d, d+1)$, respectively, and $\{d\} = d - \lfloor d \rfloor$).

Let $\pi\colon X \to S$ be a proper surjective morphism. We say that $D \in \mathrm{WDiv}(X)_\mathbb{R}$ is

(i) *free/S* if D is Cartier and the natural map $\pi^*\pi_*\mathcal{O}_X(D) \to \mathcal{O}_X(D)$ is surjective.

(ii) *nef/S* if D is $\mathbb{R}$-Cartier and $D \cdot C \geq 0$ for every proper curve $C \subset X$ such that $\pi(C)$ is a point.

(iii) *ample/S* if π is projective and the numerical class of D belongs to the real cone generated by the numerical classes of π-ample Cartier divisors.

(iv) *semiample/S* if there exists a contraction $f\colon X \to Y$, defined over S, and an ample/S $\mathbb{R}$-divisor H on Y such that $D \sim_\mathbb{R} f^*H$. If D is rational, this is equivalent to the existence of a positive integer m such that mD is free/S.

(v) *big/S* if there exists $C > 0$ such that $\mathrm{rank}\, \pi_*\mathcal{O}_X(mD) \geq Cm^d$ for m sufficiently large and divisible, where d is the dimension of the general fiber of π.

1.B. B-divisors (V.V. Shokurov [4])

Let X be a normal variety. The group of *b-divisors* of X is defined as the projective limit

$$\mathrm{BDiv}(X) = \underset{Y \to X}{\mathrm{proj\,lim}}\, \mathrm{WDiv}(Y),$$

where the index set is the set of isomorphism classes of birational contractions $Y \to X$, with the following order: $Y_1 \geq Y_2$ if the induced proper

birational map $g\colon Y_1 \dashrightarrow Y_2$ is a morphism. In this case, we have a natural push-forward homomorphism $g_*\colon \mathrm{WDiv}(Y_1) \to \mathrm{WDiv}(Y_2)$. Note that given two birational contractions $Y_1 \to X$ and $Y_2 \to X$, the normalization of the graph of the induced birational map $Y_1 \dashrightarrow Y_2$ defines a third birational contraction $Y_3 \to X$ with $Y_3 \geq Y_1$ and $Y_3 \geq Y_2$.

A *prime b-divisor* E of X is obtained as follows: let $\mu\colon Y \to X$ be a birational contraction, and let E be a prime divisor on Y. For another birational contraction $Y' \to X$, let $g\colon Y \dashrightarrow Y'$ be the induced rational map. If g induces an isomorphism between the generic point of E and the generic point of a prime divisor E' on Y', we set $\mathbf{D}_{Y'} = E'$. Otherwise, set $\mathbf{D}_{Y'} = 0$. The family $\{\mathbf{D}_{Y'}\}$ is a prime b-divisor of X, denoted by E. The center of E on X, denoted by $c_X(E)$, is the closed subset $\mu(E)$ of X. It is clear that prime b-divisors of X are in one-to-one correspondence with geometric (discrete) valuations of $k(X)$ which have non-empty center on X. In this terminology, a b-divisor of X is a $\mathbb{Z}$-valued function on the set of all prime b-divisors of X, denoted as a formal sum

$$\mathbf{D} = \sum_E \mathrm{mult}_E(\mathbf{D})E,$$

such that the sum has finite support on some (hence any) birational model of X.

A proper birational map $\Phi\colon X_1 \dashrightarrow X_2$ (i.e., the normalization of the graph of Φ induces birational contractions to X_1 and X_2), induces a push-forward isomorphism $\Phi_*\colon \mathrm{BDiv}(X_1) \xrightarrow{\sim} \mathrm{BDiv}(X_2)$.

For $L \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, an *$L$-b-divisors* of X is an element of

$$\mathrm{BDiv}(X)_L := \mathrm{BDiv}(X) \otimes_{\mathbb{Z}} L.$$

Examples 1.1.

(1) A rational function $a \in k(X)^\times$ defines the following b-divisor of X

$$\overline{(a)} = \{(f^*a)\}_{f\colon Y \to X}.$$

(2) Let $Y \to X$ be a birational contraction and let $D \in \mathrm{CDiv}(Y)_{\mathbb{R}}$. The family $\overline{D} = \{f^*D\}_{f\colon Y' \to Y}$ is an $\mathbb{R}$-b-divisor of X, called the *Cartier closure* of D.

(3) Let $\omega \in \Omega_{k(X)/k}^{\dim(X)}$ be a top rational differential form of X. The associated family of Weil divisors $\mathbf{K} = \{(f^*\omega)\}_{f\colon Y \to X}$ is called the *canonical b-divisor* of X. It is uniquely defined up to linear equivalence.

Let $\mathbf{D}$ be an $\mathbb{R}$-b-divisor of X. The round up $\lceil \mathbf{D} \rceil \in \mathrm{BDiv}(X)$ is defined componentwise. The restriction of $\mathbf{D}$ to an open subset $U \subset X$ is a well defined $\mathbb{R}$-b-divisor of U, denoted by $\mathbf{D}|_U$. Also, $\mathbf{D}$ defines an $\mathcal{O}_X$-module $\mathcal{O}_X(\mathbf{D})$ whose sections on an open subset $U \subseteq X$ are given by

$$H^0(U, \mathcal{O}_X(\mathbf{D})) = \{a \in k(X)^\times; (\overline{(a)} + \mathbf{D})|_U \geq 0\} \cup \{0\}.$$

An $\mathbb{R}$-b-divisor $\mathbf{D}$ of X is called *L-b-Cartier* if there exists a birational contraction $f\colon X' \to X$ such that $\mathbf{D}_{X'}$ is L-Cartier and $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. Note that $\mathcal{O}_X(\mathbf{D}) = f_*\mathcal{O}_{X'}(\mathbf{D}_{X'})$ in this case, in particular $\mathcal{O}_X(\mathbf{D})$ is a coherent $\mathcal{O}_X$-module.

Let $\pi\colon X \to S$ be a proper morphism. An $\mathbb{R}$-b-divisor $\mathbf{D}$ of X is called *b-free/S* (*b-nef/S*, *b-semiample/S*, *b-nef and big/S*) if there exists a birational contraction $X' \to X$ such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is free/S (nef/S, semiample/S, nef and big/S), where $X' \to S$ is the induced proper morphism.

