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Decay of Magnetic Eigenfunctions on Asymptotically Hyperbolic Plane

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Abstract. We prove a decay estimate for eigenfunctions of the magnetic Schrödinger operator on two dimensional Riemannian manifold M. We assume that M is simply connected, rotationally symmetric, and asymptotically hyperbolic. The proof is based on a method developed by S. Nakamura in [Nak1].

1. Introduction and Results

1.1. Main result

In this paper we study the decay property of eigenfunctions of the magnetic Schrödinger operators on (the trivial Hermitian line bundle over) a rotationally symmetric, asymptotically hyperbolic two-dimensional Riemannian manifolds with a pole. Here, we mean by *pole* a point at where the exponential map gives a diffeomorphism.

More precisely, we make the following assumptions (M.1)–(M.4) on manifolds.

(M.1) (M, g_M) is a two-dimensional smooth Riemannian manifold with a pole p.

(M.2) In addition, the Riemannian metric g_M is expressed as $dr \otimes dr + g(r)^2 d\theta \otimes d\theta$ in the geodesic polar coordinates $(r, \theta) \in (0, \infty) \times \mathrm{S}^1$ at p for some positive C^2-function g.

(M.3) The radial curvature $K(r) = -g''(r)/g(r)$ is bounded from below on M.

(M.4) The radial curvature $K(r)$ is non-positive and has a limit $\lim_{r\to\infty} K(r) = -1$. Moreover, the integral $\int_0^\infty |1 + K(r)| dr$ is finite.

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To formulate our result, we make the following assumptions on the electro-magnetic fields on M.

(A.1) Let ∇ be a compatible (C^1-)connection on the trivial Hermitian line bundle over M. Moreover, there exists a real-valued, continuous function B on $[0,\infty)$ such that B is positive near infinity, and the curvature 2-form ω associated with the connection ∇ takes the form $B(r)g(r)dr \wedge d\theta$ in the geodesic polar coordinates at p.

$(\mathrm{A.2})_\tau$ The scalar potential V is real-valued, continuous function on $M(\cong \{p\} \cup (0,\infty)\times \mathrm{S}^1)$ and extends to a bounded function $\tilde{V}$ on $[0,\infty)\times S_\tau$ for some $\tau > 0$, where we set

$$S_\tau = \{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \tau\}.$$

Moreover, $\tilde{V}(r,z)$ is analytic with respect to $z \in S_\tau$.

$(\mathrm{A.2})_\infty$ The scalar potential V satisfies $(\mathrm{A.2})_\tau$ for each $\tau > 0$. Moreover, $\tilde{V}(r,z)$ tends to zero as $r \to \infty$ in each S_τ.

We now introduce the magnetic Schrödinger operator with scalar potential

$$H_V = \nabla^*\nabla + V$$

starting from domain $C_0^\infty(M)$, the space of all smooth functions with compact support on M. Then H_V is essentially self-adjoint under the conditions (A.1) and $(\mathrm{A.2})_\tau$ (see Section 2 below). In what follows we shall identify any essentially self-adjoint operator with its operator closure. We shall denote by $L^2(X,\mu)$ the L^2 space on X with measure μ, and denote by $L^2(M)$ the L^2 space with the Riemannian measure dV on M. We denote by $C^k(X,Y)$ the space of all C^k-maps from X to Y.

Throughout this paper, we always assume the following:

(E) A function ψ is a (non-zero) L^2-eigenfunction of H_V corresponding to an eigenvalue E. Moreover, E belongs to the set of discrete spectrum of H_V (i.e., E is an isolated eigenvalue of finite multiplicity).

The main result of this paper is the following:

THEOREM 1.1. *Assume (M.1)–(M.4), (A.1), $(\mathrm{A.2})_\infty$, and (E). Assume that there exists a positive constant B_0 such that* $\lim_{r\to\infty} |B(r) - B_0| =$

0 and $E < B_0^2 + 1/4$. Set $\rho(r) = (B_0^2 + 1/4 - E)^{1/2} r$. Then we have the following two assertions.

1. *For any $\varepsilon > 0$, the function $e^{(1-\varepsilon)\rho}\psi$ belongs to $L^2(M)$ and the estimate $|\psi(r,\theta)| \le C_\varepsilon e^{-(1-\varepsilon)\rho(r)}$ holds on M. Moreover, there exists $c > 0$ such that*

$$\left(\int_0^{2\pi} |\psi(r,\theta)|^2 d\theta\right)^{1/2} \le c e^{-(1-\varepsilon)\rho(r)} e^{-r/2}$$

   holds for any $r \ge 0$.

2. *If we assume further that V is spherically symmetric, then for any $\varepsilon > 0$ there exist $c > 0$ and $r_0 > 0$ such that*

$$\left(\int_0^{2\pi} |\psi(r,\theta)|^2 d\theta\right)^{1/2} \ge c e^{-(1+\varepsilon)\rho(r)} e^{-r/2}$$

   holds for all $r \ge r_0$.

REMARK 1.2.

1. We need not to assume the discreteness of the eigenvalue E in the assertion 2 in Theorem 1.1.

2. The positivity of the constant B_0 is not crucial. In fact, the arguments in Sections 3–5 below are still valid in the case of $B_0 = 0$, and we can reduce the case $B_0 < 0$ to the case $B_0 > 0$ via the transform $(r,\theta) \mapsto (r,-\theta)$ as in the Euclidean case.

3. A typical example of manifold under consideration is the hyperbolic plane. In this case, the essential spectrum of the Schrödinger operator H_V with smooth, asymptotically constant magnetic field and with scalar potential satisfying $(A.2)_\infty$ consists of two parts; the continuous part $[B_0^2 + 1/4, \infty)$ and the discrete part $\{(2n+1)B_0 - n(n+1)\}$ $(0 \le n < |B_0| - 1/2)$, where the latter is empty if $|B_0| < 1/2$ (see Inahama and Shirai [I-S] and see also subsection 1.3 below). Thus, in this case, the number $B_0^2 + 1/4$ is the lower edge of the continuous spectrum, therefore our result is valid for the eigenfunctions in all spectral gaps of the essential spectrum of H_V.

4. The decay rate of the eigenfunctions as in Theorem 1.1 differs from that of the Euclidean case. As is known (see subsection 1.2 below), under conditions similar to (A.1), $(\mathrm{A.2})_\tau$, and (E), any eigenfunction of the Schrödinger operator H_V with (asymptotically) constant magnetic field has the Gaussian decay property at infinity (i.e., decays like Ce^{-cr^2}) in the Euclidian case. This means that the (asymptotically) constant magnetic field cannot bind strongly a quantum mechanical charged particle in the case of negative curvature. Note that the spectrum of the Landau Hamiltonian H_0 ($= H_V$ with $V = 0$) on the hyperbolic plane has the absolutely continuous part, and H_0 has the norm-resolvent continuity with respect to the strength B_0 of the magnetic field (see Inahama and Shirai [I-S]).

5. In the case of the constant magnetic field on the hyperbolic plane, the presence of non-zero scalar potential V can produce infinitely many discrete spectra in the spectral gaps of H_0 (see Shirai [Shi]).

The organization of the paper is as follows. In succeeding subsections 1.2 and 1.3, we recall some related works and some spectral properties of the Schrödinger operator with constant magnetic field on the hyperbolic plane, respectively. Section 2 contains some preliminary results. In Section 3 we derive a weighted L^2-estimate of the eigenfunction under a slightly general setting. The proof is based on the method developed by Nakamura [Nak1]. In Section 4 we give a proof of Theorem 1.1. The upper bound estimate for the eigenfunction follows from the result obtained in Section 3, and the lower bound estimate follows from an argument in Donnelly [Don2].

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1.2. Some related results

In this subsection, we recall some related results. The decay properties of the magnetic eigenfunctions have been studied by many authors, in particular, in the case of the Euclidean spaces. Here, we refer to a few number of works very close to ours.

First, we recall some results in the Euclidean spaces. L. Erdös [Erd] shows the Gaussian upper bound $|\psi(r,\theta)| \le Ce^{-cr^2}$ for eigenfunctions ψ of the two-dimensional Schrödinger operator $\frac{1}{2}(D-A)^2+V$ with rotationally symmetric magnetic field $B(r)$ and bounded scalar potential V. He assumes that $B(r) \geq B_0 > 0$ and $E < B_0/2$ for some constant B_0 and he put some assumptions on the Fourier coefficients $V_m(r) = \int_0^{2\pi} e^{-m\sqrt{-1}\theta} V(r,\theta) d\theta$ of scalar potential V; in particular, he requires that there exists a sequence $\{a_m\}$ such that $|V_m| \leq a_m$, $\sum_m m a_m < \infty$, and $a_m \leq C\delta^m$ for some constants $D > 0, \delta$ satisfying $0 < \delta < 1$.

S. Nakamura [Nak1] (see also [Nak2]) proves that any eigenfunctions ψ of H_V have the Gaussian upper bound in the constant magnetic field case with the assumptions $(\mathrm{A.2})_\tau$ and (E) above. In particular, the estimate $|\psi(r,\theta)| \leq Ce^{-(1-\varepsilon)B_0 r^2/4}$ holds for any $\varepsilon > 0$ under $(A.2)_\infty$, where B_0 is the strength of constant magnetic field. Nakamura's result is generalized by V. Sordoni [Sor] to a class of non-constant magnetic fields in higher dimension.

Their results require more or less some analyticity of the electro-magnetic fields. In fact, L. Erdös [Erd] also shows the eigenfunctions decay slower than Gaussian in general if we drop the condition $a_m \leq D\delta^m$ above, even in the constant magnetic field case.

In non-analytic case, H. D. Cornean and G. Nenciu [C-N] study the decay property of magnetic eigenfunctions.

