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Application of Stochastic Flows to Optimal Portfolio Strategies

By Ryuji Fukaya

Abstract. The author proposes a new algorithm using a stochastic flow technique to solve an optimal portfolio and consumption problem for a single-agent in a Markovian security market setting. In that class, optimal feedback portfolio strategies are computed by the system of stochastic differential equations, which are induced by applying the differential rule of a composite function to stochastic flows. Sufficient conditions for the existence of feedback solutions are stated using integrability of stochastic processes. In the case of power and logarithmic utility functions, more straightforward conditions are given and the continuity of optimal strategies is proved.

1. Introduction

We consider a single-agent optimal portfolio and consumption problem in a continuous-time. Optimal portfolio and consumption choice in multiperiod or in continuous-time settings were first investigated by Samuelson [15] and Merton [10] [11]. By assuming a model with constant coefficients and solving the relevant Hamilton-Jacobi-Bellman equation, [10] shows solutions when the utility function is a member of the HARA (Hyperbolic Absolute Risk Aversion) family. The "separating mutual fund theorem" in a constant coefficients environment is given in [11] . Another separating mutual fund theorem in a Markovian stochastic interest rate environment is given in [12]. These results have great impact on the investment industry. Since these seminal papers appeared, "the optimal portfolio strategy" and "the separating mutual fund theorem" have been studied

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extensively by many authors for more general settings. Especially Cvitanic-Karatzas [2] gives optimal solutions to constrained portfolio optimization problem with general settings.

However for many financial problems which practitioners tackle in daily business, it is difficult to obtain tractable solutions which are analytical and optimal, even though we have the general formula for optimal solutions such as [2]. The difficulty requires us to apply the numerical methods especially when the economy's state variables are stochastic such as in stochastic interest rate models, stochastic volatility models, bond portfolio strategies, bond-equity mix problems and so on. Recently some advanced stochastic methods using Malliavin calculus are applied extensively to obtain optimal portfolio strategies numerically. Detemple-Garcia-Rindisbacher [3] applied Malliavin calculus and the generalized Clark formula and obtained numerical results. Kunitomo-Takahashi [8] and Takahashi-Yoshida [16] used the combination of Malliavin Calculus and the asymptotic expansion approach.

In this paper, starting with the convex duality approach (see e.g. Cox-Huang [1], and Karatzas-Lehoczky-Shreve [5]), a new algorithm using stochastic flows is proposed for the determination of coefficients of the separating mutual fund theorem. A class of security market models is specified, where a wide range of financial problems are covered. Within this class, solutions are given by transition semigroups using stochastic flows.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. $\{B(t) = (B_1(t), \cdots, B_d(t));\ t \in [0,T]\}$ be a d-dimensional standard Brownian motion. The time interval is $[0,T]$. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the augmented Brownian filtration. We have the investment horizon T_0, where $T_0 < T$. The economy's state variable vector $X(t;s,x)$ at time t, starting from $x \in \mathbb{R}^n$ at time s, is given by the following $\mathbb{R}^n$-valued continuous stochastic process.

$$X(t;s,x) = (X_1(t;s,x), \cdots X_n(t;s,x)),$$

satisfying the following stochastic differential equation:

$$X(t;s,x) = x + \int_s^t \mu^X(v, X(v;s,x))\mathrm{d}v + \int_s^t \sigma^X(v, X(v;s,x))\mathrm{d}B(v). \tag{1}$$

$\mu^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$ are good continuous functions satisfying Condition **(S1)** in Section 2. From Kunita [7]

Theorem 4.6.5, we may assume that $X(t;s,x)$ is a stochastic flow of C^∞-diffeomorphisms. At time $t=0$, we choose a starting point $x_0 \in \mathbb{R}^n$ and fix it. We denote $X(t;0,x_0)$ by $X(t)$.

The risk free rate at time t is denoted by r_t, and is given by a function of state variables.

$$r_t = r(t, X(t)).$$

We define the money account $S_0(t)$ by

$$S_0(t) = \exp\left\{\int_0^t r(v, X(v))\mathrm{d}v\right\}.$$

The price processes of d given securities, $S_i(t)$, $i=1,\cdots,d$, are solutions of stochastic differential equations.

$$\begin{aligned} S_i(t) = S_{i,0} &+ \int_0^t \mu_i(v, X(v))S_i(v)\mathrm{d}v \\ &+ \sum_{j=1}^d \int_0^t \sigma_{i,j}(v, X(v))S_i(v)\mathrm{d}B_j(v), \end{aligned} \tag{2}$$

for all $i,j=1,\cdots d$. Let $S(t)=(S_1(t),\cdots,S_d(t))$. Under assumptions **(S3)**, **(S4)**, **(S5)**, and **(S6)** in Section 2, our security market model is a standard financial market in the sense of Karatzas-Shreve [6] and is complete.

We expand Karatzas-Shreve's setting of the single-agent optimal portfolio and consumption framework. Let us define a state-dependent investor's utility function as follows:

$$V(C,Z) = E^P\left[\int_0^{T_0} u(C(v), v, \omega)\mathrm{d}v + U(Z,\omega)\right],$$

where $u:(0,\infty)\times[0,T_0]\times\Omega\to\mathbb{R}$, and $U:(0,\infty)\times\Omega\to\mathbb{R}$ are measurable functions satisfying Condition **(U1)** and **(U2)**.

Our optimal portfolio and consumption problem is stated as follows:

$$J(W,x_0) = \sup_{(C,Z,\varphi)\in\mathcal{A}(W)} V(C, W^{W,C,\varphi}(T_0)), \tag{3}$$

where W is an initial endowment, φ is a trading strategy, and (C, Z, φ) is an admissible strategy whose definition is given in Section 2. $W^{W,C,\varphi}(t)$ is given as follows:

$$\begin{aligned} &W^{W,C,\varphi}(t) \\ =&W + \int_0^t \left[W^{W,C,\varphi}(v) \left(\sum_{j=1}^{d} \varphi_j(v)(\mu_j(v, X(v;x)) \right.\right. \\ &\qquad \left.\left. - r(X(v;x))) + r(X(v;x)) \right) - C(v) \right] \mathrm{d}v \\ &+ \sum_{j=1}^{d}\sum_{i=1}^{d} \int_0^t W^{W,C,\varphi}(v)\varphi_j(v)\sigma_{j,i}(v, X(v;x))\mathrm{d}B_i(v). \end{aligned}$$

We will show in Theorem 2.5 that the optimal portfolio strategy $\hat{\varphi}(t)$ is given as a rational expression of expected values of Markovian-type diffusion processes. These diffusion processes are solutions of stochastic differential equations which are induced by applying the differential rule of a composite function to stochastic flows and multiplicative functionals. Theorem 2.5 gives a fundamental framework for numerical calculations of $\hat{\varphi}(t)$ without using Malliavin derivatives. We can directly apply stochastic simulation methods such as Monte Carlo methods and quasi-Monte Carlo methods to this framework.

More advanced numerical scheme such as Kusuoka approximation [9] are expected to improve the effectiveness of the calculation, as reported in Ninomiya [13] for pricing derivatives, because our formula of optimal strategies are essentially identical to pricing formulas of derivatives.

In this paper, investors' utility functions are allowed to be state-dependent in some special form. We can cover, for example, "a present value of utility when interest rate is stochastic" and "a numeraire of utility functions is some traded security". Even in these cases, we show that optimal portfolio strategies are given as linear combinations of "mean-variance portfolios" and "hedging portfolios with respect to economy's state variables." This result is a generalization of Ocone-Karatzas [14], Cvitanic-Karatzas [2], and Takahashi-Yoshida [16].

The remainder of this paper is structured as follows. We give the main theorem in Section 2. In Section 3, assumptions and preliminary propositions are stated formally. Stochastic flow techniques are proposed and coefficients of optimal portfolios are given using transition semigroups. Section 4 gives proof of a main theorem. In Section 5, we show that in the case of power and logarithmic utility functions, alternative straightforward assumptions are enough for the main result and optimal strategies are continuous. We give some numerical examples in Section 6 and concluding remarks in Section 7.

2. Main Theorem

Throughout this paper we assume the following setting: Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\{B(t) = (B_1(t), \cdots, B_d(t)); t \in [0,T]\}$ be a d-dimensional standard Brownian motion. The time interval is $[0,T]$, where $T > 0$. Let $(\mathcal{F}_t)_{t\in[0,T]}$ be the augmented Brownian filtration. We have the investment horizon T_0, where $0 < T_0 < T$.

Definition 2.1. We say that a function $f : [0,T] \times \mathbb{R}^m \to \mathbb{R}$, where $m \in \mathbb{N}$, is a member of a class $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^m)$, if the following conditions are satisfied:

1. $f(t,x)$ is continuous in t, x, and smooth in x for all t.
2. There exists a constant $C > 0$, such that
$$|f(t,x)| \leqslant C(1 + |x|), \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^m.$$
3. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_m)$, there exists a constant depending on α, $C_\alpha > 0$, such that
$$|D^\alpha_x f(t,x)| \leqslant C_\alpha, \quad \text{for all } t \in [0,T] \text{ and } x \in \mathbb{R}^m.$$

An economy's state variable vector $X(t)$ is given by $\mathbb{R}^n$-valued continuous stochastic process $X(t) = (X_1(t), \cdots, X_n(t))$. We assume the following:

(S1): Coefficient functions $\mu^X_i(t,x)$, $\sigma^X_{i,j}(t,x)$,$i = 1, \cdots, n$, $j = 1, \cdots, d$ of $X(t)$ are in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$.

We assume that $X(t)$ is a unique solution to the following stochastic differential equation in the sense of Itô and a stochastic process with spacial parameters (see, e.g., Kunita [7]).

$$X(t; s, x) = x + \int_s^t \mu^X(v, X(v; s, x))\mathrm{d}v + \int_s^t \sigma^X(v, X(v; s, x))\mathrm{d}B(v),$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Let μ^X be an $\mathbb{R}^n$-valued function $\mu^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, and σ^X be an $\mathbb{R}^n \otimes \mathbb{R}^d$-valued function $\sigma^X : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^d$ by the following:

$$\mu^X(t,x) = \begin{pmatrix} \mu_1^X(t,x) \\ \vdots \\ \mu_n^X(t,x) \end{pmatrix}, \text{ and } \quad \sigma^X(t,x) = \begin{pmatrix} \sigma_{1,1}^X(t,x) & \cdots & \sigma_{1,d}^X(t,x) \\ \vdots & & \vdots \\ \sigma_{n,1}^X(t,x) & \cdots & \sigma_{n,d}^X(t,x) \end{pmatrix}.$$

We may assume that $X(t;x)$ is a forward stochastic flow of C^∞-diffeomorphisms (see Kunita [7] Theorem 4.6.5). We denote this stochastic flow by $X(t; s, x)$ for $0 \leqslant s \leqslant t \leqslant T$ and for $x \in \mathbb{R}^n$. At time $t = 0$, choose a starting point $x_0 \in \mathbb{R}^n$, and fix it. Let $X(t) = X(t; 0, x_0)$. Then $X(t) = X(t; s, X(s))$.

Let r be a function $r : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

(S2): $r(t,x)$ is in $C_{\mathrm{ub}}^{0,\infty}(\mathbb{R}^n)$.

We define $r_t = r(t, X(t))$ and consider r_t as the risk free rate at time t. Let

$$S_0(t) = \exp\left\{ \int_0^t r(v, X(v))\mathrm{d}v \right\}.$$

$S_0(t)$ is the money account.

Let μ_i be a function $\mu_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $\sigma_{i,j}$ be a function $\sigma_{i,j} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

(S3): $\mu_i(t,x)$, $\sigma_{i,j}(t,x)$,$i,j = 1, \cdots, d$ are in $C_{\mathrm{ub}}^{0,\infty}(\mathbb{R}^n)$.

Let us introduce d individual securities, $S_i(t)$, $i = 1, \cdots, d$, where each $S_i(t)$ is an $\mathbb{R}$-valued stochastic process and a unique solution of the following stochastic differential equation:

$$
\begin{aligned}
S_i(t) =& S_{i,0} + \int_0^t \mu_i(v, X(v)) S_i(v) \mathrm{d}v \\
&+ \sum_{j=1}^d \int_0^t \sigma_{i,j}(v, X(v)) S_i(v) \mathrm{d}B_j(v), \\
& i = 1, \cdots, d.
\end{aligned} \tag{4}
$$

Let $S(t) = (S_1(t), \cdots, S_d(t))$, and

$$
\mu(t,x) = \begin{pmatrix} \mu_1(t,x) \\ \vdots \\ \mu_d(t,x) \end{pmatrix}, \text{ and } \quad \sigma(t,x) = \begin{pmatrix} \sigma_{1,1}(t,x) & \cdots & \sigma_{1,d}(t,x) \\ \vdots & & \vdots \\ \sigma_{d,1}(t,x) & \cdots & \sigma_{d,d}(t,x) \end{pmatrix}.
$$

We assume the following condition.

(S4): The volatility matrix $\sigma(t,x)$ is invertible for all $t \in [0,T]$ and for all $x \in \mathbb{R}^n$.

Then we can define an $\mathbb{R}^d$-valued function $\lambda : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ as follows:

$$
\lambda(t,x) = \sigma(t,x)^{-1} \left(\mu(t,x) - r(t,x)\vec{1} \right),
$$

where $\vec{1} = (1, \cdots, 1) \in \mathbb{R}^d$. We denote the j-th element of $\lambda(t,x)$ by $\lambda_j(t,x)$. We assume the following:

(S5): $\lambda_j(t,x), j = 1, \cdots, d$, are in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$.

