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Remarks on Shintani's Zeta Function

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Abstract. We introduce a zeta function attached to a representation of a group. We show that the multi-dimensional zeta function due to Shintani [Sh1], which is a generalization of the multiple Hurwitz zeta function, can be obtained in this framework. We also construct a gamma function from the zeta function attached to a representation via zeta regularization. We study then a q-analogue of the Shintani zeta function and the corresponding gamma function. A sine function defined via the reflection formula of such q-Shintani gamma function is shown to be a natural generalization of the multiple elliptic function in [Ni]. Moreover, a certain non-commutative group-analogue of the Shintani zeta function is investigated.

1. Introduction

The multiple zeta function is first introduced by Barnes [Ba1] (and [Ba2]) in the beginning of 1900s for the study of the multiple gamma functions. Later, in the mid 1970s, in order to investigate the special values of the zeta functions of totally real algebraic number fields, Shintani [Sh1] (see also [Sh2]) introduced a multi-dimensional zeta function $\zeta_S(s, A, \underline{\mathbf{x}})$, where A denotes an $N \times n$-matrix with positive entries and $\underline{\mathbf{x}} \in \mathbb{R}^N$ (see §3 for the definition). Also, recently, a detailed study of multiple sine functions, which are defined through the Barnes multiple gamma functions, has been made in [KKo] (see [Sh3] and [K] about the initial study). Beside these works, q-analogue of the Barnes multiple zeta and gamma functions [KW1], and further, q-analogue of different gamma and multiple sine functions [KW3, KW2] originally introduced by Hölder [Höl], have been developed in connection with the study of the Jackson q-gamma function, multiple sine functions, and elliptic gamma functions (see, e.g., [Ruij] from the analytic difference equations viewpoint.)

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In this paper, we introduce a zeta function attached to a representation of a group. We then define a gamma function for this data by zeta regularization, like the procedure to obtaining the Lerch formula [L] for the classical gamma function, and try to understand such functions, first through a few introductory examples in §2. We observe that these definitions allow us to study several multiple zeta, multiple gamma, and multiple sine functions known so far from a unified point of view. Actaully, we prove in §3 that the Shintani zeta function can be regarded as a zeta function attached to a representation of the additive group $\mathbb{C}^n$ via the action of $SL_2(\mathbb{C})^n$ on $\mathbb{C}^{2^n}$, the n-times tensor product of $\mathbb{C}^2$. In view of this fact, we introduce a notion of a group-analogue of the Shintani zeta function. We especially define a q-analogue of the Shintani zeta function by taking the multiplicative group $\left(\mathbb{C}^\times\right)^n$ instead, and show that the corresponding gamma function can be expressed as a product of the Appell $\mathcal{O}$-functions [App] (q-shifted factorials) with some factor given by the multiple Bernoulli polynomials due to Barnes. One can find furthermore that the q-sine function defined via such q-analogue of the Shintani gamma function provides a natural generalization of the multiple elliptic gamma function in [Ni] (see also the study in [Na]). In §4, we devote ourselves to investigating a certain (non-commutative) group-analogue of the Shintani zeta function.

To introduce a zeta function attached to a representation of a group, we first recall the basic invariants of a square matrix. Let A be a complex $N \times N$ matrix. Define a conjugate invariant $L_k(A)$ of the matrix A by

$$L_k(A) := \sum_{1 \le i_1 < \cdots < i_k \le N} \begin{vmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{vmatrix}. \tag{1.1}$$

It is obvious that $L_N(A) = \det(A)$, $L_1(A) = \mathrm{tr}(A)$, and those $(-1)^{n-k} L_{n-k}(A)$ are the coefficients of t^k in the expansion of the characteristic polynomial $\Psi(t, A) := \det(tI - A)$ of A.

Let G be a group. Let (ρ, V) be a finite dimensional (linear) representation of G. Let $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_r)$ be a r-tuple of elements in G and $g \in \mathrm{End}(V)$. Let N_i be the order of the element $\rho(\gamma_i)$. Of course, if $\rho(\gamma_i)$ is an element of infinite order, then $N_i = \infty$. We define a multiple zeta function $\zeta^{(r)}_{G,\rho,j}(s, g; \underline{\gamma})$ of rank r and of height j $(1 \le j \le \dim V)$ for the representation (ρ, V) of G by the Dirichlet series

$$(1.2) \qquad \zeta^{(r)}_{G,\rho,j}(s,g;\underline{\gamma}) := \sum_{0\le n_i<N_i\ (i=1,\dots,r)} L_j(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})g))^{-s}.$$

From this definition, it is not hard to see the following ladder structure of the zeta function: Suppose that $N_k = \infty$. Then we have

$$\begin{aligned}
(1.3) \qquad & \zeta^{(r)}_{G,\rho,j}(s,\rho(\gamma_k)g;\gamma_1,\dots,\gamma_r)\\
&= -\,\zeta^{(r-1)}_{G,\rho,j}(s,g;\gamma_1,\dots,\gamma_{k-1},(\gamma_k^{-1}\gamma_{k+1}\gamma_k),\dots,(\gamma_k^{-1}\gamma_r\gamma_k))\\
&+\zeta^{(r)}_{G,\rho,j}(s,g;\gamma_1,\dots,\gamma_{k-1},\gamma_k,(\gamma_k^{-1}\gamma_{k+1}\gamma_k),\dots,(\gamma_k^{-1}\gamma_r\gamma_k)),\\
(1.4) \qquad & \zeta^{(r)}_{G,\rho,j}(s,g\rho(\gamma_k);\gamma_1,\dots,\gamma_r)\\
&= -\,\zeta^{(r-1)}_{G,\rho,j}(s,g;(\gamma_k^{-1}\gamma_1\gamma_k),\dots,(\gamma_k^{-1}\gamma_{k-1}\gamma_k),\gamma_{k+1},\dots,\gamma_r)\\
&+\zeta^{(r)}_{G,\rho,j}(s,g;(\gamma_k^{-1}\gamma_1\gamma_k),\dots,(\gamma_k^{-1}\gamma_{k-1}\gamma_k),\gamma_k,\gamma_{k+1},\dots,\gamma_r).
\end{aligned}$$

In particular, if the elements γ_k's commute with each other, it is clear that

$$\begin{aligned}
(1.5) \qquad & \zeta^{(r)}_{G,\rho,j}(s,\rho(\gamma_k)g;\gamma_1,\dots,\gamma_r) = \zeta^{(r)}_{G,\rho,j}(s,g\rho(\gamma_k);\gamma_1,\dots,\gamma_r)\\
&= -\,\zeta^{(r-1)}_{G,\rho,j}(s,g;\gamma_1,\dots,\gamma_{k-1},\gamma_{k+1},\dots,\gamma_r)\\
&+\zeta^{(r)}_{G,\rho,j}(s,g;\gamma_1,\dots,\gamma_{k-1},\gamma_k,\gamma_{k+1},\dots,\gamma_r).
\end{aligned}$$

When N_k is finite, obviously, the first term of the right hand side disappears.

As in the case of the Lerch formula of the Hurwitz zeta function for the classical gamma function [L], we introduce a gamma function attached to the data $(G,\rho,j,r,\underline{\gamma},g)$ as a coefficient of the linear term of the Laurent expansion of the attached zeta function when it can be meromorphically continued to the region containing $s = 0$.

Suppose now the zeta function $\zeta^{(r)}_{G,\rho,j}(s,g;\underline{\gamma})$ is meromorphic at the origin $s = 0$. Then one can define a (multiple) gamma function $\Gamma^{(r)}_{G,\rho,j}(g;\underline{\gamma})$ attached to the data $(G,\rho,j,r,\underline{\gamma},g)$ via the zeta regularized product as

$$(1.6) \qquad \Gamma^{(r)}_{G,\rho,j}(g;\underline{\gamma})^{-1} := \prod\!\!\!\!\!\!\bullet\;\;_{0\le n_i<N_i\ (i=1,\dots,r)} L_j(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})g).$$

Here the regularized product $\prod\!\!\!\!\!\bullet\;$ is defined as follows (see [KW1], [Ill] and [KiW]); for a sequence $\mathbf{a} = \{a_n\}_{n=0,1,\dots}$ of non-zero complex numbers, we define a zeta function attached to the sequence $\mathbf{a}$ by the Dirichlet series $\zeta_{\mathbf{a}}(s) = \sum_{n=0}^{\infty} a_n^{-s}$. We assume that the series converges absolutely when $\mathrm{Re}\, s$ is large enough, and further that the zeta function $\zeta_{\mathbf{a}}(s)$ can be meromorphically continued to the region containing $s = 0$. Then we define the (dot-)regularized product of the sequence $\mathbf{a}$ by

$$\prod_{n=0}^{\infty}\!\!\!\!\bullet\; a_n = \exp\Big(- \mathrm{Res}_{s=0} \frac{\zeta_{\mathbf{a}}(s)}{s^2}\Big). \tag{1.7}$$

This hence implies that

$$\Gamma^{(r)}_{G,\rho,j}(g;\underline{\gamma}) = \exp\Big(\mathrm{Res}_{s=0} \frac{\zeta_{G,\rho,j}(s,g;\underline{\gamma})}{s^2}\Big). \tag{1.8}$$