Next, we recall some results in the case of (non-compact) Riemannian manifolds. H. Donnelly studies the properties of eigenfunctions of the Laplace-Beltrami(-Schrödinger) operators *without* magnetic fields in a series of works [Don1]–[Don5]. In [Don1], he studies the decay properties of the eigenfunctions ϕ of $-\triangle_{\mathbb{D}}$ on the exterior domain $\{z \in \mathbb{D} | d(z,0) \geq r_0\}$, where $r_0 > 0$ and $\mathbb{D}$ is the Poincaré disk. He shows that if $E < 1/4$ there exists a real analytic function $A(\theta)$ on $[0, 2\pi)$ such that $g^{1/2}(r)h_0(r)^{-1}\phi(r,\theta)$ tends to $A(\theta)$ as $r \to \infty$ uniformly in θ, where $g(r) = \sinh r$, h_0 is a solution to the equation $-h''(r) + (F(r)-E)h(h) = 0$ and $F(r) = \frac{1}{2}g''/g - \frac{1}{4}(g'/g)^2$. Especially, the estimate $|\phi(r,\theta)| \leq C\exp\left(-(1/4-E)^{1/2}r - r/2\right)$ holds. Recently, A. Vasy and J. Wunsch [V-W] show that no eigenfunction of the Laplace-Beltrami operator (*without* electro-magnetic fields) decays super-exponentially in the case of certain class of manifolds with pinched negative curvature.

In the magnetic case on (non-compact) Riemannian manifolds, it seems that less results exist. V. Iftimie [Ift] studies the decay of eigenfunctions of the magnetic Schrödinger operators of the form $H_V = (D_i - a_i)g^{ij}(D_j - a_j) + V$ on manifold $(\mathbb{R}^n, g)$, using the Agmon-type estimates. He assumes that the metric $g^{-1} = (g^{ij})$ satisfies $0 < \lambda(|x|)|\xi|^2 \le g^{ij}\xi_i\xi_j \le \Lambda(|x|)|\xi|^2$ for some functions λ, Λ, and $\lim_{r\to\infty} r^{-2}\Lambda(r) = 0$. The assumptions on regularity of scalar and magnetic potentials are rather milder than ours; $\mathbf{a} = (a_j) \in L^2_{loc}(\mathbb{R}^n, \mathbb{R}^n)$, $g^{-1} \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^{n^2})$, and $V \in L^1_{loc}(\mathbb{R}^n, \mathbb{R})$.

Under some additional assumptions, he shows the L^2-upper bound estimate $\|e^\rho \psi\| < \infty$ and the pointwise estimate $|\psi(x)| \le Ce^{-\rho(r)}$ for the weight function $\rho(r) = (\Sigma(H_V) - E)^{1/2} r$, where r is the geodesic distance with respect to the metric g and $\Sigma(H_V)$ is the infimum of the essential spectrum of H_V (Theorem 2.4 and Theorem 4.4 in [Ift]). Here, Iftimie assumes that the eigenvalue E is located *below* $\Sigma(H_V)$, and in the proof of the pointwise upper bound he also assumes that the function λ above is bounded from below by some positive constant.

Thus, as we mentioned in Remark 1.2 (see also subsection 1.3 below), the decay estimate for the eigenfunctions corresponding to eigenvalues in the spectral gaps of the essential spectrum of H_V does not follow directly from Iftimie's result in the hyperbolic case, and the weight function in Theorem 1.1 improves Iftimie's one for such eigenfunctions.

1.3. The hyperbolic plane

The hyperbolic plane $\mathbb{H}$ is a typical example of manifolds we keep in mind in Theorem 1.1. In this case the Schrödinger operator H_0 with constant magnetic field is called the Landau Hamiltonian or the Maass Laplacian (up to gauge transform) and has been extensively studied by many authors (e.g., [Roe], [Els], [Fay], and [Com]). In this subsection we give some comments in this case. However, this subsection contains no new results.

We recall some basic spectral properties of the Landau Hamiltonian H_0 (without scalar potential) from Roelcke [Roe], Elstrodt [Els]. The spectrum of H_0 is given by

$$\mathrm{Spec}(H_0) = \begin{cases} (\cup_{n=0}^N \{E_n\}) \bigcup [B_0^2 + 1/4, \infty) & \text{if } B_0 > 1/2, \\ [B_0^2 + 1/4, \infty) & \text{if } 0 < B_0 \le 1/2, \end{cases}$$

where N is the largest integer less than $B_0 - 1/2$ and $E_n = (2n+1)B_0 - n(n+1)$. We consider the case of $B_0 > 1/2$. Each E_n is the eigenvalue of infinite multiplicity, and a complete set $\{\psi_{nk}\}_{k=-n}^{\infty}$ of the eigenfunctions corresponding to E_n is given by

$$\psi_{nk}(r,\theta) = \left(\frac{1-\tanh\frac{r}{2}e^{-\sqrt{-1}\theta}}{1-\tanh\frac{r}{2}e^{\sqrt{-1}\theta}}\right)^{-B_0}$$
$$\times\sqrt{C_{nk}}(1-\tanh^2\frac{r}{2})^{B_0-n}e^{\sqrt{-1}k\theta}\tanh^{|k|}\frac{r}{2}$$
$$\times{}_2F_1\left(\begin{matrix}B_0-n-B_0\,\mathrm{sign}(k),\ B_0-n+B_0\,\mathrm{sign}(k)+|k|\\ |k|+1\end{matrix};\tanh^2\frac{r}{2}\right)$$

in the geodesic polar coordinates, where ${}_2F_1$ is the Gauss hyper-geometric function and C_{nk} is the L^2-normalizing constant given by

$$C_{nk} = \frac{\beta_n\Gamma(k+\beta_n+n+1)\Gamma(k+n+1)}{4\pi\Gamma(n+1)\Gamma(k+1)^2\Gamma(\beta_n+n+1)}$$

with $\beta_n = 2B_0-2n-1$.

We note that the Riemannian manifold $(\mathbb{H},R^2g_{\mathbb{H}})$ "converges" to $(\mathbb{R}^2,4g_{\mathbb{R}^2})$ as $R\to\infty$ (see Section IV in Comtet [Com]). The decaying factor $(1-\tanh^2(r/2))^{B_0-n}$ in the expression of ψ_{nk}, which has the asymptotics $\exp(-(B_0^2+1/4-E_n)^{1/2}r-r/2)$ as $r\to\infty$, is transformed into $(1-\tanh^2(r/2R))^{B_0R^2-n}$ on the space $(\mathbb{H},R^2g_{\mathbb{H}})$, which converges to the Gaussian factor $e^{-B_0r^2/4}$ in the flat space limit $R\to\infty$.

In the rest of this subsection, we show that the discreteness assumption in (E) is crucial for Theorem 1.1 in general. More precisely, there exists an eigenfunction corresponding to the ground state energy E_0 which has no decay estimate as in Theorem 1.1.

Let $\{\alpha_k\}_{k=0}^\infty$ be a sequence of complex numbers satisfying $\sum_{k=0}^\infty|\alpha_k|^2<\infty$ and put $\psi_0=\sum_{k=0}^\infty\alpha_k\psi_{0k}$. In the case of $n=0$, we have $\psi_{0k}(r,\theta)=\sqrt{C_{0k}}e^{\sqrt{-1}k\theta}\rho^k(1-\rho^2)^{B_0}$ and $C_{0k}=\frac{1}{4\pi\Gamma(2B_0-1)}\frac{\Gamma(k+2B_0)}{\Gamma(k+1)}$, where we put $t=\tanh(r/2)$, and the weight function ρ in Theorem 1.1 is given by $(B_0^2+1/4-E_0)^{1/2}r=(B_0-1/2)r$. For any c satisfying $0<c<B_0-1/2$, we have

$$\begin{aligned} e^{cr}\psi_0(r,\theta) &= [(1+t)/(1-t)]^c\sum_{k=0}^\infty\alpha_k\sqrt{C_{0k}}e^{\sqrt{-1}k\theta}t^k(1-t^2)^{B_0} \\ &= \sum_{k=0}^\infty\alpha_k\sqrt{C_{0k}}e^{\sqrt{-1}k\theta}t^k(1-t^2)^{B_0-c}(1+t)^{2c}, \end{aligned}$$

where we used the relation $r = 2\tanh^{-1}(t) = \log[(1+t)/(1-t)]$. Then we have

$$
\begin{aligned}
(1.1)\quad & \|e^{cr}\psi_0\|^2_{L^2(\mathbb{H})} \\
&= \int_0^1 \frac{4tdt}{(1-t^2)^2}\int_0^{2\pi} d\theta \left|\sum_{k=0}^{\infty}\alpha_k\sqrt{C_{0k}}e^{\sqrt{-1}k\theta}t^k(1-t^2)^{B_0-c}(1+t)^{2c}\right|^2 \\
&= 4\pi\sum_{k=0}^{\infty}|\alpha_k|^2C_{0k}\int_0^1 t^{2k}(1-t^2)^{2(B_0-c-1)}(1+t)^{4c}(2t)dt \\
&\geq 4\pi\sum_{k=0}^{\infty}|\alpha_k|^2C_{0k}\int_0^1 t^{2k}(1-t^2)^{2(B_0-c-1)}(2t)dt \\
&= 4\pi\sum_{k=0}^{\infty}|\alpha_k|^2C_{0k}\int_0^1 s^{(k+1)-1}(1-s)^{(2B_0-2c-1)-1}ds \\
&= 4\pi\sum_{k=0}^{\infty}|\alpha_k|^2C_{0k}B(k+1, 2B_0-2c-1) \\
&= \frac{\Gamma(2B_0-2c-1)}{\Gamma(2B_0-1)}\sum_{k=0}^{\infty}|\alpha_k|^2(k+1)^{2c}\left((k+1)^{-2c}\frac{\Gamma(k+2B_0)}{\Gamma(k+2B_0-2c)}\right),
\end{aligned}
$$

where we used the Plancherel formula with respect to $L^2((0,2\pi), d\theta)$ in the first equality and changed the variable $s = t^2$ in the fourth equality, and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the beta function. Given $c > 0$, we can find a sequence $\{\alpha_k\}_{k=0}^{\infty}$ for which $\sum_{k=0}^{\infty}|\alpha_k|^2$ converges but $\sum_{k=0}^{\infty}|\alpha_k|^2k^{2c}$ diverges. Then, for such $\{\alpha_k\}$s, the rhs of (1.1) diverges since the Stirling formula for the gamma function yields that $\lim_{k\to\infty}k^{-2c}\Gamma(k+2B_0)/\Gamma(k+2B_0-2c) = 1$. This shows that $e^{cr}\psi$ does not belong to $L^2(\mathbb{H})$ for any $c > 0$.