Let

$$
\begin{aligned}
\Pi(t; s, x) = \exp\Bigg\{ &- \int_s^t r(v, X(v; s, x)) \mathrm{d}v \\
&- \sum_{j=1}^d \int_s^t \lambda_j(v, X(v; s, x)) \mathrm{d}B_j(v) \\
&- \frac{1}{2} \int_s^t \sum_{j=1}^d \lambda_j(v, X(v; s, x))^2 \mathrm{d}v \Bigg\}, \\
& 0 \leqslant s \leqslant t \leqslant T, \quad x \in \mathbb{R}^n.
\end{aligned} \tag{5}
$$

Let $\Pi(t) = \Pi(t; 0, x_0)$. Then we see that

$$\Pi(t) = \Pi(s)\Pi(t; s, X(s)),$$

for any $0 \leqslant s \leqslant t \leqslant T$. We see that $\Pi(t; s, x)$ is $\{\mathcal{F}_{s,t}\}_{0 \leqslant s \leqslant t \leqslant T}$-measurable, where

$$\mathcal{F}_{s,t} = \sigma(X(s)) \vee \sigma(B_j(r) - B_j(s) : j = 1, \cdots, d, s \leqslant r \leqslant t).$$

$\Pi(t)$ is the state price density process (see Duffie [4] and Karatzas-Shreve [6]). For each $j = 1, \cdots, n$, we define the following stochastic processes:

$$\begin{aligned}
\pi_j(t; s, x) = &- \int_s^t \sum_{k=1}^n \frac{\partial r}{\partial y_k}(v, X(v; s, x)) \frac{\partial X_k}{\partial x_j}(v; s, x)\mathrm{d}v \\
&- \sum_{i=1}^d \int_s^t \sum_{k=1}^n \frac{\partial \lambda_i}{\partial y_k}(v, X(v; s, x)) \frac{\partial X_k}{\partial x_j}(v; s, x)\mathrm{d}B_i(v) \\
&- \int_s^t \sum_{i=1}^d \sum_{k=1}^n \lambda_i(v, X(v; s, x)) \frac{\partial \lambda_i}{\partial y_k}(v, X(v; s, x)) \\
&\quad \times \frac{\partial X_k}{\partial x_j}(v; s, x)\mathrm{d}v, \\
&\quad j = 1, \cdots, n.
\end{aligned} \tag{6}$$

We define the following local martingale $\Xi(t)$:

$$\begin{aligned}
\Xi(t) = \exp\Bigg\{ &- \sum_{j=1}^d \int_0^t \lambda_j(v, X(v))\mathrm{d}B_j(v) \\
&- \frac{1}{2} \int_0^t \sum_{j=1}^d \lambda_j(v, X(v))^2 \mathrm{d}v \Bigg\}.
\end{aligned} \tag{7}$$

We assume the following condition:

(S6): The local martingale $\Xi(t)$ is a martingale.

Let $g_0(t, x), g_1(t, x), \cdots, g_d(t, x)$ and $h_0(t, x), h_1(t, x), \cdots, h_d(t, x)$ be functions from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}$ satisfying the following.

(S7): $g_i(t,x)$, $i = 0,1,\cdots,d$, and $h_i(t,x)$, $i = 0,1,\cdots,d$ are in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$.

We introduce the following stochastic processes with spacial parameters; for $0 \leqslant s \leqslant t \leqslant T_0$, and for all $x \in \mathbb{R}^n$,

$$\begin{aligned}\Delta(t;s,x) = \exp\Bigg\{ & \int_s^t g_0(v, X(v;s,x))\mathrm{d}v \\ & + \sum_{j=1}^d \int_s^t g_j(v, X(v;s,x))\mathrm{d}B_j(v)\Bigg\},\end{aligned} \tag{8}$$

$$\begin{aligned}\delta_j(t;s,x) = & \int_s^t \sum_{k=1}^n \frac{\partial g_0}{\partial y_k}(v, X(v;s,x))\frac{\partial X_k}{\partial x_j}(v;s,x)\mathrm{d}v \\ & + \sum_{i=1}^d \int_s^t \sum_{k=1}^n \frac{\partial g_i}{\partial y_k}(v, X(v;s,x))\frac{\partial X_k}{\partial x_j}(v;s,x)\mathrm{d}B_i(v), \\ & \text{for } j = 1,\cdots,d,\end{aligned} \tag{9}$$

$$\begin{aligned}E(t;s,x) = \exp\Bigg\{ & \int_s^t h_0(v, X(v;s,x))\mathrm{d}v \\ & + \sum_{j=1}^d \int_s^t h_j(v, X(v;s,x))\mathrm{d}B_j(v)\Bigg\},\end{aligned} \tag{10}$$

$$\begin{aligned}\eta_j(t;s,x) = & \int_s^t \sum_{k=1}^n \frac{\partial h_0}{\partial y_k}(v, X(v;s,x))\frac{\partial X_k}{\partial x_j}(v;s,x)\mathrm{d}v \\ & + \sum_{i=1}^d \int_s^t \sum_{k=1}^n \frac{\partial h_i}{\partial y_k}(v, X(v;s,x))\frac{\partial X_k}{\partial x_j}(v;s,x)\mathrm{d}B_i(v), \\ & \text{for } j = 1,\cdots,d.\end{aligned} \tag{11}$$

The following equations hold as for $\Pi(t;s,x)$: Let us define $\Delta(t)$ and $E(t)$ by

$$\Delta(t) = \Delta(t;0,x_0), \quad E(t) = E(t;0,x_0). \tag{12}$$

Then

$$\Delta(t) = \Delta(s)\Delta(t; s, X(s)), \quad E(t) = E(s)E(t; s, X(s)),$$

for all $0 \leqslant s \leqslant t \leqslant T_0$. We see that $\Delta(t; s, x)$ and $E(t; s, x)$ are $\mathcal{F}_{s,t}$-measurable.

Let $U_0 : (w_0, \infty) \to \mathbb{R}$ and $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$, where $w_0 \geqslant 0$ and $c_0 \geqslant 0$ are functions satisfying the following conditions:

(U1): $U_0 : (w_0, \infty) \to \mathbb{R}$ is a C^3-function such that

1. $U_0'(w) > 0$ for all $w \in (w_0, \infty)$, and

$$\lim_{w \to \infty} U_0'(w) = 0, \qquad \lim_{w \to w_0} U_0'(w) = +\infty,$$

2. $U_0''(w) < 0$ for all $w \in (w_0, \infty)$,
3. $U_0'''(w) > 0$ for all $w \in (w_0, \infty)$.

(U2): $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$ is a continuous function in $w \in (c_0, \infty)$ and $t \in [0, T_0]$, and for all $t \in [0, T_0]$, $u_0(w, t)$ is a C^3-function in w such that for all $t \in [0, T_0]$,

1. $\frac{\partial u_0}{\partial w}(w, t) > 0$ for all $w \in (c_0, \infty)$, and

$$\lim_{w \to \infty} \frac{\partial u_0}{\partial w}(w, t) = 0, \qquad \lim_{w \to c_0} \frac{\partial u_0}{\partial w}(w, t) = +\infty,$$

2. $\frac{\partial^2}{\partial w^2} u_0(w, t) < 0$ for all $w \in (c_0, \infty)$,
3. $\frac{\partial^3}{\partial w^3} u_0(w, t) > 0$ for all $w \in (c_0, \infty)$.

Let us define $U : (w_0, \infty) \times \Omega \to \mathbb{R}$ and $u : (c_0, \infty) \times [0, T_0] \times \Omega \to \mathbb{R}$ by

$$U(w, \omega) = \frac{U_0(w)}{\Delta(T_0)}, \quad u(w, t, \omega) = \frac{u_0(w, t)}{E(t)}.$$

Let us define $V : D \to \mathbb{R}$ by

$$V(C, Z) = E^P \left[\int_0^{T_0} u(C(v), v, \omega) \mathrm{d}v + U(Z, \omega) \right],$$

where D is given in Definition 2.3 Since U_0' and $\partial_w u_0$ are continuous, convex, positive, and strictly decreasing functions, there exist $I_1 : (0,\infty) \times [0,T_0] \to (c_0,\infty)$ and $I_2 : (0,\infty) \to (w_0,\infty)$ such that

$$\frac{\partial}{\partial w} u_0 \left(I_1(u,t),t\right) = u, \quad u \in (0,\infty), \quad U_0' \left(I_2(u)\right) = u, \quad u \in (0,\infty).$$

Then $I_1(u,t)$ and $I_2(u)$ are C^1-functions in u.

Definition 2.2. We say that $(\varphi_0(t), \varphi(t))$ is a portfolio process if $\varphi_0(t)$ is an $(\mathcal{F}_t)$-progressively measurable, $\mathbb{R}$-valued process and $\varphi(t) = (\varphi_1(t), \cdots, \varphi_d(t))$ is an $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}^d$-valued process and the followings are satisfied:

1. $\varphi_0(t) + \varphi_1(t) + \cdots + \varphi_d(t) = 1, \qquad$ for all t.
2. $\int_0^{T_0} \sum_{j=1}^d |\varphi_j(v)|^2 \mathrm{d}v < \infty, \qquad P$-a.s.

From 1. of Definition 2.2, $\varphi_0(t)$ is determined by $\varphi(t)$.

Definition 2.3. We say a triplet (C, Z, φ) is an admissible strategy at $x \geqslant 0$, if $C(t)$ is an $(\mathcal{F}_t)$-progressively measurable, non-negative stochastic process and Z is an $\mathcal{F}_{T_0}$-measurable, non-negative random variable, and $(\varphi_0(t), \varphi(t))$ is a portfolio process and the following conditions are satisfied:

1. $\int_0^{T_0} C(v)\mathrm{d}v < \infty, \quad P$-a.s.
2. Let $W^{x,C,\varphi}(t)$ be a stochastic process of a solution of the following stochastic differential equation:

$$W^{x,C,\varphi}(t) = x + \sum_{j=0}^d \int_0^t \frac{\varphi_j(v) W^{x,C,\varphi}(v)}{S_j(v)} \mathrm{d}S_j(v) - \int_0^t C(v)\mathrm{d}v.$$

   We assume that

$$W^{x,C,\varphi}(t) \geqslant 0, \qquad \text{for all } t \in [0,T_0], \quad P\text{-a.s.}$$

3. $Z = \sum_{j=0}^d \varphi_j(T_0) W^{x,C,\varphi}(T_0)$.

4. $E^P\left[\int_0^{T_0} u(C(v),v,\omega)^-\mathrm{d}v + U(Z,\omega)^-\right] < \infty.$[1]

$\mathcal{A}(x)$ and D denote the set of admissible strategies at x and denote the space of pair (C, Z) respectively.

Let us define a function $\mathcal{Y}:(0,\infty)\to\mathbb{R}$ by

$$\mathcal{Y}(x) = E^P\left[\int_0^{T_0} \Pi(v)I_1\left(x\Pi(v)E(v),v\right)\mathrm{d}v + \Pi(T_0)I_2\left(x\Pi(T_0)\Delta(T_0)\right)\right].$$

We assume the following condition.

ASSUMPTION 2.4. *For given $W>0$,*

$$\lim_{x\to 0}\mathcal{Y}(x) > W, \quad \text{and} \quad \lim_{x\to+\infty}\mathcal{Y}(x) < W.$$

Therefore, there exists $\hat{\lambda}>0$ satisfying the following equation:

$$\mathcal{Y}(\hat{\lambda}) = W. \tag{13}$$

Let $\Theta = (0,\infty)\times\mathbb{R}^n\times(0,\infty)\times(0,\infty)\times[0,T_0]$. We define functions $H:\Theta\to\mathbb{R}$, $G:\Theta\to\mathbb{R}$, and $\mathcal{X}_i:\Theta\to\mathbb{R}$, $i=1,\cdots,d$ as follows: For $(\xi,x,\zeta,\nu,t)\in\Theta$,

$$\begin{aligned} &H(\xi,x,\zeta,\nu,t)\\ &\quad = E^P\left[\int_t^{T_0} \Pi(v;t,x)^2E(v;t,x)\frac{\partial I_1}{\partial u}\left(\hat{\lambda}\xi\nu\Pi(v;t,x)E(v;t,x),v\right)\mathrm{d}v\right]. \end{aligned}$$

$$\begin{aligned} &G(\xi,x,\zeta,\nu,t)\\ &\quad = E^P\left[\Pi(T_0;t,x)^2\Delta(T_0;t,x)\frac{\mathrm{d}I_2}{\mathrm{d}u}\left(\hat{\lambda}\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x)\right)\right]. \end{aligned}$$

For $i=1,\cdots,n$,

$$\begin{aligned} \mathcal{X}_i(\xi,x,\zeta,\nu,t) = \xi E^P\Bigg[&\int_t^{T_0}\frac{\partial\Pi}{\partial x_i}(v;t,x)I_1\left(\hat{\lambda}\xi\nu\Pi(v;t,x)E(v;t,x),v\right)\mathrm{d}v\\ &+\frac{\partial\Pi}{\partial x_i}(T_0;t,x)I_2\left(\hat{\lambda}\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x)\right)\Bigg] \end{aligned}$$

[1] x^- denotes the negative part of the real number x: $x^- = -x\vee 0$.

$$
\begin{aligned}
&+\hat{\lambda}\xi^2\nu E^P\left[\int_t^{T_0}\Pi(v;t,x)\left(\frac{\partial \Pi}{\partial x_i}(v;t,x)E(v;t,x)+\Pi(v;t,x)\frac{\partial E}{\partial x_i}(v;t,x)\right)\right.\\
&\qquad\qquad\left.\times\frac{\partial I_1}{\partial u}(\hat{\lambda}\xi\nu\Pi(v;t,x)E(v;t,x),v)\mathrm{d}v\right]\\
&+\hat{\lambda}\xi^2\zeta E^P\left[\Pi(T_0;t,x)\left(\frac{\partial \Pi}{\partial x_i}(T_0;t,x)\Delta(T_0;t,x)+\Pi(T_0;t,x)\frac{\partial \Delta}{\partial x_i}(T_0;t,x)\right)\right.\\
&\qquad\qquad\qquad\qquad\left.\times\frac{\mathrm{d}I_2}{\mathrm{d}u}(\hat{\lambda}\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\right].
\end{aligned}
$$

$\frac{\partial \Pi}{\partial x_i}(s;t,x)$, $\frac{\partial \Delta}{\partial x_i}(s;t,x)$, and $\frac{\partial E}{\partial x_i}(s;t,x)$, for $i=1,\cdots,n$, in the above equations are given by applying the differential rule of a composite function to $\Pi(s;t,s)$, $\Delta(s;t,x)$ and $E(s;t,x)$. See Section 3.