LEMMA 1.2. ([5], Proposition 4.15). *Let $\pi\colon X \to S$ be a contraction. Let $k(X)$ be the constant sheaf of rational functions on X and let $\mathcal{L} \subset \pi_*k(X)$ be a non-zero coherent $\mathcal{O}_S$-module. Then there exists a unique b-free/S b-divisor $\mathbf{M}$ of X such that*

(i) $\mathcal{L} \subseteq \pi_*\mathcal{O}_X(\mathbf{M})$.

(ii) *If $\mathbf{D} \in \mathrm{BDiv}(X)_\mathbb{R}$ and $\mathcal{L} \subseteq \pi_*\mathcal{O}_X(\mathbf{D})$, then $\mathbf{M} \leq \mathbf{D}$.*

We say that $\mathbf{M}$ is the mobile b-divisor *of $\mathcal{L}$.*

Sketch of proof. Assume first that S is affine, so that $\mathcal{L}$ is generated by finitely many global sections $a_1, \dots, a_p$. Due to the inclusion $\mathcal{L} \subset \pi_*k(X)$, the a_i's are nonzero rational functions on X. The b-divisor $\mathbf{M}$ is defined by

$$\mathbf{M} = \sup\{-\overline{(a)}; a \in H^0(S, \mathcal{L}) \setminus 0\} = \max_{i=1}^{p}(-\overline{(a_i)})$$

In general, we may cover S with finitely many affine open subsets S_α, and the b-divisors $\mathbf{M}_\alpha$ of $\pi^{-1}(S_\alpha)$ constructed above glue to a b-divisor $\mathbf{M} \in \mathrm{BDiv}(X)$, which clearly satisfies (i) and (ii).

To show that $\mathbf{M}$ is b-free/S, assume again that S is affine. By the definition of $\mathbf{M}_X$, the linear system

$$\{(a) + \mathbf{M}_X; a \in H^0(S, \mathcal{L}) \setminus 0\}$$

has no fixed components. By Hironaka, there exists a birational contraction $\mu \colon Y \to X$ such that the induced linear system

$$\{(\mu^* a) + \mathbf{M}_Y; a \in H^0(S, \mathcal{L}) \setminus 0\}$$

has no fixed points. Then $\mathbf{M}_Y$ is free/S and $\mathbf{M} = \overline{\mathbf{M}_Y}$.

LEMMA 1.3. *Let* $\pi \colon X \to S$ *be a contraction. Let* $\mathbf{D} \in \mathrm{BDiv}(X)_\mathbb{R}$ *such that* $\mathbf{D}_X \leq 0$ *and* $\mathbf{D}$ *is b-nef/S. Then:*

(1) $\mathbf{D} \leq 0$.

(2) *There exists a closed subset* $S^0 \subsetneqq S$ *such that for every prime b-divisor E of X, $\mathrm{mult}_E(\mathbf{D}) \neq 0$ if and only if $\pi(c_X(E)) \subseteq S^0$.*

PROOF.   (1) Let $Y \to X$ be a birational contraction such that $\mathbf{D} = \overline{\mathbf{D}_Y}$. Since $\mathbf{D}_Y$ is nef/S, it is also nef/X. By the Negativity Lemma [3, 1.1], we obtain $\mathbf{D}_Y \leq 0$. Therefore $\mathbf{D} = \overline{\mathbf{D}_Y} \leq 0$.

(2) Let $Y \to X$ be a birational contraction such that $\mathbf{D} = \overline{\mathbf{D}_Y}$, and let $\nu \colon Y \to S$ be the induced contraction. The restriction of the $\mathbb{R}$-divisor $-\mathbf{D}_Y$ to the general fibre of ν is zero, being anti-nef and effective. Therefore

$$S^0 = \nu(\mathrm{Supp}(\mathbf{D}_Y))$$

is a closed subset of S and $S^0 \neq S$. Property (2) is equivalent to the following equality of sets

$$\mathrm{Supp}(\mathbf{D}_Y) = \nu^{-1}(S^0).$$

The inclusion $\mathrm{Supp}(\mathbf{D}_Y) \subseteq \nu^{-1}(S^0)$ is clear. For the converse, fix a point $P \in S^0$ and assume by contradiction that $\nu^{-1}(P)$ is not included in $\mathrm{Supp}(\mathbf{D}_Y)$. We may replace Y by a resolution of singularities such that

$\nu^{-1}(P)$ is the union of finitely many prime Cartier divisors $E_1, \dots, E_n$ on Y. By assumption, we have $\mathrm{mult}_{E_1}(\mathbf{D}_Y) = 0$. The fiber $\nu^{-1}(P) = \bigcup_{i=1}^{n} E_i$ is connected, hence after a possible reordering, we may assume that E_j intersects $E_1 \cup \dots \cup E_{j-1}$, for every $2 \le j \le n$.

Assume by induction that $\mathrm{mult}_{E_i}(\mathbf{D}_Y) = 0$ for $i \le j-1$. There exists a proper curve $C \subset \bigcup_{i=1}^{j-1} E_i$ such that $C \cap E_j \ne \emptyset$ and $C \nsubseteq \mathrm{Supp}(\mathbf{D}_Y)$. Since $-\mathbf{D}_Y$ is effective and $\mathbf{D}_Y \cdot C \ge 0$, we infer $\mathrm{mult}_{E_j}(\mathbf{D}_Y) = 0$. By induction, we obtain $\mathrm{mult}_{E_j}(\mathbf{D}_Y) = 0$ for $1 \le j \le n$. However, $P \in S^0$ implies that $\mathrm{Supp}(\mathbf{D}_Y) \cap \nu^{-1}(P) \ne \emptyset$. In particular, there exists a proper curve $C \subseteq \nu^{-1}(P)$ such that $C \nsubseteq \mathrm{Supp}(\mathbf{D}_Y)$ and $C \cap \mathrm{Supp}(\mathbf{D}_Y) \ne \emptyset$. Therefore $\mathbf{D}_Y \cdot C < 0$. On the other hand, $\mathbf{D}_Y$ is nef$/S$, hence $\mathbf{D}_Y \cdot C \ge 0$. Contradiction. $\square$

LEMMA 1.4. *Let $X \to S$ be a contraction, and let $\mathbf{D} \le \mathbf{H}$ be $\mathbb{R}$-b-divisors of X such that $\mathbf{D}$ is b-nef$/S$ and $\mathbf{H}$ is $\mathbb{R}$-b-semiample$/S$, that is there exists rational contraction over S*
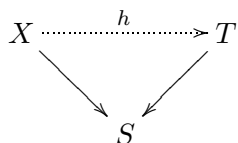
and an ample$/S$ $\mathbb{R}$-divisor A on T such that $\mathbf{H} \sim_{\mathbb{R}} \overline{h^*(A)}$. If $\mathbf{D} = \mathbf{H}$ over a neighborhood of a closed subset $X_0 \subset X$, then $\mathbf{D} = \mathbf{H}$ over a neighborhood of $h(X_0) \subset T$.