2. Preliminaries

2.1. Essential self-adjointness of H_0

We start with the essential self-adjointness of $H_0 = \nabla^*\nabla$ on $C_0^{\infty}(M)$. Throughout this paper, we always assume (M.1)–(M.3) and we identify the pole p in (M.1) with the origin 0 in the tangent space at p.

Let g and B be as in (M) and (A.1), respectively. We introduce

$$
(2.1)\qquad a(r) = \int_0^r B(t)g(t)dt.
$$

Lemma 2.1. *Assume (A.1). Let* $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ *be the geodesic coordinates around the pole* p *and let* a *be as in (2.1). Put* $\tilde{\mathbf{a}} = \tilde{a}_1 dx_1 + \tilde{a}_2 dx_2 = -x_2 a(r)/r^2\, dx_1 + x_1 a(r)/r^2\, dx_2$. *Then* $\tilde{\mathbf{a}}$ *defines a* C^1*-section of* T^*M *and* $d\tilde{\mathbf{a}} = (Bg/r)dx_1 \wedge dx_2 = Bg dr \wedge d\theta$ *in this coordinates.*

Proof. The assertion in the lemma is obvious unless $r = 0$, since $d\tilde{\mathbf{a}}(x) = (Bg/r)dx_1 \wedge dx_2$ holds if $r \neq 0$. We need to consider the behavior of $\tilde{\mathbf{a}}$ at the origin. Note that $g(0) = 0$ and $g'(0) = 1$ in the geodesic polar coordinate. The continuity of $\tilde{\mathbf{a}}$ at the origin follows from the estimate

$$|\tilde{a}_j(x) - 0| \leq \frac{|x_j|}{r^2}|a(r)| \leq \frac{1}{r}\int_0^r B(t)g(t)dt \leq \sup_{0\leq t\leq r} |B(t)|g(t) \to 0$$

as $r \to +0$. Next we show the existence of the partial derivative $\partial_{x_1}\tilde{a}_2(0) = -B(0)/2$. This follows from the definition $\lim_{h\to 0}(\tilde{a}_2(h,0) - \tilde{a}_2(0,0))/h = \lim_{h\to 0} h^{-2}\int_0^{|h|} B(t)g(t)dt$ combined with the Taylor expansion of Bg at the origin, because of the fact that $g(0) = 0$. Similarly we have $\partial_{x_2}\tilde{a}_1(0) = -B(0)/2$, $\partial_{x_1}\tilde{a}_1(0) = \partial_{x_2}\tilde{a}_2(0) = 0$. Thus we have $d\tilde{\mathbf{a}}(0) = B(0)dx_1 \wedge dx_2$, from which the lemma follows, because of the fact that $g'(0) = 1$. $\square$

Throughout this paper, we denote $-\sqrt{-1}\partial/\partial r$ by D_r, etc.

Lemma 2.2. *Under the assumption (A.1), we can find a* C^1*-section* $\tilde{\mathbf{a}}$ *of* $\Lambda^1 T^*M$ *satisfying the following properties (i)–(iii).*

(i) If we denote by $\tilde{\nabla}$ *the connection on* M *defined by* $D - \tilde{\mathbf{a}}$, *the operator* $\tilde{\nabla}^*\tilde{\nabla}$ *is expressed as*

$$\frac{1}{g}D_r g D_r + \frac{1}{g^2}(D_\theta - a(r))^2 \tag{2.2}$$

in the geodesic polar coordinate at p, *where* a *is as in (2.1).*

(ii) The operator $\tilde{\nabla}^*\tilde{\nabla}$ *is essentially self-adjoint on* $C_0^\infty(M)$.

(iii) The original Bochner Laplacian $\nabla^*\nabla$ *is also essentially self-adjoint on* $C_0^\infty(M)$ *and its operator closure is unitarily equivalent to that of* $\tilde{\nabla}^*\tilde{\nabla}$ *by some gauge transform.*

Proof. We take $\tilde{\mathbf{a}}$ as the one in the previous lemma. Then we find the expression (2.2) by a simple calculation. The essentially self-adjointness of the operator $\tilde{\nabla}^*\tilde{\nabla}$ follows from Theorem 1.1 (with $V = 0$) in Shubin

[Shu] since $\tilde{\mathbf{a}} \in C^1$ by Lemma 2.1. The assertion (iii) follows from the same argument as in the Euclidean case (see, e.g., the proof of Theorem 1.3 in Leinfelder [Lei]) since both ∇ and $\tilde{\nabla}$ give the same magnetic field $Bgdr \wedge d\theta$ by (A.1) and Lemma 2.1. $\square$

In what follows we identify ∇ with $\tilde{\nabla}$ and adopt the expression (2.2), for the assertions in the main theorems are independent of the choice of gauge. (The eigenfunction ψ is transformed into $e^{\sqrt{-1}\lambda}\psi$.) Needless to say, the perturbed operator H_V also has the unique self-adjoint extension under the condition $(V.2)_\tau$.

2.2. Diamagnetic inequality

In this subsection we show that the scalar potential V decaying at infinity is relatively compact with respect to H_0 (see Lemma 2.5 below). The so-called diamagnetic inequality for H_0 is crucial. All the results in this subsection are well-known (see, e.g., Theorem KI in Brüning, Geyler, and Pankrashkin [B-G-P]). We give, however, a proof for the sake of completeness.

First we recall a basic property of the heat kernel on M. Let $\triangle_M$ be the (negative) Laplace-Beltrami operator on M and let $p(t, x, y)$ is the heat kernel on M.

LEMMA 2.3. *Assume (M.1)–(M.4). Then, for each $t > 0$, there exist positive constants C_1 and C_2, which may depend on t, such that the estimate*

$$0 \leq p(t, x, y) \leq C_1 e^{-C_2 d(x,y)^2}$$

holds for all $(x, y) \in M \times M$, where $d(x, y)$ is the Riemannian distance on M. Moreover, $\int_M p(t, x, y)dV(y) = 1$ holds for any $x \in M$. Here, $B_r(x)$ stands for the geodesic ball centred at x of radius r.

PROOF. This is a well-known fact in Riemannian geometry. However, we give a proof for the sake of completeness. Under the assumptions (M.1)–(M.4), the manifold M is complete by the Hopf-Rinow theorem, and the Ricci curvature Ric_M of M is bounded below. In fact, Ric_M is given by $-(g''/g)g_M$. Then M is a Cartan-Hadamard manifold, i.e., a complete, simply connected manifold with non-positive sectional curvature. In particular, this implies that the injectivity radius of M is $+\infty$. The completeness of M implies that the essential self-adjointness of $-\triangle_M$ on $C_0^\infty(M)$ by the classical result of Chernoff [Che], and in this case, the heat kernel $p(t,x,y)$ of $e^{t\triangle_M}$ exists.

It follows from a result at the end of Section 5 in Davies [Dav] that for any $\delta > 0$ there exists a positive constant c_δ such that

$$0 \leq p(t,x,y) \leq c_\delta [\mathrm{Vol}(B_{t^{1/2}}(x))\mathrm{Vol}(B_{t^{1/2}}(y))]^{-1/2} e^{(\delta-\lambda)t} e^{-d(x,y)^2/(4+\delta)t}$$

holds for any $x, y \in M$ and all $t > 0$, where λ is the infimum of the spectrum of $-\triangle_M$. It is enough to show that for any $t > 0$ there exists a positive constant C_t such that $\mathrm{Vol}(B_{t^{1/2}}(x)) \geq C_t$ holds for all $x \in M$.

The Ricci curvature of the two dimensional manifold M coincides with the sectional curvature of M, so we can use Bishop's volume comparison theorem (see e.g., [Cha], Theorem 3 in Section III). Since $K(r) \leq 0$, it follows that for fixed $t > 0$ the quantity $\mathrm{Vol}(B_{t^{1/2}}(x))$ is bounded from below by the volume of a ball of radius $t^{1/2}$ in $\mathbb{R}^2$, which is given by πt and this does not depend on the location of the ball.

Finally, it is well-known that $\int_M p(t,x,y)dV(y) = 1$ holds if M is complete and the Ricci curvature is bounded from below (see, e.g., [Cha], Theorem 5 in Section VIII). $\square$

Let $L_0^2(M)$ be the space of all L^2-functions on M with compact support. We say f *belongs to* $L_0^2(M)+L_\varepsilon^\infty(M)$ if for any $\delta > 0$ there exist $f_1 \in L_0^2(M)$ and $f_2 \in L^\infty(M)$ such that $f = f_1 + f_2$ and $\|f_2\|_{L^\infty} \leq \delta$.