Also we define functions, $F:\Theta\to\mathbb{R}$, $F_\xi:\Theta\to\mathbb{R}$, $F_\nu:\Theta\to\mathbb{R}$, and $F_\zeta:\Theta\to\mathbb{R}$ as follows:

$$
\begin{aligned}
F(\xi,x,\zeta,\nu,t)=\xi E^P&\left[\int_t^{T_0}\Pi(v;t,x)I_1\left(\hat{\lambda}\xi\nu\Pi(v;t,x)E(v;t,x),v\right)\mathrm{d}v\right.\\
&\left.+\Pi(T_0;t,x)I_2\left(\hat{\lambda}\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x)\right)\right],
\end{aligned}
$$

$$
F_\xi(\xi,x,\zeta,\nu,t)=\frac{1}{\xi}F(\xi,x,\zeta,\nu,t)+\hat{\lambda}\xi\nu H(\xi,x,\zeta,\nu,t)+\hat{\lambda}\xi\zeta G(\xi,x,\zeta,\nu,t),
$$

$$
F_\nu(\xi,x,\zeta,\nu,t)=\hat{\lambda}\xi^2 H(\xi,x,\zeta,\nu,t),
$$

$$
F_\zeta(\xi,x,\zeta,\nu,t)=\hat{\lambda}\xi^2 G(\xi,x,\zeta,\nu,t).
$$

We consider the following conditions:

(A1): For any compact set $K\subset\mathbb{R}^n$,

$$
\sup_{x\in K}E^P\left[\int_0^{T_0}\Pi(v;0,x)\mathrm{d}v+\Pi(T_0;0,x)\right]<\infty. \tag{14}
$$

(A2): For any compact set $K \subset \mathbb{R}^n$, $y \in \mathbb{R}$, and $t \in [0, T_0]$, the following equations hold:

$$(15)\quad \sup_{x\in K} E^P\Bigg[\int_0^{T_0}\Bigg(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\Bigg)\Pi(v;0,x) \times I_1(y\Pi(v;0,x)E(v;0,x),v)\mathrm{d}v\Bigg] < \infty,$$

$$(16)\quad \sup_{x\in K}\sup_{t\in[0,T_0]} E^P\Bigg[\Bigg(1+\sum_{j=1}^n(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\Bigg) \times \Pi(T_0;t,x)I_2(y\Pi(T_0;t,x)\Delta(T_0;t,x))\Bigg] < \infty,$$

$$(17)\quad \sup_{x\in K} E^P\Bigg[\int_0^{T_0}\Bigg(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\Bigg) \times \Pi(v;0,x)^2E(v;0,x) \times \left|\frac{\partial I_1}{\partial u}(y\Pi(v;0,x)E(v;0,x),v)\right|\mathrm{d}v\Bigg] < \infty,$$

$$(18)\quad \sup_{x\in K}\sup_{t\in[0,T_0]} E^P\Bigg[\Bigg(1+\sum_{j=1}^n(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\Bigg) \times \Pi(T_0;t,x)^2 \times \Delta(T_0;t,x)\left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\Bigg] < \infty,$$

$$(19)\quad \sup_{x\in K} E^P\Bigg[\int_0^{T_0}\Bigg(1+\sum_{j=1}^n(|\pi_j(v;0,x)|+|\eta_j(v;0,x)|)\Bigg) \times \Pi(v;0,x)^2E(v;0,x) \times I_1(y\Pi(v;0,x)E(v;0,x),v)\mathrm{d}v\Bigg] < \infty,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\Bigg[\Bigg(1+\sum_{j=1}^{n}(|\pi_j(T_0;t,x)|+|\delta_j(T_0;t,x)|)\Bigg) \times \Pi(T_0;t,x)^2 \times \Delta(T_0;t,x) I_2(y\Pi(T_0;t,x)\Delta(T_0;t,x))\Bigg] < \infty. \tag{20}$$

Then we have the following theorem:

THEOREM 2.5. *We assume Conditions* **(U1)**, **(U2)**, **(S1)**, **(S2)**, **(S3)**, **(S4)**, **(S5)**, **(S6)**, **(S7)**, **(A1)**, **(A2)**, *and* Assumption 2.4. *Then there exists an optimal portfolio strategy* $\hat{\varphi}(t)$ *of Equation* (3). $\hat{\varphi}(t)$ *is given by the following feedback form:*

$$\begin{aligned}\hat{\varphi}(t) = {} & \left(1-\frac{1}{\Pi(t)}\right)(\sigma(t,X(t))^*)^{-1}\lambda(t,X(t)) \\ & - \hat{\lambda}\frac{\Pi(t)E(t)}{W(t)} H(\Pi(t),X(t),\Delta(t),E(t),t)(\sigma(t,X(t))^*)^{-1} \\ & \times (\lambda(t,X(t)) - h(t,X(t))) \\ & - \hat{\lambda}\frac{\Pi(t)\Delta(t)}{W(t)} G(\Pi(t),X(t),\Delta(t),E(t),t)(\sigma(t,X(t))^*)^{-1} \\ & \times (\lambda(t,X(t)) - g(t,X(t))) \\ & + \frac{1}{W(t)}\frac{1}{\Pi(t)}(\sigma(t,X(t))^*)^{-1}(\sigma^X(t,X(t))^*) \\ & \times \begin{pmatrix} \mathcal{X}_1(\Pi(t),X(t),\Delta(t),E(t),t) \\ \vdots \\ \mathcal{X}_n(\Pi(t),X(t),\Delta(t),E(t),t)\end{pmatrix},\end{aligned} \tag{21}$$

where $W(t) = W^{W,\hat{C},\hat{\varphi}}(t)$ *and* $\hat{C}(t)$ *is an optimal consumption strategy.*

REMARK 2.6. Confirming **(S3)** may not be feasible when bonds or other derivative securities are included in tradable securities. In that case, using $\mu^X(t,x)$ and $\sigma^X(t,x)$, we calculate the following:

$$\hat{\mu}(t,x) = \begin{pmatrix} \mu_1(t,x) \\ \vdots \\ \mu_{d-n}(t,x) \\ \mu_1^X(t,x) \\ \vdots \\ \mu_n^X(t,x) \end{pmatrix}, \hat{\sigma}(t,x) = \begin{pmatrix} \sigma_{1,1}(t,x) & \cdots & \sigma_{1,d}(t,x) \\ \vdots & & \vdots \\ \sigma_{d-n,1}(t,x) & \cdots & \sigma_{d-n,d}(t,x) \\ \sigma_{1,1}^X(t,x) & \cdots & \sigma_{1,d}^X(t,x) \\ \vdots & & \vdots \\ \sigma_{n,1}^X(t,x) & \cdots & \sigma_{n,d}^X(t,x) \end{pmatrix},$$

$$\text{and} \quad \lambda(t,x) = \hat{\sigma}(t,x)^{-1} \left(\hat{\mu}(t,x) - \begin{pmatrix} r(t,x) \\ \vdots \\ r(t,x) \\ \bar{\mu}_1^X(t,x) \\ \vdots \\ \bar{\mu}_n^X(t,x) \end{pmatrix} \right),$$

where $\bar{\mu}_j^X(t,x)$, $j = 1, \cdots, n$ are drift terms of $X(t)$ with respect to the equivalent martingale measure ([4] and [6]). Then **(S3)** and **(S5)** will be satisfied with these processes, and Theorem 2.5 also holds.

3. Preliminaries

We introduce the static optimal problem equivalent to Equation (3) as follows:

$$J_1(W, x_0) = \sup_{(C,Z)\in D} V(C,Z), \tag{22}$$

with a constraint

$$E^P\left[\int_0^{T_0} \Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z\right] = W.$$

Proposition 3.1. *The optimal solution of Equation* (22) *is given as follows:*

$$\hat{C}(t) = I_1\left(\hat{\lambda}\Pi(t)E(t), t\right),$$

$$\hat{Z} = I_2\left(\hat{\lambda}\Pi(T_0)\Delta(T_0)\right),$$

where $I_1(u,t), I_2(u)$ *are inverse functions of* $\partial_w u_0(w)$ *and* $U_0'(w)$ *respectively, and* $\hat{\lambda}$ *is the constant given in Equation* (13). *It also holds that,*

$$E^P\left[\int_0^{T_0} u(\hat{C}(v),v,\omega)^- \mathrm{d}v + U(\hat{Z},\omega)^-\right] < \infty.$$

Proof. We prove the proposition using the same argument as that in Cvitanic-Karatzas [2] Section 7. Regarding $u_0(w,t)$, from Lemma 4.3 in Chapter 3 of [6], for any $y,w>0$, $u_0(I_1(y,t),t) \geqslant u_0(w,t) + y(I_1(y,t)-w)$. Therefore,

$$u_0(I_1(\hat{\lambda}\Pi(t)E(t),t),t) \geqslant u_0(1,t) + \hat{\lambda}\Pi(t)E(t)(I_1(\hat{\lambda}\Pi(t)E(t),t)-1), \quad \text{a.s.},$$

and

$$u_0(\hat{C}(t),t) \geqslant u_0(1,t) + \hat{\lambda}\Pi(t)E(t)(\hat{C}(t)-1), \ \text{a.s.}$$

Therefore,

$$u(\hat{C}(t),t,\omega) \geqslant u(1,t,\omega) + \hat{\lambda}\Pi(t)(\hat{C}(t)-1), \ \text{a.s.}$$

Similarly,

$$U(\hat{Z},\omega) \geqslant U(1,\omega) + \hat{\lambda}\Pi(T_0)(\hat{Z}-1), \ \text{a.s.}$$

With Condition **(A1)**, these yield

$$\begin{aligned}
&E^P\left[\int_0^{T_0} u(\hat{C}(v),v,\omega)^-\mathrm{d}v + U(\hat{Z},\omega)^-\right] \\
\leqslant &E^P\left[\int_0^{T_0} \frac{|u_0(1,v)|}{E(v)}\mathrm{d}v\right] + E^P\left[\frac{|U_0(1)|}{\Delta(T_0)}\right] + \hat{\lambda}E^P\left[\int_0^{T_0}\Pi(v)\mathrm{d}v + \Pi(T_0)\right] \\
= &\int_0^{T_0} \frac{|u_0(1,v)|}{E^P[E(v)]}\mathrm{d}v + \frac{|U_0(1)|}{E^P[\Delta(T_0)]} + \hat{\lambda}E^P\left[\int_0^{T_0}\Pi(v)\mathrm{d}v + \Pi(T_0)\right] < \infty.
\end{aligned}$$

Also, for any $C(t) \in \mathcal{F}_t$,

$$u_0\left(I_1(\hat{\lambda}\Pi(t)E(t),t),t\right)$$
$$\geqslant u_0(C(t),t) + \hat{\lambda}\Pi(t)E(t)\left(I_1(\hat{\lambda}\Pi(t)E(t),t) - C(t)\right), \text{ a.s.,}$$

and

$$u(\hat{C}(t),t,\omega) \geqslant u(C(t),t,\omega) + \hat{\lambda}\Pi(t)(\hat{C}(t) - C(t)), \text{ a.s.}$$

Similarly we have, for any $Z \in \mathcal{F}_{T_0}$,

$$U(\hat{Z},\omega) \geqslant U(Z,\omega) + \hat{\lambda}\Pi(T_0)(\hat{Z} - Z), \text{ a.s.}$$

Therefore,

$$\int_0^{T_0} u(\hat{C}(v),v,\omega)\mathrm{d}v + U(\hat{Z},\omega) + \hat{\lambda}\left(\int_0^{T_0} \Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z\right)$$
$$\geqslant \int_0^{T_0} u(C(v),v,\omega)\mathrm{d}v + U(Z,\omega) + \hat{\lambda}\left(\int_0^{T_0} \Pi(v)\hat{C}(v)\mathrm{d}v + \Pi(T_0)\hat{Z}\right) \quad \text{a.s.}$$

This yields

$$\begin{aligned}
&E^P\left[\int_0^{T_0} u(\hat{C}(v),v,\omega)\mathrm{d}v + U(\hat{Z},\omega)\right] \\
\geqslant &E^P\left[\int_0^{T_0} u(C(v),v,\omega)\mathrm{d}v + U(Z,\omega)\right] \\
&\quad + \hat{\lambda}\Bigg\{E^P\left[\int_0^{T_0} \Pi(v)\hat{C}(v)\mathrm{d}v + \Pi(T_0)\hat{Z}\right] \\
&\quad - E^P\left[\int_0^{T_0} \Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z\right]\Bigg\} \\
= &E^P\left[\int_0^{T_0} u(C(v),v,\omega)\mathrm{d}v + U(Z,\omega)\right]
\end{aligned}$$

$$+\hat{\lambda}\left(W - E^P\left[\int_0^{T_0}\Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z\right]\right)$$
$$\geqslant J_1(W, x_0),$$

and we have the conclusion. □

To calculate optimal strategies we use the following theorem.

THEOREM 3.2 (Karatzas-Shreve [6]). *Let $W \geqslant 0$ be given, $C(v)$, $v \in [0, T_0]$ be a consumption process, and Z be a non-negative, $\mathcal{F}_{T_0}$-measurable random variable such that*

$$E^P\left[\int_0^{T_0}\Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z\right] = W.$$

Then there exists a portfolio process $(\varphi_0(t), \varphi(t))$ such that $(C, Z, \varphi) \in \mathcal{A}(W)$ and $Z = W^{C,\varphi}(T_0)$.

