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Displacement Exponents of Self-Repelling Walks and Self-Attracting Walks on the Pre-Sierpiński Gasket

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Abstract. We construct a family of self-repelling and self-attracting walks (stochastic chains) on the (infinite) pre-Sierpiński gasket. The family continuously interpolates the simple random walk and a self-avoiding walk. The asymptotic behavior of the walks is given in terms of the displacement exponent.

1. Introduction

In [3] Ben Hambly and the authors considered a family of self-repelling walks with fixed endpoints on the finite pre-Sierpiński gaskets. We proved the existence of the continuum limit, i.e., reducing the unit length to 0 (with suitable scaling of time parameter). The limit is a family of continuous self-repelling processes with specific fixed endpoints ('pinned processes'). We studied their sample path properties, such as Hölder continuity, short-time speed and a generalized law of the iterated logarithm.

In this paper, we consider the same family of walks on the finite pre-Sierpiński gaskets, which we recall in Section 2, but instead of taking continuum limit, we fix the unit length and extend the walks to the (infinite) pre-Sierpiński gasket, thus remove the pinning condition and construct (infinite length) stochastic chains, and study their properties.

In Section 3 (Theorem 5) we construct a family of stochastic chains on the pre-Sierpiński gasket consistent with the pinned self-repelling walks on the finite pre-Sierpiński gaskets studied in [3]. Our family of walks is parametrized by u which indicates the strength of self-repulsion. The walk corresponding to $u = 1$ is the standard simple random walk, and for $u = 0$ the corresponding walk is self-avoiding and of infinite length. The path measure for the self-avoiding ($u = 0$) case is rather complex, as is for all cases

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other than the simple random walk ($u \neq 1$). In particular, it is supported on walks without sharp turns for $u = 0$. The measure is natural, however, from the renormalization group point of view [4], in that it corresponds to the unique fixed point of the renormalization group recursion equation (10) associated with the self-avoiding walks on the pre-Sierpiński gasket which respect symmetries of the Sierpiński gasket. This family of walks on the finite pre-Sierpiński gaskets parametrized by u is introduced in [3] to interpolate the simple random walk and the measure on self-avoiding paths by the path measures corresponding to the fixed points of the (u dependent) renormalization group, which, hereafter we refer to as the fixed point theories. (We can of course consider more general class of walks by introducing more parameters, but to make clear the aim of this paper, we will stick to the family studied in [3].) We can also construct the walks for $u > 1$, which correspond to self-attracting walks.

It may also be worthwhile to note that the extension of pinned walks to the infinite pre-Sierpiński gasket is not trivial for the walks. The continuum limit continuous processes of [3] have exact self-similarities which can be used to obtain extentions to large scales. The walk (chain), on the other hand, has a finite unit, so that there is no exact self-similarity. It turns out, as we see in Section 3, that the fixed point condition of the renormalization group serves as a consistency condition in applying the extension theorems. In this sense, we may say that the renormalization group fixed point theory is an extended notion of a self-similar processes. Note also that since our family of walks lack Markov properties, constructions based on analytic approach cannot be applied in general.

In Section 4, we prove in Theorem 8 an asymptotic bahavior of the self-repelling and self-attracting walks, in terms of the (u dependent) displacement exponent γ. (The exponent is equal to the exponent for the 'mean-square displacement' in physics literatures, defined by $E[|w(n)|^2] \sim n^{2\gamma}$. Our proof implies that the exponent is the same for all the moments $E[|w(n)|^s] \sim n^{s\gamma}$, $s > 0$.)

Main tools for the proof are a reflection principle and an estimate on short and long paths. These tools have also been employed in [6], where we proved the existence of displacement exponent for a self-avoiding walk. We would like to emphasize that the reflection principle introduced in [6] is similar in spirit to the reflection principle used in Section 4 but is actually entirely different. In fact, by comparing the definition of reflection principles in [6] and that in this paper for the self-avoiding ($u = 0$) case, one should notice that they are absolutely different reflections. A main reason for the difference is that, in [6] we considered equal weights for self-avoiding walks with a fixed number of steps, hence in applying a reflection principle we only needed to compare the numbers of certain sets of walks and their reflections, whereas we here consider fixed point theories, whose weights are natural from renormalization point of view but complex from walks' point of view and also depends on u, so that a very delicate coupling type argument is necessary. On the other hand, since we are working on fixed point theories, the estimates on short and long paths are considerably easier than those in [6].

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2. Self-Repelling Walks on the Finite Pre-Sierpiński gasket

The pre-Sierpiński gasket is defined as follows. Let $O = (0,0)$, $a_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $b_0 = (1,0)$, $c_0 = (-1,0)$, $d_0 = (\frac{-1}{2}, \frac{\sqrt{3}}{2})$, and $a_N = 2^N a_0$, $b_N = 2^N b_0$, $c_N = 2^N c_0$, $d_N = 2^N d_0$, $N \in \mathbb{N}$. Let F'_0 be the set of all the points on the vertices and edges of $\triangle O a_0 b_0$. We define a sequence of sets F'_0, F'_1, F'_2, $\dots$, inductively by

$$F'_{N+1} = F'_N \cup (F'_N + a_N) \cup (F'_N + b_N), \qquad N \in \mathbb{Z}_+ = \{0,1,2,\dots\},$$

where $A + a = \{x + a \ : \ x \in A\}$ and $kA = \{kx \ : \ x \in A\}$. Let $F''_N = F'_N \cup (F'_N + c_N)$ and $F_0 = \bigcup_{N=1}^{\infty} F''_N$. We call F_0 the (infinite) pre-Sierpiński gasket. For $N \in \mathbb{Z}_+$, let $F_N = 2^N F_0$ and denote the set of vertices in F_N by G_N.

For $n \in \mathbb{Z}_+$ we call $w = (w(0), w(1), \cdots, w(n))$ an n-step path (or a path of length n), if

$$w(i) \in G_0,\ |w(i+1) - w(i)| = 1,\ \overline{w(i)w(i+1)} \in F_0,\ i = 0, 1, \cdots, n-1.$$

Similarly, we call $w = (w(0), w(1), w(2), \cdots)$ an infinite path (or a path of infinite length), if

$$w(i) \in G_0,\ |w(i+1) - w(i)| = 1,\ \overline{w(i)w(i+1)} \in F_0,\ i = 0, 1, 2, \cdots.$$

We denote the length of path w by $L(w)$.

For a path w and $A \subset F_0$, we define the hitting time $T_A(w)$ of A for w, by $T_A(w) = \min\{j \geqq 0 :\ w(j) \in A\}$. If the minimum does not exist, we put $T_A(w) = \infty$. For a path w on F_0 and $M \in \mathbb{Z}_+$, define $T_i^M(w)$, $i = 0, 1, 2, \cdots$, by induction as follows: $T_0^M(w) = T_{G_M}(w)$, and for $i \geqq 1$, let $T_i^M(w) = \min\{j > T_{i-1}^M(w) :\ w(j) \in G_M \setminus \{w(T_{i-1}^M(w))\}\}$, if the minimum exists, otherwise $T_i^M(w) = \infty$. $T_i^M(w)$ is the time when the path w hits a vertex of G_M for the $i+1$-th time (including the case $i = 0$), under the condition that if w hits the same element of G_M more than once in a row, we consider it as once.

We mainly consider walks starting at the origin. (Notion for general paths introduced so far are also used when we consider cutting and reflecting procedures of paths in the proofs of our results.) For each $n \in \mathbb{Z}_+$, denote a set of n-step paths on F_0 starting at the origin O by $W(n)$. Namely,

$$\begin{aligned} W(n) \ = \ & \{\ (w(0), w(1), \cdots, w(n)) :\ w(0) = O,\ w(i) \in G_0, \\ & |w(i+1) - w(i)| = 1, \\ & \overline{w(i)w(i+1)} \in F_0,\ i, i+1 \in \{0, 1, \cdots, n\}\ \}. \end{aligned}$$

For $w \in W(n)$, $L(w) = n$. Let $W^* = \displaystyle\bigcup_{n=1}^{\infty} W(n)$.

Fix $N \in \mathbb{Z}_+$ for the rest of this section. Let $A_N = \{a_N, b_N, c_N, d_N\}$, and define

$$\begin{aligned} W_{N,a} &= \{w \in W^* : L(w) = T_{A_N} = T_{\{a_N\}}\}, \\ W_{N,b} &= \{w \in W^* : L(w) = T_{A_N} = T_{\{b_N\}}\}, \\ W_{N,c} &= \{w \in W^* : L(w) = T_{A_N} = T_{\{c_N\}}\}, \\ W_{N,d} &= \{w \in W^* : L(w) = T_{A_N} = T_{\{d_N\}}\}, \end{aligned}$$

and

$$W_N = W_{N,a} \cup W_{N,b} \cup W_{N,c} \cup W_{N,d}.$$

For a path $w \in W_N$ and $M \in \mathbb{Z}_+$, define a 'decimation' map Q_M by setting $(Q_M w)(i) = w(T_i^M(w))$ for $i = 0, 1, 2, \ldots, j$, where j is the smallest integer such that $T_{j+1}^M(w) = \infty$. $Q_M w$ may be regarded as a path on F_M. If we write $(2^{-M} Q_M w)(i) = 2^{-M} w(T_i^M(w))$, then $2^{-M} Q_M w$ is a path on F_0 and $L(2^{-M} Q_M w) = j$. We will write $L(Q_M w) = L(2^{-M} Q_M w)$ for the length of decimated path (with a unit step normalized to be 1), and $T_i^N(Q_M w) = T_i^N(2^{-M} Q_M w)$ for the hitting times.