Lemma 2.4. *Let $\triangle_M$ be the (negative) Laplace-Beltrami operator on M. The operator $Ve^{t\triangle_M}$ is compact if $V \in L_0^2(M)+L_\varepsilon^\infty(M)$ and $0 < t < 1$. In particular, $Ve^{t\triangle_M}$ is a Hilbert-Schmidt operator if $V \in L_0^2(M)$.*

Proof. It follows from Lemma 2.3 that

$$\begin{aligned}\int_M p(t,x,y)^2 dV(y) &= \int_M p(t,x,y)p(t,y,x)dV(y) \\ &= p(2t,x,x) \\ &\leq Ct^{-1}(\mathrm{Vol}(B_1(x)))^{-1},\end{aligned}$$

where we used, in the first equality, the symmetricity of the heat kernel, which follows from the self-adjointness of the Laplacian, used the semi-group property in the second inequality and the estimate mentioned above in the last inequality. Thus the integral kernel of $Ve^{t\triangle_M}$ satisfies the estimate

$$\begin{aligned} & \iint_{M\times M} |V(x)p(t,x,y)|^2 dV(x)dV(y) \\ \leq \;& Ct^{-1}\int_M dV(x)|V(x)|^2(\mathrm{Vol}(B_1(x)))^{-1}, \end{aligned}$$

which is finite if $V \in L_0^2(M)$ and $0 < t < 1$. Thus, $Ve^{t\triangle_M}$ is a Hilbert-Schmidt operator.

In the case of general $V = V_1+V_2 \in L_0^2(M)+L_\varepsilon^\infty(M)$, the lemma follows from the uniform estimate $\|Ve^{t\triangle_M} - V_1e^{t\triangle_M}\|_{op} \leq \|V_2\|_{L^\infty}$. $\square$

LEMMA 2.5. *Let z belong the resolvent set of H_0. Let $V \in L_0^2(M) + L_\varepsilon^\infty(M)$. Then the operator $V(H_0 - z)^{-1}$ is compact on $L^2(M)$.*

PROOF. It is enough to show that Ve^{-tH_0} is compact for any $t > 0$ because of the formula $(A-z)^{-1} = \int_0^\infty e^{-t}e^{-tA}dt$.

We first show the so-called diamagnetic inequality for H_0, following the line of argument in the proof of Theorem 1 in Simon [Sim]. In order that, we use Kato's inequality $-\triangle_M|f| \leq \mathrm{Re}(\mathrm{sgn} f)\nabla^*\nabla$ for any f satisfying f, $\nabla^*\nabla f \in L^1_{loc}(M)$, which is a special case of Theorem 5.7 in Braverman, Milatovic and Shubin [B-M-S]. For any $\phi \in D(H_0)$ satisfying $\phi \geq 0$ a.e. and any $u \in C_0^\infty(M)$, we have

$$\begin{aligned} (2.3) \qquad (\phi, (-\triangle_M + 1)|u|)_{L^2} &\leq \mathrm{Re}(\phi, (\mathrm{sgn} u)(\nabla^*\nabla + 1)u)_{L^2} \\ &\leq (\phi, |(\nabla^*\nabla+1)u|)_{L^2} \end{aligned}$$

by Kato's inequality. If we set $\phi = (-\triangle_M + 1)\psi$ for any $\psi \in C_0^\infty(M)$ satisfying $\psi \geq 0$, then $\phi \geq 0$ since $(-\triangle_M + 1)^{-1}$ is positivity preserving, which follows from the non-negativity of the heat kernel of $-\triangle_M$. Set $f = (\nabla^*\nabla + 1)u$ for any $u \in C_0^\infty(M)$. Then (2.3) implies

$$(2.4) \qquad (\psi, |(\nabla^*\nabla+1)^{-1}f|)_{L^2} \leq (\psi, (-\triangle_M + 1)^{-1}|f|)_{L^2}.$$

Since $\nabla^*\nabla$ is essentially self-adjoint on $C_0^\infty(M)$ by Lemma 2.2, the range $(\nabla^*\nabla+1)C_0^\infty(M)$ is dense in $L^2(M)$. Then, by a simple density argument, we obtain $|(\nabla^*\nabla+1)^{-1}f| \leq (-\triangle_M + 1)^{-1}|f|$ for any $f \in L^2(M)$. Using the positivity preserving property of $(-\triangle_M + 1)^{-1}$ repeatedly, we have $|(\nabla^*\nabla + 1)^{-n} f| \leq (-\triangle_M + 1)^{-1}|f|$ for any $f \in L^2(M)$ and any $n \in \mathbb{N}$. Then we obtain the inequality $|e^{-tH_0} f| \leq e^{t\triangle_M}|f|$ by the formula $e^{-tA} = \text{s-lim}_{n\to\infty}(A + n/t)^{-n}$.

Then it follows from Proposition 3.1 in Doi, Iwatsuka, and Mine [D-I-M] and Lemma 2.4 that Ve^{-tH_0} is a Hilbert-Schmidt operator if $0 < t < 1$ and $V \in L^2_0(M)$ since $|Ve^{-tH_0} f| \leq |V|e^{t\triangle_M}|f|$. For general $V \in L^2_0(M) + L^\infty(M)$, by the same argument as that at the end of the proof of Lemma 2.4, we conclude that Ve^{-tH_0} is compact if $0 < t < 1$. In fact, the conclusion is true for all $t > 0$ because of the semi-group property of e^{-tH_0}. $\square$

2.3. Fourier transform

Define the Fourier transform $\mathcal{F}$ from $L^2(\mathrm{S}^1, d\theta)$ to $l^2(\mathbb{Z})$ by

$$\mathcal{F}f(n) = (2\pi)^{-1/2} \int_0^{2\pi} e^{-\sqrt{-1}n\theta} f(\theta) d\theta.$$

Naturally, $\mathcal{F}$ extends to a unitary operator $\mathrm{Id} \otimes \mathcal{F}$ from $L^2(M)$ to $L^2((0,\infty)) \otimes l^2(\mathbb{Z})$, etc. In the sequel we write $\mathcal{F}$ also for such extended operators for simplicity. (Note that $\mathcal{F}$ commutes with any "radial" operator.) We find that

$$\mathcal{F}H_V\mathcal{F}^{-1} = \sum_{n\in\mathbb{Z}} \oplus \left(\frac{1}{g} D_r g D_r + \frac{1}{g^2}(n-a)^2 \right) + \mathcal{V}$$

on $\mathcal{F}C_0^\infty(M)$, where

$$(\mathcal{V}f)(r,n) = (2\pi)^{-1/2} \sum_{m\in\mathbb{Z}} \mathcal{F}V(r, n-m) f(r,m). \tag{2.5}$$

Here, $\sum \oplus$ stands for the direct sum with respect to the direct sum decomposition of Hilbert space $L^2((0,\infty)) \otimes l^2(\mathbb{Z}) \cong \sum_{n\in\mathbb{N}} \oplus L^2((0,\infty))$.

As in Nakamura [Nak1], we can show the following two lemmata.

LEMMA 2.6. *Assume (A.1), (A.2)$_\tau$, and (E). Then there exists $\sigma_0 > 0$ such that*

$$e^{\sigma|n|}\mathcal{F}\psi \quad \textit{belongs to} \quad L^2((0,\infty), g dr) \otimes l^2(\mathbb{Z}) \tag{2.6}$$

if $0 < \sigma \leq \sigma_0$.

If we assume further $(A.2)_\infty$*, it is true that (2.6) for any* $\sigma > 0$.

PROOF. This follows from the standard translation-analytic argument as in the proof of Lemma 3.1 and Lemma 3.2 in Nakamura [Nak1]: For any $\varepsilon > 0$, define a unitary operator T_ε on $L^2(M)$ by $T_\varepsilon f(r,\theta) = f(r,\theta-\varepsilon)$, where we regard $\theta - \varepsilon$ as an element of $\mathrm{S}^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$. We write $H_V(\varepsilon)$ for $T_\varepsilon H_V T_\varepsilon^{-1}$, then it follows that $H_V(\varepsilon) = H_V + (V(r,\theta-\varepsilon) - V(r,\theta))$. Under $(\mathrm{A.2})_\tau$, the family $\{H_V(\varepsilon)\}_\varepsilon$ extends to an analytic family of type (A) on $\{\varepsilon \in \mathbb{C} \,|\, |\mathrm{Im}\ \varepsilon| < \tau\}$ (see, e.g., [R-S4]), and the eigenvalue E is stable for all ε in a small region S_{σ_0}. Moreover, Lemma 2.5 and $(\mathrm{A.2})_\infty$ ensure the stability of the essential spectrum of $H_V(\varepsilon)$ for any $\varepsilon \in \mathbb{C}$. The argument is still valid for any ε under $(\mathrm{A.2})_\infty$. Note that there exists a dense set of translation-analytic vectors, which is in fact obtained by the pull back of translation-analytic vectors in $L^2(\mathbb{R}^2)$, via the unitary operator U_g. □

LEMMA 2.7. *Let* $0 < \sigma < \tau$. *Assume* $(A.2)_\tau$. *Then* $e^{\sigma|n|}|(\mathcal{F}V)(r,n)|$ *is uniformly bounded with respect to* (r,n). *If we assume further* $(A.2)_\infty$*, then* $\lim_{r\to\infty}\sup_n e^{\sigma|n|}|(\mathcal{F}V)(r,n)| = 0$ *holds.*

PROOF. This follows from the standard Payley–Wiener type argument: $\mathcal{F}V(r,n)$ has the expression $(2\pi)^{-1/2}e^{-\sigma|n|}\int_0^{2\pi} e^{\sqrt{-1}nt}\tilde{V}(r,t\mp\sqrt{-1}\sigma)dt$ if $\pm n \leq 0$, respectively. By Cauchy's integral formula, we can show that

$$e^{\sigma|n|}|(\mathcal{F}V)(r,n)| \leq (2\pi)^{-1/2}\int_0^{2\pi}\max\{|\tilde{V}(r,t+\sqrt{-1}\sigma)|, |\tilde{V}(r,t-\sqrt{-1}\sigma)|\}dt,$$

using the fact that $\tilde{V}(r,z+2\pi) = \tilde{V}(r,z)$ holds for any $(r,z) \in (0,\infty)\times S_\tau$ by the analytic continuation theorem. Then the lemma follows immediately. □

LEMMA 2.8. *Let* $r_0 > 0$. *Assume that the radial curvature* $K(r) = -g''(r)/g(r)$ *is bounded from below on some interval* $[r_0,\infty)$ *and assume that*

$$k > \max\{|g'(r_0)/g(r_0)|, \ \sup_{r\geq r_0} K_-(r)^{1/2}\}.$$

Here, we set $K_- = -\min\{0,K\}$. *Then we have the estimate* $|g'(r)/g(r)| \leq k$ *for any* $r \geq r_0$.