Also $\varphi(t)$ is given as

$$\varphi(t) = \frac{1}{W(t)\Pi(t)}(\sigma(t, X(t))^*)^{-1}\psi(t) + (\sigma(t, X(t))^*)^{-1}\lambda(t, X(t)),$$

where $\psi(t)$ is a stochastic process given by the martingale representation theorem:

$$\begin{aligned} M(t) &= E^P\left[\int_0^{T_0}\Pi(v)C(v)\mathrm{d}v + \Pi(T_0)Z \,\Big|\, \mathcal{F}_t\right] \\ &= W + \sum_{j=1}^{d}\int_0^t \psi_j(v)\mathrm{d}B_j(v), \qquad 0 \leqslant t \leqslant T_0, \end{aligned}$$

and $W(t) = W^{W,C,\varphi}(t)$.

Using Proposition 3.1 and Theorem 3.2, we can show that Equation (3) has the optimal solution which is given in Theorem 2.5.

Regarding $\partial\Pi/\partial x_j, \partial\Delta/\partial x_j$, and $\partial E/\partial x_j$, we have the following lemma.

Lemma 3.3.

1. *For any* $0 \leqslant s \leqslant t \leqslant T$, *and for any* $x \in \mathbb{R}^n$, *we have*

$$\frac{\partial \Pi}{\partial x_j}(t;s,x) = \pi_j(t;s,x)\Pi(t;s,x), \quad \frac{\partial \Delta}{\partial x_j}(t;s,x) = \delta_j(t;s,x)\Delta(t;s,x),$$

$$\frac{\partial E}{\partial x_j}(t;s,x) = \eta_j(t;s,x)E(t;s,x).$$

2. *Also, for any compact set* K *in* $\mathbb{R}^n$,

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\pi_j(t;0,x)^2\right] < \infty, \quad j = 1,\cdots,n,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\delta_j(t;0,x)^2\right] < \infty, \quad j = 1,\cdots,n,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\eta_j(t;0,x)^2\right] < \infty, \quad j = 1,\cdots,n.$$

Proof. $X(t;s,x)$ is a stochastic flow of diffeomorphisms. Therefore, from the differential rule of composite function, the first part is proved. Regarding the second part, from **(S1)**, $X(t)$ and $\partial X/\partial x_j$ have finite moments for any order. From **(S1)**,**(S2)**,**(S3)**,**(S5)**, and **(S7)**, derivatives of r, λ_j, g_j, and h_j are bounded. Therefore, we have the conclusion. □

Lemma 3.4. *For* $j = 1,\cdots,d$,

$$E^P\left[\int_0^{T_0}\left(\lambda_j(t,X(t))^2 + \sum_{k=1}^d \sigma_{k,j}(t,X(t))^2 \right.\right.$$
$$\left.\left. + g_j(t,X(t))^2 + h_j(t,X(t))^2\right)\mathrm{d}t\right] < \infty.$$

Proof. From **(S1)**, $X(t)$ has finite moments for any order. From **(S3)**,**(S5)** and **(S7)**, $\lambda_j, \sigma_{k,j}, g_j$, and h_j are linear growth order and the statement is proved easily. □

4. Proof of Theorem 2.5

1. From Proposition 3.1, we have

$$\hat{C}(t) = I_1\left(\hat{\lambda}\Pi(t)E(t), t\right), \quad \hat{Z} = I_2\left(\hat{\lambda}\Pi(T_0)\Delta(T_0)\right).$$

Therefore $M(t)$ in Theorem 3.2 for the optimal solution is given by the following equation:

$$M(t) = N(t) + \int_0^t \Pi(v)\hat{C}(v)\mathrm{d}v,$$

$$\begin{aligned} N(t) = E^P\Bigg[\int_t^{T_0} & \Pi(t)\Pi(v;t,X(t)) \\ & \times I_1\left(\hat{\lambda}\Pi(t)\Pi(v;t,X(t))E(t)E(v;t,X(t)), v\right)\mathrm{d}v \\ & + \Pi(t)\Pi(T_0;t,X(t))I_2\left(\hat{\lambda}\Pi(t)\Pi(T_0;t,X(t))\Delta(t)\Delta(T_0;t,X(t))\right)\Bigg|\mathcal{F}_t\Bigg]. \end{aligned}$$

Note that $\Pi(t), X(t), E(t)$, and $\Delta(t)$ are $\mathcal{F}_t$-measurable. Also $\Pi(s;t,X(t))$, $E(s;t,X(t))$ and $\Delta(s;t,X(t))$ are independent of $\mathcal{F}_t$. Therefore

$$N(t) = F(\Pi(t), X(t), \Delta(t), E(t), t).$$

Let $J(x)$ be a function from $\mathbb{R}$ to $\mathbb{R}_+$ and $J(x) \in C_0^\infty(\mathbb{R})$, such that $J(x) = 0$, if $x \leqslant -1$ or $x \geqslant 0$, and $\int_{-\infty}^{\infty} J(x)\mathrm{d}x = 0$. For any $h > 0$, let us define a mollifier $J_h : \mathbb{R} \to \mathbb{R}$ by

$$J_h(x) = \frac{1}{h}J\left(\frac{x}{h}\right).$$

Then the following are easily proved.

Lemma 4.1. *Suppose $f(x)$ is a decreasing function. Then*

$$f(x) - J_h * f(x) \geqslant 0, \quad \text{for any } x \in (0,\infty).$$

We define $I_1^{(h)} : (0,\infty) \times [0,T_0] \to \mathbb{R}$ and $I_2^{(h)} : (0,\infty) \to \mathbb{R}$ as follows:

$$I_1^{(h)}(u,s) = \frac{1}{h}\int_{-\infty}^{\infty} J\left(\frac{u-y}{h}\right) I_1(y,s)\mathrm{d}y, \quad I_2^{(h)}(u) = J_h * I_2(u).$$

Then $I_1^{(h)} \in C^{\infty,0}((0,\infty)\times[0,T_0])$ and $I_2^{(h)} \in C^{\infty}((0,\infty))$. Let us define $\phi_1(y) \in C^{\infty}(\mathbb{R})$ and $\phi_2(t)(y) \in C^{\infty}(\mathbb{R})$ by

$$\phi_1(y) = \begin{cases} 0, & y \leqslant 1, \\ 1, & y > 2, \end{cases} \quad \phi_2(y) = \begin{cases} 1, & y \leqslant 3, \\ 0, & y > 4. \end{cases}$$

Then there exist constants $C_{(1)}, C_{(2)} > 0$ such that

$$0 \leqslant \phi_1'(y) \leqslant C_{(1)}, \quad \text{for any } y \in \mathbb{R},$$

$$0 \leqslant -\phi_2'(y) \leqslant C_{(2)}, \quad \text{for any } y \in \mathbb{R}.$$

We define $\mathcal{I}_1^{(h)} : \mathbb{R}\times[0,T_0] \to \mathbb{R}$ and $\mathcal{I}_2^{(h)} : \mathbb{R} \to \mathbb{R}$ by the following equation.

$$\mathcal{I}_1^{(h)}(u,s) = I_1^{(h)}(u,s)\phi_1\left(\frac{u}{h}\right)\phi_2(hu), \tag{23}$$

$$\mathcal{I}_2^{(h)}(u) = I_2^{(h)}(u)\phi_1\left(\frac{u}{h}\right)\phi_2(hu). \tag{24}$$

Then $\mathcal{I}_1^{(h)} \in C_0^{\infty,0}(\mathbb{R}\times[0,T_0])$ and $\mathcal{I}_2^{(h)} \in C_0^{\infty}(\mathbb{R})$. From Lemma 4.1, we have the following properties.

$$\mathcal{I}_1^{(h)}(u,s) \leqslant I_1(u,s), \quad \text{for any } u \in (0,\infty), s \in [0,T_0],$$

$$\mathcal{I}_2^{(h)}(u) \leqslant I_2(u), \quad \text{for any } u \in (0,\infty).$$

Also, for any $u \in (0,\infty), s \in [0,T_0]$,

$$\lim_{h\downarrow 0}\mathcal{I}_1^{(h)}(u,s) = I_1(u,s), \quad \lim_{h\downarrow 0}\mathcal{I}_2^{(h)}(u) = I_2(u).$$

We define $F^{(h)} : \Theta \to \mathbb{R}$ by

$$F^{(h)}(\xi, x, \zeta, \nu, t) = \xi E^P \Bigg[\int_t^{T_0} \Pi(v; t, x) \mathcal{I}_1^{(h)}(\hat{\lambda}\xi\nu\Pi(v; t, x)E(v; t, x), v)\mathrm{d}v + \Pi(T_0; t, x)\mathcal{I}_2^{(h)}(\hat{\lambda}\xi\zeta\Pi(T_0; t, x)\Delta(T_0; t, x)) \Bigg].$$

By the monotone convergence theorem,

$$\lim_{h \downarrow 0} F^{(h)}(\Pi(t), X(t), \Delta(t), E(t), t) = F(\Pi(t), X(t), \Delta(t), E(t), t). \tag{25}$$

By Itô's formula,

$$\begin{aligned} F^{(h)}(\Pi(t), X(t), \Delta(t), E(t), t) &= \int_0^t \frac{\partial F^{(h)}}{\partial \xi}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}\Pi(v) \\ &+ \sum_{j=1}^d \int_0^t \frac{\partial F^{(h)}}{\partial x_j}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}X_j(v) \\ &+ \int_0^t \frac{\partial F^{(h)}}{\partial \zeta}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}\Delta(v) \\ &+ \int_0^t \frac{\partial F^{(h)}}{\partial \nu}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}E(v) + \int_0^t \mathrm{d}A^{(h)}(v), \end{aligned}$$

where $A^{(h)}(t)$ is a process of bounded variation.

2. The following equation holds

$$\lim_{h \downarrow 0} \frac{\partial F^{(h)}}{\partial \xi}(\xi, x, \zeta, \nu, t) = F_\xi(\xi, x, \zeta, \nu, t). \tag{26}$$

In fact, by Fubini's Theorem

$$\frac{\partial F^{(h)}}{\partial \xi}(\xi, x, \zeta, \nu, t) = \frac{1}{\xi} F^{(h)}(\xi, x, \zeta, \nu, t) + f_{1,1}^{(h)}(\xi, x, \zeta, \nu, t) + f_{1,2}^{(h)}(\xi, x, \zeta, \nu, t).$$

$f_{1,1}^{(h)}$ and $f_{1,2}^{(h)}$ are defined as

$$\begin{aligned} &f_{1,1}^{(h)}(\xi, x, \zeta, \nu, t) \\ &= E^P \left[\int_t^{T_0} \frac{\partial}{\partial \xi} \left(\Pi(v; t, x)\mathcal{I}_1^{(h)}(\hat{\lambda}\xi\nu\Pi(v; t, x)E(v; t, x), v) \right) \mathrm{d}v \right], \end{aligned}$$

and

$$f_{1,2}^{(h)}(\xi,x,\zeta,\nu,t)=E^P\left[\frac{\partial}{\partial\xi}\Big(\Pi(T_0;t,x)\mathcal{I}_2^{(h)}(\hat\lambda\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\Big)\right].$$

Regarding $f_{1,2}^{(h)}(\xi,x,\zeta,\nu,t)$, we have the following.

Lemma 4.2.

$$\lim_{h\downarrow 0} f_{1,2}^{(h)}(\xi,x,\zeta,\nu,t)=\hat\lambda\zeta G(\xi,x,\zeta,\nu,t).$$

Proof. We have the following decomposition:

$$f_{1,2}^{(h)}(\xi,x,\zeta,\nu,t)=f_{1,2,1}^{(h)}(\xi,x,\zeta,\nu,t)+f_{1,2,2}^{(h)}(\xi,x,\zeta,\nu,t)+f_{1,2,3}^{(h)}(\xi,x,\zeta,\nu,t),$$

$$\begin{aligned}f_{1,2,1}^{(h)}(\xi,x,\zeta,\nu,t)=E^P\bigg[&\frac{1}{\xi}\Pi(T_0;t,x)Y_2(T_0;t,x)\frac{dI_2^{(h)}}{du}(Y_2(T_0;t,x))\\&\times\phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\bigg],\end{aligned}$$

$$\begin{aligned}f_{1,2,2}^{(h)}(\xi,x,\zeta,\nu,t)=E^P\bigg[&\frac{1}{\xi}\Pi(T_0;t,x)\frac{Y_2(T_0;t,x)}{h}I_2^{(h)}(Y_2(T_0;t,x))\\&\times\phi_1'\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\bigg],\end{aligned}$$

$$\begin{aligned}f_{1,2,3}^{(h)}(\xi,x,\zeta,\nu,t)=E^P\bigg[&\frac{1}{\xi}\Pi(T_0;t,x)hY_2(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\\&\times\phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2'(hY_2(T_0;t,x))\bigg],\end{aligned}$$

$$Y_2(s;t,x)=\hat\lambda\xi\zeta\Pi(s;t,x)\Delta(s;t,x).$$

Note that

$$0 \leqslant -\frac{dI_2^{(h)}}{du}(u) \leqslant -\frac{dI_2}{du}(u), \quad \text{for any } u \in (0,\infty).$$

Therefore, by the monotone convergence theorem,

$$\lim_{h\downarrow 0} f_{1,2,1}^{(h)}(\xi,x,\zeta,\nu,t) = \hat{\lambda}\zeta G(\xi,x,\zeta,\nu,t).$$

Regarding $f_{1,2,2}^{(h)}(\xi,x,\zeta,\nu,t)$,

$$\begin{aligned}
0 &\leqslant f_{1,2,2}^{(h)}(\xi,x,\zeta,\nu,t) \\
&\leqslant E^P\left[\frac{1}{\xi}\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\frac{Y_2(T_0;t,x)}{h}\phi_1'\left(\frac{Y_2(T_0;t,x)}{h}\right)\right] \\
&\leqslant \frac{2C_{(1)}}{\xi}E^P\left[\Pi(T_0;t,x)I_2(Y_2(T_0;t,x)), Y_2(T_0;t,x) \leqslant 2h\right].
\end{aligned}$$

From **(A2)**, we have $E^P\Bigg[\Pi(T_0;t,x)I_2(Y_2(T_0;t,x))\Bigg] < \infty$, and

$$\lim_{h\downarrow 0} E^P\Bigg[\Pi(T_0;t,x)I_2(Y_2(T_0;t,x)), Y_2(T_0;t,x) \leqslant 2h\Bigg] = 0.$$

Therefore, $\lim_{h\downarrow 0} f_{1,2,2}^{(h)}(\xi,x,\zeta,\nu,t) = 0$. Similarly, $\lim_{h\downarrow 0} f_{1,2,3}^{(h)}(\xi,x,\zeta,\nu,t) = 0$. $\square$

Regarding $f_{1,1}^{(h)}(\xi,x,\zeta,\nu,t)$, we have the following.