For a path $w \in W_N$, define the reversing number $N_K(w)$ and the returning number $M_K(w)$ for level $K \in \{0, 1, \cdots, N-1\}$ in the following manner. For $\ell = 1, \cdots, L(Q_{K+1} w)$, let

$$\begin{aligned} N_K(\ell)(w) = \sharp\{ & i \in \mathbb{Z}_+ :\ T_{\ell-1}^{K+1}(Q_K w) < i < T_\ell^{K+1}(Q_K w)\ : \\ & \overrightarrow{(Q_K w)(i-1)(Q_K w)(i)} \cdot \overrightarrow{(Q_K w)(i)(Q_K w)(i+1)} < 0, \\ & (Q_K w)(i) \neq (Q_K w)(T_{\ell-1}^{K+1}(Q_K w))\ \}, \end{aligned} \tag{1}$$

where $\vec{a} \cdot \vec{b}$ denotes the inner product of $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^2$, and

$$\begin{aligned} M_K(\ell)(w) &= \sharp\{ i \in \mathbb{Z}_+ :\ T_{\ell-1}^{K+1}(Q_K w) < i < T_\ell^{K+1}(Q_K w)\ : \\ &\qquad (Q_K w)(i) = (Q_K w)(T_{\ell-1}^{K+1}(Q_K w))\}, \\ N_K(w) &= \sum_{\ell=1}^{L(Q_{K+1} w)} N_K(\ell)(w), \\ M_K(w) &= \sum_{\ell=1}^{L(Q_{K+1} w)} M_K(\ell)(w). \end{aligned} \tag{2}$$

Thus $N_K(w)$ counts the number of times the path $Q_K w$ makes U-turns or sharp-angle turns at vertices in $G_K \setminus G_{K+1}$, and $M_K(w)$ counts the number of times $Q_K w$ revisits a vertex in G_{K+1}. It is these types of steps that we will suppress or enhance in our path measures.

For $p, q \in \{a, b, c, d\}$, we define bijections $R_{p,q} : W_{N,p} \to W_{N,q}$ as follows. $R_{a,d}$, $R_{d,a}$, $R_{b,c}$, and $R_{c,b}$ are defined as the reflection with regard to y-axis. Consider the reflection of the parts of path within $\triangle O a_N b_N$ with regard to the line $y = \dfrac{1}{\sqrt{3}} x$, and that of the parts within $\triangle O c_N d_N$ with regard to the line $y = -\frac{1}{\sqrt{3}}x$. This defines $R_{a,b}$, $R_{b,a}$, $R_{c,d}$, and $R_{d,c}$. Then $R_{a,c} = R_{b,c} \circ R_{a,b}$ defines $R_{a,c}$, and other cases are defined in a similar way. Under these bijections, $N_K(w)$, $M_K(w)$ and $L(w)$ remain invariant.

Let $x > 0$ and $u \geqq 0$. For $w \in W_N$, define

$$f_N(u,w) = f_N(w) = \prod_{K=0}^{N-1} u^{N_K(w)+M_K(w)}, \tag{3}$$

and

$$\Phi_N(x,u) = \sum_{w \in W_{N,a}} f_N(w) x^{L(w)}. \tag{4}$$

Owing to the one-to-one correspondence shown above, if the summation in (4) is taken over $W_{N,b}$, $W_{N,c}$ or $W_{N,d}$ instead of $W_{N,a}$, it gives the same value. Thus, in the rest of this section, we work on $W_{N,a}$. We will often write $\Phi(x,u)$ instead of $\Phi_1(x,u)$. For the explicit form of Φ, see [3]. We do not use it here.

If $w \in W_{N,a}$ and $M \leqq N$, then $2^{-M} Q_M w \in W_{N-M,a}$. For $w \in W_{N+1,a}$, put $w' = 2^{-N} Q_N w \in W_{1,a}$, and for each $j = 1, 2, \cdots, L(w')$, consider a path segment w_j of the path w

$$w_j = (w(T_{j-1}^N(w)), w(T_{j-1}^N(w)+1), w(T_{j-1}^N(w)+2), \cdots, w(T_j^N(w))). \tag{5}$$

This path segment is the 'fine structure' of the j-th step of the decimated path $Q_N w$. (a) It is a path on G_0 starting from $w(T_{j-1}^N(w)) \in G_N$ and stopping at $w(T_j^N(w))$, a neighboring point of $w(T_{j-1}^N(w))$ in G_N, and (b) it has no common point with G_N other than the starting point $w(T_{j-1}^N(w))$ before it reaches its endpoint. A path with properties (a) and (b) can be identified, via reflection, with a path $\tilde{w}_j \in W_{N,a}$, in such a way that $w(T_{j-1}^N(w))$ and $w(T_j^N(w))$ correspond to O and a_N, respectively. Conversely, given arbitrarily $w' \in W_{1,a}$, any $\tilde{w} \in W_{N,a}$ can be the j-th path segment (5) of some $w \in W_{N+1,a}$ such that $2^{-N} Q_N w = w'$. Thus there is a one-to-one correspondence $w_j \mapsto \tilde{w}_j \in W_{N,a}$. With this correspondence, there is a natural one-to-one mapping

$$W_{N+1,a} \ni w \ \mapsto \ (w', \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{L(w')}) \in W_{1,a} \times W_{N,a} \times \cdots \times W_{N,a}. \tag{6}$$

Also we have

$$L(\tilde{w}_j) = T_j^N(w) - T_{j-1}^N(w), \tag{7}$$

and

$$L(w) = \sum_{j=1}^{L(2^{-N}Q_N w)} (T_j^N(w) - T_{j-1}^N(w)) = \sum_{j=1}^{L(w')} L(\tilde{w}_j). \tag{8}$$

For any $w \in W_{N+1,a}$, by considering the path decomposition $(w', \tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{L(w')})$ determined by the correspondence (6), we have from (3)

$$f_{N+1}(w) = f_1(w') \prod_{j=1}^{L(w')} f_N(\tilde{w}_j). \tag{9}$$

Combining (3), (4), (8) and (9), we have the recursion relation of Φ_N,

$$\Phi_{N+1}(x,u) = \Phi(\Phi_N(x,u),u) = \Phi \circ \cdots \circ \Phi(x,u). \tag{10}$$

This implies that for any $M < N$,

$$\Phi_N(x,u) = \Phi_{N-M}(\Phi_M(x,u),u). \tag{11}$$

Let r_u be the radius of convergence for $\Phi(x,u)$ as a power series in x.

Proposition 1.

(1) For each $u \geqq 0$, $r_u > 0$ and there is a unique fixed point x_u of the mapping $\Phi(\cdot,u) : (0,r_u) \to (0,\infty)$, that is, $\Phi(x_u,u) = x_u, x_u > 0$. As a function in u, x_u is continuous and strictly decreasing on $[0,1]$.

(2) Let $\lambda_u = \dfrac{\partial \Phi}{\partial x}(x_u,u)$. Then λ_u is continuous in u and $\lambda_u > 2$.

Proof. The case of $0 \leqq u \leqq 1$ corresponds to Proposition 2.3 in [3]. For $u > 1$, it is sufficient to show $r_u > 0$. The rest follows just as in the case of $0 \leqq u \leqq 1$. Note that from the definitions of $N_0(w)$ and $M_0(w)$, we have $L(w) \geqq N_0(w) + M_0(w)$. Thus if $u > 1$,

$$\Phi(x,u) = \sum_{w \in W_{1,a}} u^{N_0(w)+M_0(w)} x^{L(w)} \leqq \sum_{w \in W_{1,a}} (ux)^{L(w)} = \Phi(ux,1).$$

Since we already know that $r_1 > 0$, we have $r_u \geqq r_1/u > 0$. □

(10) implies that $\Phi_N(x_u, u) = x_u$ for all $N \in \mathbb{N}$. In the two extreme cases, we know that $x_0 = \dfrac{\sqrt{5}-1}{2}$, $\lambda_0 = \dfrac{7-\sqrt{5}}{2}$ (see [4, 6]), and $x_1 = \dfrac{1}{4}$, $\lambda_1 = 5$ (see [1, 7]).