PROOF. We may replace X by a higher birational model so that h becomes a morphism. By assumption, $\mathbf{D} - \mathbf{H} \le 0$ is b-nef$/T$. By Lemma 1.3, there exists a closed subset $T^0 \subset T$ such that $\mathrm{mult}_E(\mathbf{D} - \mathbf{H}) \ne 0$ if and only if $c_X(E) \subseteq h^{-1}(T^0)$, for every prime b-divisor E of X. By assumption, there exists an open set $U \supset X_0$ such that $\mathrm{mult}_E(\mathbf{D} - \mathbf{H}) = 0$ for every prime b-divisor E of X with $c_X(E) \cap U \ne \emptyset$. Therefore $U \subseteq X \setminus h^{-1}(T^0)$. In particular, $X_0 \cap h^{-1}(T^0) = \emptyset$. Therefore $V = T \setminus T^0$ is an open neighborhood of $h(X_0)$, and $\mathbf{D}|_{h^{-1}(V)} = \mathbf{H}|_{h^{-1}(V)}$. $\square$

1.C. Log pairs

A *log pair* (X, B) is a normal variety X endowed with an $\mathbb{R}$-Weil divisor B such that $K + B$ is $\mathbb{R}$-Cartier. A *log variety* is a log pair (X, B) such that B is effective.

Let (X, B) be a log pair and let $\mu\colon Y \to X$ be a birational contraction. Choose a top rational differential form $\omega \in \Omega^{\dim(X)}_{k(X)/k}$ with $K = (\omega)$. The Weil divisor $K_Y = (\mu^*\omega)$ is a canonical divisor of Y, and there exists a unique $\mathbb{R}$-Weil divisor B_Y on Y satisfying the adjunction formula

$$\mu^*(K + B) = K_Y + B_Y.$$

We say that (Y, B_Y) is the *crepant log pair structure* induced by (X, B) via μ. It is independent of the choice of ω and in fact it is independent of the choice of K in its linear equivalence class. There exists a unique $\mathbb{R}$-b-divisor $\mathbf{A}(X, B)$ of X, called the *discrepancy $\mathbb{R}$-b-divisor* of the log pair (X, B), whose trace on birational modifications $Y \to X$ is

$$\mathbf{A}(X, B)_Y = -B_Y.$$

A log pair (X, B) is said to have *Kawamata log terminal singularities* if $\mathrm{mult}_E(\mathbf{A}(X, B)) > -1$ for every geometric valuation E of X. We denote by $\mathrm{LCS}(X, B)$ the largest closed subset of X on whose complement (X, B) has Kawamata log terminal singularities.

For a log pair (X, B), the $\mathcal{O}_X$-module $\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil)$ is coherent. Indeed, let $\mu\colon Y \to X$ be a resolution of singularities such that $B_Y = \sum_l b_l E_l$ and $\sum_l E_l$ is a simple normal crossings divisor. Then (see the proof of [5], Proposition 4.46)

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B) \rceil) = \mu_* \mathcal{O}_Y(\lceil -B_Y \rceil).$$

Also, $\mathrm{LCS}(X, B) = \mu(\mathrm{LCS}(Y, B_Y))$ and $\mathrm{LCS}(Y, B_Y) = \bigcup_{b_l \geq 1} E_l$.

1.D.   Functional algebras (V.V. Shokurov [5])

Let $\pi\colon X \to S$ be a contraction. A *functional algebra of X/S* is an $\mathbb{N}$-graded $\mathcal{O}_S$-subalgebra

$$\mathcal{L} \subset \bigoplus_{i=0}^{\infty} \pi_* k(X)$$

such that $\mathcal{L}_0 = \mathcal{O}_S$ and $\mathcal{L}_i$ is a coherent $\mathcal{O}_S$-module for every $i \geq 1$. Here, $k(X)$ is the constant sheaf of rational functions on X, and the multiplication in $\bigoplus_{i=0}^{\infty} \pi_* k(X)$ is induced by that of $k(X)$.

Denote by $\mathbb{N}(\mathcal{L})$ the semigroup of positive integers i with $\mathcal{L}_i \neq 0$. For $i \in \mathbb{N}(\mathcal{L})$, let $\mathbf{M}_i$ be the mobile b-divisor of $\mathcal{L}_i$ (see Lemma 1.2) and set

$\mathbf{D}_i = \mathbf{M}_i/i$. By construction, $\mathbf{M}_i$ is b-free$/S$ and $\mathbf{D}_i$ is a b-semiample$/S$ $\mathbb{Q}$-b-divisor. The sequences $(\mathbf{M}_i)_{i \in \mathbb{N}(\mathcal{L})}$ and $(\mathbf{D}_i)_{i \in \mathbb{N}(\mathcal{L})}$ are called the *mobile sequence* and *characteristic sequence* of $\mathcal{L}$. Since $\mathcal{L}$ is an algebra, the following properties hold:

(i) $\mathbf{M}_i + \mathbf{M}_j \leq \mathbf{M}_{i+j}$.

(ii) $\mathbf{M}_0 = 0$.

In particular, $\bar{\mathcal{L}} = \bigoplus_{i \in \{0\} \cup \mathbb{N}(\mathcal{L})} \pi_* \mathcal{O}_X(\mathbf{M}_i)$ is a functional algebra of X/S. The $\mathcal{O}_S$-algebra $\mathcal{L}$ is finitely generated if and only if $\bar{\mathcal{L}}$ is, and $\mathrm{Proj}_S(\bar{\mathcal{L}}) \to \mathrm{Proj}_S(\mathcal{L})$ is the normalization morphism in this case ([5], Example 4.8, Proposition 4.15).