LEMMA 4.3.

$$\lim_{h\downarrow 0} f_{1,1}^{(h)}(\xi,x,\zeta,\nu,t) = \hat{\lambda}\nu H(\xi,x,\zeta,\nu,t).$$

PROOF. We have the following decomposition:

$$f_{1,1}^{(h)}(\xi,x,\zeta,\nu,t) = f_{1,1,1}^{(h)}(\xi,x,\zeta,\nu,t) + f_{1,1,2}^{(h)}(\xi,x,\zeta,\nu,t) + f_{1,1,3}^{(h)}(\xi,x,\zeta,\nu,t),$$

where

$$f^{(h)}_{1,1,1}(\xi,x,\zeta,\nu,t) = E^P\left[\int_t^{T_0} \frac{1}{\xi}\Pi(v;t,x)Y_1(v;t,x)\frac{\partial I_1^{(h)}}{\partial u}(Y_1(v;t,x),v) \right.$$
$$\left. \times\, \phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\right],$$

$$f^{(h)}_{1,1,2}(\xi,x,\zeta,\nu,t) = E^P\left[\int_t^{T_0} \frac{1}{\xi}\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)\frac{Y_1(v;t,x)}{h} \right.$$
$$\left. \times\, \phi_1'\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\right],$$

$$f^{(h)}_{1,1,3}(\xi,x,\zeta,\nu,t) = E^P\left[\int_t^{T_0} \frac{1}{\xi}\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)hY_1(v;t,x) \right.$$
$$\left. \times\, \phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2'(hY_1(v;t,x))\mathrm{d}v\right],$$

and

$$Y_1(v;t,x) = \hat{\lambda}\xi\nu\Pi(s;t,x)E(s;t,x).$$

From the similar argument as in Lemma 4.2, we have

$$\lim_{h\downarrow 0} f^{(h)}_{1,1,1}(\xi,x,\zeta,\nu,t) = \hat{\lambda}\nu H(\xi,x,\zeta,\nu,t).$$

Regarding $f^{(h)}_{1,1,2}(\xi,x,\zeta,\nu,t)$,

$$\begin{aligned}
0 &\leqslant f^{(h)}_{1,1,2}(\xi,x,\zeta,\nu,t) \\
&\leqslant E^P\left[\int_t^{T_0} \frac{1}{\xi}\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)\frac{Y_1(v;t,x)}{h}\phi_1'\left(\frac{Y_1(v;t,x)}{h}\right)\mathrm{d}v\right] \\
&\leqslant \frac{2C_{(1)}}{\xi}E^P\left[\int_t^{T_0} \Pi(v;t,x)I_1(Y_1(v;t,x),v)\mathrm{d}v, B_{2h}\right],
\end{aligned}$$

where

$$B_h = \left\{ \omega; \inf_{s \in [t,T_0]} Y_1(s;t,x) \leqslant h \right\}.$$

From the similar arguments as in Lemma 4.2, **(A2)** yields $\lim_{h\downarrow 0} f^{(h)}_{1,1,2}(\xi, x, \zeta, \nu, t) = 0$. Similarly, $\lim_{h\downarrow 0} f^{(h)}_{1,1,3}(\xi, x, \zeta, \nu, t) = 0$. □

From the above arguments, we have

$$\begin{aligned}
&\lim_{h\downarrow 0} \frac{\partial F^{(h)}}{\partial \xi}(\xi, x, \zeta, \nu, t) \\
&= \frac{1}{\xi} F(\xi, x, \zeta, \nu, t) + \hat{\lambda}\xi\nu H(\xi, x, \zeta, \nu, t) + \hat{\lambda}\xi\zeta G(\xi, x, \zeta, \nu, t).
\end{aligned}$$

Also,we have the following equations.

$$\begin{aligned}
\xi \frac{\partial F^{(h)}}{\partial \xi}&(\xi, x, \zeta, \nu, t) = F^{(h)}(\xi, x, \zeta, \nu, t) \\
&+ \xi E^P\left[\int_t^{T_0} \Pi(v;t,x) Y_1(v;t,x) \frac{\partial I_1^{(h)}}{\partial u}(Y_1(v;t,x), v) \right. \\
&\qquad \left. \times \phi_1\left(\frac{Y_1(s;t,x)}{h}\right) \phi_2(hY_1(v;t,x)) \mathrm{d}v \right] \\
&+ \xi E^P\left[\int_t^{T_0} \Pi(v;t,x) I_1^{(h)}(Y_1(v;t,x), v) \frac{Y_1(v;t,x)}{h} \right. \\
&\qquad \left. \times \phi_1'\left(\frac{Y_1(v;t,x)}{h}\right) \phi_2(hY_1(v;t,x)) \mathrm{d}v \right] \\
&+ \xi E^P\left[\int_t^{T_0} \Pi(v;t,x) I_1^{(h)}(Y_1(v;t,x), v) h Y_1(v;t,x) \right. \\
&\qquad \left. \times \phi_1\left(\frac{Y_1(v;t,x)}{h}\right) \phi_2'(hY_1(v;t,x)) \mathrm{d}v \right] \\
&+ \xi E^P\left[\Pi(T_0;t,x) Y_2(T_0;t,x) \frac{dI_2^{(h)}}{du}(Y_2(T_0;t,x)) \right. \\
&\qquad \left. \times \phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right) \phi_2(hY_2(T_0;t,x)) \right]
\end{aligned} \tag{27}$$

$$
\begin{aligned}
&+\xi E^P\Bigg[\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\frac{Y_2(T_0;t,x)}{h}\\
&\qquad\times\phi_1'\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\Bigg]\\
&+\xi E^P\Bigg[\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))hY_2(T_0;t,x)\\
&\qquad\times\phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2'(hY_2(T_0;t,x))\Bigg],
\end{aligned}
$$

(28)
$$
\begin{aligned}
&\xi F_\xi(\xi,x,\zeta,\nu,t)\\
=&F(\xi,x,\zeta,\nu,t)+\xi E^P\left[\int_t^{T_0}\Pi(v;t,x)Y_1(v;t,x)\frac{\partial I_1}{\partial u}(Y_1(v;t,x),v)\mathrm{d}v\right]\\
&+\xi E^P\left[\Pi(T_0;t,x)Y_2(T_0;t,x)\frac{dI_2}{du}(Y_2(T_0;t,x))\right].
\end{aligned}
$$

3. From similar arguments as in (2), the following equations hold.

(29)
$$
\lim_{h\downarrow 0}\frac{\partial F^{(h)}}{\partial\nu}(\xi,x,\zeta,\nu,t)=F_\nu(\xi,x,\zeta,\nu,t).
$$

(30)
$$
\begin{aligned}
&\nu\frac{\partial F^{(h)}}{\partial\nu}(\xi,x,\zeta,\nu,t)\\
=&E^P\Bigg[\int_t^{T_0}\xi\Pi(v;t,x)Y_1(v;t,x)\frac{\partial I_1^{(h)}}{\partial u}(Y_1(v;t,x),v)\\
&\qquad\times\phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\Bigg]\\
&+E^P\Bigg[\int_t^{T_0}\xi\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)\frac{Y_1(v;t,x)}{h}\\
&\qquad\times\phi_1'\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\Bigg]
\end{aligned}
$$

$$+ E^P\left[\int_t^{T_0} \xi\Pi(v;t,x) I_1^{(h)}(Y_1(v;t,x),v) hY_1(v;t,x) \right.$$
$$\left. \times \phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2'(hY_1(v;t,x))\mathrm{d}v\right],$$

(31)
$$\nu F_\nu(\xi,x,\zeta,\nu,t)$$
$$= E^P\left[\int_t^{T_0} \xi\Pi(v;t,x) Y_1(v;t,x)\frac{\partial I_1}{\partial u}(Y_1(v;t,x),v)\mathrm{d}v\right].$$

(32)
$$\lim_{h\downarrow 0}\frac{\partial F^{(h)}}{\partial \zeta}(\xi,x,\zeta,\nu,t) = F_\zeta(\xi,x,\zeta,\nu,t).$$

(33)
$$\zeta\frac{\partial F^{(h)}}{\partial \zeta}(\xi,x,\zeta,\nu,t)$$
$$= E^P\left[\xi\Pi(T_0;t,x)Y_2(T_0;t,x)\frac{dI_2^{(h)}}{du}(Y_2(T_0;t,x)) \right.$$
$$\left. \times \phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\right]$$
$$+ E^P\left[\xi\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\frac{Y_2(T_0;t,x)}{h} \right.$$
$$\left. \times \phi_1'\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\right]$$
$$+ E^P\left[\xi\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))hY_2(T_0;t,x) \right.$$
$$\left. \times \phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2'(hY_2(T_0;t,x))\right],$$

(34)
$$\zeta F_\zeta(\xi,x,\zeta,\nu,t) = E^P\left[\xi\Pi(T_0;t,x)Y_2(T_0;t,x)\frac{dI_2}{du}(Y_2(T_0;t,x))\right].$$

$$\lim_{h\downarrow 0}\frac{\partial F^{(h)}}{\partial x_j}(\xi,x,\zeta,\nu,t)=\mathcal{X}_j(\xi,x,\zeta,\nu,t),\quad \text{for } j=1,\cdots,n. \tag{35}$$

$$\begin{aligned}
&\frac{\partial F^{(h)}}{\partial x_k}(\xi,x,\zeta,\nu,t)\\
&=\xi E^P\left[\int_t^{T_0}\pi_k(v;t,x)\Pi(v;t,x)\mathcal{I}_1^{(h)}(Y_1(v;t,x),v)\mathrm{d}v\right]\\
&+\xi E^P\left[\int_t^{T_0}(\pi_k(v;t,x)+\eta(v;t,x))\Pi(v;t,x)Y_1(v;t,x)\right.\\
&\left.\times\frac{\partial I_1^{(h)}}{\partial u}(Y_1(v;t,x),v)\phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\right]\\
&+\xi E^P\left[\int_t^{T_0}(\pi_k(v;t,x)+\eta(v;t,x))\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)\right.\\
&\left.\times\frac{Y_1(v;t,x)}{h}\phi_1'\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2(hY_1(v;t,x))\mathrm{d}v\right]\\
&+\xi E^P\left[\int_t^{T_0}(\pi_k(v;t,x)+\eta(v;t,x))\Pi(v;t,x)I_1^{(h)}(Y_1(v;t,x),v)\right.\\
&\left.\times hY_1(v;t,x)\phi_1\left(\frac{Y_1(v;t,x)}{h}\right)\phi_2'(hY_1(v;t,x))\mathrm{d}v\right]\\
&+\xi E^P[\pi_k(T_0;t,x)\Pi(T_0;t,x)\mathcal{I}_2^{(h)}(Y_2(T_0;t,x))]\\
&+\xi E^P\left[(\pi_k(T_0;t,x)+\delta_k(T_0;t,x))\Pi(T_0;t,x)Y_2(T_0;t,x)\right.\\
&\left.\times\frac{\mathrm{d}I_2^{(h)}}{\mathrm{d}u}(Y_2(T_0;t,x))\phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\right]\\
&+\xi E^P\left[(\pi_k(T_0;t,x)+\delta_k(T_0;t,x))\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\right.\\
&\left.\times\frac{Y_2(T_0;t,x)}{h}\phi_1'\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2(hY_2(T_0;t,x))\right]
\end{aligned}\tag{36}$$

$$+\xi E^P\left[(\pi_k(T_0;t,x)+\delta_k(T_0;t,x))\Pi(T_0;t,x)I_2^{(h)}(Y_2(T_0;t,x))\right.$$
$$\left.\times hY_2(T_0;t,x)\phi_1\left(\frac{Y_2(T_0;t,x)}{h}\right)\phi_2'(hY_2(T_0;t,x))\right],$$

(37)
$$\begin{aligned}
&\mathcal{X}_k(\xi,x,\zeta,\nu,t)\\
=&\xi E^P\left[\int_t^{T_0}\frac{\partial\Pi}{\partial x_k}(v;t,x)I_1(Y_1(v;t,x),v)\mathrm{d}v\right]\\
&+\xi E^P\left[\int_t^{T_0}\frac{\partial\Pi}{\partial x_k}(v;t,x)Y_1(v;t,x)\frac{\partial I_1}{\partial u}(Y_1(v;t,x),v)\mathrm{d}v\right]\\
&+\xi E^P\left[\int_t^{T_0}\eta_k(v;t,x)\Pi(v;t,x)Y_1(v;t,x)\frac{\partial I_1}{\partial u}(Y_1(v;t,x),v)\mathrm{d}v\right]\\
&+\xi E^P\left[\frac{\partial\Pi}{\partial x_k}(T_0;t,x)I_2(Y_2(T_0;t,x))\right]\\
&+\xi E^P\left[\frac{\partial\Pi}{\partial x_k}(T_0;t,x)Y_2(T_0;t,x)\frac{\mathrm{d}I_2}{\mathrm{d}u}(Y_2(T_0;t,x))\right]\\
&+\xi E^P\left[\delta_k(T_0;t,x)\Pi(T_0;t,x)Y_2(T_0;t,x)\frac{\mathrm{d}I_2}{\mathrm{d}u}(Y_2(T_0;t,x))\right]
\end{aligned}$$