Of course, if the product indicates only over a finite range, the regularized product gives a usual (finite) product. Note that the aforementioned ladder relation of the zeta function induces automatically a functional equation, i.e. the translation law of the corresponding gamma function. We can define a left and right (and also a central) multiple sine functions (because of the non-commutativity of γ_j's) from the multiple gamma function for a representation of a group by means of the idea originated from the reflection formula of $\Gamma(x)$ (see [Sh3], [KW3], [KW4]). Then, it can be shown that those multiple sine functions possess periodicity, or rather, they satisfy a difference equation with some restriction (see, e.g., §4). It is, therefore, important to note that the ladder equations (1.3) and (1.4) can be the basis of the meromorphic continuation of those gamma and sine functions. We will treat these problems in [KW5]. We only give here the simplest example; let $G = SL_2(\mathbb{R})$. Consider the natural representation of G on $\mathbb{C}^2$. Let $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$ and $g = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2)$. Then $\zeta^{(1)}_{G,\rho,1}(s,g;\gamma) = \zeta(s,x) := \sum_{n=0}^{\infty}(n+x)^{-s}$, the Hurwitz zeta function. By the Lerch formula [L], the corresponding gamma function is actually the classical gamma function $\Gamma(x)$ (times $\sqrt{2\pi}$) and the corresponding sine function is a classical sine function; $\sin(\pi x) = \pi\Gamma(x)^{-1}\Gamma(1-x)^{-1}$, i.e. the so-called reflection formula of $\Gamma(x)$. The translation law, $\Gamma(x+1) = x\Gamma(x)$, follows from the ladder equation $\zeta^{(1)}_{G,\rho,1}(s,\gamma g;\gamma) = -x^{-s} + \zeta^{(1)}_{G,\rho,1}(s,g;\gamma)$.

Throughout the paper, we fix the log branch as $\log z = \log|z| + i \arg z$ ($-\pi \leq \arg z < \pi$). Also, when the zeta function $\zeta_{\mathbf{a}}(s)$ is holomorphic at

$s = 0$, we occasionally use the notation $\prod$ (according to [De] for indicating the holomorphic situation) in place of $\boxdot$ by dropping the central dot.

2. Preliminary Examples

The examples we give below can illustrate the situation for developing a study of multi-dimensional zeta functions and gamma functions in the following sections.

Example 2.1. Let $G = GL_2(\mathbb{C})$. Let ρ be the natural representation of G on $\mathbb{C}^2$. Let γ be an element of infinite order in G. Then, for $g \in \mathrm{End}(\mathbb{C}^2) = \mathrm{Mat}_2(\mathbb{C})$, we have

$$\zeta^{(1)}_{G,\rho,1}(s, g; \gamma) = \sum_{n=0}^{\infty} \mathrm{tr}(\gamma^n g)^{-s}, \tag{2.1}$$

$$\zeta^{(1)}_{G,\rho,2}(s, g; \gamma) = \sum_{n=0}^{\infty} \det(\gamma^n g)^{-s} = (\det g)^{-s} \sum_{n=0}^{\infty} (\det \gamma)^{-sn}. \tag{2.2}$$

• For the second (2.2), we assume that $|\det \gamma| > 1$. Then, it is easy to see that $\zeta^{(1)}_{G,\rho,2}(s, g; \gamma) = \frac{(\det g)^{-s}}{1-(\det \gamma)^{-s}}$ for $\mathrm{Re}\, s > 0$. This shows that $\zeta^{(1)}_{G,\rho,2}(s, g; \gamma)$ is meromorphic in $\mathbb{C}$ and the corresponding gamma function is $\Gamma^{(1)}_{G,\rho,2}(g; \gamma) = (\det \gamma)^{\frac{1}{12}} (\det \gamma g^{-1})^{\frac{1}{2}\left(\frac{\log \det \gamma g^{-1}}{\log \det \gamma} - 1\right)}$.

• To study the zeta function in (2.1), recall first the Cayley-Hamilton theorem for $h \in GL_2(\mathbb{C})$:

$$h^2 - \mathrm{tr}(h)h + \det(h)I = 0. \tag{2.3}$$

Multiplying $h^{n-1}g$ to this formula, and then further, taking the trace of the both sides, we obtain

$$\mathrm{tr}(h^{n+1}g) - \mathrm{tr}(h)\,\mathrm{tr}(h^n g) + \det(h)\,\mathrm{tr}(h^{n-1}g) = 0. \tag{2.4}$$

Put $a_n = \mathrm{tr}(h^n g)$. Let α and β be the eigenvalues of the matrix h. Since $a_{n+1} - (\alpha + \beta)a_n + \alpha\beta a_{n-1} = 0$ from (2.4), it is easy to see that

$$a_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} a_1 - \frac{\beta^n \alpha - \alpha^n \beta}{\beta - \alpha} a_0 = \frac{\beta^n}{\beta - \alpha}(a_1 - \alpha a_0) - \frac{\alpha^n}{\beta - \alpha}(a_1 - \beta a_0).$$

We now assume that $|\alpha| < |\beta|$. Then, if $a_1 - \alpha a_0 \neq 0$, for instance, we can write

$$a_n^{-s} = \beta^{-ns}\Big(\frac{a_1-\alpha a_0}{\beta-\alpha}\Big)^{-s}\Big\{1 - \frac{a_1-\beta a_0}{a_1-\alpha a_0}\cdot\Big(\frac{\alpha}{\beta}\Big)^n\Big\}^{-s}.$$

For simplicity, we assume that $\left|\frac{a_1-\beta a_0}{a_1-\alpha a_0}\right| < 1$. (This assumption can be actually removed.) Then, the binomial expansion yields

$$\begin{aligned}&\zeta^{(1)}_{G,\rho,1}(s,g;\gamma) = \sum_{n=0}^{\infty}\mathrm{tr}(h^n g)^{-s}\\ =&\Big(\frac{a_1-\alpha a_0}{\beta-\alpha}\Big)^{-s}\sum_{m=0}^{\infty}\binom{s+m-1}{m}\Big(\frac{a_1-\beta a_0}{a_1-\alpha a_0}\Big)^m\frac{1}{1-\beta^{-s}(\alpha/\beta)^m}.\end{aligned}$$

From this expression, we find that $\zeta^{(1)}_{G,\rho,1}(s,g;\gamma)$ is meromorphic in $s \in \mathbb{C}$ and has a simple pole at $s = 0$. Therefore, the dot-product $\prod\!\!\!\!\bullet_{n=0}^{\infty}\mathrm{tr}(h^n g)$ exists and is calculated as

$$\begin{aligned}&\Gamma^{(1)}_{G,\rho,1}(g;\gamma)\\ =&\beta^{\frac{1}{12}}\Big(\frac{\mathrm{tr}(hg)-\alpha\,\mathrm{tr}(g)}{\beta-\alpha}\Big)^{-\frac{1}{2}+\frac{1}{2}\log_\beta\left(\frac{\mathrm{tr}(hg)-\alpha\,\mathrm{tr}(g)}{\beta-\alpha}\right)}G\Big(\frac{\alpha}{\beta},\frac{\mathrm{tr}(hg)-\beta\,\mathrm{tr}(g)}{\mathrm{tr}(hg)-\alpha\,\mathrm{tr}(g)}\Big)^{-1},\end{aligned}$$

where $G(q,z) := \prod_{n=0}^{\infty}(1-q^n z)$ for $|q| < 1$. We can also calculate the zeta and gamma function in this way when ρ is an arbitrary finite dimensional (irreducible) representation of $GL_2(\mathbb{C})$ (see [KW5]).

Example 2.2. Let $G = \mathbb{C}$. Let $(n, \mathbb{C}^2)$ be a 2-dimensional representation of the additive group $\mathbb{C}$ defined by $n(z) := \begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix} \in SL_2(\mathbb{C})$. For a r-tuple of complex numbers $\underline{\omega} = (\omega_1, \dots, \omega_r)$, put $\underline{\gamma} = (\gamma_1, \dots, \gamma_r)$, where $\gamma_j = n(\omega_j)$. Then we have $\gamma_1^{n_1}\cdots\gamma_r^{n_r} = n(\sum_{j=1}^r n_j\omega_j)$. Hence, for $g = \begin{pmatrix}a & 0\\ 1 & d\end{pmatrix} \in \mathrm{End}(\mathbb{C}^2)$ with $a + d = x$, we obtain $\mathrm{tr}\,\big(n(\gamma_1^{n_1}\cdots\gamma_r^{n_r})g\big) = x + \sum_{j=1}^r n_j\omega_j$, that is, the zeta function $\zeta^{(r)}_{\mathbb{C},n,1}(n(\gamma_1^{n_1}\cdots\gamma_r^{n_r})g)$ attached to this data agrees with the multiple Hurwitz zeta function of weight $\underline{\omega}$; $\zeta_r(s,x;\underline{\omega}) := \sum_{n_1,n_2,\dots,n_r=0}^{\infty}(n_1\omega_1+\cdots+n_r\omega_r+x)^{-s}$. The corresponding gamma function $\Gamma_r(x;\underline{\omega}) = \exp(\zeta_r'(0,x;\underline{\omega}))$ is the Barnes multiple gamma function of weight $\underline{\omega}$ (see, e.g., [Sh3] and [KKo]). The so-called multiple sine function is then expressed as $S_r(x;\underline{\omega}) = \Gamma_r(x;\underline{\omega})^{-1}\Gamma_r(\omega_1 - \cdots - \omega_r - x;\underline{\omega})^{(-1)^r}$. On the other hand, we note that the ring sine function $S_{\mathbb{Z}[\tau]}(x)$ introduced in [KMOW] for the ring of integers of the imaginary quadratic field $\mathbb{Q}(\tau)$, which is explicitly given in terms of the theta function as a variation of the Kronecker limit formula, is expressed as $S_{\mathbb{Z}[\tau]}(x) = \prod_{m,n=-\infty}^{\infty} \operatorname{tr}\left(n(1)^m n(\tau)^n \begin{pmatrix} -x-d & 0 \\ 1 & d \end{pmatrix}\right)$.