The functional algebra $\mathcal{L}$ is called *bounded* if the supremum

$$\mathbf{D} = \sup_{i \in \mathbb{N}(\mathcal{L})} \mathbf{D}_i$$

is an $\mathbb{R}$-b-divisor of X, called the *characteristic limit* of $\mathcal{L}$. In this case, the convexity property (i) implies that the supremum is a limit

$$\mathbf{D} = \lim_{\mathbb{N}(\mathcal{L}) \ni i \to \infty} \mathbf{D}_i.$$

The $\mathcal{O}_S$-algebra $\mathcal{L}$ is finitely generated if and only if $\mathbf{D}_i = \mathbf{D}$ for some $i \in \mathbb{N}(\mathcal{L})$ ([5], Theorem 4.28).

Let $\mathbf{A} \in \mathrm{BDiv}(X)_{\mathbb{R}}$. The functional algebra $\mathcal{L}$ is *asymptotically* $\mathbf{A}$-*saturated* if there exists a positive integer I such that $\mathcal{L}_i \neq 0$ for $I | i$, and the following inclusion holds for $I | i, j$:

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A} + j \mathbf{D}_i \rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}_j).$$

We say that $\mathcal{L}$ is *asymptotically saturated with respect to a log pair* (X, B) if $\mathcal{L}$ is asymptotically $\mathbf{A}(X, B)$-saturated.

Example 1.5. Let $\pi \colon X \to S$ be a contraction.

(1) Let $\mathbf{D}$ be an $\mathbb{R}$-b-divisor of X. Then

$$\mathcal{R}_{X/S}(\mathbf{D}) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(i\mathbf{D})$$

is a bounded functional algebra of X/S.

(2) Let (X, B) be a log variety structure on X and let $D \in \mathrm{CDiv}(X)_{\mathbb{Q}}$. Then the normal bounded functional algebra of X/S

$$\mathcal{R}_{X/S}(D) = \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD)$$

is asymptotically $\mathbf{A}(X, B)$-saturated.

1.E.   Kawamata's nonvanishing

For a real number r, the *round off* $\langle r \rangle$ is defined as the unique integer lying in the unit interval $[r - 1/2, r + 1/2)$. The following properties are easy to check:

(1) $\lfloor r \rfloor \leq \langle r \rangle \leq \lceil r \rceil$.

(2) $\langle -r \rangle = -\langle r \rangle$ unless $\{r\} = \frac{1}{2}$.

(3) $\langle r \rangle = 0$ for $r \in (-\frac{1}{2}, \frac{1}{2}]$.

LEMMA 1.6.   [2] *Let $x \in \mathbb{R}^d$, let I be a positive integer and let $\epsilon > 0$. Then there exists a positive multiple m of I such that $mx_1 \leq \langle mx_1 \rangle$ and $\max_{i=1}^{d} |mx_i - \langle mx_i \rangle| < \epsilon$.*

For an $\mathbb{R}$-divisor D on a normal variety X, the *round off* $\langle D \rangle$ is defined componentwise. Also, the *absolute value* of D is defined by

$$\|D\| = \max\{|\mathrm{mult}_P(D)|; P \subset X, \mathrm{codim}(P, X) = 1\}.$$

THEOREM 1.7.   [2] *Let X be a nonsingular variety and let $\pi \colon X \to S$ be a proper morphism. Let D, B be $\mathbb{R}$-divisors on X, satisfying the following properties:*

(1) *D is nef/S.*

(2) *The $\mathbb{R}$-divisor $rD - (K + B)$ is nef/S and big/S, for some $r \in \mathbb{Q}$.*

(3) *$\mathrm{Supp}\{B\}$ is a simple normal crossings divisor and $\lfloor B \rfloor \leq 0$.*

Then there exists $t_0, \epsilon > 0$ such that $\pi_* \mathcal{O}_X(\lceil -B \rceil + \langle tD \rangle) \neq 0$ for every $t \in \mathbb{R}$ with $t \geq t_0$ and $\|\langle tD \rangle - tD\| < \epsilon$.

COROLLARY 1.8. Let (X, B) be a log pair with Kawamata log terminal singularities and let $\pi \colon X \to S$ be a proper morphism. Let $\mathbf{D}$ be a b-nef/S $\mathbb{R}$-b-divisor of X such that the $\mathbb{R}$-b-divisor $r\mathbf{D} - \overline{K + B}$ is b-nef/S and b-big/S for some $r \in \mathbb{R}$.

Then there exists $t_0, \epsilon > 0$ and a birational contraction $Y \to X$ such that $\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) + t\mathbf{D} \rceil) \neq 0$ for every $t \in \mathbb{R}$ with $t \geq t_0$ and $\|t\mathbf{D}_Y - \langle t\mathbf{D}_Y \rangle\| < \epsilon$.

PROOF. Let $Y \to X$ be a resolution of singularities such that $\mathbf{D} = \overline{\mathbf{D}_Y}$, and $\mathrm{Supp}(B_Y) \cup \mathrm{Supp}(\mathbf{D}_Y)$ is included in a simple normal crossings divisor $\sum_l E_l$ on Y. Let $\pi' \colon Y \to S$ be the induced proper morphism. In particular,

$$\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) + t\mathbf{D} \rceil) = \pi'_* \mathcal{O}_Y(\lceil -B_Y + t\mathbf{D}_Y \rceil) \text{ for } t \in \mathbb{R}.$$

By Theorem 1.7, there exist $t_0, \epsilon' > 0$ such that

$$\pi'_* \mathcal{O}_Y(\lceil -B_Y \rceil + \langle t\mathbf{D}_Y \rangle) \neq 0$$

for $t \geq t_0$ and $\|t\mathbf{D}_Y - \langle t\mathbf{D}_Y \rangle\| < \epsilon'$. Choose $\epsilon \in (0, \epsilon')$ and

$$\epsilon \leq \min_l (1 - \{\mathrm{mult}_{E_l}(B_Y)\}).$$

Then for every $t \in \mathbb{R}$ with $\|t\mathbf{D}_Y - \langle t\mathbf{D}_Y \rangle\| < \epsilon$, we have

$$\lceil -B_Y \rceil + \langle t\mathbf{D}_Y \rangle \leq \lceil -B_Y + t\mathbf{D}_Y \rceil.$$

Therefore $\pi'_* \mathcal{O}_Y(\lceil -B_Y + t\mathbf{D}_Y \rceil) \neq 0$ for $t \geq t_0$ and $\|t\mathbf{D}_Y - \langle t\mathbf{D}_Y \rangle\| < \epsilon$. $\square$

2. A Semiampleness Criterion

THEOREM 2.1. Let (X, B) be a log pair, let $\pi \colon X \to S$ be a proper morphism and let D be a nef/S $\mathbb{R}$-divisor on X, satisfying the following properties:

(i) $rD - (K + B)$ is nef/S and big/S, for some $r \in \mathbb{Q}$.