PROOF. Let ϵ denote either $+1$ or -1 and let k be as above. Put $G_\epsilon(r) = \epsilon e^{\epsilon k(r-r_0)}$. We have

$$(g'G_\epsilon - gG'_\epsilon)' = -(K + k^2)gG_\epsilon. \tag{2.7}$$

In the case of $\epsilon = +1$, we have $(g'G_\epsilon - gG'_\epsilon)(r) \le (g'G_\epsilon - gG'_\epsilon)(r_0)$ for any $r \ge r_0$, since the rhs of (2.7) is non-positive. Then we have

$$\begin{aligned} g'(r)/g(r) &\le G'_\epsilon(r)/G_\epsilon(r) + \frac{g(r_0)}{g(r)}(g'(r_0)/g(r_0) - k) \\ &\le G'_\epsilon(r)/G_\epsilon(r) = k, \end{aligned}$$

where we used the fact that $k > g'(r_0)/g(r_0)$ in the second inequality.

Similarly, in the case of $\epsilon = -1$, it follows from (2.7) that

$$\begin{aligned} g'(r)/g(r) &\ge G'_\epsilon(r)/G_\epsilon(r) + \frac{g(r_0)}{g(r)}(g'(r_0)/g(r_0) + k) \\ &\ge G'_\epsilon(r)/G_\epsilon(r) = -k. \end{aligned}$$

This completes the proof. □

3. L^2-Exponential Estimate of the Eigenfunction

In this section, following the line of argument as in Nakamura [Nak1], we derive an exponential estimate for the eigenfunctions, assuming (A.1), $(A.2)_\tau$, (E), and

(A.3) Positive, monotone increasing, continuous functions a_1 and a_2 on $[0,\infty)$ satisfy the following conditions (i) and (ii):

(i) $0 < a_1(r) < a_2(r) < a(r)$ holds for all $r > 0$, where a is as in (2.1).

(ii) There exist $R > 0$ and $C_0 > 0$ such that

$$\int_{a_2^{-1}(y)}^{a_2^{-1}(x)} \left[\frac{(a(t) - a_2(t))^2}{g(t)^2} + F(t) - E\right]_+^{1/2} dt \le C_0|x - y|$$

holds if $x \ge y \ge R$, where $[x]_+$ stands for the non-negative part of x and

$$F(r) = \frac{1}{2}\frac{g''(r)}{g(r)} - \frac{1}{4}\left(\frac{g'(r)}{g(r)}\right)^2. \tag{3.1}$$

Given a_1, a_2 and a as in (A.3), we take and fix a monotone increasing, continuous function a_3 on $[0,\infty)$ satisfying $0 < a_1 < a_2 < a_3 < a$ on $[0,\infty)$. Note that each of a_js has the inverse function a_j^{-1} on $[0,\infty)$. In what follows we set $a_j^{-1}(n) = 0$ for all $n \leq 0$ for notational convenience.

For any real-valued function $\tilde{a}$ on $[0,\infty)$, we introduce the set

$$\Omega(\tilde{a}) = \{(r,n) \in [0,\infty) \times \mathbb{Z} | n < \tilde{a}(r)\}.$$

Then the inequality $(n-a)^2 \geq (a-a_j)^2$ holds on $\Omega(a_j)$ for any $j = 1,2,3$, since the condition $n < a_j$ implies that $n - a < -(a - a_j)$, i.e., $|n-a| \geq |a - a_j|$.

Under the conventions above, we define a weight function

$$\rho(r,n) = \begin{cases} \int_{a_2^{-1}(n)}^{r} [(a(t)-a_2(t))^2/g(t)^2 + F(t) - E]_+^{1/2}\, dt & \text{if } (r,n) \in \Omega(a_2), \\ 0 & \text{if } (r,n) \notin \Omega(a_2). \end{cases}$$

LEMMA 3.1. *Let $\mathcal{V}$ be as in (2.5). Then the operator $e^{\delta\rho(r,n)}\mathcal{V}e^{-\delta\rho(r,n)}$ is bounded on $L^2((0,\infty)) \otimes l^2(\mathbb{Z})$ if $\delta C_0 < \tau$, where C_0 is the constant as in (A.3) (ii). In addition, if we assume $(A.2)_\infty$, then $\lim_{R\to\infty} \|\chi_{\{r\geq R\}} e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\|_{op} = 0$, where $\chi_{\{r>R\}}$ stands for the characteristic function of $\{(r,n) | r > R, n \in \mathbb{Z}\}$ and $\|\cdot\|_{op}$ stands for the operator norm.*

PROOF. We prove this lemma as in the proof of Lemma 4.1 in Nakamura [Nak1]. First, we claim that

$$(3.2) \quad |\rho(r,n) - \rho(r,m)| \leq \int_{a_2^{-1}(m)}^{a_2^{-1}(n)} [(a(t)-a_2(t))^2/g(t)^2 + F(t) - E]_+^{1/2}\, dt$$

holds if $n \geq m$. Indeed, $\rho(r,n) - \rho(r,m) = 0$ if $a_2(r) \leq m \leq n$, and $|\rho(r,n) - \rho(r,m)| = |0 - \int_{a_2^{-1}(m)}^{r}| \leq |\int_{a_2^{-1}(m)}^{a_2^{-1}(n)}|$ if $m \leq a_2(r) \leq n$, and $\rho(r,n) - \rho(r,m) = \int_{a_2^{-1}(m)}^{a_2^{-1}(n)}$ if $m \leq n \leq a_2(r)$.

Next, for any σ satisfying $\delta C_0 < \sigma < \tau$, we have

$$\begin{aligned} &|e^{\delta\rho}\mathcal{V}e^{-\delta\rho} f(r,n)|^2 \\ =&(2\pi)^{-1} \left(\sum_{m\in\mathbb{Z}} e^{\delta(\rho(r,n)-\rho(r,m))} |(\mathcal{F}V)(r,n-m) f(r,m)| \right)^2 \end{aligned}$$

$$\begin{aligned}
&\leq (2\pi)^{-1}\left(\sum_{m\in\mathbb{Z}} e^{\delta C_0|n-m|}|(\mathcal{F}V)(r,n-m)f(r,m)|\right)^2 \\
&\leq (2\pi)^{-1}\left(\sup_{m\in\mathbb{Z}}|e^{\sigma|m|}(\mathcal{F}V)(r,m)|\sum_{m\in\mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|}|f(r,m)|\right)^2 \\
&\leq (2\pi)^{-1}\sup_{m\in\mathbb{Z}}|e^{\sigma|m|}(\mathcal{F}V)(r,m)|^2 \\
&\quad\times\left(\sum_{m\in\mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|}\right)\left(\sum_{m\in\mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|}|f(r,m)|^2\right) \\
&= C\sup_{m\in\mathbb{Z}}|e^{\sigma|m|}(\mathcal{F}V)(r,m)|^2\sum_{m\in\mathbb{Z}} e^{-(\sigma-\delta C_0)|n-m|}|f(r,m)|^2,
\end{aligned}$$

where we used (A.3) (ii) and (3.2) in the first inequality and the Schwarz inequality in the third inequality. Then it follows that

$$\|\chi_{\{r>R\}}e^{\delta\rho}\mathcal{V}e^{-\delta\rho}f\| \leq C\sup_{r>R}\sup_{m\in\mathbb{Z}}|e^{\sigma|m|}(\mathcal{F}V)(r,m)|\|f\|,$$

from which we have the result because of Lemma 2.7. $\square$

Define a unitary operator U_g from $L^2(M,dV)$ to $L^2(M,drd\theta)$ by $U_g f = g^{1/2}f$. Then a simple calculation shows that

$$U_g H_V U_g^{-1} = D_r^2 + \frac{1}{g^2}(D_\theta - a)^2 + F + V. \tag{3.3}$$

Lemma 3.2. *Let χ_Ω be the characteristic function of Ω and let a_js be as above. Assume that $0<\delta<1$. Then there exists $R>0$ such that*

$$\begin{aligned}
&\operatorname{Re}\ \chi_{\{r>R\}}\chi_{\Omega(a_3)}e^{\delta\rho(r,n)}\mathcal{F}(H_V-E)\mathcal{F}^{-1}e^{-\delta\rho(r,n)}\chi_{\Omega(a_3)}\chi_{\{r>R\}} \\
\geq\ &\min\{(1-\delta^2)\left((a-a_2)^2/g^2+F-E\right), \\
&\quad\times(a-a_3)^2/g^2+F-E\}\chi_{\Omega(a_3)}\chi_{\{r>R\}} \\
&\quad-\operatorname{Re}\ \chi_{\{r>R\}}\chi_{\Omega(a_3)}e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\chi_{\Omega(a_3)}\chi_{\{r>R\}},
\end{aligned} \tag{3.4}$$

where Re *stands for the real part.*

PROOF. We have

$$
\begin{aligned}
(3.5)\qquad & \mathrm{Re}\, e^{\delta\rho}\mathcal{F}(H_V - E)\mathcal{F}^{-1}e^{-\delta\rho} \\
= \quad & \mathrm{Re}\ U_g^{-1}e^{\delta\rho}\left\{\sum_{n\in\mathbb{Z}}\oplus\left(D_r^2+(n-a)^2/g^2+F-E\right)+\mathcal{V}\right\}e^{-\delta\rho}U_g.
\end{aligned}
$$