Example 2.3. Suppose $q \in \mathbb{C}$ satisfies $|q| \neq 1$. Let G be the additive group $\mathbb{C}$. Let $(\pi_q, \mathbb{C}^2)$ be a representation of G given by $\pi_q(t) = \begin{pmatrix} q^{t/2} & 0 \\ 0 & q^{-t/2} \end{pmatrix} \in SL_2(\mathbb{C})$. For a r-tuple of complex numbers $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, put $\underline{\gamma} = (\gamma_1, \ldots, \gamma_r)$, where $\gamma_j = \rho(\omega_j)$. Set $g = \frac{1}{q^{1/2}-q^{-1/2}} \times \begin{pmatrix} q^{x/2} & 0 \\ 0 & q^{-x/2} \end{pmatrix} \in \operatorname{End}(\mathbb{C}^2)$. Then it is clear that $\operatorname{tr}\left(\pi_q(\gamma_1^{n_1}\cdots\gamma_r^{n_r})g\right) = [x + \sum_{j=1}^r n_j\omega_j]_q$, where $[x]_q := \frac{q^{x/2}-q^{-x/2}}{q^{1/2}-q^{-1/2}}$. Consequently, the zeta function is a q-analogue of the multiple Hurwitz zeta function $\zeta^{(r)}_{\mathbb{C},\pi_q,1}(s,x;\underline{\omega}) = \sum_{\underline{\mathbf{n}}\geq\underline{0}} [\underline{\mathbf{n}} \cdot \underline{\omega} + x]^{-s}$. Hence the attached gamma function is essentially the q-shifted factorial (see e.g., [Ruij]) or Appell's $\mathcal{O}$-function [App] (for the definitions, see (3.9) and (3.10)) up to a factor expressed by the multiple Bernoulli polynomials (see [KW1]).

Example 2.4. Let $G = SL_2(\mathbb{R})$ and ρ again be the natural representation of G. Put $\gamma_1 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a > 1$ and $\gamma_2 = n(1)$ defined in Example 2.2. For $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \operatorname{Mat}_2(\mathbb{C})$, we have $\operatorname{tr}(\gamma_1^m\gamma_2^n g) = a^m x + a^{-m}w + nza^m$. We show that the zeta function $\zeta^{(2)}_{G,\rho,1}(s,g;\gamma_1,\gamma_2) := \sum_{m,n=0}^{\infty}(a^m x + a^{-m}w + nza^m)^{-s}$ can be meromorphically extended to $s \in \mathbb{C}$ as follows. For simplicity, we assume that $0 < w < \min(x,1)$ and $z \geq 1$. We have then, for $\operatorname{Re} s > 1$ initially,

$$\zeta^{(2)}_{G,\rho,1}(s,g;\gamma_1,\gamma_2) = \sum_{m,n=0}^{\infty} \left\{a^m(x+nz)\right\}^{-s}\left\{1 + a^{-2m}w(x+nz)^{-1}\right\}^{-s}$$

$$
\begin{aligned}
&= \sum_{m,n=0}^{\infty} \left\{a^m(x+nz)\right\}^{-s} \sum_{\ell=0}^{\infty} \binom{-s}{\ell} a^{-2m\ell} w^{\ell} (x+nz)^{-\ell} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left(\because 0 < \frac{w}{x+nz} < 1\right) \\
&= \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \binom{-s}{\ell} w^{\ell} z^{-s-\ell} \left(\frac{x}{z}+n\right)^{-s-\ell} \sum_{m=0}^{\infty} a^{-m(s+2\ell)} \\
&= \frac{\zeta(s,\frac{x}{z})z^{-s}}{1-a^{-s}} - \frac{s\zeta(s+1,\frac{x}{z})wz^{-s-1}}{1-a^{-(s+2)}} \\
&\qquad + \sum_{\ell=2}^{\infty} \binom{-s}{\ell} \frac{\zeta(s+\ell,\frac{x}{z})w^{\ell}z^{-s-\ell}}{1-a^{-(s+2\ell)}}.
\end{aligned}
$$

Since $\zeta(s,\frac{x}{z})$, $s\zeta(s+1,\frac{x}{z})$, and $\zeta(s+\ell,\frac{x}{z})$ $(\ell \geq 2)$ are holomorphic at $s=0$, the meromorphy of $\zeta^{(2)}_{G,\rho,1}(s,g;\gamma_1,\gamma_2)$ follows from the fact $|\zeta(\ell,\frac{x}{z})| < 1$ for $\ell \geq 2$. Hence, we have the (non-constant) gamma function attached to this data; $\Gamma(g;\gamma_1,\gamma_2) := \left\{ \boxdot_{n,m=0}^{\infty} (a^m x + a^{-m} w + nza^m) \right\}^{-1}$. Then by the ladder equations (1.3) and (1.4), it can be shown that the attached gamma function $\Gamma(g;\gamma_1,\gamma_2)$ satisfies the functional equations

$$
\begin{aligned}
\text{(2.5)}\qquad \Gamma(g\gamma_1;\gamma_1,\gamma_2) &= \Gamma(g,1,\gamma_2)^{-1}\Gamma(g;\gamma_1,\gamma_2) \\
&= \prod_{n=0}^{\infty} (x+w+nz) \times \Gamma(g;\gamma_1,\gamma_2), \\
\text{(2.6)}\qquad \Gamma(\gamma_2 g;\gamma_1,\gamma_2) &= \Gamma(g,\gamma_1,1)^{-1}\Gamma(g;\gamma_1,\gamma_2) \\
&= \boxdot_{m=0}^{\infty} (a^m x + a^{-m} w) \times \Gamma(g;\gamma_1,\gamma_2),
\end{aligned}
$$

where the factors $\left\{ \prod_{n=0}^{\infty} (x+w+nz) \right\}^{-1}$ and $\left\{ \boxdot_{m=0}^{\infty} (a^m x + a^{-m} w) \right\}^{-1}$ are essentially the gamma function $\Gamma\left(\frac{x+w}{z}\right)$ and Jackson's q gamma function $\Gamma_{a^2}(\log_a x)$ when $xw = 1$, respectively (see [KW5] in detail). See, e.g., [AAR] for Jackson's q-gamma function $\Gamma_q(s)$.

3. Shintani's Zeta Function and Its q-Analogue

Recall a multi-dimensional zeta function $\zeta_S(s, A, \underline{\mathbf{x}})$, with $N \times n$ matrix

$A = (a_{ij})$ and $\underline{\mathbf{x}} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ due to Shintani ([Sh1]):

(3.1) $$\zeta_S(s, A, \underline{\mathbf{x}}) := \sum_{m_1,\ldots,m_N=0}^{\infty} \prod_{j=1}^{n} \Big\{ \big(\sum_{i=1}^{N} (m_i + x_i) a_{ij} \big) \Big\}^{-s}, \qquad \mathrm{Re}\, s > N/n.$$

We call this Dirichlet series the Shintani zeta function. This is obviously a generalization of the Hurwitz zeta function We first show that $\zeta_S(s, A, \underline{\mathbf{x}})$ can be obtained in our framework, i.e. it is considered as a zeta function for a representation of $\mathbb{C}^n$. Moreover, we introduce a group analogue of the Shintani zeta function, and in particular, study a q-analogue of the Shintani zeta function by considering the group of diagonal matrices in $SL_2(\mathbb{C})$.