(ii) $\pi_* \mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\overline{D} \rceil) \subseteq \pi_* \mathcal{O}_X(jD)$ for $I | j$, where I is some positive integer.

(iii) *For every point $P \in \mathrm{LCS}(X, B)$, there exists a non-zero rational function $a \in k(X)^\times$ and there exists $q \in \mathbb{Q}$ such that the $\mathbb{R}$-divisor*

$$q(a) + D$$

is effective in a neighborhood of $\pi^{-1}(\pi(P))$ and it does not contain P in its support.

Then D is a semiample/S $\mathbb{Q}$-divisor.

PROOF.
Step 1.   There exists a positive multiple I_1 of I such that

$$\pi_*\mathcal{O}_X(mD) \neq 0 \text{ for } I_1|m.$$

Indeed, if (X, B) does not have Kawamata log terminal singularities, the claim follows from (iii). If (X, B) has Kawamata log terminal singularities, we infer by Corollary 1.8 and Lemma 1.6 that there exists a multiple I_1 of I such that

$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X, B) + I_1\overline{D}\rceil) \neq 0.$$

By (ii), we obtain $\pi_*\mathcal{O}_X(I_1 D) \neq 0$. The same holds for multiples of I_1.

Step 2.   Let $(\mathbf{D}_i)_i$ be the characteristic sequence of the functional algebra of X/S

$$\mathcal{R}_{X/S}(D) = \bigoplus_{i=0}^{\infty} \pi_*\mathcal{O}_X(iD).$$

By (iii), there exists a multiple I_2 of I_1 such that $\overline{D} - \mathbf{D}_m$ is zero above a neighborhood of $\mathrm{LCS}(X, B)$, for every $m \in I_2\mathbb{Z}_{>0}$. We replace D, r by I_2D, rI_2^{-1}, respectively, so that we can assume from now that $I = I_1 = I_2 = 1$. We will also shrink S to an affine neighborhood of some fixed point of S, without further notice.

Step 3.   Fix a positive integer m such that $\mathbf{D}_m \neq \overline{D}$. There exists a resolution $\mu\colon Y \to X$ with the following properties (cf. [1], Corollary 0-3-6):

(1) $\mathbf{D}_m = \overline{(\mathbf{D}_m)_Y}$. Denote $F = \mu^*D - \mathbf{D}_{m,Y}$.

(2) $\mathrm{Supp}(B_Y) \cup \mathrm{Supp}(F)$ is included in a simple normal crossings divisor $\sum_\alpha G_\alpha$ on Y, where $K_Y + B_Y = \mu^*(K + B)$.

(3) $r\mu^*D - K_Y - B_Y - \sum_\alpha \epsilon_\alpha G_\alpha$ is an ample/S $\mathbb{R}$-divisor, with $0 < \epsilon_\alpha \ll 1$.

Let $\pi' = \pi \circ \mu\colon Y \to S$ be the induced proper morphism. Define c to be the largest real number such that the log pair

$$(Y, B_Y + cF + \sum_\alpha \epsilon_\alpha G_\alpha)$$

has log canonical singularities above $X \setminus \mathrm{LCS}(X, B)$. It follows from above that F is supported over $X \setminus \mathrm{LCS}(X, B)$, hence

$$c = \min_\alpha \frac{1 - \mathrm{mult}_{G_\alpha}(B_Y) - \epsilon_\alpha}{\mathrm{mult}_{G_\alpha}(F)}.$$

By perturbing the coefficients ϵ_α if necessary, we may assume that the minimum is attained at a unique prime component of F, denoted by E. By definition, the $\mathbb{R}$-divisor

$$B' = B_Y + cF + \sum_\alpha \epsilon_\alpha G_\alpha - E.$$

does not contain E in its support, and $\lceil -B' \rceil$ is effective above $X \setminus \mathrm{LCS}(X, B)$. The identity

$$(r + c)\mu^*D - (K_Y + E + B') = r\mu^*D - K_Y - B_Y - \sum_\alpha \epsilon_\alpha G_\alpha + c\mathbf{D}_{m,Y}$$

implies that the $\mathbb{R}$-divisor $t\mu^*D - (K_Y + E + B')$ is ample/S for every $t \geq r + c$.

Let $\mu^*D = \sum_i d_i D_i$ be the decomposition of μ^*D into prime components. There exists a π'-very ample divisor L on Y such that $L + D_i$ is π'-very ample for every i. Choose *general* elements $D_i^1 \in |L + D_i|$ and $D_i^2 \in |L|$ and set $D' = \sum_i d_i(D_i^1 - D_i^2)$. The following properties hold:

(4) $\mu^*D \sim_{\mathbb{R}} D'$.

(5) $\langle t\mu^*D \rangle \sim \langle tD' \rangle$ for every $t \in \mathbb{R}$ with $\|\langle t\mu^*D \rangle - t\mu^*D\| < 1/2$.

(6) $\mathrm{Supp}(D') \cup \mathrm{Supp}(E) \cup \mathrm{Supp}(B')$ is a simple normal crossings divisor on Y.