We next define a probability measure P_N^u on $W_{N,a}$ by assigning to each $w \in W_{N,a}$,

$$(12) \quad P_N^u[\{w\}] = (\prod_{K=0}^{N-1} u^{N_K(w)+M_K(w)})\, x_u^{L(w)}/\Phi_N(x_u, u) = f_N(w) x_u^{L(w)-1}$$

P_N^1 corresponds to the simple random walk on F_N'' conditioned that $T_{A_N} = T_{\{a_N\}}$. Under P_N^0, only self-avoiding paths survive. (To be precise, the measure corresponds to the fixed point of the renormalization group [4], hence the measure is supported on the self-avoiding paths with no sharp turns.) Let us denote by E_N^u the expectation with regard to P_N^u.

We cite the following Proposition 2 – Proposition 4 from [3]. They hold true also for $u > 1$.

For $M \leqq N$, let $Q_M P_N^u$ be the image measure of P_N^u induced by $2^{-M} Q_M$. Combining (9), (11) and (12), we have

PROPOSITION 2. *If $w \in W_{N,a}$ and $M \leqq N$, then $2^{-M} Q_M w \in W_{N-M,a}$ and $Q_M P_N^u = P_{N-M}^u$.*

For $M \in \mathbb{Z}_+$ and $k \in \mathbb{N}$, put $S_k^M = T_k^M - T_{k-1}^M$.

PROPOSITION 3. *Assume $N \geqq M$ and $k \in \mathbb{N}$.*

(a) Let $w \in W_{N,a}$. Under the conditional probability $P_N^u[\ \cdot \mid S_k^M(w) < \infty\ ]$, the random variables $S_i^M(w)$ for $i = 1, \dots, k$ are i.i.d. and they are jointly independent of $Q_M w$.

(b) The law of S_1^M under P_N^u is equal to that of S_1^M under P_M^u. $E_M^u[S_1^M] = \lambda_u^M$ and the Laplace transform of S_1^M is given by

$$g_M^u(t) = E_M^u[e^{-tS_1^M}] = \frac{1}{x_u} \Phi_M(x_u e^{-t}, u), \quad t \geqq 0.$$

PROPOSITION 4. *The law of* $\lambda_u^{-N} S_1^N$ *under* P_N^u *converges weakly as* $N \to \infty$ *to that of a random variable* S^**, with properties* $P_u[\, S^* > 0 \,] = 1$*,* $E^u[\, S^* \,] = 1$*, and its Laplace transform* $g_u(t) = E^u[\, \exp(-tS^*) \,]$ *being the unique solution to*

$$g_u(\lambda_u t) = \frac{1}{x_u} \Phi(x_u g_u(t), u), \quad g_u'(0) = 1.$$

3. Existence of Stochastic Chain Consistent with the Renormalization Group

Let $N \in \mathbb{N}$. The probability measure P_N^u in the preceeding section is defined on the set $W_{N,a}$, which is a set of paths on F_N'' with fixed endpoints O and a_N. In deducing displacement exponents, we need to consider probability measures on sets of paths with fixed length (steps) n for all $n \in \mathbb{N}$. We prove the existence of a probability measure P^u on the set of paths of infinite length, for which the probability of the paths up to the first hitting time of A_N coincides with P_N^u (The precise statement is given in Corollary 7).

Let $P_{N,a}^u$ be a probability measure on W_N such that $P_{N,a}^u[A] = P_N^u[A]$ for any $A \subset W_{N,a}$, and define probability measures $P_{N,b}^u$, $P_{N,c}^u$, $P_{N,d}^u$, on W_N supported on $W_{N,b}$, $W_{N,c}$, $W_{N,d}$, by the same formula as (12), with a_N replaced by b_N, c_N, d_N, respectively. Define

$$\begin{aligned} W(\infty) \;\; = \;\; & \{(w(0), w(1), w(2), \cdots) : \; w(0) = O, \; w(i) \in G_0, \\ & |w(i+1) - w(i)| = 1, \\ & \overline{w(i)w(i+1)} \in F_0, \; i \in \mathbb{Z}_+\}, \end{aligned}$$

and let $\mathcal{F}$ be the σ-algebra on $W(\infty)$ generated by cylinder sets

$$\begin{aligned} C_n(w) = \{ & w' = (w'(0), w'(1), w'(2), \cdots) \in W(\infty) : \\ & w'(j) = w(j)\,, \; j = 0, 1, 2, \cdots, n\}, \\ & w \in W(n), \; n \in \mathbb{N}, \end{aligned} \tag{13}$$

consisting of infinite-length paths whose first n steps are identical to w.

THEOREM 5. *There exists a probability measure* P^u *on* $(W(\infty), \mathcal{F})$ *satisfying the following: For each* $n \in \mathbb{N}$ *and* $w = (w(0), w(1), w(2), \cdots,$

$w(n)) \in W(n)$, it holds that

$$
\begin{aligned}
&P^u[\{w' \in W(\infty)\ :\ w'(j) = w(j),\ j = 0,1,2,\cdots,n\}] \\
(14)\quad &= \frac{1}{4} \sum_{p \in \{a,b,c,d\}} P^u_{N,p}[\{w' \in W_{N,p} :\ w'(j) = w(j),\ j = 0,1,2,\cdots,n\}],
\end{aligned}
$$

for any integer N satisfying

$$
|w(j)| < 2^N,\ j = 0,1,2,\cdots,n-1, \tag{15}
$$

where $|\cdot|$ denotes the Euclidean metric.

We remark that 2^N or more steps are required for a path starting from O to hit $A_N = \{a_N, b_N, c_N, d_N\}$, hence the condition (15) holds if $2^N \geqq n$. Also, for each $j \in \mathbb{Z}_+$, $X(j,\cdot) :\ W(\infty) \to G_0$ defined by $X(j,w) = w(j)$ is a G_0-valued stochastic variable on $(W(\infty), \mathcal{F}, P^u)$.

To prove Theorem 5, we first note the following. For a path in W_{N+1}, the first hit of $G_N \setminus \{O\}$ occurs at one of A_N. By restricting the original path to $[0, T_{A_N}]$, we have a correspondence $W_{N+1} \to W_N$.

In Section 2, we introduced natural bijections $R_{p,q} : W_{N,p} \to W_{N,q}$, $p,q \in \{a,b,c,d\}$ which maps $w \in W_{N,p}$ to $R_{p,q}w \in W_{N,q}$ in such a way that its shape (modulo partial reflection) does not change, and in particular,

$$
f_N(w) = f_N(R_{p,q}(w)). \tag{16}
$$

For simplicity of notation, we may write $R_{p,p}$ for an identity map, and, in the proof of Theorem 5, we will fix u and write $P_{N,q}$ for $P^u_{N,q}$.

PROPOSITION 6. *Let N be a positive integer and let $p \in \{a,b,c,d\}$. Consider a path*

$$
\hat{w} = (\hat{w}(0), \hat{w}(1), \hat{w}(2), \cdots, \hat{w}(L(\hat{w}))) \in W_{N,p}\,.
$$

Then for any integer N' satisfying $N' > N$,

$$
\begin{aligned}
&\sum_{q \in \{a,b,c,d\}} P_{N',q}[\{w \in W_{N',q} :\ w(j) = \hat{w}(j)\,,\ j = 0,1,2,\cdots,L(\hat{w})\}] \\
&\qquad = P_{N,p}[\{\hat{w}\}].
\end{aligned}
$$

PROOF. We prove the case $N' = N+1$: The general case follows by induction in N'. We also assume $\hat{w} \in W_{N,a}$, since other cases are similar.