Since $\rho(r,n)$ vanishes on $\Omega(a_3)\setminus\Omega(a_2)$, the rhs of (3.5) is bounded from below by

$$
\sum_n\oplus\left((a-a_3)^2/g^2+F-E\right)+\mathrm{Re}\ e^{\delta\rho}\mathcal{V}e^{-\delta\rho},
$$

where we used the fact that $(n-a)^2\geq(a-a_3)^2$ holds on $\Omega(a_3)$. On $\Omega(a_2)\cap\{(r,n)|r>R\}$, the rhs of (3.5) is bounded from below by

$$
\begin{aligned}
& \mathrm{Re}\ U_g^{-1}\sum_n\oplus\left((n-a)^2/g^2+F-E-\delta^2(\partial_r\rho)^2\right)U_g \\
& +\mathrm{Re}\ U_g^{-1}e^{\delta\rho}\mathcal{V}e^{-\delta\rho}U_g \\
\geq\ & \sum_n\oplus(1-\delta^2)\left((a-a_2)^2/g^2+F-E\right)-\mathrm{Re}\ e^{\delta\rho}\mathcal{V}e^{-\delta\rho}
\end{aligned}
$$

if we take $R>0$ so large that$((a-a_2)/g)^2+F-E>0$ holds for all $r>R$. Then the result follows. $\square$

Let $R>0$ and let f_R be a smooth function on $(0,\infty)\times\mathbb{Z}$ satisfying the following conditions: $|\partial_r f_R|+|\partial_r^2 f_R|$ is bounded, $0\leq f_R(r,n)\leq 1$ holds for all (r,n) and

$$
f_R(r,n)=\begin{cases}1 & \text{if } (r,n)\in\Omega(a_2) \text{ and } r\geq 2R,\\ 0 & \text{if } (r,n)\notin\Omega(a_3) \text{ or } r\leq R.\end{cases}
$$

LEMMA 3.3. *Let $0<\delta<1$ and let f_R be as above. Then there exists a constant $C=C(R,\rho,\delta,g,V,E)>0$ such that*

$$
C_R\|e^{\delta\rho}f_R\mathcal{F}\psi\|\leq C\|\psi\|
$$

holds for any large $R>0$, where we define C_R by

$$
\begin{aligned}
\inf_{r>R}\min\Bigg\{ & (1-\delta^2)\left(\frac{(a(r)-a_2(r))^2}{g(r)^2}+F(r)-E\right), \\
& \qquad\qquad \frac{(a(r)-a_3(r))^2}{g(r)^2}+F(r)-E\Bigg\} \\
& -3\|\chi_{\{r>R\}}e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\|_{op}.
\end{aligned}
$$

PROOF. In this proof we put

$$Q = \mathrm{Re}\,(e^{\delta\rho} f_R \mathcal{F}\psi, e^{\delta\rho}\mathcal{F}(H_V - E)\mathcal{F}^{-1}e^{-\delta\rho}e^{\delta\rho} f_R \mathcal{F}\psi).$$

Then it follows from Lemma 3.2 that Q is bounded from below by

$$\begin{aligned} (3.6)\qquad & (\min\big\{(1-\delta^2)\left((a-a_2)^2/g^2 + F - E\right), \\ & \qquad (a-a_3)^2/g^2 + F - E\big\} e^{\delta\rho} f_R \mathcal{F}\psi, e^{\delta\rho} f_R \mathcal{F}\psi) \\ & -\|\chi_{\{r>R\}} e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\|_{op}\|e^{\delta\rho} f_R \mathcal{F}\psi\|^2, \end{aligned}$$

where we used the fact that supp $f_R \subset \Omega(a_3)$ and $f_R \cdot \chi_{\Omega(a_3)} = f_R$ hold.

On the other hand, Q is bounded from above by

$$\begin{aligned} (3.7)\qquad & |(e^{2\delta\rho} f_R \mathcal{F}\psi, [\mathcal{F}(H_V - E)\mathcal{F}^{-1}, f_R]\mathcal{F}\psi)| \\ \leq\ & |(e^{2\delta\rho} f_R \mathcal{F}\psi, [\sum_n \oplus \frac{1}{g} D_r g D_r, f_R]\mathcal{F}\psi)| + |(e^{2\delta\rho} f_R \mathcal{F}\psi, [\mathcal{V}, f_R]\mathcal{F}\psi)| \\ =\ & |(e^{\delta\rho} f_R \mathcal{F}\psi, e^{\delta\rho}[\sum_n \oplus \frac{1}{g} D_r g D_r, f_R]\mathcal{F}\psi)| \\ & +|(e^{\delta\rho} f_R \mathcal{F}\psi, [e^{\delta\rho}\mathcal{V}e^{-\delta\rho}, f_R]e^{\delta\rho}\mathcal{F}\psi)|, \end{aligned}$$

where we used the eigen-equation. Note that

$$\rho(r,n) \leq \int_0^{2R} \sqrt{[(a-a_2)^2/g^2 + F - E]_+}\, dt$$

holds on the support of $\partial_r f_R$ or on the support of $1 - f_R$. The first term on the rhs of (3.7) is bounded from above by

$$\begin{aligned} (3.8)\qquad & C(\|\chi_{\mathrm{supp}\,\partial_r f_R}\mathcal{F}D_r\psi\| + \|\psi\|)\|e^{\delta\rho} f_R \mathcal{F}\psi\| \\ \leq\ & C(\|H_0\psi\| + \|\psi\|)\|e^{\delta\rho} f_R \mathcal{F}\psi\|, \end{aligned}$$

where we used the fact that $\|D_r\psi\| \leq C(\|H_0\psi\| + \|\psi\|)$ and used Lemma 2.8 and the boundedness of $\partial_r f_R$ and $\partial_r^2 f_R$. The second term on the rhs of (3.7) is bounded from above by

$$\begin{aligned} (3.9)\quad & 2\|\chi_{\{r>R\}} e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\|\|e^{\delta\rho} f_R \mathcal{F}\psi\|(\|e^{\delta\rho} f_R \mathcal{F}\psi\| + \|e^{\delta\rho}(1-f_R)\mathcal{F}\psi\|) \\ \leq\ & 2\|\chi_{\{r>R\}} e^{\delta\rho}\mathcal{V}e^{-\delta\rho}\|\|e^{\delta\rho} f_R \mathcal{F}\psi\|(\|e^{\delta\rho} f_R \mathcal{F}\psi\| + (\sup_{0\leq r\leq 2R} e^{\delta\rho(r,0)})\|\psi\|). \end{aligned}$$

Then the result follows from (3.6)–(3.9).

In fact, we need to regularize the weight function ρ appropriately, since ρ is not bounded. However, the argument above remains valid for such regularized weights. (For instance, we can use $\rho_N(r,n) = \min\{\rho(r,n), N\}$ as in [Nak1]. See also Section 4 in [Sor].) $\square$

Let $0 < \sigma < \tau$ and $0 < \delta < 1$ and put

$$(3.10)\quad \rho_1(r) = \min\{\sigma a_1(r),\ \delta \int_{a_2^{-1}(a_1(r))}^{r} \left[(a(t) - a_2(t))^2/g(t)^2 + F(t) - E\right]_+^{1/2} dt\}.$$

The main result in this section is the following:

PROPOSITION 3.4. *Assume (M.1)–(M.3), (A.1), (A.2)$_\tau$, (A.3), and (E). Let δ, σ, and ρ_1 be as above. Assume that the constant C_R as in Lemma 3.3 is positive for some $R > 0$. Then $e^{\rho_1}\psi$ belongs to $L^2(M)$.*

PROOF. Since $a_2^{-1}(n) \le a_2^{-1}(a_1(r))$ holds on $\Omega(a_1) = \{(r,n) | n < a_1(r)\}$, we have

$$\begin{aligned} \rho(r,n) &= \int_{a_2^{-1}(n)}^{r} [(a - a_2)^2/g^2 + F - E]_+^{1/2} \\ &\ge \int_{a_2^{-1}(a_1(r))}^{r} [(a - a_2)^2/g^2 + F - E]_+^{1/2} \\ &=: \tilde{\rho}(r). \end{aligned}$$

Let f_R be as before and take $R > 0$ sufficiently large. Then we obtain

$$(3.11)\quad \begin{aligned} \|\chi_{\Omega(a_1)} e^{\delta\tilde{\rho}} \mathcal{F}\psi\| &\le \|\chi_{\Omega(a_1)} e^{\delta\rho(r,n)} \mathcal{F}\psi\| \\ &\le \|f_R \chi_{\Omega(a_1)} e^{\delta\rho(r,n)} \mathcal{F}\psi\| \\ &\quad + \|(1 - f_R)\chi_{\Omega(a_1)} e^{\delta\rho(r,n)} \mathcal{F}\psi\| \\ &\le \|f_R e^{\delta\rho(r,n)} \mathcal{F}\psi\| + \sup_{0 \le r \le 2R} e^{\delta\rho(r,0)} \|\psi\|, \end{aligned}$$

which is finite because of Lemma 3.3 and the positivity assumption on C_R.