There exists $\epsilon_1 > 0$ such that the $\mathbb{R}$-divisor

$$\langle tD' \rangle - K_Y - E - B'$$

is ample/S for $t \geq r + c$ and $\|\langle tD' \rangle - tD'\| < \epsilon_1$. We infer by Kawamata-Viehweg vanishing ([1], Theorem 1-2-3) that the restriction map

$$\pi'_* \mathcal{O}_Y(\lceil -B' \rceil + \langle tD' \rangle) \to \pi'_* \mathcal{O}_E(\lceil -B'|_E \rceil + \langle tD'|_E \rangle)$$

is surjective for $t \geq r + c$ and $\|\langle tD' \rangle - tD'\| < \epsilon_1$. On the other hand, the restrictions of $(E, B'|_E)$ and $D'|_E$ to the generic fiber of $E \to \pi'(E)$ satisfy the assumptions of Theorem 1.7. Indeed, (1) and (3) are clear, whereas (2) follows from the π'-ampleness of the $\mathbb{R}$-divisor $(r + c)D'|_E - (K_E + B'|_E)$. Therefore there exists $t_0 \geq r + c$ and $0 < \epsilon_2 < \epsilon_1$ such that

$$\pi'_* \mathcal{O}_E(\lceil -B'|_E \rceil + \langle tD'|_E \rangle) \neq 0$$

for every t with $t \geq t_0$ and $\|\langle tD' \rangle - tD'\| < \epsilon_2$. In particular,

$$E \nsubseteq \mathrm{Bsl}_{\pi'} |\lceil -B' \rceil + \langle tD' \rangle| \text{ for } t \geq t_0, \|\langle tD' \rangle - tD'\| < \epsilon_2.$$

Choose $0 < \epsilon_3 < \min(\epsilon_2, \frac{1}{2})$. We obtain

$$E \nsubseteq \mathrm{Bsl}_{\pi'} |\lceil -B' \rceil + \langle t\mu^* D \rangle| \text{ for } t \geq t_0, \|\langle t\mu^* D \rangle - t\mu^* D\| < \epsilon_3.$$

It is easy to see that there exists $\epsilon_4 < \epsilon_3$ such that

$$\lceil -B' \rceil + \langle t\mu^* D \rangle \leq \lceil -B' + t\mu^* D \rceil \text{ for } \|\langle t\mu^* D \rangle - t\mu^* D\| < \epsilon_4.$$

By Lemma 1.6, there exists a multiple j of m such that $j \geq t_0$, $\|j\mu^* D - \langle j\mu^* D \rangle\| < \epsilon_4$ and $\mathrm{mult}_E(j\mu^* D - \langle j\mu^* D \rangle) \leq 0$. In particular, there exists a rational function $a \in k(Y)^\times$ such that the divisor

$$(a) + \lceil -B' \rceil + \langle j\mu^* D \rangle$$

is effective and it does not contain E in its support. Since $\mathrm{mult}_E(B') = 0$, the latter property is equivalent to

$$\mathrm{mult}_E((a) + \langle j\mu^* D \rangle) = 0.$$

On the other hand, the following inclusions hold

$$\pi'_* \mathcal{O}_Y(\lceil -B' \rceil + \langle j\mu^* D \rangle) \subseteq \pi'_* \mathcal{O}_Y(\lceil -B_Y + j\mu^* D \rceil) \subseteq \pi'_* \mathcal{O}_Y(j\mu^* D).$$

Therefore $(a) + j\mu^*D \geq 0$. In particular, $\mathrm{mult}_E(j\mu^*D - \langle j\mu^*D\rangle) \geq 0$. The opposite inclusion holds by assumption, hence $\mathrm{mult}_E(j\mu^*D) \in \mathbb{Z}$. Therefore the effective $\mathbb{R}$-divisor $(a) + j\mu^*D$ has multiplicity zero at E. In particular, $\mathrm{mult}_E(\mathbf{D}_j - \overline{D}) = 0$. By Lemma 1.3, there exists an open set V in X such that $V \cap \mu(E) \neq \emptyset$ and $(\overline{D} - \mathbf{D}_j)|_V = 0$.

Step 4. Let $m \geq 1$. By Lemma 1.3, there exists a closed subset $\mathrm{Bsl}_\pi |mD|$ of X such that for every prime b-divisor E of X, $\mathrm{mult}_E(\overline{D} - \mathbf{D}_j) = 0$ if and only if $c_X(E) \subset \mathrm{Bsl}_\pi |mD|$. For $m|m'$, we have $\mathbf{D}_m \leq \mathbf{D}_{m'}$, hence $\mathrm{Bsl}_\pi |mD| \supseteq \mathrm{Bsl}_\pi |m'D|$. From above, the following properties hold:

(1) $\mathrm{Bsl}_\pi |mD| \subsetneq X$ for $m \geq 1$.

(2) For every m, there exists $m|m'$ such that $\mathrm{Bsl}_\pi |mD| \supsetneq \mathrm{Bsl}_\pi |m'D|$.

By Noetherian induction, there exists a positive integer m such that $\mathrm{Bsl}_\pi |mD| = \emptyset$, that is $\mathbf{D}_m = \overline{D}$. Therefore D is rational, and the Cartier divisor mD is π-free. $\square$

We can use Theorem 2.1 to reprove Kawamata's result ([2], Theorem 1) on the finite generation of the log canonical ring modulo the existence of a Cutkosky-Kawamata-Moriwaki decomposition:

THEOREM 2.2. [2] *Let* (X, B) *be a log variety with Kawamata log terminal singularities such that* $K + B$ *is* $\mathbb{Q}$-*Cartier and let* $\pi\colon X \to S$ *be a proper morphism. Assume that* $\mathbf{P}$ *is a b-nef/S and b-big/S* $\mathbb{R}$-*b-divisor of* X *such that*

(1) $\mathbf{P} \leq \overline{K + B}$.

(2) $\mathcal{R}_{X/S}(\mathbf{P}) = \mathcal{R}_{X/S}(K + B)$.

Then $\mathbf{P}$ *is rational and b-semiample/S. In particular, the log canonical* $\mathcal{O}_S$-*algebra* $\mathcal{R}_{X/S}(K + B)$ *is finitely generated.*

PROOF.
Step 1. Let $\mu : Y \to X$ be a resolution of singularities such that $\mathbf{P} = \overline{\mathbf{P}_Y}$. Let $\mu^*(K+B) = K_Y + B_Y$ be the log pullback, and let $B_Y = B_Y^+ - B_Y^-$ be the decomposition of B_Y into its positive and negative part, respectively. The effective $\mathbb{Q}$-divisor B_Y^- is μ-exceptional, and

$$K_Y + B_Y^+ = \mu^*(K+B) + B_Y^-.$$

In particular, $\mathbf{P}_Y \leq K_Y + B_Y^+$ and $\mathcal{R}_{Y/S}(K_Y + B_Y^+) = \mathcal{R}_{X/S}(K + B)$. Therefore (Y, B_Y^+) is a log variety with Kawamata log terminal singularities, $\mathbf{P} \leq \overline{K_Y + B_Y^+}$, $\mathcal{R}_{Y/S}(\mathbf{P}) = \mathcal{R}_{Y/S}(K_Y + B_Y^+)$. Furthermore, $\mathbf{P} = \overline{\mathbf{P}_Y}$.