We decompose $w \in W_{N+1,q}$ into $(w', \tilde{w}_1, \cdots, \tilde{w}_{L(w')}) \in W_{1,q} \times W_{N,a} \times \cdots \times W_{N,a}$, in the same way as in (6), where $w' = 2^{-N} Q_N w$ and $\tilde{w}_j$ is the j-th path segment identified, via appropriate reflection, with a path in $W_{N,a}$. Using (12), (8), (9) and (16), for the first equality, the condition that $w(j) = \hat{w}(j)\,,\ j = 0,1,2,\cdots,L(\hat{w})$ for the second, (4) for the third, and finally, $\Phi_N(x_u, u) = x_u$ and $\hat{w}(L(\hat{w})) = a_N$, we have,

$$
\begin{aligned}
&P_{N+1,q}[\{w \in W_{N+1,q} :\ w(j) = \hat{w}(j)\,,\ j = 0,1,2,\cdots,L(\hat{w})\}] \\
&= \sum_{\substack{w \in W_{N+1,q};\\ w(j)=\hat{w}(j)\,,\ j=0,1,2,\cdots,L(\hat{w})}} f_1(w') x_u^{-1} \prod_{j=1}^{L(w')} f_N(\tilde{w}_j) \prod_{j=1}^{L(w')} x_u^{L(\tilde{w}_j)} \\
&= f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{\substack{w' \in W_{1,q};\\ w'(1)=\hat{w}(L(\hat{w}))}} f_1(w') \prod_{j=2}^{L(w')} \sum_{\tilde{w}_j \in W_{N,a}} f_N(\tilde{w}_j) x_u^{L(\tilde{w}_j)} \\
&= f_N(\hat{w}) x_u^{L(\hat{w})-1} \sum_{\substack{w' \in W_{1,q};\\ w'(1)=\hat{w}(L(\hat{w}))}} f_1(w') \Phi_N(x_u, u)^{L(w')-1} \\
&= f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{\substack{w' \in W_{1,q};\\ w'(1)=a_0}} f_1(w') x_u^{L(w')-1}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\sum_{q \in \{a,b,c,d\}} P_{N+1,q}[\{w \in W_{N+1,q} :\ w(j) = \hat{w}(j)\,,\ j = 0,1,2,\cdots,L(\hat{w})\}] \\
&= f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{q \in \{a,b,c,d\}} \sum_{\substack{w' \in W_{1,q};\\ w'(1)=a_0}} f_1(w') x_u^{L(w')-1}.
\end{aligned}
$$

According to the definition of $R_{p,q}$, $w' \in W_{1,q}$ is mapped by $R_{q,a}$ to $R_{q,a}(w') \in W_{1,a}$, while if $w'(1) = a_0$ then this point is mapped to $R_{q,a}(w')(1) = q$. On the other hand, $L(w') = L(R_{q,a}(w'))$, and (16) implies $f_N(w') = f_N(R_{q,a}(w'))$. Note also that the first step $w'(1)$ is a point in A_0. Therefore

$$
f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{q \in \{a,b,c,d\}} \sum_{\substack{w' \in W_{1,q};\\ w'(1)=a_0}} f_1(w') x_u^{L(w')-1}
$$

$$= f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{q\in\{a,b,c,d\}} \sum_{\substack{w'\in W_{1,a};\\ w'(1)=q}} f_1(w') x_u^{L(w')-1}$$

$$= f_N(\hat{w})\, x_u^{L(\hat{w})-1} \sum_{w'\in W_{1,a}} f_1(w') x_u^{L(w')-1} = P_{N,p}[\{\hat{w}\}]. \ \square$$

PROOF OF THEOREM 5. Let $n \in \mathbb{N}$. Take an n step path $w \in W(n)$ and a cylinder set $C_n(w)$ defined in (13). Take N satisfying (15), and put

$$\tilde{P}_n[C_n(w)] = \frac{1}{4} \sum_{p\in\{a,b,c,d\}} P_{N,p}[\{w' \in W_{N,p} : w'(j) = w(j)\,,\ j = 0,1,2,\cdots,n\}]. \tag{17}$$

We first prove that the right-hand side is independent of N. Let $N' > N$. For any $w' \in W_{N'}$, it holds that $T_1^N(w') < T_1^{N'}(w')$. By restricting w' up to $T_1^N(w')$, we have a path in W_N. Since (15) implies $n \leqq T_1^N(w')$, we can classify the paths in $\{w' \in W_{N',p} :\ w'(j) = w(j)\,,\ j = 0,1,2,\cdots,n\}$ by the behavior up to $T_1^N(w')$ and we have

$$\begin{aligned}
&\frac{1}{4} \sum_{q\in\{a,b,c,d\}} P_{N',q}[\{w' \in W_{N',q} :\ w'(j) = w(j)\,,\ j = 0,1,2,\cdots,n\}] \\
&= \frac{1}{4} \sum_{p\in\{a,b,c,d\}} \sum_{\substack{\hat{w}\in W_{N,p};\\ \hat{w}(j)=w(j)\,,\ j=0,1,2,\cdots,n}} \\
&\times \sum_{q\in\{a,b,c,d\}} P_{N',q}[\{w' \in W_{N',q} :\ w'(j) = \hat{w}(j)\,,\ j = 0,1,2,\cdots,L(\hat{w})\}],
\end{aligned}$$

which, by Proposition 6, is equal to

$$\frac{1}{4} \sum_{p\in\{a,b,c,d\}} P_{N,p}[\{\hat{w} \in W_{N,p} :\ \hat{w}(j) = w(j)\,,\ j = 0,1,2,\cdots,n\}],$$

which proves that the right-hand side of (17) gives the same value for all N satisfying (15).

We next extend $\tilde{P}_n$, defined in (17) on the cylinder sets $C_n(w)$, to a probability measure. Let $\mathcal{F}_n$ be the σ-algebra on $W(\infty)$ generated by the cylinder sets $C_n(w)$ i.e., a family of sets which are determined by the first n steps of the paths in $W(\infty)$. We extend $\tilde{P}_n$ to $\mathcal{F}_n$ by

$$\tilde{P}_n[V] = \sum_{w \in W(n);\ C_n(w) \subset V} \tilde{P}_n[C_n(w)], \quad V \in \mathcal{F}_n\,. \tag{18}$$

To prove that $\tilde{P}_n$ is a probability measure, it is sufficient to prove $\tilde{P}_n[W(\infty)] = 1$. Let N be a positive integer satisfying $2^N \geqq n$. If $w' \in W_N$ then $L(w') \geqq 2^N$ ($\geqq n$), hence there exists a unique $w \in W(n)$ satisfying $w'(j) = w(j)\,,\ j = 0, 1, 2, \cdots, n$. Using also (17), we therefore have

$$\tilde{P}_n[W(\infty)] = \sum_{w \in W(n)} \tilde{P}_n[C_n(w)] = 1\,.$$

Lastly, we note that it is a standard argument of the extension theorem that if $\tilde{P}_n$, $n \in \mathbb{N}$, satisfies the consistency condition

$$\tilde{P}_{n+1}[C_n(w)] = \tilde{P}_n[C_n(w)]\,, \quad w \in W(n)\,, \quad n = 1, 2, \cdots, \tag{19}$$

then there exists a probability measure P^u on $(W(\infty), \mathcal{F})$ satisfying (14) for all $n \in \mathbb{N}$ and $w \in W(n)$. To prove the consistency condition (19), let $n \in \mathbb{N}$ and $w \in W(n)$. Note that $C_n(w) \in \mathcal{F}_n \subset \mathcal{F}_{n+1}$. If $2^N \geqq n+1$ then (18), and (17) imply

$$\begin{aligned}
&\tilde{P}_{n+1}[C_n(w)] = \sum_{\substack{w'' \in W(n+1); \\ w''(j) = w(j),\ j = 0,1,2,\cdots,n}} \tilde{P}_{n+1}[C_{n+1}(w'')] \\
&= \frac{1}{4} \sum_{p \in \{a,b,c,d\}} \sum_{\substack{w'' \in W(n+1); \\ w''(j) = w(j),\ j = 0,1,2,\cdots,n}} \\
&\times P_{N,p}[\{w' \in W_{N,p} :\ w'(j) = w''(j)\,,\ j = 0, 1, 2, \cdots, n+1\}] \\
&= \frac{1}{4} \sum_{p \in \{a,b,c,d\}} P_{N,p}[\{w' \in W_{N,p} :\ w'(j) = w(j)\,,\ j = 0, 1, 2, \cdots, n\}] \\
&= \tilde{P}_n[C_n(w)].\ \square
\end{aligned}$$

COROLLARY 7. *If $N \in \mathbb{N}$ and $w \in W_N$, then*

$$P^u[\{w' \in W(\infty) : w'(j) = w(j),\ j = 0, 1, 2, \cdots, L(w)\}] = \frac{1}{4} \sum_{p \in \{a,b,c,d\}} P^u_{N,p}[\{w\}]. \tag{20}$$

PROOF. Put $n = L(w)$. Then the definition of W_N implies (15), hence by Theorem 5 and Proposition 6 we have the statement. □

4. Displacement Exponents

In this section, we prove the following.

THEOREM 8 (Displacement exponent). *Let $\gamma_u = \dfrac{\log 2}{\log \lambda_u}$, $u \geqq 0$, where λ_u is a continuous function of u defined in Proposition 1. Then, for any $s > 0$,*

$$\lim_{n \to \infty} (\log n)^{-1} \log E^u[\ |w(n)|^s\ ] = s\gamma_u\,,$$

where $|\cdot|$ denotes the Euclidean metric.

Our proof below implies an additional statement on the correction to the 'leading term' $\log E[|w(n)|^s] \sim s\gamma_u \log n$ in Theorem 8. See (37) and (38) for details.