Proposition 3.1. *Let $G = \Big\{ n(z) := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \; ; \; z \in \mathbb{C} \Big\} \cong \mathbb{C}$ be the group of all upper triangular matrices with complex coefficients. Let $A = (\underline{\mathbf{a}}_1, \ldots, \underline{\mathbf{a}}_n) = \{a_{ij}\}_{1 \le i \le N, 1 \le j \le n}$ be a complex $N \times n$-matrix whose entries have positive real parts and let $w_j \, (1 \le j \le n)$ be real positive numbers. Put $\gamma_{ij} = n(a_{ij})$, $g_j = n(w_j)$ and $\underline{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_{>0}$. Form a tensor product $V = \big(\mathbb{C}^2\big)^{\otimes n}$ by the G-action on $\mathbb{C}^2$. We consider the n-times product group $G^n = G \times \cdots \times G$ which acts on V in an obvious way. Let $\gamma_i := (\gamma_{i1}, \cdots, \gamma_{in}) \in G^n$ $(i = 1, 2, \ldots, N)$. Further, put $\rho(\gamma_i) := \gamma_{i1} \otimes \cdots \otimes \gamma_{in} \in \mathrm{End}(V)$ and $g = g_1 S \otimes \cdots \otimes g_n S \in \mathrm{End}(V)$ with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then the zeta function $\zeta^{(N)}_{G^n,\rho,1}(s, g; \underline{\gamma})$ attached to the data $(G, \rho, 1, N, \underline{\gamma}, g)$ gives Shintani's zeta function $\zeta_S(s, A, \underline{\mathbf{x}})$.*

Proof. We have

$$\zeta^{(N)}_{G^n,\rho,1}(s, g; \gamma_1, \gamma_2, \ldots, \gamma_n) = \sum_{m_1,\ldots,m_N=0}^{\infty} \big\{ \mathrm{tr}(\rho(\gamma_1^{m_1} \gamma_2^{m_2} \cdots \gamma_N^{m_N}) g) \big\}^{-s}$$

$$= \sum_{m_1,\ldots,m_N=0}^{\infty} \Big\{ \mathrm{tr} \Big(\big(\prod_{i=1}^{N} \gamma_{i1}^{m_i} \otimes \cdots \otimes \prod_{i=1}^{N} \gamma_{in}^{m_i} \big) (g_1 S \otimes \cdots \otimes g_n S) \Big) \Big\}^{-s}$$

$$= \sum_{m_1,\ldots,m_N=0}^{\infty} \Big\{ \mathrm{tr} \Big(n(w_1 + \sum_{i=1}^{N} m_i a_{i1}) S \otimes \cdots \otimes n(w_n + \sum_{i=1}^{N} m_i a_{in}) S \Big) \Big\}^{-s}$$

$$= \sum_{m_1,\dots,m_N=0}^{\infty} \left\{ \operatorname{tr} \left(\begin{pmatrix} w_1 + \sum_{i=1}^N m_i a_{i1} & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} w_n + \sum_{i=1}^N m_i a_{in} & 1 \\ 1 & 0 \end{pmatrix} \right) \right\}^{-s}$$

$$= \sum_{m_1,\dots,m_N=0}^{\infty} \Big\{ \prod_{j=1}^{n} \big(w_j + \sum_{i=1}^{N} m_i a_{ij}\big) \Big\}^{-s} =: \zeta(s, A, \underline{\omega}).$$

Here we used the fact; for square matrices A_j, the trace of $A_1 \otimes \cdots \otimes A_r$ is given by

$$\operatorname{tr}(A_1 \otimes \cdots \otimes A_r) = \prod_{j=1}^{r} \operatorname{tr}(A_j). \tag{3.2}$$

The Dirichlet series $\zeta(s, A, \underline{\omega})$ clearly agrees with the Shintani zeta function $\zeta_S(s, A, \underline{\mathbf{x}})$. Actually, it is enough to write the present variables w_j as $w_j = \sum_{i=1}^N x_i a_{ij}$. This proves the assertion. □

Shintani [Sh1] showed that $\zeta_S(s, A, \underline{\mathbf{x}})$ admits a meromorphic continuation to the entire plane $\mathbb{C}$ as a function of s. Also, he showed that the exponential of $\frac{\partial}{\partial s}\zeta_S(0, A, \underline{\mathbf{x}})$, i.e. the gamma function associated with $\zeta_S(s, A, \underline{\mathbf{x}})$ can be expressed in terms of Bernoulli polynomials and Barnes' multiple gamma functions (see also [Sh3]). In the present formulation of the Shintani zeta functions, which is slightly different from the original one, we have the following simple expression of the corresponding gamma function (see also Remark 3 below).

Example 3.2. Retain the notation in the proposition above. Recall the property of the zeta regularized product; $\prod\!\!\!\!\cdot_{n\in I \amalg J}\, a_n = \prod\!\!\!\!\cdot_{n\in I}\, a_n \prod\!\!\!\!\cdot_{n\in J}\, a_n$ (see, e.g., [KiW]). By this fact, it is easily shown that the gamma function $\Gamma_S(\underline{\omega}, A) = \Gamma_S^{(N,n)}(\underline{\omega}, A) := \exp\Big(\operatorname{Res}_{s=0} \zeta(s, A, \underline{\omega})/s^2\Big)$ attached to the Shintani zeta function is expressed as a product of the Barnes multiple gamma functions:

$$\Gamma_S^{(N,n)}(\underline{\omega}, A) := \Big\{ \prod_{\underline{\mathbf{m}}\geq\underline{0}} \prod_{j=1}^{n} \big(w_j + \sum_{i=1}^{N} m_i a_{ij}\big) \Big\}^{-1} = \prod_{j=1}^{n} \Gamma_N(\omega_j, \underline{\mathbf{a}}_j). \tag{3.3}$$

Since the Shintani zeta function is holomorphic at $s = 0$, we have used $\prod$ in place of $\boxdot$. The translation law of the gamma function inherited from the ladder relation (1.5) of the zeta function $\zeta(s, A, \underline{\mathbf{x}})$ can be expressed as

$$\Gamma_S^{(N,n)}(\underline{\omega} + \underline{\mathbf{a}}^{i_0}, A) = \Gamma_S^{(N-1,n)}(\underline{\omega}, \check{A}_{i_0})^{-1}\Gamma_S^{(N,n)}(\underline{\omega}, A), \tag{3.4}$$

where $\underline{\mathbf{a}}^{i_0}$ is the i_0-th row of the matrix A and $\check{A}_{i_0}$ is the $(N-1) \times n$ matrix obtained from A by deleting the i_0-th row of A. For the detailed discussion about the meromorphic continuation of the zeta function, see [Sh1, Sh2]. (See also §4.) Moreover, if we define a Shintani sine function by $S_S^{(N,n)}(\underline{\omega}, A) := \Gamma_S^{(N,n)}(\underline{\omega}, A)^{-1}\Gamma_S^{(N,n)}(-\underline{\omega} + \sum_{i=1}^N \underline{\mathbf{a}}^i, A)^{(-1)^N}$, then the periodicity

$$S_S^{(N,n)}(\underline{\omega} + \underline{\mathbf{a}}^{i_0}, A) = S_S^{(N-1,n)}(\underline{\omega}, \check{A}_{i_0})^{-1} S_S^{(N,n)}(\underline{\omega}, A) \tag{3.5}$$

follows from (3.4). This is a generalization of Theorem 2.1 in [KKo].

Remark 1. In view of the construction of $\zeta_S(s, A, \underline{\omega})$ from the group data, the natural variable $\underline{\omega}$ of Shintani gamma and sine functions should be taken in $\{g = \otimes_{j=1}^n g_j \ ; \ g_j \in \mathrm{Mat}_2(\mathbb{C})\}$ or much bigger $\mathrm{End}(V) \cong \mathrm{Mat}_{\mathbb{C}}(2^n) \cong \mathbb{C}^{2^{2n}}$ (in place of the limited domain $\mathbb{R}^n$).

Remark 2. If we take the minor sum L_2 of degree 2 in place of the trace function L_1, the first few examples of the zeta functions $\zeta_{G^n,\rho,2}^{(N)}(s, g : \underline{\gamma}) := \sum_{\underline{\mathbf{m}} \geq \underline{0}} L_2(\rho(\underline{\gamma}^{\underline{\mathbf{m}}})g)^{-s}$, where $\underline{\gamma}^{\underline{\mathbf{m}}} = \gamma_1^{m_1} \cdots \gamma_r^{m_r}$, of height 2 attached to the same data with the Shinatani zeta function above are given as

$$\begin{aligned}
&\zeta_{G^2,\rho,2}^{(N)}(s, g; \gamma_1, \gamma_2) \\
&= e^{\pi s} \sum_{m_1, \dots, m_N = 0}^{\infty} \left\{ \Big(w_1 + \sum_{i=1}^N m_i a_{i1}\Big)^2 + \Big(w_2 + \sum_{i=1}^N m_i a_{i2}\Big)^2 + 2 \right\}^{-s}, \\
&\zeta_{G^3,\rho,2}^{(N)}(s, g; \gamma_1, \gamma_2, \gamma_3) \\
&= e^{\pi s} \sum_{m_1, \dots, m_N = 0}^{\infty} \left\{ \Big(w_1 + \sum_{i=1}^N m_i a_{i1}\Big)^2 \Big(w_2 + \sum_{i=1}^N m_i a_{i2}\Big)^2 \right. \\
&\qquad + \Big(w_2 + \sum_{i=1}^N m_i a_{i2}\Big)^2 \Big(w_3 + \sum_{i=1}^N m_i a_{i3}\Big)^2
\end{aligned}$$

$$+\Big(w_3+\sum_{i=1}^N m_i a_{i3}\Big)^2\Big(w_1+\sum_{i=1}^N m_i a_{i1}\Big)^2$$
$$+2\Big(w_1+\sum_{i=1}^N m_i a_{i1}\Big)^2+2\Big(w_2+\sum_{i=1}^N m_i a_{i2}\Big)^2$$
$$+2\Big(w_3+\sum_{i=1}^N m_i a_{i3}\Big)^2+4\Big\}^{-s}, \text{ etc.}$$

Note that, in general, $L_2(\rho(\underline{\gamma}^{\mathbf{m}})g)$ is of degree 2^{n-1} with respect to the variables m_i's.