Step 2. By Step 1, we may assume that X is nonsingular, $\mathbf{P} = \overline{P}$ for some $\mathbb{R}$-divisor P on X, and if we denote $N = K + B - P$, then $\mathrm{Supp}(B) \cup \mathrm{Supp}(N) \cup \mathrm{Supp}(P)$ is included in a simple normal crossings divisor on X. Define $B' = B - N$ and let I be the smallest positive integer such that $I(K + B)$ is an integral divisor. For $I|j$, the following inequalities hold:

$$\lceil jP - B' \rceil \leq \lceil jP + N \rceil \leq \lceil jP + jN \rceil = j(K + B).$$

Since $\pi_* \mathcal{O}_X(jP) = \pi_* \mathcal{O}_X(j(K + B))$, we obtain

$$\pi_* \mathcal{O}_X(\lceil jP - B' \rceil) \subseteq \pi_* \mathcal{O}_X(jP) \text{ for } I|j.$$

Furthermore, $P = K + B'$, hence $2P - (K + B') = P$ is nef/S and big/S. By Theorem 2.1, P is rational and semiample/S. $\square$

3. Finite Generation of Some Algebras

PROPOSITION 3.1. *Let $\pi: X \to S$ be a proper surjective morphism and let D be a nef/S and big/S $\mathbb{R}$-divisor on X. Then:*

(1) *The characteristic limit of the $\mathcal{O}_S$-algebra $\mathcal{R}_{X/S}(D)$ is $\overline{D}$.*

(2) *The $\mathcal{O}_S$-algebra $\mathcal{R}_{X/S}(D)$ is finitely generated if and only if D is rational and semiample/S.*

(3) *Let (X, B) be a log pair structure on X, and let $\mathbf{D}_{\bullet}$ be the characteristic sequence of $\mathcal{R}_{X/S}(D)$. Then for every $j \geq 1$, there exists a positive integer $I(j)$ such that the natural inclusion of sheaves*

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\mathbf{D}_i \rceil) \subseteq \mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\overline{D} \rceil)$$

is an equality for $I(j)|i$.

PROOF. If $\mu: Y \to X$ is a birational contraction, it is clear that we may replace X and D by Y and $\mu^* D$, respectively. In particular, we may assume that X is nonsingular and π is projective. We may also assume that S is affine.

We claim that there exists a sequence of ample/S $\mathbb{Q}$-divisors Q_k, supported by some fixed reduced divisor on X, such that $Q_k \leq D$ and $\lim_{k \to \infty} Q_k = D$. Indeed, by Kodaira's Lemma, there exists an effective $\mathbb{R}$-divisor E on X such that $D - E$ is an ample/S $\mathbb{Q}$-divisor. Since D is nef/S, it follows that $D - \frac{1}{k}E$ is an ample/S $\mathbb{R}$-divisor for every $k \geq 1$. Since ampleness is an open condition and X is $\mathbb{Q}$-factorial, we may find for each k an ample/S $\mathbb{Q}$-divisor D_k, supported by $\mathrm{Supp}(D) \cup \mathrm{Supp}(E)$, such that $Q_k \leq D - \frac{1}{k}E$ and $\|D - \frac{1}{k}E - Q_k\| \leq \frac{1}{k}$.

(1) Consider the sequence $(Q_k)_{k \geq 1}$ constructed above. For each $k \geq 1$, there exists a positive integer m_k such that $m_k Q_k$ is free/S. Since $m_k Q_k \leq m_k D$, we obtain $\overline{m_k Q_k} \leq \mathbf{M}_{m_k}$, that is $\overline{Q_k} \leq \mathbf{D}_{m_k}$. In particular,

$$\overline{Q_k} \leq \mathbf{D}.$$

Since $\lim_{k \to \infty} Q_k = D$, we obtain

$$\overline{D} = \lim_{k \to \infty} \overline{Q_k} \leq \mathbf{D}.$$

The opposite inequality always holds, hence $\mathbf{D} = \overline{D}$.

(2) If $\mathcal{R}_{X/S}(D)$ is finitely generated, then the characteristic limit $\mathbf{D}$ is rational and semiample/S. By (1), D is rational and semiample/S. The converse is clear.

(3) If $\mu \colon Y \to X$ is a birational contraction, let $\mu^*(K + B) = K_Y + B_Y$. Then $\mathbf{A}(X, B) = \mathbf{A}(Y, B_Y)$ and we may replace (X, B) by (Y, B_Y). In particular, using [1], Corollary 0-3-6, we may assume that X is nonsingular and contains a simple normal crossings divisor supporting B, D and Q_k, for every $k \geq 1$.

Fix $j \geq 1$. Since $jQ_k \leq jD$ and $\lim_{k \to \infty} jQ_k = jD$, there exists $k = k(j)$ such that

$$\lceil -B + jQ_k \rceil = \lceil -B + jD \rceil.$$

We have $\overline{Q_k} \leq \mathbf{D}_{m_k}$, hence for $m_k | i$ the following inclusion holds:

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\overline{Q_k} \rceil) \subseteq \mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\mathbf{D}_i \rceil) \subseteq \mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\overline{D} \rceil)$$

On the other hand, by the simple normal crossings assumption, we have

$$\mathcal{O}_X(\lceil \mathbf{A}(X, B) + j\overline{Q_k} \rceil) = \mathcal{O}_X(\lceil -B + jQ_k \rceil)$$

and
$$\mathcal{O}_X(\lceil \mathbf{A}(X,B) + j\overline{D} \rceil) = \mathcal{O}_X(\lceil -B + jD \rceil).$$
The claim holds if we set $I(j) = m_{k(j)}$. $\square$

THEOREM 3.2. *Let (X,B) be a log pair with Kawamata log terminal singularities and let $\pi\colon X \to S$ be a proper morphism. Assume that D is a nef/S and big/S $\mathbb{R}$-divisor on X such that*

(1) *$\mathcal{R}_{X/S}(D)$ is $\mathbf{A}(X,B)$-asymptotically saturated.*

(2) *$rD - (K+B)$ is nef/S, for some $r \in \mathbb{Q}$.*

Then $\mathcal{R}_{X/S}(D)$ is a finitely generated $\mathcal{O}_S$-algebra.