Let us study the location of the walk after n-steps. Define for $w \in W(\infty) \cup \bigcup_{k \geqq n} W(k)$

$$D_n(w) = \min\{\ M \geqq 0\ :\ |w(i)| \leqq 2^M,\ 0 \leqq i \leqq n\ \},$$

and

$$||w||_n = \max_{0 \leqq i \leqq n} |w(i)|.$$

Then

$$2^{D_n(w)-1} < ||w||_n \leqq 2^{D_n(w)} \tag{21}$$

holds. Let $K(n)$ be the positive integer such that

$$\lambda_u^{K(n)} \leqq n < \lambda_u^{K(n)+1}. \tag{22}$$

Proposition 9 (Long-path estimate). *There exist positive constants $C_1 = C_1(u)$ and $C_2 = C_2(u)$ such that for any positive integers n and M, $P^u[\ D_n(w) \geqq K(n) + M\ ] \leqq C_2 e^{-C_1 2^M}$.*

To prove this proposition, we prepare a few lemmas. In the following we fix $u \geqq 0$ arbitrarily and simply write $\Phi_N(\cdot, u) = \Phi_N(\cdot)$, $\Phi_1(\cdot, u) = \Phi(\cdot)$ and $\lambda_u = \lambda$.

Lemma 10. *If $x < x_u$, then there exist positive constants $C_3 = C_3(u, x)$ and $C_4 = C_4(u, x)$ such that $\Phi_N(x) < C_4 e^{-C_3 2^N}$ for all $N \in \mathbb{N}$.*

Proof. We use the fact that $\Phi(x)$ is a power series of x without constant and linear terms, with non-negative coefficients. We can easily see that for $x < x_u$, $\{\Phi_N(x)\}_{N=1,2,\cdots}$ is a decreasing sequence and since 0 is the only fixed point of Φ in $[0, x_u)$, we see that $\Phi_N(x, u) \to 0$ as $N \to \infty$. We also see

$$\Phi_{N+1}(x) = \Phi(\Phi_N(x)) = \Phi_N(x)^2 P(\Phi_N(x)),$$

where P is expressed as a power series with non-negative coefficients. This combined with $P(\Phi_N(x)) < P(x_u) = 1/x_u$, implies

$$2^{-(N+1)}\{\log \Phi_{N+1}(x) + \log \frac{1}{x_u}\} < 2^{-N}\{\log \Phi_N(x) + \log \frac{1}{x_u}\}.$$

By repeated use of this inequality, we have

$$\begin{aligned}&\limsup_{N\to\infty} 2^{-N} \log \Phi_N(x) = \limsup_{N\to\infty} 2^{-N}\{\log \Phi_N(x) + \log \frac{1}{x_u}\}\\ &\leqq \log \frac{x}{x_u} = -C_3 < 0.\end{aligned}$$

This implies that there is an $N_0 \in \mathbb{N}$ such that $\Phi_N(x) \leqq e^{-C_3 2^N}$ for any $N > N_0$. Taking C_4 large enough, we have the statement. □

Lemma 11. *For any $\delta > 0$, there exist positive constants $C_3' = C_3'(u, \delta)$ and $C_4' = C_4'(u, \delta)$ such that $\Phi_{N+M}(x_u^{1+\delta\lambda^{-N}}) \leqq C_4' e^{-C_3' 2^M}$, for any $N, M \in \mathbb{N}$.*

Proof. Since $\Phi_{N+M}(x_u^{1+\delta\lambda^{-N}}) = \Phi_M(\Phi_N(x_u^{1+\delta\lambda^{-N}}))$, the statement is proved from Lemma 10 if we show that there exists a positive constant ε

such that

$$\Phi_N(x_u^{1+\delta\lambda^{-N}}) < x_u - \varepsilon \tag{23}$$

for all $N \in \mathbb{N}$. Let $\tilde{g}_N(t) = E[e^{-t\lambda^{-N}S_1^N}] = \frac{1}{x_u}\Phi_N(x_u e^{-\lambda^{-N}t})$, $t \geqq 0$, be the Laplace transform of $\lambda^{-N}S_1^N$. Note that

$$\Phi_N(x_u^{1+\delta\lambda^{-N}}) = x_u \tilde{g}_N(-\delta \log x_u). \tag{24}$$

Proposition 3 and Proposition 4 imply that $\tilde{g}_N(t)$ converges to $g(t) = E[e^{-tS^*}]$ as $N \to \infty$. Since $P[S^* > 0] = 1$, we have $g(t) < 1$ for any $t > 0$. Thus there exist an $\varepsilon' > 0$ and $N_1 \in \mathbb{N}$ such that $\Phi_N(x_u^{1+\delta\lambda^{-N}}) < x_u - \varepsilon'$ for all $N > N_1$. Since it holds that $\Phi_N(x_u^{1+\delta\lambda^{-N}}) < x_u$ also for $N = 1, \cdots, N_1$, there exists $\varepsilon > 0$ satisfying (23). □

PROOF OF PROPOSITION 9. The equations (21) and (22) imply

$$\begin{aligned} P^u[\ D_n(w) \geqq K(n) + M\ ] &\leqq P^u[\ S_1^{K(n)+M-1} < n\ ] \\ &\leqq P^u[\ S_1^{K(n)+M-1} < \lambda^{K(n)+1}\ ]. \end{aligned} \tag{25}$$

Corollary 7 and (12) imply that, if $0 \leqq M - 2 \leqq J$,

$$\begin{aligned} &P^u[\ S_1^J < \lambda^{J-M+2}\ ] = \frac{1}{4} \sum_{p \in \{a,b,c,d\}} P^u_{J,p}[\ S_1^J < \lambda^{J-M+2}\ ] \\ &= P^u_{J,a}[\ S_1^J < \lambda^{J-M+2}\ ] \\ &= \frac{1}{x_u} \sum_{w \in W_{J,a}, L(w) < \lambda^{J-M+2}} x_u^{L(w)} f_J(w) \leqq \frac{1}{x_u^2}\ \Phi_J(x_u^{(1+\lambda^{-(J-M+2)})}), \end{aligned}$$

where we used $S_1^J(w) = L(w)$ for $w \in W_{J,a}$ and $x_u^{L(w)(1+1/L(w))} \leqq (x_u^{1+\lambda^{-(J-M+2)}})^{L(w)}$. This combined with Lemma 11 and (25) leads to $P^u[\ D_n(w) \geqq K(n) + M\ ] \leqq C_2 e^{-C_1 2^M}$. □

PROPOSITION 12 (Short-path estimate). *There exists a positive constant $C_5 = C_5(u)$ such that for any positive integers n and M satisfying $M < K(n)$,*

$$P^u[\ D_n(w) < K(n) - M\ ] \leqq \frac{1}{x_u} e^{-C_5 \lambda^M}.$$

PROOF. Put $N = K(n) - M$. Taylor's theorem implies, for $|z| \leqq 1$, $|\Phi(x_u + z) - x_u| \leqq \lambda|z|(1 + b|z|)$, where $b = \frac{1}{2\lambda} \max_{|y| \leqq 1} |\Phi''(x_u + y)|$. Let a be a positive number such that $a \cdot \prod_{k=0}^{\infty}(1 + b\lambda^{-k}) \leqq 1$. Then by induction we can show that for $|z| \leqq a\lambda^{-N}$ and any $K \leqq N$,

$$|\Phi_K(x_u + z) - x_u| \leqq \lambda^K |z| \prod_{\ell=0}^{K-1}(1 + b\lambda^{-(N-\ell)}) \leqq \frac{\lambda^K |z|}{a}. \tag{26}$$

Since $x_u < 1$, we can choose $0 < C_5 < \frac{1}{2}$ so that $(1 + \frac{2C_5}{a})x_u \leqq 1$. Since $e^{x/2} \leqq 1 + x$ for $0 \leqq x \leqq 1$, (26) implies

$$\Phi_N(x_u e^{\lambda^{-N} C_5}) \leqq \Phi_N(x_u(1 + 2C_5\lambda^{-N})) \leqq x_u(1 + \frac{2C_5}{a}) \leqq 1.$$

Thus we have

$$E_N^u[\ e^{\lambda^{-N} S_1^N C_5}\ ] = \frac{1}{x_u}\Phi_N(x_u e^{\lambda^{-N} C_5}) \leqq \frac{1}{x_u}.$$

Using Chebyshev's inequality, we obtain

$$P_{N,a}^u[\ \frac{S_1^N}{\lambda^N} \geqq \lambda^M\ ] \leqq \frac{1}{x_u} e^{-C_5 \lambda^M}. \tag{27}$$

Note that $D_n(w) < N$ implies that $S_1^N > n$. Therefore

$$\begin{aligned} P^u[D_n(w) < N] &= \frac{1}{4} \sum_{q \in \{a,b,c,d\}} P_{N+1,q}^u[D_n(w) < N] \\ &= P_{N,a}^u[D_n(w) < N] \leqq P_{N,a}^u[S_1^N > n]. \end{aligned}$$

This combined with (22) and (27) implies the statement. □

We move on to the reflection argument. The exponent γ_u in Theorem 8 takes the same value as the one that governs the short-time speed of the corresponding continuum limit process, $E[|X(t)|^s] \sim t^{s\gamma}$, $t \to 0$, obtained in [3]. The reason is that both exponents are derived from the same renormalization group analysis. However, to relate the renormalization group results to the asymptotic behaviors of the walks, we need different methods, as may be easily anticipated from the fact that the continuum limit processes have self-similarity, while the walks (discrete chains) do not. One of the main tools here is a somewhat complicated use of a reflection principle, which we will explain in detail.