On the other hand, on $\Omega(a_1)^c$, we have

$$(3.12)\quad \|\chi_{\Omega(a_1)^c} e^{\sigma a_1(r)} \mathcal{F}\psi\| \le \|\chi_{\Omega(a_1)^c} e^{\sigma |n|} \mathcal{F}\psi\|,$$

which is finite because of Lemma 2.6. Then the results follows from (3.11) and (3.12). □

The following result is a straightforward generalization of the L^2-estimate of eigenfunctions obtained by Nakamura [Nak1] in the Euclidean case.

Corollary 3.5. *Assume (A.1), (A.2)$_\tau$, (A.3) and (E). Moreover, assume that*

$$\lim_{r\to\infty}(a(r)-a_2(r))/g(r)=\infty. \tag{3.13}$$

Then $e^{\rho_1}\psi$ belongs to $L^2(M)$.

Furthermore, if we assume (A.2)$_\infty$, then we have the same conclusion replaced ρ_1 by

$$\rho_2(r)=\delta\int_{a_2^{-1}(a_1(r))}^{r}[(a(t)-a_2(t))^2/g(t)^2+F(t)-E]_+^{1/2}\,dt.$$

Proof. The assumption (3.13) implies that the constant C_R is positive for any large $R>0$. Under (A.2)$_\infty$, the assertion follows since we can take τ (and so σ) arbitrary large by Lemma 2.6. □

Remark 3.6. In the case of spherically symmetric scalar potential V, the conclusions in Theorem 3.5 are still valid if we replace the function F by $F+V$ and the condition (3.13) by

$$E<\liminf_{r\to\infty}\left((a(r)-a_2(r))^2/g(r)^2+F(r)+V(r)\right).$$

Indeed, in the proof of Lemma 3.3 above, we can replace F by $F+V$, and the last term in (3.6) and the commutator in (3.7) vanish.

4. Proof of Theorem 1.1

4.1. Upper bound

In this section we assume (M.1)–(M.4), (A.1), (A.2)$_\infty$, (E), and assume further that the magnetic field $B=B(r)$ approaches to a positive constant B_0 satisfying $E<B_0^2+1/4$ at infinity. It is known that the assumptions

on the radial curvature $K = -g''/g$ in Theorem 1.1 ensures the existence of the two limits

$$c_1 = \lim_{r\to\infty} e^{-r} g(r) \quad \text{and} \quad \lim_{r\to\infty} F(r) = 1/4 \tag{4.1}$$

(see Lemma 3.5 in [Don2]).

By the assumption on B, for any $\varepsilon > 0$, there exists $R_0 > 0$ such that $|B(r) - B_0| < \varepsilon$ for all $r \ge R_0$, so we have

$$\begin{aligned} |a(r) - B_0 \int_0^r g(t)dt| &\le (\int_0^{R_0} + \int_{R_0}^r)|B(t) - B_0|g(t)dt \\ &\le C_\varepsilon + \varepsilon \int_0^r g(t)dt \end{aligned}$$

for some constant C_ε and for all $r \ge R_0$. By (4.1), for any $\varepsilon > 0$, there exists $R_1 > 0$ such that $(1-\varepsilon)c_1 e^r \le g(r) \le (1+\varepsilon)c_1 e^r$ holds for all $r \ge R_1$. Hence, for any $\varepsilon > 0$, we find that

$$(1-\varepsilon)B_0 c_1 e^r + C_-(\varepsilon) \le a(r) \le (1+\varepsilon)B_0 c_1 e^r + C_+(\varepsilon)$$

holds for all $r \ge R_1$, where $C_\pm(\varepsilon)$ are independent of r.

Let $R_2 > 0$ and $0 < \varepsilon_1 < \varepsilon_2 < 1$, which are appropriately chosen below. We choose a_1, a_2 in (A.3) so that

$$a_j(r) = \begin{cases} B_0 c_1 e^{\varepsilon_j r} & \text{if } r \ge 2R_2, \\ (j/3)a(r) & \text{if } 0 \le r \le R_2, \end{cases} \tag{4.2}$$

and a_js are continuous and monotone increasing. Then it follows that

$$\lim_{r\to\infty} \big((a(r) - a_j(r))^2/g(t)^2 + F(r)\big) = B_0^2 + 1/4 \tag{4.3}$$

for $j = 1, 2$. Note that $a_2^{-1}(x) = \varepsilon_2^{-1}(\log x - \log B_0 c_1)$ for large x.

Lemma 4.1. *For any $\varepsilon > 0$, there exists $N > 0$ such that*

$$\begin{aligned} &\int_{a_2^{-1}(m)}^{a_2^{-1}(n)} \left[\frac{(a(t) - a_2(t))^2}{g(t)^2} + F(t) - E\right]_+^{1/2} dt \\ &\le \frac{1+\varepsilon}{\varepsilon_2}(B_0^2 + 1/4 - E)^{1/2}|n - m| \end{aligned} \tag{4.4}$$

holds if $n \geq m \geq N$.

PROOF. It follows from (4.3) that, for any $\varepsilon > 0$, there exists $N > 0$ such that, if $n \geq m \geq N$, the lhs of (4.4) is bounded from above by

$$\begin{aligned} & (1+\varepsilon)(B_0^2 + 1/4 - E)^{1/2}(a_2^{-1}(n) - a_2^{-1}(m)) \\ = \; & (1+\varepsilon)(B_0^2 + 1/4 - E)^{1/2}(\varepsilon_2^{-1}\log n - \varepsilon_2^{-1}\log m) \\ = \; & \frac{1+\varepsilon}{\varepsilon_2}(B_0^2 + 1/4 - E)^{1/2}\log(n/m) \\ \leq \; & \frac{1+\varepsilon}{\varepsilon_2}(B_0^2 + 1/4 - E)^{1/2}|n-m|, \end{aligned}$$

since $\log(X/x) = \log(1 + (X-x)/x) \leq (X-x)/x \leq X - x$ holds if $X \geq x > 1$. $\square$

All the conditions in (A.3) are now satisfied for our choice of a_js, therefore, we can apply Proposition 3.4. In this case, there exists $R_3 > 0$ such that $(a(r) - a_2(r))^2/g(r)^2 + F(r) - E > 0$ and therefore

$$\rho_2(r) = \delta \int_{(\varepsilon_1/\varepsilon_2)r}^{r} \left((a(t) - a_2(t))^2/g(t)^2 + F(t) - E\right)^{1/2} dt$$

for all $r \geq R_3$, because of (4.3), (4.2) and the assumption $E < B_0^2 + 1/4$.

Moreover, there exists $R_4 > 0$ such that

$$\left((a(t) - a_2(t))^2/g(t)^2 + F(t) - E\right)^{1/2} \geq (1-\varepsilon_1)(B_0^2 + 1/4 - E)^{1/2}$$

holds for all $t \geq R_4$. Finally, it follows that

$$\rho_2(r) \geq \delta(1-\varepsilon_1)\left(1 - (\varepsilon_1/\varepsilon_2)\right)(B_0^2 + 1/4 - E)^{1/2} r$$

holds for large $r \geq \max\{R_1, R_2, R_3, R_4\}$, from which we obtain the upper estimate of ψ in Theorem 1.1 since we can choose positive numbers ε_1 and $1 - \delta$ arbitrarily small.

4.2. Upper bound of the L^2-spherical average

In this subsection we obtain the upper bound of the L^2-spherical average of the eigenfunction ψ as in Theorem 1.1 and we obtain also the pointwise upper bound of ψ.

LEMMA 4.2. *Under the same assumption as in Theorem 1.1, the function* $\triangle_M(e^{(1-\varepsilon)\rho}\psi)$ *belongs to* $L^2(M)$ *for any* $\varepsilon > 0$.

PROOF. In this proof we denote the weight function $(1-\varepsilon)\rho$ by ρ for simplicity. By Kato's inequality, we have

$$
\begin{aligned}
(4.5)\qquad -\triangle_M(e^\rho|\psi|) &\leq \mathrm{Re}[(\mathrm{sgn}(\psi))\nabla^*\nabla(e^\rho\psi)] \\
&= \mathrm{Re}[(\mathrm{sgn}(\psi))(e^\rho\nabla^*\nabla\psi + [\nabla^*\nabla,\, e^\rho]\psi)] \\
&= \mathrm{Re}[(\mathrm{sgn}(\psi))(e^\rho(E-V)\psi + [\nabla^*\nabla,\, e^\rho]\psi)],
\end{aligned}
$$

where we used the eigen-equation in the last equality. We can find that

$$
\begin{aligned}
(4.6)\qquad [\nabla^*\nabla, e^\rho] &= [g^{-1}D_r g D_r, e^\rho] \\
&= 2(D_r\rho)e^\rho D_r + [(D_r^2\rho) + (D_r\rho)^2]e^\rho \\
&\quad + g^{-1}(D_r g)(D_r\rho)e^\rho.
\end{aligned}
$$

Note that g'/g is bounded on M by Lemma 2.8, and also $D_r\rho$, $D_r^2\rho$ and V are bounded. Then the rhs of (4.5) is bounded from above by $Ce^\rho(|D_r\psi| + |\psi|)$ for some constant $C > 0$.