4. We define $\psi_j^{(h)}(t)$ and $\hat{\psi}_j(t)$ for $j=1,\cdots,d$, $t\in[0,T_0]$ as follows:

$$\begin{aligned}
\psi_j^{(h)}(t)=&-\Pi(t)\frac{\partial F^{(h)}}{\partial\xi}(\Pi(t),X(t),\Delta(t),E(t),t)\lambda_j(t,X(t))\\
&+\sum_{k=1}^n\frac{\partial F^{(h)}}{\partial x_k}(\Pi(t),X(t),\Delta(t),E(t),t)\sigma^X_{k,j}(t,X(t))\\
&+\Delta(t)\frac{\partial F^{(h)}}{\partial\zeta}(\Pi(t),X(t),\Delta(t),E(t),t)g_j(t,X(t))\\
&+E(t)\frac{\partial F^{(h)}}{\partial\nu}(\Pi(t),X(t),\Delta(t),E(t),t)h_j(t,X(t)),
\end{aligned}$$

$$\begin{aligned}\hat{\psi}_j(t) &= -\Pi(t)F_\xi(\Pi(t),X(t),\Delta(t),E(t),t)\lambda_j(t,X(t))\\ &\quad+\sum_{k=1}^{n}\mathcal{X}_k(\Pi(t),X(t),\Delta(t),E(t),t)\sigma^X_{k,j}(t,X(t))\\ &\quad+\Delta(t)F_\zeta(\Pi(t),X(t),\Delta(t),E(t),t)g_j(t,X(t))\\ &\quad+E(t)F_\nu(\Pi(t),X(t),\Delta(t),E(t),t)h_j(t,X(t)).\end{aligned}$$

Let us define $N^{(h)}(t)$ and $\hat{N}(t)$ as follows:

$$N^{(h)}(t)=\sum_{j=1}^{d}\int_0^t\psi_j^{(h)}(v)\mathrm{d}B_j(v),\quad \hat{N}(t)=\sum_{j=1}^{d}\int_0^t\hat{\psi}_j(v)\mathrm{d}B_j(v).$$

We claim that as a local martingale, $N^{(h)}(t)$ converges to $\hat{N}(t)$. To show this claim, it is enough to show that for each $j=1,\cdots,d$, $\langle N^{(h)}-\hat{N}\rangle_{T_0}$ converges to 0 in probability. It is enough to show that for each $j=1,\cdots,d$, $\lim_{h\downarrow 0}\int_0^{T_0}\|\psi_j^{(h)}(v)-\hat{\psi}_j(v)\|^2\mathrm{d}v=0$ in probability. Let us define a sequence of stopping times $\{\tau_N;N=1,2,\cdots\}$ as follows: $\tau_N=\inf\{t\in[0,T_0];\Pi(t)+|X(t)|+\Delta(t)+E(t)\geqslant N\}\wedge T_0$, Using these stopping times, we define stopped processes, $\psi_j^{(h),\tau_N}(t)=\psi_j^{(h)}(t\wedge\tau_N)$, and $\hat{\psi}_j^{\tau_N}(t)=\hat{\psi}_j(t\wedge\tau_N)$.

Let $y\in(0,\infty)$ and $(\xi,x,\zeta,\nu,t)\in\Theta$. Let us define functions $L_1, L_2^j, j=1,\cdots,n, L_3$, and L_4 as follows:

$$\begin{aligned}L_1(\xi,x,\zeta,\nu,t;y)&=E^P\left[\int_t^{T_0}\xi\Pi(v;t,x)|I_1(y\xi\nu\Pi(v;t,x)E(v;t,x),v)|\mathrm{d}v\right]\\ &\quad+E^P\left[\xi\Pi(T_0;t,x)|I_2(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))|\right]\\ &+E^P\left[\int_t^{T_0}y(\xi\Pi(v;t,x))^2(\nu E(v;t,x))\left|\frac{\partial I_1}{\partial u}(y\xi\nu\Pi(v;t,x)E(v;t,x),v)\right|\mathrm{d}v\right]\\ &\quad+E^P\left[y(\xi\Pi(T_0;t,x))^2(\zeta\Delta(T_0;t,x))\left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\right],\end{aligned}$$

$$\begin{aligned}&L_2^j(\xi,x,\zeta,\nu,t;y)\\ &\quad=E^P\left[\int_t^{T_0}|\pi_j(v;t,x)|\xi\Pi(v;t,x)|I_1(y\xi\nu\Pi(v;t,x)E(v;t,x),v)|\mathrm{d}v\right]\end{aligned}$$

$$
\begin{aligned}
&+ E^P\left[|\pi_j(T_0;t,x)|\xi\Pi(T_0;t,x)|I_2(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))|\right] \\
&\quad + E^P\left[\int_t^{T_0} y|\pi_j(v;t,x)|(\xi\Pi(v;t,x))^2(\nu E(v;t,x)) \right.\\
&\qquad \left. \times \left|\frac{\partial I_1}{\partial u}(y\xi\nu\Pi(v;t,x)E(v;t,x),v)\right| \mathrm{d}v\right] \\
&\quad + E^P\left[y|\pi_j(T_0;t,x)|(\xi\Pi(T_0;t,x))^2(\zeta\Delta(T_0;t,x)) \right.\\
&\qquad \left. \times \left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\right] \\
&\quad + E^P\left[\int_t^{T_0} y|\eta_j(v;t,x)|(\xi\Pi(v;t,x))^2(\nu E(v;t,x)) \right.\\
&\qquad \left. \times \left|\frac{\partial I_1}{\partial u}(y\xi\nu\Pi(v;t,x)E(v;t,x),v)\right| \mathrm{d}v\right] \\
&\quad + E^P\left[y|\delta_j(T_0;t,x)|(\xi\Pi(T_0;t,x))^2(\zeta\Delta(T_0;t,x)) \right.\\
&\qquad \left. \times \left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\right] \\
&\quad + E^P\left[\int_t^{T_0} y|\eta_j(v;t,x)|(\xi\Pi(v;t,x))^2(\nu E(v;t,x)) \right.\\
&\qquad \left. \times |I_1(y\xi\nu\Pi(v;t,x)E(v;t,x),v)| \mathrm{d}v\right] \\
&\quad + E^P\left[y|\delta_j(T_0;t,x)|(\xi\Pi(T_0;t,x))^2(\zeta\Delta(T_0;t,x)) \right.\\
&\qquad\qquad \left. \times |I_2(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))|\right],
\end{aligned}
$$

$$L_3(\xi,x,\zeta,\nu,t;y)=E^P\Bigg[\xi\Pi(T_0;t,x)|I_2(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))|\Bigg]$$
$$+E^P\Bigg[y(\xi\Pi(T_0;t,x))^2(\zeta\Delta(T_0;t,x))$$
$$\times\left|\frac{\mathrm{d}I_2}{\mathrm{d}u}(y\xi\zeta\Pi(T_0;t,x)\Delta(T_0;t,x))\right|\Bigg],$$

$$L_4(\xi,x,\zeta,\nu,t;y)=E^P\left[\int_t^{T_0}\xi\Pi(v;t,x)|I_1(y\xi\nu\Pi(v;t,x)E(v;t,x),v)|\mathrm{d}v\right]$$
$$+E^P\Bigg[\int_t^{T_0}y(\xi\Pi(v;t,x))^2(\nu E(v;t,x))$$
$$\times\left|\frac{\partial I_1}{\partial u}(y\xi\nu\Pi(v;t,x)E(v;t,x),v)\right|\mathrm{d}v\Bigg].$$

Also, we define the following processes.

$$\ell_1(t)=L_1(\Pi(t),X(t),\Delta(t),E(t),t;\hat{\lambda}),$$

$$\ell_2^j(t)=L_2^j(\Pi(t),X(t),\Delta(t),E(t),t;\hat{\lambda}),\quad j=1,\cdots,n,$$

$$\ell_3(t)=L_3(\Pi(t),X(t),\Delta(t),E(t),t;\hat{\lambda}),$$

$$\ell_4(t)=L_4(\Pi(t),X(t),\Delta(t),E(t),t;\hat{\lambda}).$$

Then, for $j=1,\cdots,n$, from Equations (27), (28), (30), (31), (33), (34), (36), and (37) there exits a constant $C_1>0$ such that

$$|\psi_j^{(h),\tau_N}(t)-\hat{\psi}_j^{\tau_N}(t)|^2$$
$$=\Bigg|-\Pi(t\wedge\tau_N)\left(\frac{\partial F^{(h)}}{\partial\xi}(t\wedge\tau_N)-F_\xi(t\wedge\tau_N)\right)\lambda_j(t\wedge\tau_N,X(t\wedge\tau_N))$$
$$+\sum_{k=1}^n\left(\frac{\partial F^{(h)}}{\partial x_k}(t\wedge\tau_N)-\mathcal{X}_k(t\wedge\tau_N)\right)\sigma_{k,j}(t\wedge\tau_N,X(t\wedge\tau_N))$$

$$
\begin{aligned}
&+ \Delta(t\wedge\tau_N)\left(\frac{\partial F^{(h)}}{\partial\zeta}(t\wedge\tau_N) - F_\zeta(t\wedge\tau_N)\right) g_j(t\wedge\tau_N, X(t\wedge\tau_N)) \\
&+ E(t\wedge\tau_N)\left(\frac{\partial F^{(h)}}{\partial\nu}(t\wedge\tau_N) - F_\nu(t\wedge\tau_N)\right) h_j(t\wedge\tau_N, X(t\wedge\tau_N))\Bigg|^2 \\
\leqslant C_1\Bigg\{ &\lambda_j(t\wedge\tau_N, X((t\wedge\tau_N)))^2\ell_1(t\wedge\tau_N)^2 \\
&+ \sum_{k=1}^n \sigma_{k,j}(t\wedge\tau_N, X(t\wedge\tau_N))^2\ell_2^k(t\wedge\tau_N)^2 \\
&+ g_j(t\wedge\tau_N, X(t\wedge\tau_N))^2\ell_3(t\wedge\tau_N)^2 \\
&+ h_j(t\wedge\tau_N, X(t\wedge\tau_N))^2\ell_4(t\wedge\tau_N)^2\Bigg\},
\end{aligned}
$$

where $F_\xi(t)$ means $F_\xi(\Pi(t), X(t), \Delta(t), E(t), t)$ and the remaining terms are defined similarly.

From **(A2)**, there exits some function C_2 such that

$$\ell_1(t\wedge\tau_N)^2 + \sum_{k=1}^n \ell_2^k(t\wedge\tau_N)^2 + \ell_3(t\wedge\tau_N)^2 + \ell_4(t\wedge\tau_N)^2 < C_2.$$

Also, there exists $C_3 > 0$ such that

$$
\begin{aligned}
\lambda_j(t\wedge\tau_N, X(t\wedge\tau_N))^2 &+ \sum_{k=1}^n \sigma_{k,j}(t\wedge\tau_N, X(t\wedge\tau_N))^2 \\
&+ g_j(t\wedge\tau_N, X(t\wedge\tau_N))^2 + h_j(t\wedge\tau_N, X(t\wedge\tau_N))^2 < C_3.
\end{aligned}
$$

Therefore for some constant $C > 0$, $|\psi_j^{(h),\tau_N}(t) - \hat{\psi}_j^{\tau_N}(t)|^2 \leqslant C$. From $\lim_{h\downarrow 0}|\psi_j^{(h),\tau_N}(t) - \hat{\psi}_j^{\tau_N}(t)|^2 = 0$, and Lebesgue convergence theorem,

$$\lim_{h\downarrow 0} E^P\left[\int_0^{T_0}\left|\psi_j^{(h),\tau_N}(v) - \hat{\psi}_j^{\tau_N}(v)\right|^2 \mathrm{d}v\right] = 0.$$

Also

$$\int_0^t \psi_j^{(h),\tau_N}(v)dv \to \int_0^t \psi_j^{(h)}(v)\mathrm{d}v \text{ as a local martingale,}$$

and

$$\int_0^t \hat{\psi}_j^{\tau_N}(v)dv \to \int_0^t \hat{\psi}_j(v)\mathrm{d}v \text{ as a local martingale.}$$

Therefore

$$\lim_{h\downarrow 0}\int_0^{T_0} |\psi_j^{(h)}(v) - \hat{\psi}_j(v)|^2\mathrm{d}v = 0, \qquad \text{in probability.}$$

5. From the above arguments,

$$\begin{aligned}
&F(\Pi(t), X(t), \Delta(t), E(t), t) = \lim_{h\downarrow 0} F^{(h)}(\Pi(t), X(t), \Delta(t), E(t), t)\\
=&\lim_{h\downarrow 0}\Bigg\{\int_0^t \frac{\partial F^{(h)}}{\partial \xi}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}\Pi(v)\\
&+\sum_{k=1}^n \int_0^t \frac{\partial F^{(h)}}{\partial x_k}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}X_k(v)\\
&+\int_0^t \frac{\partial F^{(h)}}{\partial \zeta}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}\Delta(v)\\
&+\int_0^t \frac{\partial F^{(h)}}{\partial \nu}(\Pi(v), X(v), \Delta(v), E(v), v)\mathrm{d}E(v)\Bigg\}\\
=&\lim_{h\downarrow 0}\sum_{j=1}^d \int_0^t \psi_j^{(h)}(v)\mathrm{d}B_j(v) = \sum_{j=1}^d \int_0^t \hat{\psi}_j(v)\mathrm{d}B_j(v).
\end{aligned}$$

Therefore, $\psi(t)$ in Theorem 3.2 is given as

$$\begin{aligned}
\psi(t) =& \hat{\psi}(t)\\
=& -W(t)\lambda(t, X(t))\\
&-\hat{\lambda}\Pi(t)^2 E(t) H(\Pi(t), X(t), \Delta(t), E(t), t)\left(\lambda(t, X(t)) - h(t, X(t))\right)\\
&-\hat{\lambda}\Pi(t)^2 \Delta(t) G(\Pi(t), X(t), \Delta(t), E(t), t)\left(\lambda(t, X(t)) - g(t, X(t))\right)\\
&+\sigma^X(t, X(t))^* \begin{pmatrix} \mathcal{X}_1(\Pi(t), X(t), \Delta(t), E(t), t)\\ \vdots \\ \mathcal{X}_n(\Pi(t), X(t), \Delta(t), E(t), t)\end{pmatrix},
\end{aligned}$$

and finally we have the conclusion.