From the observation above, we now introduce a group analogue of the Shintani zeta function as follows. Let $G = GL_m(\mathbb{C})$ (or its subgroup) and consider the natural representation of G on $\mathbb{C}^m$. Let $\gamma_{ij} \in G$ $(1 \le i \le N, 1 \le j \le n)$. Let $\gamma_i := (\gamma_{i1}, \cdots, \gamma_{in}) \in G^n$ $(i = 1, 2, \dots, N)$. Also, put $\rho(\gamma_i) := \gamma_{i1} \otimes \cdots \otimes \gamma_{in} \in \mathrm{End}(V)$, where $V = (\mathbb{C})^{\otimes n}$. Let $g \in \mathrm{End}(V)$. Then we define a G-analogue of the Shintani zeta function $\zeta_G(s, \{\gamma_{ij}\}_{1\le i\le N, 1\le j\le n}, g)$ by

$$\zeta_G(s, \{\gamma_{ij}\}_{1\le i\le N, 1\le j\le n}, g) := \zeta^{(N)}_{G,\rho,1}(s, g; \underline{\gamma}). \tag{3.6}$$

We study now a special case.

Let q be a non-zero complex number such that $|q| \ne 1$. Let $A = \{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ be a complex $N \times n$-matrix and let w_j $(1 \le j \le n)$ be complex numbers. Let $G(\cong \mathbb{C}^\times)$ be a group of diagonal matrices in $SL_2(\mathbb{C})$. Put $\gamma_{ij} = \begin{pmatrix} q^{a_{ij}/2} & 0 \\ 0 & q^{-a_{ij}/2}\end{pmatrix} \in G$ and $g = \frac{1}{q^{1/2}-q^{-1/2}}\begin{pmatrix} q^{w_1/2} & 0 \\ 0 & -q^{-w_1/2}\end{pmatrix} \otimes \cdots \otimes \frac{1}{q^{1/2}-q^{-1/2}}\begin{pmatrix} q^{w_n/2} & 0 \\ 0 & -q^{-w_n/2}\end{pmatrix} \in \mathrm{End}(V)$. We call this $\zeta_G(s, \{\gamma_{ij}\}_{1\le i\le N, 1\le j\le n}, g)$ a q-analogue of the Shintani zeta function and denote it simply by $\zeta_q(s, A, \underline{\omega})$. Then, from (3.2) we have

$$\zeta_q(s, A, \underline{\omega}) = \sum_{m_1,\dots,m_N=0}^{\infty}\Big\{\prod_{j=1}^n [\omega_j + \sum_{i=1}^N m_i a_{ij}]_q\Big\}^{-s}. \tag{3.7}$$

We study a meromorphy of the q-analogue of the Shintani zeta function, and moreover, show that the corresponding gamma function can be expressed in terms of the Appell $\mathcal{O}$-functions with Barnes' multiple Bernoulli polynomials.

THEOREM 3.2. *Let $q \in \mathbb{R}$. Let $A = \{a_{ij}\}_{1\le i\le N, 1\le j\le n}$ be a complex $N \times n$-matrix whose entries have positive real parts and let w_j $(1 \le j \le n)$ be complex numbers with positive real parts. Then the q-analogue of the Shintani zeta function $\zeta_q(s, A, \underline{\omega})$ can be meromorphically extended to the whole plane $\mathbb{C}$.*

PROOF. We may assume $q > 1$. Write $\underline{\mathbf{a}}_j = (a_{1j}, a_{2j}, \dots, a_{Nj})$, i.e. $A = {}^t(\underline{\mathbf{a}}_1, \dots, \underline{\mathbf{a}}_n)$. Then, for $\operatorname{Re} s > 0$, we have

$$\begin{aligned}\zeta_q(s, A, \underline{\omega}) &= \sum_{\underline{\mathbf{m}}\ge\underline{0}} \Big\{ \prod_{j=1}^{n} [\omega_j + \underline{\mathbf{m}} \cdot \underline{\mathbf{a}}_j]_q \Big\}^{-s} \\ &= (q^{1/2} - q^{-1/2})^{ns} \sum_{\underline{\mathbf{m}}\ge\underline{0}} \prod_{j=1}^{n} \Big\{ q^{-s(\omega_j+\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j)/2} \big(1 - q^{-(\omega_j+\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j)}\big)^{-s} \Big\}.\end{aligned}$$

Therefore, it follows from the binomial expansion that

$$\begin{aligned}\zeta_q(s, A, \underline{\omega}) &= (q^{1/2} - q^{-1/2})^{ns} \sum_{\underline{\mathbf{m}}\ge\underline{0}} \\ &\qquad \times \prod_{j=1}^{n} \Big\{ q^{-s(\omega_j+\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j)/2} \sum_{k_j=0}^{\infty} \binom{s+k_j-1}{k_j} q^{-(\omega_j+\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j)k_j} \Big\} \\ &= (q^{1/2} - q^{-1/2})^{ns} \sum_{\underline{\mathbf{m}}\ge\underline{0}} \sum_{k_1,\dots,k_n=0}^{\infty} \\ &\qquad \times \prod_{j=1}^{n} \binom{s+k_j-1}{k_j} q^{-\sum_{j=1}^{n}(s/2+k_j)\omega_j} q^{-\underline{\mathbf{m}}\cdot\sum_{j=1}^{n}(s/2+k_j)\underline{\mathbf{a}}_j} \\ &= (q^{1/2} - q^{-1/2})^{ns} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} \binom{s+k_j-1}{k_j} q^{-\sum_{j=1}^{n}(s/2+k_j)\omega_j} \\ &\qquad \times \Big\{ \sum_{m_1=0}^{\infty} q^{-m_1\sum_{j=1}^{n}(s/2+k_j)a_{1j}} \Big\} \\ &\qquad\quad \cdots \Big\{ \sum_{m_N=0}^{\infty} q^{-m_N\sum_{j=1}^{n}(s/2+k_j)a_{Nj}} \Big\}\end{aligned}$$

$$= (q^{1/2} - q^{-1/2})^{ns} \sum_{k_1,\dots,k_n=0}^{\infty} \prod_{j=1}^{n} \binom{s+k_j-1}{k_j} q^{-\sum_{j=1}^{n}(s/2+k_j)\omega_j}$$
$$\times \frac{1}{\left(1-q^{-\sum_{j=1}^{n}(s/2+k_j)a_{1j}}\right)\cdots\left(1-q^{-\sum_{j=1}^{n}(s/2+k_j)a_{Nj}}\right)}.$$

This shows that $\zeta_q(s, A, \underline{\omega})$ is meromorphic in $s \in \mathbb{C}$. The remaining part of the assertion is clear from the last expression. This proves the theorem. □

As a corollary of the proof, we can determine an expression of the corresponding gamma function by the Appell $\mathcal{O}$-functions. To describe the result, it is necessary to recall the multiple Bernoulli polynomials $B_m(\omega; \underline{\mathbf{a}})$ due to Barnes [Ba2]:

$$\frac{e^{-\omega t}}{(1-e^{a_1 t})\cdots(1-e^{a_r t})} = \sum_{m=0}^{\infty} \frac{B_m(\omega; \underline{\mathbf{a}})}{m!} t^{m-r}, \tag{3.8}$$

where $\underline{\mathbf{a}} = (a_1, \dots, a_r)$. Recall further the Appell $\mathcal{O}$-function $\mathcal{O}_q(\omega, \underline{\mathbf{a}})$ defined by the product

$$\mathcal{O}_q(\omega, \underline{\mathbf{a}}) := \prod_{\underline{\mathbf{n}} \geq 0} (1 - q^{-(\underline{\mathbf{n}} \cdot \underline{\mathbf{a}} + \omega)}) \tag{3.9}$$

when $|q| < 1$. Obviously, $O_q(\omega, \underline{\mathbf{a}})$ can be (equivalently) expressed by the q-shifted factorial $G(\underline{\mathbf{q}}; z) = G(q_1, \dots, q_r; z)$ defined by

$$G(\underline{\mathbf{q}}; z) := \prod_{n_1,\dots,n_r=0}^{\infty} (1 - q_1^{n_1} \cdots q_r^{n_r} z) \tag{3.10}$$

for $|q_j| < 1$. In fact, we have $O_q(\omega, \underline{\mathbf{a}}) = G(q^{-a_1}, \dots, q^{-a_r}; q^{-\omega})$.