In the reflection argument, we split paths into parts, hence we have to consider paths starting from points other than O. Let

$$W = \bigcup_{n=1}^{\infty} \{(w(0), w(1), \cdots, w(n)) : \ w(i) \in G_0, \ |w(i+1) - w(i)| = 1,$$
$$\overline{w(i)w(i+1)} \in F_0, \ i \in \{0, 1, \cdots, n-1\}\}.$$

Namely, W is a set of finite-length paths whose starting points are not fixed at O. We extend the definitions of the reversing number $N_K(w)$ and the returning number $M_K(w)$ so that they hold also for any $w \in W$. Define

$$N_K(w) = \sum_{\ell=0}^{L(Q_{K+1}w)+1} N_K(\ell)(w) \ \text{ and } \ M_K(w) = \sum_{\ell=1}^{L(Q_{K+1}w)+1} M_K(\ell)(w),$$

where, for $\ell = 1, \cdots, L(Q_{K+1}w)$, $N_K(\ell)$ and $M_K(\ell)$ are defined by (1) and (2), and the term for $\ell = 0$ is counted only if $T_0^{K+1}(Q_K w) > 1$, with

$$N_K(0)(w) = \sharp\{i \in \mathbb{Z}_+ : \ 0 < i < T_0^{K+1}(Q_K w) :$$
$$\overrightarrow{(Q_K w)(i-1)(Q_K w)(i)} \cdot \overrightarrow{(Q_K w)(i)(Q_K w)(i+1)} < 0 \ \},$$

and the term for $\ell = L(Q_{K+1}w) + 1$ is counted only if $(Q_K w)(L(Q_K w)) \notin G_{K+1}$, with

$$\begin{aligned} &N_K(L(Q_{K+1}w) + 1)(w) \\ &= \sharp\{i \in \mathbb{Z}_+ : \ T^{K+1}_{L(Q_{K+1}w)}(Q_K w) < i \leqq L(Q_K w) - 1 : \\ &\quad \overrightarrow{(Q_K w)(i-1)(Q_K w)(i)} \cdot \overrightarrow{(Q_K w)(i)(Q_K w)(i+1)} < 0, \\ &\quad (Q_K w)(i) \neq (Q_K w)(T^{K+1}_{L(Q_{K+1}w)}(Q_K w)) \ \}, \end{aligned}$$

and

$$\begin{aligned} &M_K(L(Q_{K+1}w) + 1)(w) \\ &= \sharp\{i \in \mathbb{Z}_+ : T^{K+1}_{L(Q_{K+1}w))}(Q_K w) < i \leqq L(Q_K w) : \\ &\qquad (Q_K w)(i) = (Q_K w)(T^{K+1}_{L(Q_{K+1}w))}(Q_K w))\}. \end{aligned}$$

The definition of $f_N(w)$ given by (3) is unchanged.

Let $w \in W_N$. Consider splitting w into two parts at a time $t < L(w)$. Let

$$w_1(i) = w(i), \quad 0 \leqq i \leqq t, \quad \text{and} \quad w_2(i) = w(t+i), \quad 0 \leqq i \leqq L(w) - t.$$

Then

$$L(w_1) = t, \quad L(w_2) = L(w) - t. \tag{28}$$

Note that for $K \leqq N$,

$$T_0^K(w_2) = \inf\{i \geqq 0 : w_2(i) \in G_K\} = \inf\{i \geqq 0 : w(t+i) \in G_K\}.$$

Proposition 13 (Path splitting). *Let $w \in W_N$. Assume that w is split into two parts at some time $t < L(w)$.*

(1) If $w(t) \in G_M \setminus G_{M+1}$ for some $M < N$, then

$$f_N(w) \leqq f_N(w_1) \cdot f_N(w_2) \leqq u^{-3(N-M)} f_N(w) \quad \textit{for} \;\; 0 \leqq u \leqq 1,$$

$$u^{-3(N-M)} f_N(w) \leqq f_N(w_1) \cdot f_N(w_2) \leqq f_N(w) \quad \textit{for} \;\; u > 1,$$

$$L(w) = L(w_1) + L(w_2).$$

(2) If $w(t) = O$, then

$$f_N(w) = f_N(w_1) \cdot f_N(w_2),$$

$$L(w) = L(w_1) + L(w_2).$$

Proof. (1) First consider $N_K(w)$ with $M \leqq K \leqq N-1$. There is an integer $\ell(K)$ such that $T^K_{\ell(K)-1}(w) \leqq t < T^K_{\ell(K)}(w)$. $w_2(T_0^K(w_2))$ coincides either with $w(T^K_{\ell(K)}(w))$ or $w(T^K_{\ell(K)-1}(w))$. When we count sharp turns and U-turns of w_1 and w_2, at most two turns of $(Q_K w)(i)$ at $i = \ell(K) - 1$ or $\ell(K)$ may elude the counting (see Fig. 1). Thus

$$N_K(w) - 2 \leqq N_K(w_1) + N_K(w_2) \leqq N_K(w).$$

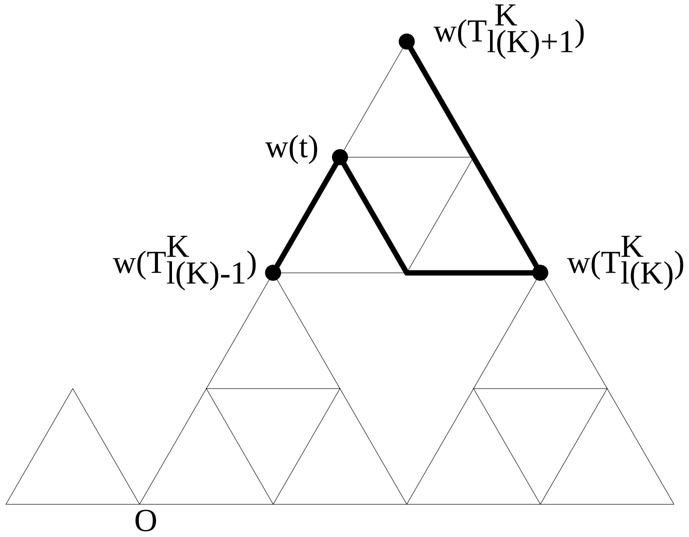

Fig. 1.

As for $M_K(w)$ with $M \leqq K \leqq N-1$, there is an integer $\ell'(K+1)$ such that $T^{K+1}_{\ell'(K+1)-1}(w) < t < T^{K+1}_{\ell'(K+1)}(w)$. $w_2(T^{K+1}_0(w_2))$ coincides either with $w(T^{K+1}_{\ell'(K+1)}(w))$ or $w(T^{K+1}_{\ell'(K+1)-1}(w))$. In the latter case, the first return of $Q_K w$ to $w(T_{\ell'(K+1)-1}(w))$ after t eludes the counting in $M_K(w_2)$. Thus

$$M_K(w) - 1 \leqq M_K(w_1) + M_K(w_2) \leqq M_K(w).$$

Thus, for $M \leqq K$ we have

$$\begin{aligned} N_K(w) + M_K(w) - 3 &\leqq N_K(w_1) + M_K(w_1) + N_K(w_2) + M_K(w_2) \\ &\leqq N_K(w) + M_K(w). \end{aligned} \tag{29}$$

For $N_K(w)$ and $M_K(w)$ with $1 \leqq K < M$, from the fact that $w_2(0) \in G_{K+1} \subset G_M$, it holds that

$$N_K(w) = N_K(w_1) + N_K(w_2), \tag{30}$$

and

$$M_K(w) = M_K(w_1) + M_K(w_2). \tag{31}$$

(29), (31) and (30) combined with (3) prove the proposition. (2) is immediate if we note that splitting at O does not affect $N_K(w)$ or $M_K(w)$. □

Proposition 14 (Reflection principle). *There exists $C_6 = C_6(u) > 0$ such that for any $n \in \mathbb{N}$ and $s > 0$*

$$C_6\ E^u[\ 2^{D_n(w)s},\ \ |w(n)| \geqq 2^{D_n(w)-2}\ ] \geqq E^u[\ 2^{D_n(w)s},\ \ |w(n)| < 2^{D_n(w)-2}\ ]$$

Proof. First, fix arbitrarily $N \in \mathbb{N}$ and condition on $\{D_n(w) = N\}$. Then it is enough to study the behavior of paths within F''_{N+1}.