PROOF. By Proposition 3.1, (1) is equivalent to the following property:
$$\pi_*\mathcal{O}_X(\lceil \mathbf{A}(X,B) + j\overline{D} \rceil) \subseteq \pi_*\mathcal{O}_X(jD) \text{ for } I|j.$$

By Theorem 2.1, D is rational and semiample/S. Therefore the algebra $\mathcal{R}_{X/S}(D)$ is finitely generated. $\square$

THEOREM 3.3. *Let (X,B) be a log pair and let $\pi\colon X \to S$ be a contraction. Let $\mathcal{L}$ be a bounded functional algebra of X/S, with characteristic sequence $(\mathbf{D}_i)_{i\in\mathbb{N}(\mathcal{L})}$ and characteristic limit $\mathbf{D}$. Assume that the following properties hold:*

(1) *$\mathcal{L}$ is asymptotically $\mathbf{A}(X,B)$-saturated.*

(2) *$\mathbf{D}_i = \overline{\mathbf{D}_{i,X}}$ for every $i \in \mathbb{N}(\mathcal{L})$.*

(3) *$r\mathbf{D}_X - (K+B)$ is nef/S and big/S, for some $r \in \mathbb{Q}$.*

(4) *There exists $i \in \mathbb{N}(\mathcal{L})$ and an open neighborhood U of $\mathrm{LCS}(X,B)$ such that $\mathbf{D}_i|_U = \mathbf{D}|_U$.*

Then $\mathcal{L}$ is a finitely generated $\mathcal{O}_S$-algebra.

PROOF. If $\mu\colon Y \to X$ is a birational contraction, let $\mu^*(K+B) = K_Y + B_Y$ be the induced crepant log pair structure on Y. We have $\mathbf{A}(X,B) = \mathbf{A}(Y,B_Y)$. By (2) and (3), we have $\mathbf{D} = \overline{\mathbf{D}_X}$. Therefore the asumptions are

preserved if we replace (X, B) by (Y, B_Y). In particular, we may assume that X is nonsingular and admits a normal crossings divisor supporting B and every $\mathbf{D}_{i,X}$.

By (2), we have $\mathbf{D}_i = \overline{\mathbf{D}_{i,X}}$ for every i. In particular, every $\mathbb{Q}$-divisor $\mathbf{D}_{i,X}$ is nef/S and their limit $D = \lim_{i\to\infty} \mathbf{D}_{i,X}$ is a nef/S $\mathbb{R}$-divisor. By the normal crossings assumption and (2), asymptotic saturation is equivalent to the following inclusions

$$\pi_* \mathcal{O}_X(\lceil -B + j\mathbf{D}_{i,X}\rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}_j) \text{ for } I|i, j.$$

For each j, we have $\lceil -B + j\mathbf{D}_{i,X}\rceil = \lceil -B + jD\rceil$ for i sufficiently large and divisible. Therefore

$$\pi_* \mathcal{O}_X(\lceil -B + jD\rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}_j) \subseteq \pi_* \mathcal{O}_X(jD) \text{ for } I|j.$$

Since $D - \mathbf{D}_{i,X}$ is effective and zero on the open neighborhood U of $\mathrm{LCS}(X, B)$, and $i\mathbf{D}_{i,X}$ is free/S, the $\mathbb{R}$-divisor iD is free/S on U. Furthermore, $rD - (K + B)$ is nef/S and big/S. Theorem 2.1 implies that D is a semiample/S $\mathbb{Q}$-divisor.

Let j be a multiple of I such that jD is free/S. There exists a contraction $\Phi\colon X \to T$, defined over S, and an ample/S Cartier divisor A on T such that $jD \sim \Phi^*(A)$.

Set $C = \lceil -B^{\geq 1}\rceil \leq 0$, so that $\mathrm{LCS}(X, B) = \mathrm{Supp}(C)$. By assumption, $\mathbf{D}_j = \mathbf{D}$ over the neighborhood U of $\mathrm{Supp}(C)$. By Lemma 1.4, there exist an open set $V_j \supset \Phi(\mathrm{Supp}(C))$ such that

$$\mathbf{D}_j|_{\Phi^{-1}(V_j)} = \mathbf{D}|_{\Phi^{-1}(V_j)}.$$

If $\Phi(\mathrm{Supp}(C)) = T$, then $V_j = T$ and we are done. Assume now that $\Phi(\mathrm{Supp}(C)) \subsetneq T$.

Since A is ample/S, we may replace j by a multiple, so that the subsheaf

$$\Phi_* \mathcal{O}_Y(C) \otimes \mathcal{O}_T(jA) \subset \mathcal{O}_T(jA)$$

is relatively generated by global sections. In particular, the divisor $\lceil C + jD\rceil$ is free on $X \setminus \Phi^{-1}\Phi(\mathrm{Supp}(C))$. By asymptotic saturation, we have

$$\pi_* \mathcal{O}_X(\lceil C + jD\rceil) \subseteq \pi_* \mathcal{O}_X(\lceil -B + jD\rceil) \subseteq \pi_* \mathcal{O}_X(\mathbf{M}_j).$$

Therefore $\overline{jD} \leq j\mathbf{D}_j$ above $X \setminus \Phi^{-1}\Phi(\mathrm{Supp}(C))$. The opposite inclusion always holds, hence $\mathbf{D}_j = \mathbf{D}$ over $X \setminus \Phi^{-1}\Phi(\mathrm{Supp}(C))$. The two open sets $\Phi^{-1}(V_j)$ and $X \setminus \Phi^{-1}\Phi(\mathrm{Supp}(C))$ cover X, hence we obtain $\mathbf{D}_j = \mathbf{D}$.

Therefore the $\mathcal{O}_S$-algebra $\mathcal{L}$ is finitely generated. $\square$

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