COROLLARY 3.3. *The gamma function $\Gamma_q(\underline{\omega}, A)$ attached to the zeta function $\zeta_q(s, A, \underline{\omega})$ is essentially given by the product of the multiple elliptic gamma functions. Precisely,*

$$\Gamma_q(\underline{\omega}, A)^{-1} := \prod\!\!\!\!\!\bullet_{\underline{\mathbf{m}} \geq 0} \prod_{j=1}^{n} [\omega_j + \underline{\mathbf{m}} \cdot \underline{\mathbf{a}}_j]_q \tag{3.11}$$
$$= q^{-\frac{B_{N+1}\left(\frac{1}{2}\sum_{j=1}^{n}\omega_j - n\frac{\log(q^{1/2}-q^{-1/2})}{\log q}; \frac{1}{2}\sum_{j=1}^{n}\underline{\mathbf{a}}_j\right)}{(N+1)!}}$$
$$\times \prod_{j=1}^{n} \mathcal{O}_q(\omega_j, \underline{\mathbf{a}}_j).$$

Moreover, we have the functional equation of $\Gamma_q(\underline{\omega}, A) =: \Gamma_q^{(N,n)}(\underline{\omega}, A)$*:*

$$\Gamma_q^{(N,n)}(\underline{\omega} + \underline{\mathbf{a}}^{i_0}, A) = \Gamma_q^{(N-1,n)}(\underline{\omega}, \check{A}_{i_0})^{-1}\Gamma_q^{(N,n)}(\underline{\omega}, A), \tag{3.12}$$

where $\underline{\mathbf{a}}^{i_0}$ *is the* i_0*-th row of the matrix* A *and* $\check{A}_{i_0}$ *is the* $(N-1) \times n$ *matrix obtained from* A *by removing the* i_0*-th row of* A*.*

Proof. From the last expression of $\zeta_q(s, A, \underline{\omega})$ in the proof of the theorem above, we see easily that

$$\begin{aligned}
&\zeta_q(s, A, \underline{\omega}) \\
=&(q^{1/2} - q^{-1/2})^{ns}\Bigg\{\frac{q^{-s\sum_{j=1}^{n}\omega_j/2}}{(1-q^{-s\sum_{j=1}^{n}a_{1j}/2})\cdots(1-q^{-s\sum_{j=1}^{n}a_{Nj}/2})} \\
&\qquad + s\sum_{j=1}^{n}\sum_{k_j=1}^{\infty}\frac{1}{k_j}\frac{q^{-k_j\omega_j}}{(1-q^{-k_ja_{1j}})\cdots(1-q^{-k_ja_{Nj}})} + O(s^2)\Bigg\}.
\end{aligned}$$

Here we used the fact $\binom{s+k-1}{k} = \frac{1}{k}\cdot s + O(s^2)$ for $k \geq 1$. As to the first term of this equation, by the definition (3.8) of the multiple Bernoulli polynomials, we get

$$\begin{aligned}
&\frac{(q^{1/2} - q^{-1/2})^{ns}\cdot q^{-s\sum_{j=1}^{n}\omega_j/2}}{(1-q^{-s\sum_{j=1}^{n}a_{1j}/2})\cdots(1-q^{-s\sum_{j=1}^{n}a_{Nj}/2})} \\
=&\frac{e^{-s(\log q)\left(\sum_{j=1}^{n}\omega_j/2 - n\frac{\log(q^{1/2}-q^{-1/2})}{\log q}\right)}}{(1-q^{-s\sum_{j=1}^{n}a_{1j}/2})\cdots(1-q^{-s\sum_{j=1}^{n}a_{Nj}/2})} \\
=&\sum_{m=0}^{\infty}\frac{B_m\left(\sum_{j=1}^{n}\omega_j/2 - n\frac{\log(q^{1/2}-q^{-1/2})}{\log q}; (\frac{1}{2}\sum_{j=1}^{n}a_{1j}, \dots, \frac{1}{2}\sum_{j=1}^{n}a_{Nj})\right)}{m!} \\
&\qquad \times (s\log q)^{m-N}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\operatorname{Res}_{s=0}\frac{\zeta_q(s, A, \underline{\omega})}{s^2} \\
=&\frac{B_{N+1}\left(\sum_{j=1}^{n}\omega_j/2 - n\frac{\log(q^{1/2}-q^{-1/2})}{\log q}; (\frac{1}{2}\sum_{j=1}^{n}a_{1j}, \dots, \frac{1}{2}\sum_{j=1}^{n}a_{Nj})\right)}{(N+1)!}\log q \\
&\qquad + \sum_{j=1}^{n}\sum_{\underline{\mathbf{m}}\geq 0}\sum_{k_j=1}^{\infty}\frac{1}{k_j}q^{-k_j(\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j + \omega_j)}
\end{aligned}$$

$$
=\frac{B_{N+1}\big(\sum_{j=1}^{n}\omega_j/2-n\frac{\log(q^{1/2}-q^{-1/2})}{\log q};\frac{1}{2}\sum_{j=1}^{n}\underline{\mathbf{a}}_j\big)}{(N+1)!}\log q
-\sum_{j=1}^{n}\sum_{\underline{\mathbf{m}}\geq 0}\log\big(1-q^{-k_j(\underline{\mathbf{m}}\cdot\underline{\mathbf{a}}_j+\omega_j)}\big).
$$

Hence the first assertion follows immediately from the definition of the gamma function by the zeta regularized product $\prod\!\!\!\!\!\!\cdot\;$ in (1.8). Furthermore, the second assertion simply follows from the general property $\prod\!\!\!\!\!\!\cdot\;_{n\in I\amalg J}a_n=\prod\!\!\!\!\!\!\cdot\;_{n\in I}a_n\prod\!\!\!\!\!\!\cdot\;_{n\in J}a_n$. This proves the corollary. □

In view of the construction, it is reasonable to define a sine function from the q-analogue of the Shintani gamma function by the equation

$$
S_q^{(N,n)}(\underline{\omega},A):=\Gamma_q^{(N,n)}(\underline{\omega},A)^{-1}\Gamma_q^{(N,n)}(-\underline{\omega}+\sum_{i=1}^{N}\underline{\mathbf{a}}^i,A)^{(-1)^N}. \tag{3.13}
$$

Then, like the formula (3.5) for the Shintani sine function $S_S^{(N,n)}(\underline{\omega},A)$, the following periodicity $S_q^{(N,n)}(\underline{\omega},A)$ follows immediately from the functional equation (3.12).

Corollary 3.4. *Retain the notation in the corollary above. Then*

$$
S_q^{(N,n)}(\underline{\omega}+\underline{\mathbf{a}}^{i_0},A)=S_q^{(N-1,n)}(\underline{\omega},\check{A}_{i_0})^{-1}S_q^{(N,n)}(\underline{\omega},A). \tag{3.14}
$$

Remark 3. It is immediate to see that the multiple elliptic gamma function [Ni] is the sine function $S_q^{(N,n)}(\underline{\omega},A)^{(-1)^N}$ when $n=1$. Hence, one may obtain the modular properties of $S_q^{(N,n)}(\underline{\omega},A)$ from the one recently proved in [Na].

Remark 4. Suppose $n=1$. Then, as to the original Barnes' cases, the expression of the multiple sine function $S_N(\omega,\underline{\mathbf{a}}):=\Gamma(\omega,\underline{\mathbf{a}})^{-1}\Gamma(-\omega+|\underline{\mathbf{a}}|,\underline{\mathbf{a}})^{(-1)^N}\big(=S_N^{(N,1)}(\omega,\underline{\mathbf{a}})\big)$ by $G(\underline{\mathbf{q}};z)$ has been obtained in Proposition 5 of [Sh3] for $N=2$ and in Theorem 1.4 of [KW4] for a general N.

Remark 5. The appearance of the factor in front of the product of the q-shifted factorials in (3.11) is, in contrast with the none of such in (3.4), due to the presence of a pole at $s=0$ of the q-analogue $\zeta_q(s,A,\underline{\omega})$ of the Shintani zeta function.

4. A Group Analogue of Shintani's Zeta Function

We notice that the Shintani zeta function and its q-analogue defined in the previous section are both obtained from abelian groups. Thus, in this section, we try to provide an example of a G-analogue of the Shintani zeta function when G is a non-commutative group. The analysis of the zeta function of this type seems, however, much harder. Actually, it can not be kept away from the study of forms of higher degree.

Let $G = \left\{ n(x,y,z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;;\; x,y,z \in \mathbb{C} \right\}$. Let $A = \{a_{ij}\}_{1\le i\le N, 1\le j\le n}$, $B = \{b_{ij}\}_{1\le i\le N, 1\le j\le n}$, and $C = \{c_{ij}\}_{1\le i\le N, 1\le j\le n}$ be complex $N \times n$-matrices and let $x_j\,(1 \le j \le n)$ be complex numbers with positive real parts. Put $\gamma_{ij} = n(a_{ij},\, b_{ij},\, c_{ij})$ and $\gamma_i = (\gamma_{i1}, \dots, \gamma_{in}) \in G^n$. The action of G on $\mathbb{C}^3$ induces the action of G^n on $V = (\mathbb{C}^3)^{\otimes n}$. Denote the map of this action by ρ and let $\rho(\gamma_i) = \gamma_{i1} \otimes \ldots \otimes \gamma_{in} \in \mathrm{End}(V)$. Set $g = \otimes_{j=1}^{n} \begin{pmatrix} x_j & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in SL(V)$. It is immediate that $n(a_1,b_1,c_1)\cdots n(a_N,b_N,c_N) = n\big(\sum_{i=1}^N a_i,\, \sum_{i=1}^N b_i,\, \sum_{i=1}^N c_i + \sum_{1\le i<k\le N} a_i b_k\big)$, and in particular, $n(a,b,c)^m = n(ma,\, mb,\, mc + \binom{m}{2}ab)$. Then, since