For $w \in W(n)$ such that $D_n(w) = N$, let $T(n, A_{N-2}) = \sup\{i < n : w(i) \in A_{N-2}\}$, where $A_M = \{a_M, b_M, c_M, d_M\}$. Define

$$\begin{aligned} U_N(z) &= \{w \in W(n) : D_n(w) = N,\ |w(n)| < 2^{N-2},\ w(T(n, A_{N-2})) = z\},\\ V_N(z) &= \{w \in W(n) : D_n(w) = N,\ |w(n)| \geqq 2^{N-2},\\ &\qquad w(T(n, A_{N-2})) = z\},\ z \in A_{N-2}. \end{aligned}$$

For $w \in U_N(b_{N-2})$, let us denote by w_R the n-step path obtained from w by reflecting the part $\{w(i) : T(n, A_{N-2}) < i \leqq n\}$ with respect to a line parallel to the y-axis that passes b_{N-2} (see Fig. 2). This mapping is an injection from $U_N(b_{N-2})$ to $V_N(b_{N-2})$. Also for the case that $z \in \{a_{N-2}, c_{N-2}, d_{N-2}\}$, we can define an injection from $U_N(z)$ to $V_N(z)$ in the following way. For $w \in U_N(z)$, we first reflect the part $\{w(i) : T(n, A_{N-2}) < i \leqq n\}$ at z, and if the reflected path leaks out of F_0 , then reflect the leaking part appropriately so that it lands on F_0 (see Fig. 3). We denote the reflected path for the cases $z \in \{a_{N-2}, c_{N-2}, d_{N-2}\}$ also by w_R .

For $\tilde{w} \in U_N(z) \cup V_N(z)$, define

$$p_{N+1}(\tilde{w}) = \sum f_{N+1}(w')x_u^{L(w')-1},$$

where the summation is taken over $w' \in W_{N+1}$ with $w'(i) = \tilde{w}(i)$ for $i = 0, 1, \cdots, n$. We claim that there exists a positive constant M_3 that depends only on u such that

$$p_{N+1}(w_R) \geqq M_3 p_{N+1}(w) \tag{32}$$

holds for all $w \in U_N(z)$, $z \in A_{N-2}$ and $N \in \mathbb{N}$.

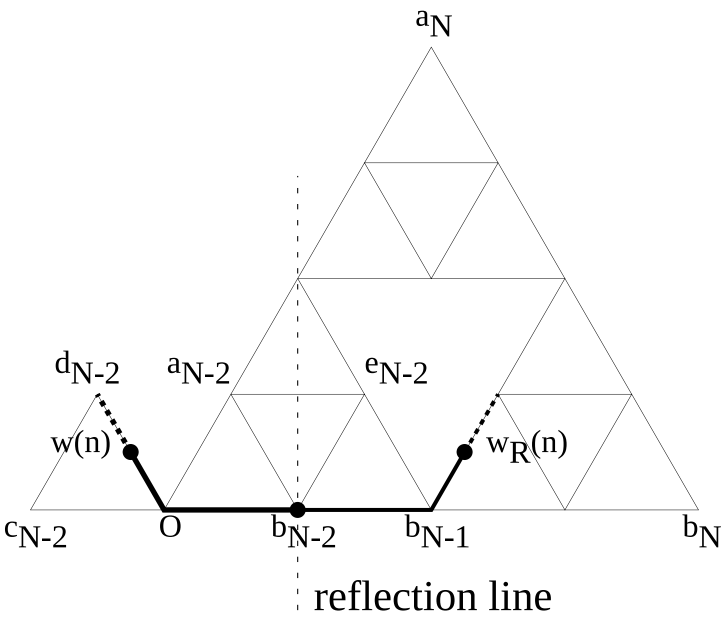

Fig. 2.

With (32), the proposition is proved as follows. Let

$$U_N = \{w \in W_{N+1} \ : \ D_n(w) = N,\ |w(n)| < 2^{N-2}\},$$
$$V_N = \{w \in W_{N+1} \ : \ D_n(w) = N,\ |w(n)| \geqq 2^{N-2}\}.$$

Corollary 7 and (32) imply

$$\begin{aligned} & E^u[\ 2^{D_n s},\ |w(n)| < 2^{D_n - 2}\ ] \\ & = \sum_{N=1}^{\infty} 2^{Ns} P^u[\ D_n(w) = N,\ |w(n)| < 2^{N-2}\ ] \\ & = \sum_{N=1}^{\infty} 2^{Ns}\ \frac{1}{4} \sum_{p \in \{a,b,c,d\}} \sum_{w \in U_N \cap W_{N+1,p}} P^u_{N+1,p}[\{w\}] \\ & \leqq \sum_{N=1}^{\infty} 2^{Ns}\ \frac{1}{4M_3} \sum_{p \in \{a,b,c,d\}} \sum_{w \in V_N \cap W_{N+1,p}} P^u_{N+1,p}[\{w\}] \end{aligned}$$

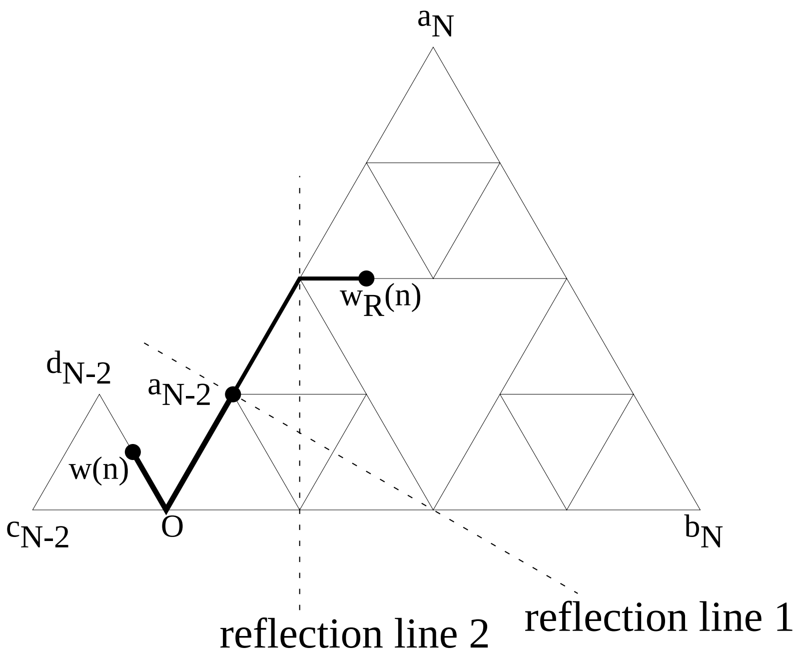

Fig. 3.

$$= \frac{1}{M_3} \sum_{N=1}^{\infty} 2^{Ns} P^u[\ D_n(w) = N,\ |w(n)| \geqq 2^{N-2}\ ]$$
$$= \frac{1}{M_3} E^u[\ 2^{D_n s},\ |w(n)| \geqq 2^{D_n - 2}\ ].$$

This implies the statement, with $C_6 = \dfrac{1}{M_3}$.

It remains to prove (32). We prove for the case $w \in U_N(b_{N-2})$ and $0 \leqq u \leqq 1$. The other cases can be proved in a similar manner. Put $e_M = \dfrac{a_{M+1} + b_{M+1}}{2}$, $M \in \mathbb{Z}_+$. For a path $w \in U_N(b_{N-2})$, $w(n)$ can lie either in $\triangle O a_{N-2} b_{N-2}$ or in $\triangle O c_{N-2} d_{N-2}$. Let us consider the first case. Note that $w_R(n) \in \triangle b_{N-2} e_{N-2} b_{N-1}$. We will prepare some inequalities relating w and w_R. For $w' \in \bigcup_{k \geqq n} W(k)$, put $T(n+, G_M) = \inf\{i \geqq n : w'(i) \in G_M\}$, $M \in \mathbb{N}$. Note that if $w'(n) \in \triangle O a_{N-2} b_{N-2}$, then $w'(T(n+, G_{N-2})) \in \{a_{N-2}, b_{N-2}, O\}$ and if $w'(n) \in \triangle b_{N-2} e_{N-2} b_{N-1}$ then $w'(T(n+, G_{N-2})) \in$

$\{e_{N-2}, b_{N-2}, b_{N-1}\}$. We will extend w and w_R up to time $T(n+, G_{N-2})$. To this end, we define for $\tilde{w} \in U_N(b_{N-2}) \cup V_N(b_{N-2})$

$$H(z)[\tilde{w}] = \sum f_{N+1}(w') x_u^{L(w')-1}, \quad z \in G_{N-2},$$

where the summation is taken over $w' \in \bigcup_{k \geqq n} W(k)$ satisfying $w'(i) = \tilde{w}(i)$ for $i = 0, \cdots, n$, and $w'(L(w')) = w'(T(n+, G_{N-2})) = z$.