5. HARA Utility Functions

Our goal of this section is to show that in the case of some HARA utility functions, **(A1)** and **(A2)** are replaced by more straightforward conditions and optimal strategies are continuous.

5.1. Power utility functions

Let $g_0(t,x), g_1(t,x), \cdots, g_d(t,x)$ and $h_0(t,x), h_1(t,x), \cdots, h_d(t,x)$ be in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$. Let us define $\Delta(t)$ and $E(t)$ as Equations (8),(10), and (12). Let $U_0 : (w_0, \infty) \to \mathbb{R}$, $w_0 \geqslant 0$ and $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$, $c_0 \geqslant 0$ be given by the following equations:

$$U_0(w) = \frac{(w - w_0)^{1-\gamma}}{1-\gamma}, \quad u_0(w,t) = \beta\frac{(w - c_0)^{1-\gamma}}{1-\gamma},$$

for some common $0 < \gamma < 1$ and $\beta > 0$. Let us define $U : (w_0, \infty) \times \Omega \to \mathbb{R}$, $u : (c_0, \infty) \times [0, T_0] \times \Omega \to \mathbb{R}$ by

$$U(w, \omega) = \frac{U_0(w)}{\Delta(T_0)}, \quad u(w,t,\omega) = \frac{u_0(w,t)}{E(t)},$$

and $V : D \to \mathbb{R}$ by $V(C, Z) = E^P\left[\int_0^{T_0} u(C(v), v)\mathrm{d}v + U(Z)\right]$. Then, Conditions **(U1)** and **(U2)** hold. We say that V is a utility function of power type $(\gamma, \beta, w_0, c_0)$.

Let us define a stochastic process $K(t)$ by $K(t) = E^P\left[\int_t^{T_0} c_0\Pi(v)\mathrm{d}v + w_0\Pi(T_0) \,\middle|\, \mathcal{F}_t\right]$. We assume that $W > K(0)$. Then Assumption 2.4 holds.

Regarding Conditions **(A1)** and **(A2)**, we have alternative conditions which are easily checked compared to the original conditions.

Assumption 5.1. *For any compact set K of $\mathbb{R}^n$ and any $p \in \mathbb{R} \setminus \{0\}$,*

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\Pi(t;0,x)^p\right] < \infty, \quad \sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[E(t;0,x)^p\right] < \infty,$$

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\Delta(t;0,x)^p\right] < \infty.$$

As a corollary of Theorem 2.5 we can show the following.

COROLLARY 5.2.

1. *Let V be a utility function of power type $(\gamma, \beta, w_0, c_0)$, and* Assumption 5.1 *be satisfied. Then Conditions* **(A1)** *and* **(A2)** *hold. Therefore, there exists an optimal strategy to Equation* (3).

2. *Let us define following processes:*

$$A_1(t) = E^P\left[\int_t^{T_0} \Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v \,\middle|\, \mathcal{F}_t\right],$$
$$A_2(t) = E^P\left[\Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \,\middle|\, \mathcal{F}_t\right].$$

For a utility function of power type $(\gamma, \beta, w_0, c_0)$,

$$\hat{\lambda}^{-\frac{1}{\gamma}} = \frac{W - K(0)}{\beta^{1/\gamma} A_1(0) + A_2(0)}.$$

3. *The optimal portfolio strategy $\hat{\varphi}(t)$ is a continuous process and given by*

$$\begin{aligned}
\hat{\varphi}(t) =& \left(1 - \left(1 - \frac{1}{\gamma}\right)\frac{1}{\Pi(t)}\right)(\sigma(t, X(t))^*)^{-1}\lambda(t, X(t)) \\
&- \frac{1}{\gamma}\frac{W}{W(t)\Pi(t)}\frac{1}{\beta^{1/\gamma}A_1(0) + A_2(0)}(\sigma(t, X(t))^*)^{-1} \\
&\qquad \times \sigma^X(t, X(t))^* \begin{pmatrix} D_1(t) \\ \vdots \\ D_n(t) \end{pmatrix} \\
&- \frac{1}{\gamma}\frac{W}{W(t)\Pi(t)}\frac{\beta^{1/\gamma}A_1(t)}{\beta^{1/\gamma}A_1(0) + A_2(0)}(\sigma(t, X(t))^*)^{-1}h(t, X(t)) \\
&- \frac{1}{\gamma}\frac{W}{W(t)\Pi(t)}\frac{A_2(t)}{\beta^{1/\gamma}A_1(0) + A_2(0)}(\sigma(t, X(t))^*)^{-1}g(t, X(t))
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\gamma} \frac{K(t)}{W(t)\Pi(t)} (\sigma(t, X(t))^*)^{-1} \lambda(t, X(t)) \\
&+ \frac{1}{W(t)\Pi(t)} (\sigma(t, X(t))^*)^{-1} \sigma^X(t, X(t))^* \begin{pmatrix} K_1(t) \\ \vdots \\ K_n(t) \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
D_k(t) = (1-\gamma) E^P \Bigg[& \beta^{\frac{1}{\gamma}} \int_t^{T_0} \pi_k(v; t, X(t)) \Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v \\
& + \pi_k(T_0; t, X(t)) \Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \Bigg| \mathcal{F}_t \Bigg] \\
+ E^P \Bigg[& \beta^{\frac{1}{\gamma}} \int_t^{T_0} \eta_k(v; t, X(t)) \Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}} \mathrm{d}v \\
& + \delta_k(T_0; t, X(t)) \Pi(T_0)^{1-\frac{1}{\gamma}} \Delta(T_0)^{-\frac{1}{\gamma}} \Bigg| \mathcal{F}_t \Bigg], k = 1, \cdots, n,
\end{aligned}
$$

and

$$
\begin{aligned}
K_k(t) = E^P \Bigg[& \int_t^{T_0} c_0 \pi_k(v; t, X(t)) \Pi(v) \mathrm{d}v \\
& + w_0 \pi_k(T_0; t, X(t)) \Pi(T_0) \Bigg| \mathcal{F}_t \Bigg], k = 1, \cdots, n.
\end{aligned}
$$

The optimal value function $J(W, x_0)$ of the portfolio problem is given by

$$
J(W, x_0) = \frac{(W - K(0))^{1-\gamma}}{1-\gamma} \left(\beta^{1/\gamma} A_1(0) + A_2(0) \right)^{\gamma}.
$$

Proof. Statement 1 is shown by using Schwartz's inequality. Statement 2 is shown from the definition of $\hat{\lambda}$. Using arguments of uniform integrability, the continuity of optimal solutions is shown. The remaining statements are shown easily. □

5.2. Logarithmic utility functions

Let $g_0(t,x), g_1(t,x), \cdots, g_d(t,x)$ and $h_0(t,x), h_1(t,x), \cdots, h_d(t,x)$ be in $C^{0,\infty}_{\mathrm{ub}}(\mathbb{R}^n)$. Let us define $\Delta(t)$ and $E(t)$ as Equations (8),(10), and (12). Let $U_0 : (w_0, \infty) \to \mathbb{R}$, $w_0 \geqslant 0$ and $u_0 : (c_0, \infty) \times [0, T_0] \to \mathbb{R}$, $c_0 \geqslant 0$ be given by the following equations:

$$U_0(w) = \log(w - w_0), \quad u_0(w,t) = \beta \log(w - c_0),$$

for some $\beta \in \mathbb{R}_{>0}$. Let us define $U : (w_0, \infty) \times \Omega \to \mathbb{R}$, $u : (c_0, \infty) \times [0, T_0] \times \Omega \to \mathbb{R}$ by

$$U(w, \omega) = \frac{U_0(w)}{\Delta(T_0)}, \quad u(w, t, \omega) = \frac{u_0(w,t)}{E(t)},$$

and $V : D \to \mathbb{R}$ by $V(C,Z) = E^P\left[\int_0^{T_0} u(C(v), v)\mathrm{d}v + U(Z)\right]$. Then, Conditions **(U1)** and **(U2)** hold. We say that V is a utility function of logarithmic type (β, w_0, c_0). Regarding stochastic processes, we assume Condition **(S1)**–**(S7)**.

Let us define a stochastic process $K(t)$ by $K(t) = E^P\left[\int_t^{T_0} c_0 \Pi(v)\mathrm{d}v + w_0 \Pi(T_0) \,\middle|\, \mathcal{F}_t\right]$, and we assume that $W > K(0)$. Then Assumption 2.4 holds.

Let us define stochastic processes

$$A_1(t) = E^P\left[\int_t^{T_0} E(v)^{-1}\mathrm{d}v \,\middle|\, \mathcal{F}_t\right], \quad A_2(t) = E^P\left[\Delta(T_0)^{-1} \,\middle|\, \mathcal{F}_t\right],$$

$$\begin{aligned} D_k(t) = E^P\Bigg[\beta \int_t^{T_0} \eta_k(v; t, X(t)) E(v)^{-1}\mathrm{d}v \\ + \delta_k(T_0; t, X(t))\Delta(T_0)^{-1} \,\Bigg|\, \mathcal{F}_t\Bigg], \quad k = 1, \cdots, n, \end{aligned}$$

$$\begin{aligned} K_k(t) = E^P\Bigg[\int_t^{T_0} c_0 \pi_k(v; t, X(t))\Pi(v)\mathrm{d}v \\ + w_0 \pi_k(T_0; t, X(t))\Pi(T_0) \,\Bigg|\, \mathcal{F}_t\Bigg], \quad k = 1, \cdots, n. \end{aligned}$$

Then as in the case of power utility functions, the following corollary holds.

COROLLARY 5.3.

1. *Let V be a utility function of logarithmic type (β, w_0, c_0), and Assumption 5.1 be satisfied. Then Condition* **(A1)** *and* **(A2)** *hold. Therefore, there exists an optimal strategy to Equation* (3).

2. *The following equation holds:*

$$\hat{\lambda}^{-1} = \frac{W - K(0)}{\beta A_1(0) + A_2(0)}.$$

3. *The optimal strategy $\hat{\varphi}(t)$ is a continuous process and given by*

$$\begin{aligned}\hat{\varphi}(t) &= (\sigma(t, X(t))^*)^{-1}\lambda(t, X(t)) \\ &\quad - \frac{W}{W(t)\Pi(t)} \frac{1}{\beta A_1(0) + A_2(0)} (\sigma(t, X(t))^*)^{-1} \\ &\quad \times \sigma^X(t, X(t))^* \begin{pmatrix} D_1(t) \\ \vdots \\ D_n(t) \end{pmatrix} \\ &\quad - \frac{W}{W(t)\Pi(t)} \frac{\beta A_1(t)}{\beta A_1(0) + A_2(0)} (\sigma(t, X(t))^*)^{-1} h(t, X(t)) \\ &\quad - \frac{W}{W(t)\Pi(t)} \frac{A_2(t)}{\beta A_1(0) + A_2(0)} (\sigma(t, X(t))^*)^{-1} g(t, X(t)) \\ &\quad - \frac{K(t)}{W(t)\Pi(t)} (\sigma(t, X(t))^*)^{-1}\lambda(t, X(t)) \\ &\quad + \frac{1}{W(t)\Pi(t)} (\sigma(t, X(t))^*)^{-1} \sigma^X(t, X(t))^* \begin{pmatrix} K_1(t) \\ \vdots \\ K_n(t) \end{pmatrix}.\end{aligned}$$

   The optimal value function $J(W, x_0)$ of the portfolio problem is given by

$$J(W, x_0) = (\beta A_1(0) + A_2(0)) \log (W - K(0)).$$

6. Numerical Examples

This section gives examples of optimal portfolio strategies. An investor has an initial endowment W at time 0. Her utility function is of power type $(\gamma, \beta, 0, 0)$. Also her utilities of consumptions are discounted by a proportion of interest rate and utilities of terminal wealths are discounted by a linear combination of interest rates and stock returns. In this setting, her terminal wealth may be hedged partially against stock returns.

6.1. Settings and optimal portfolio strategies

The market is modeled as follows. Let $d = 2$ and $n = 1$. Let $X(t)$ be

$$X(t) = x_0 - a\int_0^t X(v)\mathrm{d}v + b\int_0^t \mathrm{d}B_1(v) = x_0e^{-at} + be^{-at}\int_0^t e^{av}\mathrm{d}B_1(v),$$

where $a > 0, b \neq 0$. We set $m_t = x_0e^{-at}$, $V_t = b^2(1 - e^{-2at})/(2a)$, and thus

$$P(X(t) \in dx) = \frac{1}{\sqrt{2\pi V_t}}\exp\left\{-\frac{(x - m_t)^2}{2V_t}\right\}\mathrm{d}x.$$

The short rate is modeled by $r_t = r(X(t)) = c\left(\log\left(1 + e^{X(t)}\right)\right)^{\alpha}$, for some $\alpha \in (0, 1)$ and $c > 0$. Because $x^+ \leqslant \log(1 + e^x) \leqslant 1 + x^+$, where $x^+ = x \vee 0$, we have $0 \leqslant (X(t)^+)^{\alpha} \leqslant r(X(t)) \leqslant (1 + X(t)^+)^{\alpha}$.