$$
\begin{aligned}
\overrightarrow{\prod_{i=1}^{N}}\gamma_{ij}^{m_i} :=& n(a_{1j},\, b_{1j},\, c_{1j})^{m_1} \cdots n(a_{Nj},\, b_{Nj},\, c_{Nj})^{m_N} \\
=& n\Big(m_1 a_{1j},\, m_1 b_{1j},\, m_1 c_{1j} + \binom{m_1}{2} a_{1j} b_{1j}\Big) \\
& \qquad \cdots n\Big(m_N a_{Nj},\, m_N b_{Nj},\, m_N c_{Nj} + \binom{m_N}{2} a_{Nj} b_{Nj}\Big) \\
=& n\Big(\sum_{i=1}^{N} m_i a_{ij},\, \sum_{i=1}^{N} m_i b_{ij},\, \sum_{i=1}^{N} \Big\{m_i c_{ij} + \binom{m_i}{2} a_{ij} b_{ij}\Big\} \\
& \qquad + \sum_{1\le i<k\le N} m_i m_k a_{ij} b_{kj}\Big),
\end{aligned}
$$

the G-analogue of the Shintani zeta function $\zeta_G(s, \{\gamma_{ij}\}_{1\le i\le N, 1\le j\le n}, g)$ can be calculated as

$$\zeta_G(s,\{\gamma_{ij}\}_{1\le i\le N,1\le j\le n},g)=\sum_{m_1,\dots,m_N=0}^{\infty}\Big\{\operatorname{tr}(\rho(\gamma_1^{m_1}\gamma_2^{m_2}\cdots\gamma_N^{m_N})g)\Big\}^{-s}$$

$$=\sum_{m_1,\dots,m_N=0}^{\infty}\Big\{\operatorname{tr}\Big(\Big(\overrightarrow{\prod_{i=1}^{N}}\gamma_{i1}^{m_i}\otimes\cdots\otimes\overrightarrow{\prod_{i=1}^{N}}\gamma_{in}^{m_i}\Big)g\Big)\Big\}^{-s}$$

$$=\sum_{m_1,\dots,m_N=0}^{\infty}\Big\{\operatorname{tr}\Big(n\big(\sum_{i=1}^{N}m_ia_{i1},\sum_{i=1}^{N}m_ib_{i1},\sum_{i=1}^{N}\{m_ic_{i1}+\binom{m_i}{2}a_{i1}b_{i1}\}$$
$$+\sum_{1\le i<k\le N}m_im_ka_{i1}b_{k1}\big)\otimes\cdots$$
$$\cdots\otimes n\big(\sum_{i=1}^{N}m_ia_{in},\sum_{i=1}^{N}m_ib_{in},\sum_{i=1}^{N}\{m_ic_{in}+\binom{m_i}{2}a_{in}b_{in}\}$$
$$+\sum_{1\le i<k\le N}m_im_ka_{in}b_{kn}\big)g\Big)\Big\}^{-s}.$$

Since

$$n(A,B,C)\begin{pmatrix}x&0&1\\1&0&0\\1&1&0\end{pmatrix}=\begin{pmatrix}1&A&C\\0&1&B\\0&0&1\end{pmatrix}\begin{pmatrix}x&0&1\\1&0&0\\1&1&0\end{pmatrix}$$
$$=\begin{pmatrix}x+A+C&C&1\\1+B&B&0\\1&1&0\end{pmatrix}$$

and

$$\operatorname{tr}\left\{\begin{pmatrix}x_1+A_1+C_1&C_1&1\\1+B_1&B_1&0\\1&1&0\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}x_n+A_n+C_n&C_n&1\\1+B_n&B_n&0\\1&1&0\end{pmatrix}\right\}$$
$$=\prod_{j=1}^{n}(x_j+A_j+B_j+C_j),$$

we obtain

$$\zeta_G(s,\{\gamma_{ij}\}_{1\le i\le N,1\le j\le n},g)=\sum_{\underline{\mathbf{m}}\ge\underline{0}}^{\infty}\prod_{j=1}^{n}\phi_j(\underline{\mathbf{m}},\underline{\mathbf{x}},A,B,C)^{-s}\tag{4.1}$$

$$=: \zeta(s, \underline{\mathbf{x}}, A, B, C).$$

Here we put

$$\phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) := x_j + \sum_{i=1}^{N} \Big\{ m_i(a_{ij} + b_{ij} + c_{ij}) + \binom{m_i}{2} a_{ij} b_{ij} \Big\} + \sum_{1 \le i < k \le N} m_i m_k a_{ij} b_{kj}. \tag{4.2}$$

Note that if either $A = 0$ or $B = 0$, then the present zeta function reproduces the Shintani zeta function. Also, since the term $\phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C)$ is linear in the Shintani zeta function case, the theta series attached to the zeta function is given by a product of the generating functions of the multiple Bernoulli polynomials (see [Sh1]). Hence, the behavior of the theta function in the Shintani case is relatively simple, actually, it can have a Laurent expansion at the origin. To study a meromorphy of the zeta function $\zeta(s, \underline{\mathbf{x}}, A, B, C) = \zeta_G(s, \{\gamma_{ij}\}_{1 \le i \le N, 1 \le j \le n}, g)$, we first need the following lemma to find a positivity condition of the quadratic forms ϕ_j.

LEMMA 4.1. *Let a_i and b_i $(1 \le i \le N)$ be positive real numbers. Define an $N \times N$ symmetric matrix $Q = Q(N)$ by $Q_{ik} := a_i b_k$ for $1 \le i \le k \le N$. Suppose that*

$$\begin{vmatrix} b_k & a_k \\ b_{k+1} & a_{k+1} \end{vmatrix} > 0 \quad \textit{for any} \quad 1 \le k < N.$$

Then the real symmetric matrix Q (and the corresponding quadratic form) is positive definite, whence all the eigenvalues are positive.

PROOF. It suffices to show that every principal minor of the form $\det Q(k)$ is positive. Clearly $\det Q(1) = a_1 b_1 > 0$. Since it is not difficult to see the relation

$$\begin{aligned} \det Q(k+1) &= b_{k+1}\Big(a_{k+1} - a_k \frac{b_{k+1}}{b_k}\Big) \det Q(k) \\ &= \frac{b_{k+1}}{b_k} \begin{vmatrix} b_k & a_k \\ b_{k+1} & a_{k+1} \end{vmatrix} \det Q(k), \end{aligned}$$

one concludes that the assertion is true for all N by induction. This proves the lemma. □

Theorem 4.2. *Suppose that the entries of the $N \times n$ matrices A and B are positive and satisfy*

$$\begin{vmatrix} b_{kj} & a_{kj} \\ b_{k+1\,j} & a_{k+1\,j} \end{vmatrix} > 0 \quad \textit{for any} \quad 1 \le k < N \textit{ and } 1 \le j \le n.$$

Then the Dirichlet series defined in (4.1) *is absolutely convergent for* $\mathrm{Re}\, s > \frac{N}{2n}$, *whence the zeta function* $\zeta(s, \underline{\mathbf{x}}, A, B, C)$ *is holomorphic in this half plane. Furthermore,* $\zeta(s, \underline{\mathbf{x}}, A, B, C)$ *can be meromorphically extended to the entire plane* $\mathbb{C}$.

Proof. Define a real symmetric matrix $Q^{(j)}$ by putting $Q^{(j)}_{ik} = a_{ij}b_{kj}$ when $i \le k$. We may assume $x_j \in \mathbb{R}_{>0}$. Then it is easy to see that there exists M_0 such that

$$C_1\, Q^{(j)}[\underline{\mathbf{m}}] + x_j \ge \phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) \ge C_2\, Q^{(j)}[\underline{\mathbf{m}}] + x_j, \tag{4.3}$$

holds if $m_k \ge M_0$ for some k $(1 \le k \le N)$. Here, C_1 and C_2 are positive constants, and $Q^{(j)}[\underline{\mathbf{m}}] = {}^t\underline{\mathbf{m}} Q^{(j)} \underline{\mathbf{m}} = \sum_{i,k=1}^{N} Q^{(j)}_{ik} m_i m_k$, the quadratic form of $Q^{(j)}$. In fact, this follows from the equation

$$\begin{aligned}\phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) = x_j &+ \frac{1}{2} Q^{(j)}[\underline{\mathbf{m}}] \\ &+ \sum_{i=1}^{N} m_i \big(a_{ij} + b_{ij} + c_{ij} - \frac{1}{2} a_{ij} b_{ij}\big).\end{aligned} \tag{4.4}$$

Moreover, since $Q^{(j)}$ are positive definite by Lemma 4.1, when $\underline{\mathbf{x}} = (x_1, \dots, x_n)$ lies in a compact subset U of $\mathbb{R}^n_{>0}$, there exist positive constants D_1, D_2 (which are respectively the product of the maximum eigenvalues and the one of the minimum eigenvalues of $Q^{(j)}$ over $1 \le j \le n$) and $M_0(U)$ such that

$$D_1\, I_N[\underline{\mathbf{m}}]^n \ge \prod_{j=1}^{n} \phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) \ge D_2\, I_N[\underline{\mathbf{m}}]^n \tag{4.5}$$

is true when $m_k \ge M_0(U)$ for some k $(1 \le k \le N)$. Here I_N denotes the identity matrix of degree N. Hence it is easy to see that the Dirichlet series (4.1) is absolutely convergent for $\mathrm{Re}\, s > N/(2n)$. Therefore, if $\mathrm{Re}\, s > N/(2n)$, we have

$$\zeta(s, \underline{\mathbf{x}}, A, B, C) = \sum_{\underline{\mathbf{m}} \geq \underline{0}} \Big\{ \prod_{j=1}^{n} \phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) \Big\}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Theta(t, \underline{\mathbf{x}}, A, B, C) dt,$$

where we put $\Theta(t, \underline{\mathbf{x}}, A, B, C) := \sum_{\underline{\mathbf{m}} \geq \underline{0}} \exp \Big\{ -t \prod_{j=1}^{n} \phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) \Big\}$. Now we need the following lemma.