Then it is enough to show that $e^\rho D_r\psi$ belongs to $L^2(M)$ since we have already shown that $e^\rho\psi$ belongs to $L^2(M)$. To see this, we first consider the case where ρ is smooth and bounded. Then we have $\|e^\rho D_r\psi\| \leq \|D_r(e^\rho\psi)\| + C\|e^\rho\psi\|$ and

$$
\begin{aligned}
\|D_r(e^\rho\psi)\|^2 &\leq \|D_r(e^\rho\psi)\|^2 + \|g^{-1}(D_\theta - a)(e^\rho\psi)\|^2 \\
&= (e^\rho\psi, \nabla^*\nabla(e^\rho\psi)) \\
&= (e^\rho\psi, e^\rho\nabla^*\nabla\psi) + (e^\rho\psi, [\nabla^*\nabla, e^\rho]\psi) \\
&\leq C\|e^\rho D_r\psi\|\|e^\rho\psi\| + C\|e^\rho\psi\|^2 \\
&\leq \frac{1}{2}\|e^\rho D_r\psi\|^2 + C\|e^\rho\psi\|^2,
\end{aligned}
$$

where we used (4.6) in the second inequality and used the elementary inequality $2XY \leq X^2 + Y^2$ in the last inequality. Thus we have shown that $\|e^\rho D_r\psi\| \leq C\|e^\rho\psi\|$. Finally, we have the same conclusion by approximating the original weight function ρ by smooth, bounded ones. □

LEMMA 4.3. *Assume that* (M, g) *satisfies (M.1)–(M.4). Assume that both* f *and* $\triangle_M f$ *belong to* $L^2(M)$. *Then there exists a constant* $C > 0$ *such that* $\int_0^{2\pi}|f(r,\theta)|^2 d\theta \leq Ce^{-r/2}$ *for all large* $r \geq 0$.

Proof. By (4.1), there exists $R > 0$ such that $g(r) \geq e^r/2$ holds for any $r \geq R$. Let n be a positive integer. Fix $\varepsilon > 0$. Let $\phi \in C_0^\infty(\mathbb{R})$ satisfy the following conditions: $1 \leq \phi \leq 1$, $\phi(t) = 1$ if $n \leq t \leq n+1$, $\phi(t) = 0$ outside $[n-\varepsilon, n+1+\varepsilon]$. For any $r \in [n, n+1]$, we have

$$\begin{aligned}
|f(r,\theta)| &\leq \int_{n-\varepsilon}^{r} |\partial_t[f(t,\theta)\phi(t)]|dt \\
&\leq \left(\int_{n-\varepsilon}^{r} \frac{dt}{g(t)}\right)^{1/2} \left(\int_{n-\varepsilon}^{r} |\partial_t[f(t,\theta)\phi(t)]|^2 g(t)dt\right)^{1/2} \\
&\leq \left(2\int_{n-\varepsilon}^{r} e^{-t}dt\right)^{1/2} \left(\int_{n-\varepsilon}^{r} |\partial_t[f(t,\theta)\phi(t)]|^2 g(t)dt\right)^{1/2} \\
&\leq (2(\varepsilon+1)e^{\varepsilon+1})^{1/2} e^{-r/2} \left(\int_{0}^{\infty} |\partial_t[f(t,\theta)\phi(t)]|^2 g(t)dt\right)^{1/2},
\end{aligned}$$

where we used Schwarz' inequality in the second inequality and used the fact that $n \leq r \leq n+1$ in the last inequality. Then we have

$$\begin{aligned}
(4.7) \quad \left(\int_0^{2\pi} |f(r,\theta)|d\theta\right)^{1/2} &\leq C_\varepsilon e^{-r/2} \|D_r(f\phi)\|_{L^2(M)} \\
&\leq C'_\varepsilon e^{-r/2}(\|\triangle_M f\|_{L^2(M)} + \|f\|_{L^2(M)}),
\end{aligned}$$

where we used the facts that $\|D_r f\|^2 \leq \|\triangle_M f\|\|f\|$ and that each of the derivatives of ϕ are bounded in the last inequality. The estimate (4.7) holds for any $r \geq R+\varepsilon$ since the constant C'_ε is independent of n. Then the result obeys. □

Then the upper bound of the L^2-spherical average of the eigenfunction ψ follows from Lemma 4.2 and Lemma 4.3 with $f = e^{(1-\varepsilon)\rho}\psi$. Note that a local elliptic argument shows that the eigenfunction ψ is continuous, hence bounded on each compact subset of M.

Finally, we show the pointwise estimate of $|\psi(r,\theta)|$. We recall Sobolev's embedding theorem from Hebey [Heb], Theorem 3.4: Let (M, g) be a smooth, complete n-dimensional Riemannian manifold with Ricci curvature bounded from below and positive injectivity radius. Assume that $q \geq 1$, $0 \leq m < k$, integers, and $1/q < (k-m)/n$. Then $H_k^q(M) \subset C_B^m(M)$. Here, the spaces $H_k^q(M)$, $C_B^m(M)$ are defined by the norms

$$\begin{aligned} \|u\|_{k,q} &= \left(\sum_{j=0}^{k} \int_M |\nabla^j u|^q dV\right)^{1/q}, \\ \|u\|_{C_B^m(M)} &= \sum_{j=0}^{m} \sup_{x\in M} |(\nabla^j u)(x)|, \end{aligned}$$

respectively.

We use this with $u = e^{(1-\varepsilon)\rho}\psi$, $q = 2$, $k = 2$, $n = 2$, and $m = 0$. Then the conclusion is that $|e^{(1-\varepsilon)\rho}\psi|$ is bounded on M. (This estimate can be also obtained as in the proof of Theorem 5.3 in Donnelly [Don5] using Lemma 2.3.) This completes the proof of the assertion 1 in Theorem 1.1.

4.3. Lower bound of the L^2-spherical average

In this subsection we obtain a lower bound of the L^2-spherical average of the eigenfunction ψ as in Theorem 1.1, following the line of argument as in the proof of Theorem 3.6 in Donnelly [Don2]. We may assume that the eigenfunction ψ is real-valued; otherwise we consider the real and complex parts of ψ. We note that the discreteness assumption on the eigenvalue E is not needed in the proof below.

LEMMA 4.4. *Let $r_0 \in \mathbb{R}$. Let $u \in C^2([r_0,\infty),\mathbb{R}) \cap L^2((r_0,\infty),dr)$ and $u(r_0) > 0$. Let $f, q \in C([r_0,\infty))$, $f \geq 0$ and $q > 0$. Assume that u satisfies the ODE: $u'' = qu - f$ on $[r_0,\infty)$. Then u is positive on $[r_0,\infty)$.*

PROOF. This lemma is elementary and well-known in the theory of ODE. However, we give a proof for the sake of completeness. We show this by contradiction. Assume that there exists the first zero r_1 ($> r_0$) of u. Then we can deduce that $u'(r_1) \leq 0$ and u is concave at r_1, which implies that u is concave and strictly negative on (r_1,∞) by the ODE. This contradicts the fact that $u \in L^2$. □

In what follows we assume that V is spherically symmetric in addition to $(\mathrm{A}.2)_\infty$. The Plancherel formula with respect to $L^2(\mathrm{S}^1)$ yields that

$$\begin{aligned} \int_0^{2\pi} |\psi(r,\theta)|^2 d\theta &= \sum_{n\in\mathbb{Z}} |(\mathcal{F}\psi)(r,n)|^2 \\ &\geq |(\mathcal{F}\psi)(r,j)|^2 \end{aligned} \tag{4.8}$$

for any $j \in \mathbb{Z}$. Put $\psi_j(r) = g^{1/2}(r)(\mathcal{F}\psi)(r,j)$. Then, by the assumption on V, each ψ_j satisfies the equation

$$\left(D_r^2 + (j - a(r))^2/g(r)^2 + F(r) + V(r) - E\right)\psi_j(r) = 0$$

on $[0,\infty)$ by (3.3). We rewrite this equation as $\left(D_r^2 + \mu - E + q_1\right)\psi_j = 0$, where we put $\mu = B_0^2 + 1/4$, $q_1 = (j-a)^2/g^2 + F + V - \mu$. We can find and fix an integer j for which $(\mathcal{F}\psi)(\cdot, j)$ is not identically zero, since $\psi \neq 0$. Then ψ_j is real-valued, smooth on $[0,\infty)$ and belongs to $L^2((0,\infty), dr)$.

We claim that there exists $r_0 > 0$ such that $\psi_j(r) > 0$ holds for all $r \geq r_0$. To see this we take $\tilde{r}_0 > 0$ such that $\mu - E + q_1 > 0$ holds for all $r \geq \tilde{r}_0$, since $\mu - E > 0$, $q_1 \to 0$ as $r \to \infty$. Moreover, we can find r_0 $(\geq \tilde{r}_0)$ so that $\psi_j(r_0) > 0$ because of the choice of j; otherwise, we take $-\psi_j$ instead of ψ_j. Then the claim follows from Lemma 4.4 with $u = \psi_j$, $f = 0$, $q = \mu - E + q_1$.

Let r_0 be as above and let $\varepsilon > 0$ be fixed small enough. We introduce the auxiliary function $v(r) = (1/2)\psi_j(r_0)\exp\left(-(\mu - E + \varepsilon)^{1/2}(r - r_0)\right)$, which solves the equation $(D_r^2 + \mu - E + \varepsilon)v = 0$ on $[r_0, \infty)$.

We put $\gamma = \psi_j - v$, then γ satisfies the equation $\gamma'' = (\mu - E + q_1)\gamma - (\varepsilon - q_1)v$ with the boundary condition $\gamma(r_0) = \psi_j(r_0)/2 > 0$. Again, we apply Lemma 4.4 with $u = \gamma$, $q = \mu - E + q_1$, and $f = (\varepsilon - q_1)v$. All the assumptions in Lemma 4.4 are satisfied for large $r_0 > 0$, and so Lemma 4.4 implies that $\gamma(r) > 0$ for all large r. This means that $\psi_j(r) > (1/2)\psi_j(r_0)\exp\left(-(\mu - E + \varepsilon)^{1/2}(r - r_0)\right)$ if r is large enough. Then, by (4.8), there exists $c = c(r_0) > 0$ such that

$$\int_0^{2\pi} |\psi(r,\theta)|^2 d\theta \geq c \exp\left(-(\mu - E + \varepsilon)^{1/2} r\right)\exp\left(-r/2\right)$$

holds for any large $r > 0$, from which the lower bound estimate in Theorem 1.1 follows. Now we complete the proof of Theorem 1.1.

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