Money account $S_0(t)$ is given by $S_0(t) = \exp\{\int_0^t r(X(v))\mathrm{d}v\}$. A stock, $S_1(t)$, is traded in the market. $S_1(t) = S\exp\{(\mu - \frac{1}{2}(\rho^2 + \sigma^2))t + \rho B_1(t) + \sigma B_2(t)\}$, where $\sigma > 0$ and $\rho \neq 0$.

Let us introduce a zero bond $S_2(t)$ whose maturity is T. $S_2(t) = E^Q\left[\exp\{-\int_t^T r(X(v))\mathrm{d}v\}|\mathcal{F}_t\right]$, where Q is the equivalent martingale measure, which is supposed to be defined by the following market price of risk processes $(\lambda_1(t), \lambda_2(t))$:

$$\lambda_1(t, X(t)) = \lambda = \text{constant}, \quad \lambda_2(t, X(t)) = c_1 - c_2r(X(t)),$$

where $c_1 = (\mu - \rho\lambda)/\sigma$ and $c_2 = 1/\sigma$. Then, the state price deflator $\Pi(t; x)$

is given by

$$\Pi(t;x) = \exp\biggl\{ -\int_0^t r(X(v;x))\mathrm{d}v - \int_0^t \lambda \mathrm{d}B_1(v)$$
$$- \int_0^t (c_1 - c_2 r(X(v;x)))\mathrm{d}B_2(v) - \frac{1}{2}\int_0^t (\lambda^2 + (c_1 - c_2 r(X(v;x)))^2)\mathrm{d}v \biggr\}.$$

Using $\partial X(t;x)/\partial x = e^{-at}$, we have

$$\begin{aligned}\pi(t;x) = & -\int_0^t \{c_1 c_2 - 1 - {c_2}^2 r(X(v;x))\} r'(X(v;x)) e^{-av}\mathrm{d}v \\ & + c_2 \int_0^t r'(X(v;x)) e^{-av}\mathrm{d}B_2(v).\end{aligned}$$

The volatility matrix of $S_1(t)$ and $S_2(t)$ at time 0 is given by

$$\sigma(0,x) = \begin{pmatrix} \rho & \sigma \\ \sigma_2 & 0 \end{pmatrix}, \text{ where } \sigma_2 = \frac{b}{S_2(0;x)}\frac{\partial S_2}{\partial x}(0;x),$$

$$S_2(0;x) = E^Q\left[\exp\biggl\{ -\int_0^T r(X(v;x))\mathrm{d}v \biggr\}\right],$$

and

$$\frac{\partial S_2}{\partial x}(0;x) = E^Q\left[\biggl\{ -\int_0^T r'(X(v;x))e^{-av}\mathrm{d}v \biggr\} \exp\biggl\{ -\int_0^T r(X(v;x))\mathrm{d}v \biggr\}\right].$$

In this example, we assume that an investor has a utility function of power type $(\gamma, \beta, 0, 0)$, $\gamma \in (0,1), \beta > 0$:

$$u(w,t,\omega) = \beta\frac{w^{1-\gamma}}{1-\gamma}\frac{1}{E(t)}, \quad U(w,\omega) = \frac{w^{1-\gamma}}{1-\gamma}\frac{1}{\Delta(T_0)},$$

where for some $0 < \beta_1, \beta_2, \beta_3$,

$$E(t;x) = \exp\biggl\{ \beta_1 \int_0^t r(X(v;x))dv \biggr\},$$

$$\Delta(t;x) = \exp\left\{\beta_2 \int_0^t r(X(v;x))dv\right\} \\ \times \exp\left\{\beta_3\left((\mu - \frac{1}{2}(\rho^2+\sigma^2))t + \rho B_1(t) + \sigma B_2(t)\right)\right\}.$$

Here, her utilities of consumption and terminal wealth are measured against $E(t;x)$ and $\Delta(T_0;x)$ respectively, to compensate prices and interest rates increases.

$\Delta(t;x)$. Let $h(t,x) = (0,0)$, $\quad g(t,x) = (\beta_3\rho, \beta_3\sigma)$. In this case, we have

$$\eta(t;x) = \beta_1 \int_0^t r'(X(v;x))e^{-av}\mathrm{d}v, \text{ and } \quad \delta(t;x) = \beta_2 \int_0^t r'(X(v;x))e^{-av}\mathrm{d}v.$$

Let K be any compact set of $\mathbb{R}$. Then the following lemma can be shown using Jensen's inequality.

LEMMA 6.1. *For any $p \in \mathbb{R} \setminus \{0\}$,*

$$\sup_{x\in K}\sup_{t\in[0,T_0]} E^P\left[\Pi(t;x)^p\right] < \infty, \quad \sup_{x\in K}\sup_{t\in[0,T_0]} E^P[E(t;x)^p] < \infty, \\ \sup_{x\in K}\sup_{t\in[0,T_0]} E^P[\Delta(t;x)^p] < \infty.$$

From Corollary 5.2, we have the following formula.

$$\hat{\varphi}(0) = \begin{pmatrix}\varphi_s \\ \varphi_b\end{pmatrix} = \frac{1}{\gamma}\begin{pmatrix} \frac{1}{\sigma}(c_1 - c_2 r(x_0)) - \dfrac{\beta_3 A_2(0)}{\beta^{1/\gamma}A_1(0) + A_2(0)} \\ \dfrac{\lambda}{\sigma_2} - \dfrac{\rho(c_1 - c_2 r(x_0))}{\sigma\sigma_2} - \dfrac{b}{\sigma_2}\dfrac{D(0)}{\beta^{1/\gamma}A_1(0) + A_2(0)}\end{pmatrix},$$

where

$$A_1(0) = E^P\left[\int_0^{T_0} \Pi(v)^{1-\frac{1}{\gamma}} E(v)^{-\frac{1}{\gamma}}\mathrm{d}v\right],$$

$$A_2(0) = E^P\left[\Pi(T_0)^{1-\frac{1}{\gamma}}\Delta(T_0)^{-\frac{1}{\gamma}}\right],$$

$$
\begin{aligned}
D(0) = (1-\gamma)E^P\Bigg[\beta^{1/\gamma}\int_0^{T_0} \pi(v)\Pi(v)^{1-\frac{1}{\gamma}}E(v)^{-\frac{1}{\gamma}}\mathrm{d}v \\
+ \pi(T_0)\Pi(T_0)^{1-\frac{1}{\gamma}}\Delta(T_0)^{-\frac{1}{\gamma}}\Bigg] \\
+ E^P\Bigg[\beta^{1/\gamma}\int_0^{T_0} \eta(v)\Pi(v)^{1-\frac{1}{\gamma}}E(v)^{-\frac{1}{\gamma}}\mathrm{d}v \\
+ \delta(T_0)\Pi(T_0)^{1-\frac{1}{\gamma}}\Delta(T_0)^{-\frac{1}{\gamma}}\Bigg].
\end{aligned}
$$

6.2. Monte Carlo simulations

We apply Euler-Maruyama scheme for calculations of optimal strategies. A base case of parameters is presented in Table 1. The time length of each step Δt is 0.01.

1. **Convergence of simulation:** We check the convergence of simulations. Cases of samples are 10, 10^2, 10^3, 10^4, 10^5, and 10^6. Zero bond yields(y) and σ_2 are reported in Table 2. Optimal holding ratios of stock (φ_s) and bond (φ_b) are reported in Table 3. Also, values of objective functions, J, are shown. For each case, 10 trials are performed.

   It seems that convergence speed is proportional to an inverse of number of sample, if we see standard deviations of concerned terms. Therefore, to guarantee precise holding ratios, we have to perform quite large sample size Monte Carlo simulations. This is quite critical in terms of optimal strategies. Even though errors of $A_1(0)$, $A_2(0)$ and so on are small, errors of optimal strategies might be large, because these optimal strategies are ratios of these estimated numbers. Therefore, to

Table 1. Base case parameters.

x_0	a	b	α	c	μ	ρ	σ
1.0	1.0	2.0	0.9	0.01	0.08	-0.14	0.20
T	T_0	λ	γ	β	$\beta 1$	$\beta 2$	$\beta 3$
2	1	-0.165	0.90	2.0	0.1	0.05	0.05

Table 2. Zero bond yields and volatilities.

No. samples	$E[y]$	SD$[y]$	$E[\sigma_2]$	SD$[\sigma_2]$
10	0.00004	6.3247×10^{-2}	-0.01513	1.3409×10^{-2}
10^2	0.01080	1.8464×10^{-2}	-0.01285	4.1714×10^{-3}
10^3	0.00885	6.4658×10^{-3}	-0.01196	1.3846×10^{-3}
10^4	0.01294	2.1558×10^{-3}	-0.01131	4.7085×10^{-4}
10^5	0.01185	6.6245×10^{-4}	-0.01145	1.7123×10^{-4}
10^6	0.01154	2.0531×10^{-4}	-0.01148	6.5880×10^{-5}

Table 3. Values of objective functions and optimal portfolios.

No. samples	$E[J]$	SD$[J]$	$E[\varphi_{\rm s}]$	SD$[\varphi_{\rm s}]$	$E[\varphi_{\rm b}]$	SD$[\varphi_{\rm b}]$
10	28.20662	1.5253×10^{-1}	1.208000	4.4472×10^{-5}	0.64152	6.9523×10^{-1}
10^2	28.21650	4.6766×10^{-2}	1.208009	1.2867×10^{-5}	1.08033	4.9412×10^{-1}
10^3	28.20934	1.3629×10^{-2}	1.208006	5.1640×10^{-6}	1.03338	1.2854×10^{-1}
10^4	28.21811	6.0493×10^{-3}	1.208009	3.1622×10^{-6}	1.08178	4.5354×10^{-2}
10^5	28.21646	1.0157×10^{-3}	1.208010	1.4049×10^{-8}	1.06623	1.6909×10^{-2}
10^6	28.21551	5.9151×10^{-4}	1.208010	0.0000×10^{0}	1.06278	6.5457×10^{-3}

apply our methods in more realistic financial problems, we need more advanced simulation methods such as Kusuoka approximation([9]) or an application of low discrepancy sequences.

2. **Sensitivities with respect to γ:** Table 4 shows values of objective functions and the optimal portfolio for various $\gamma, \beta_1, \beta_2, \beta_3$. Regarding these constants, we set the following relations,

$$\beta_1 = 1 - \gamma = 2\beta_2 = 2\beta_3,$$

which means that the investor's consumptions are discounted by short rates and the terminal wealth is discounted by the average of short rates and returns of stocks to measure her utilities. Number of samples is 10^6.

As γ increases, holding ratios of stock and bond decrease. This is quite reasonable because γ represents a risk aversion tendency of this investor. Also, it is quite interesting that as γ increases, J, the value of objective function increases.

Table 4. Optimal portfolios for various γ.

γ	β_1	β_2	β_3	J	φ_s	φ_b
0.10	0.90	0.45	0.45	2.89211	11.0289	50.1905
0.20	0.80	0.40	0.40	2.74641	5.4635	12.8109
0.30	0.70	0.35	0.35	3.08050	3.5770	6.0761
0.40	0.60	0.30	0.30	3.66091	2.6477	3.6999
0.50	0.50	0.25	0.25	4.55488	2.1074	2.5655
0.60	0.40	0.20	0.20	5.96267	1.7591	1.9263
0.70	0.30	0.15	0.15	8.37761	1.5180	1.5261
0.80	0.20	0.10	0.10	13.29610	1.3419	1.2565
0.90	0.10	0.05	0.05	28.21590	1.2080	1.0648

Table 5. Optimal portfolio for various x_0.

x_0	$r(x_0)$	y	J	φ_{s}	φ_{b}
0.70	0.01092	0.01100	28.21800	1.25955	0.43429
0.80	0.01153	0.01124	28.21730	1.24279	0.63943
0.90	0.01215	0.01148	28.21660	1.22560	0.84947
1.00	0.01278	0.01172	28.21590	1.20801	1.06476
1.10	0.01343	0.01197	28.21520	1.19003	1.28567
1.20	0.01409	0.01222	28.21440	1.17171	1.51261
1.30	0.01476	0.01247	28.21370	1.15305	1.74598

3. **Sensitivities with respect to x_0:** Table 5 shows values of objective functions and the optimal portfolio for various x_0. Number of samples is 10^6.

   As x_0 increases, an initial short rate r_0 increases, and in our setting this means that an expected return of bond increases. Therefore, the holding ratio of bond increases. It is quite interesting that the holding ratio of stock decreases as x_0 increases.

7. Concluding Remarks

This paper gives the mathematical validity of the stochastic flow technique for the calculation of optimal strategies when the market is modeled by a Markovian setting. When investors' utility functions are of power types and logarithmic types, we give straightforward conditions and the continuous solution formula. A simple Cash-Bond-Equity problem is studied as a numerical example. Because optimal solutions are expressed by rational equations of expected values of diffusion processes, efficient simulation methods such as Kusuoka approximation or an application of low discrepancy sequences may improve the speed for calculating solutions. Using those methods we could expect more accurate optimal strategies for realistic financial problems.

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Quantitative Strategy Group
DLIBJ Asset Management Co., Ltd
New Tokyo Building 7F, 3-1
Marunouchi 3-chome, Chiyoda-ku
Tokyo 100-0005, Japan
E-mail: ryuji-fukaya@diam.co.jp