Taking into account the possibility that w_R makes a 2^{N-2}-scale sharp turn at the reflection point b_{N-2} while w does not, we have

$$H(e_{N-2})[w_R] \geqq uH(a_{N-2})[w], \tag{33}$$

$$H(b_{N-2})[w_R] \geqq uH(b_{N-2})[w], \tag{34}$$

$$H(b_{N-1})[w_R] \geqq uH(O)[w]. \tag{35}$$

Next we introduce two quantities Ξ and Ξ'. Let $W_M^a = \{w \in W : w(0) = a_M,\ L(w) = T_{A_{M+1}}(w)\}$, $M \in \mathbb{Z}_+$, and $\Xi = \sum_{w \in W_0^a} f_1(w) x_u^{L(w)}$. We will show that for any $M \in \mathbb{N}$, $\sum_{w \in W_M^a} f_{M+1}(w) x_u^{L(w)} = \Xi$ holds. Note that if $w \in W_M^a$, then $2^{-M} Q_M w \in W_0^a$. We split w into segments w_i such that $w_i(t) = w(T_{i-1}^M(w) + t)$, $0 \leqq t \leqq T_i^M(w) - T_{i-1}^M(w)$, $i = 1, \cdots, L(2^{-M} Q_M w)$, and apply (30) and (31). Noting that each w_i can be identified, via reflection, with a path in $W_{M,a}$, we have

$$\begin{aligned}
\sum_{w \in W_M^a} f_{M+1}(w) x_u^{L(w)} &= \sum_{w \in W_M^a} x_u^{L(w)} \prod_{K=0}^{M-1} u^{N_K(w)+M_K(w)} \cdot u^{N_M(w)+M_M(w)} \\
&= \sum_{v \in W_0^a} \sum_{\substack{w \in W_M^a \\ 2^{-M} Q_M w = v}} \prod_{i=1}^{L(v)} \Big(x_u^{L(w_i)} \prod_{K=0}^{M-1} u^{N_K(w_i)+M_K(w_i)} \Big) \cdot u^{N_1(v)+M_1(v)} \\
&= \sum_{v \in W_0^a} \prod_{i=1}^{L(v)} \Big(\sum_{w_i \in W_{M,a}} x_u^{L(w_i)} \prod_{K=0}^{M-1} u^{N_K(w_i)+M_K(w_i)} \Big) \cdot u^{N_1(v)+M_1(v)} \\
&= \sum_{v \in W_0^a} (\Phi_M(x_u, u))^{L(v)} u^{N_1(v)+M_1(v)} \qquad = \Xi.
\end{aligned}$$

Moreover, symmetry arguments imply that if the summation is taken over paths starting at any other point in A_M, instead of a_M, the correspoinding value is equal to Ξ. Let $W^e_{,M} = \{w \in W \ : \ w(0) = e_M, \ L(w) = T_{A_{M+1}}(w)\}$, and $\Xi' = \sum_{w \in W^e_{,0}} f_1(w) x_u^{L(w)}$. In a similar manner to the above argument, we see that for any $M \in \mathbb{N}$, it holds that $\sum_{w \in W^e_{,M}} f_{M+1}(w) x_u^{L(w)} = \Xi'$.

Now we are ready to prove (32). Let $w \in U_N(b_{N-2})$. We divide the path w' in the summation into segments by splitting at $T(n+, G_{N-2})$, and if necessary, also at $T(n+, G_{N-1})$ and at $T(n+, G_N)$. Then Proposition 13 gives,

$$p_{N+1}(w) \leqq (H(a_{N-2})[w] + H(b_{N-2})[w])\Xi^3 + H(O)[w] \cdot 4x_u,$$

where we used $\Phi_M(x_u, u) = x_u$. Splitting w_R at $T(n+, G_{N-2})$, and, if necessary, also at $T(n+, G_{N-1})$ and $T(n+, G_N)$, we have from Proposition 13 and (33) – (35),

$$\begin{aligned} p_{N+1}(w_R) &\geqq u^{9+6+3}(H(e_{N-2})[w_R] \cdot \Xi' \, \Xi^2 + H(b_{N-2})[w_R]\Xi^3) \\ &\quad + u^9 H(b_{N-1})[w_R]\Xi^2 \\ &\geqq u^{19}(H(a_{N-2})[w] \cdot \Xi' \, \Xi^2 + H(b_{N-2})[w]\Xi^3) + u^{10} H(O)[w]\Xi^2 \\ &\geqq u^{19} M_1 \{(H(a_{N-2})[w] + H(b_{N-2})[w])\Xi^3 + 4H(O)[w]x_u\} \\ &\geqq u^{19} M_1 p_{N+1}(w), \end{aligned}$$

where we put $M_1 = \min\{1, \frac{\Xi'}{\Xi}, \frac{\Xi^2}{4x_u}\} > 0$.

The case $w(n) \in \triangle O c_{N-2} d_{N-2}$ can be handled in the same way to give $p_{N+1}(w_R) \geqq u^{13} M_2 p_{N+1}(w)$, where M_2 is a positive constant depending only on u. Thus (32) holds with $M_3 = \min\{u^{19}M_1, u^{13}M_2\} > 0$. $\square$

Let $C_6' = \dfrac{1}{1 + C_6}$. Proposition 14 implies

$$\begin{aligned} E^u[\ |w(n)|^s\ ] &\geqq E^u[\ 2^{(D_n(w)-2)s}, \ \ |w(n)| \geqq 2^{D_n(w)-2}\ ] \\ &\geqq \frac{1}{1 + C_6} E^u[\ 2^{(D_n(w)-2)s}\ ] = 2^{-2s} C_6' \, E^u[\ 2^{D_n(w)s}\ ], \end{aligned}$$

which, with the definitions of $||w||_n$ and $D_n(w)$, further implies

PROPOSITION 15. $2^{-2s}C_6' E^u[\, 2^{D_n(w)s}\,] \leqq E^u[\, |w(n)|^s\,] \leqq E^u[\, ||w||_n^s\,] \leqq E^u[\, 2^{D_n(w)s}\,]$.

PROOF OF THEOREM 8. Assume $\beta > 1$.

$$\begin{aligned} E^u[\, ||w||_n^s\,] &= E^u[\, ||w||_n^s\,,\ ||w||_n < (\log n)^{\beta} n^{\gamma}\,] \\ &\quad + E^u[\, ||w||_n^s\,,\ ||w||_n \geqq (\log n)^{\beta} n^{\gamma}\,] \\ &\leqq \{(\log n)^{\beta} n^{\gamma}\}^s + n^s P^u[||w||_n \geqq (\log n)^{\beta} n^{\gamma}], \end{aligned} \tag{36}$$

where we used $||w||_n \leqq n$. Also, (21), (22), and $\gamma = \log 2/\log\lambda$ imply

$$\begin{aligned} P^u[||w||_n \geqq (\log n)^{\beta} n^{\gamma}] &\leqq P^u[D_n(w) \geqq \frac{\beta \log\log n}{\log 2} + \frac{\log n}{\log\lambda}] \\ &\leqq P^u[D_n(w) \geqq \frac{\beta \log\log n}{\log 2} + K(n)] \\ &\leqq C_2 \exp\{-C_1 (\log n)^{\beta}\}, \end{aligned}$$

where in the last inequality Proposition 9 was used. This combined with (36) gives

$$\begin{aligned} (\log n)^{-s\beta} n^{-s\gamma} E^u[\, ||w||_n^s\,] &\leqq 1 + (\log n)^{-s\beta} n^{s(1-\gamma)} C_2 \exp\{-C_1 (\log n)^{\beta}\} \\ &\leqq 1 + C_2 (\log n)^{-s\beta} n^{-\gamma s}, \end{aligned}$$

for any large n such that $C_1(\log n)^{\beta} \geqq s \log n$. Thus

$$\limsup_{n\to\infty} (\log n)^{-s\beta} n^{-s\gamma} E^u[\, ||w||_n^s\,] \leqq 1. \tag{37}$$

On the other hand, combining (21), (22) and Proposition 12 in a similar way to the above argument, we see that for any $\alpha > 0$ there exists a constant $C' > 0$ such that

$$\begin{aligned} E^u[\, ||w||_n^s\,] &\geqq \{(\log n)^{-\alpha} n^{\gamma}\}^s \{1 - P^u[||w||_n < (\log n)^{-\alpha} n^{\gamma}]\} \\ &\geqq \{(\log n)^{-\alpha} n^{\gamma}\}^s \{1 - \frac{1}{x_u} \exp\{-C' (\log n)^{\alpha/\gamma}\}\}. \end{aligned}$$

Thus

$$\liminf_{n\to\infty} (\log n)^{s\alpha} n^{-s\gamma} E^u[\, ||w||_n^s\,] \geqq 1. \tag{38}$$

(37) and (38) imply $\lim_{n\to\infty}(\log n)^{-1}\log E^u[\,||w||_n^s\,] = s\gamma$, which combined with Proposition 15, further implies the assertion of the theorem. □

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