LEMMA 4.3. *When $t \downarrow 0$, it holds that*

$$\sum_{\underline{\mathbf{m}} \geq \underline{0}} \exp \big\{ -t I_N[\underline{\mathbf{m}}]^n \big\} = d_{-1}(N) t^{-\frac{N}{2n}} + O(t^{-\frac{N-1}{2n}}), \tag{4.6}$$

with the constant $d_{-1}(N) := \mathcal{A}(S^{N-1}) \int_0^\infty e^{-\rho^{2n}} \rho^{N-1} d\rho$. Here $\mathcal{A}(S^{N-1})$ denotes the area of the $(N-1)$-dimensional unit sphere S^{N-1}.

PROOF. It is immediate to see that the relation

$$\sum_{\underline{\mathbf{m}} \geq \underline{0}} \exp \big\{ -t I_N[\underline{\mathbf{m}}]^n \big\} = \sum_{k=0}^{N} \binom{N}{k} \theta_k(t, n) \tag{4.7}$$

holds. Here we put

$$\theta_k(t, n) = \sum_{m_1, \ldots, m_k = 1}^{\infty} e^{-t(m_1^2 + \cdots + m_k^2)^n} \quad \text{for} \quad k = 0, \ldots, N. \tag{4.8}$$

Thus, to prove the lemma, since $t^{-\frac{k}{2n}} < t^{-\frac{N}{2n}}$ for $0 \leq k < N$ provided $0 < t < 1$, it is enough to show the estimate

$$\theta_k(t, n) = d_{-1}(k) t^{-\frac{k}{2n}} + O(t^{-\frac{k-1}{2n}}) \quad \text{when} \quad t \downarrow 0. \tag{4.9}$$

We prove this asymptotic estimate (4.9). First, it is clear to see that

$$\theta_k(t, n) \leq \int_0^\infty \cdots \int_0^\infty e^{-t(x_1^2 + \cdots + x_k^2)^n} dx_1 \cdots dx_k.$$

To obtain the desired estimate of this integral, we employ the spherical coordinates $0 < r < \infty$, $\omega \in S^{k-1}$ ($r^2 = x_1^2 + \cdots + x_k^2$). Then, for any k we have

$$\begin{aligned}\theta_k(t,n) &\le \int_0^\infty \int_{\omega\in S^{N-1}} e^{-tr^{2n}} r^{k-1} dr d\omega \\ &= \mathcal{A}(S^{k-1}) t^{-k/2n} \int_0^\infty e^{-\rho^{2n}} \rho^{k-1} d\rho =: d_{-1}(k) t^{-\frac{k}{2n}}.\end{aligned} \tag{4.10}$$

On the other hand, we find that

$$\begin{aligned}\theta_k(t,n) &\ge \int_0^\infty \cdots \int_0^\infty e^{-t(x_1^2+\cdots+x_k^2)^n} dx_1 \cdots dx_k - k\theta_{k-1}(t,n) \\ &\ge d_{-1}(k) t^{-\frac{k}{2n}} - k d_{-1}(k-1) t^{-\frac{k-1}{2n}}.\end{aligned} \tag{4.11}$$

Here we have used the estimate (4.10) above for $k-1$. Hence the asymptotic estimate (4.9) follows from (4.10) and (4.11). This completes the proof. □

By Lemma 4.3 and the estimate (4.3), we obtain

$$\Theta(t, \underline{\mathbf{x}}, A, B, C) = d_N t^{-\frac{N}{2n}} + O(t^{-\frac{N-1}{2n}}), \qquad t \downarrow 0 \tag{4.12}$$

for some positive constant d_N. Consequently, the standard procedure gotten by dividing the integral representation of the zeta function

$$\begin{aligned}&\zeta(s, \underline{\mathbf{x}}, A, B, C) \\ =&\frac{1}{\Gamma(s)} \Big\{ \int_0^1 t^{s-1} \Theta(t, \underline{\mathbf{x}}, A, B, C) dt + \int_1^\infty t^{s-1} \Theta(t, \underline{\mathbf{x}}, A, B, C) dt \Big\}\end{aligned}$$

yields the analytic continuation of the zeta function. This completes the proof of the theorem. □

Remark. The asymptotic estimate (4.12) indicates that the zeta function $\zeta(s, \underline{\mathbf{x}}, A, B, C)$ has a simple pole at $s = \frac{N}{2n}$.

Since the zeta function $\zeta(s, \underline{\mathbf{x}}, A, B, C)$ is holomorphic at $s = 0$, one can define a corresponding gamma function $\Gamma_N(\underline{\mathbf{x}}, A, B, C)$ as

$$
\begin{aligned}
(4.13) \qquad \Gamma_N(\underline{\mathbf{x}}, A, B, C) &:= \Big\{ \prod_{\underline{\mathbf{m}} \geq \underline{0}} \prod_{j=1}^{n} \phi_j(\underline{\mathbf{m}}, \underline{\mathbf{x}}, A, B, C) \Big\}^{-1} \\
&= \Bigg[\prod_{\underline{\mathbf{m}} \geq \underline{0}} \prod_{j=1}^{n} \Big\{ x_j + \sum_{i=1}^{N} \{ m_i (a_{ij} + b_{ij} + c_{ij}) + \binom{m_i}{2} a_{ij} b_{ij} \} \\
&\qquad + \sum_{1 \leq i < k \leq N} m_i m_k a_{ij} b_{kj} \Big\} \Bigg]^{-1}.
\end{aligned}
$$

Then, we verify that $\Gamma_N(\underline{\mathbf{x}}, A, B, C)$ can be meromorphically continued to the entire plane in $x_j \in \mathbb{C}$ and has poles at the places indicated by the product expression (4.13) (see [Vo] or [KiW]). Remark that, when either $A = 0$ or $B = 0$, this gamma function coincides with the Shintani gamma function in Example 3.2. Hence it is true that the present gamma function is not a constant function. The translation law inherited from the ladder relation (1.3), however, can not be seen in this simple choice of the variable $g = \otimes_{j=1}^{n} \begin{pmatrix} x_j & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in \mathrm{End}(V)$.

To explain this implicit translation law of the gamma function $\Gamma_N(\underline{\mathbf{x}}, A, B, C)$, we have to take into account the number of the free variables in a choice of $g \in \mathrm{End}(V)$. We give the following example. (See also (5.2) in §5 for another definition of a zeta function attached to the same data which is invariant under a compact subgroup of $GL(V)$. It may allow us to determine a necessary and sufficient number of the essential variables for describing the translation law explicitly.)

Example 4.1. We keep the notation above. If we take a variable g in $\mathrm{End}(V)$ as $g(x) = \otimes_{j=1}^{n} g_j(x)$, where $g_j(x) = \{x_{st}^{(j)}\}_{1 \leq s,t \leq 3} \in \mathrm{Mat}_3(\mathbb{C})$, then the translation property of the gamma function as a function of $g(x)$ can be seen from the ladder equation of the zeta function. In fact, we can prove similarly the holomorphy of the zeta function $\zeta(s, \{\gamma_{ij}\}_{1 \leq i \leq N, 1 \leq j \leq n}, g(x))$ at $s = 0$ when $x_{31}^{(j)}$ has a positive real part as in the theorem above, whence the existence of the corresponding gamma function $\Gamma_N(g(x), A, B, C)$ follows.

Here, the gamma function is written as

$$\Gamma_N(g(x),A,B,C)$$
$$=\Bigg[\prod_{\underline{\mathbf{m}}\geq\underline{0}}\prod_{j=1}^{n}\Bigg\{x_{11}^{(j)}+x_{22}^{(j)}+x_{33}^{(j)}+x_{21}^{(j)}\sum_{i=1}^{N}m_ia_{ij}+x_{32}^{(j)}\sum_{i=1}^{N}m_ib_{ij}$$
$$+x_{31}^{(j)}\Big(\Big[\sum_{i=1}^{N}m_ic_{ij}+\binom{m_i}{2}a_{ij}b_{ij}\Big]+\sum_{1\leq i<k\leq N}m_im_ka_{ij}b_{kj}\Big)\Bigg\}\Bigg]^{-1}.$$

Then, for instance, when $k=1$, the ladder equation (1.4) for $\zeta(s,\{\gamma_{ij}\}_{1\leq i\leq N,1\leq j\leq n},g(x))$ yields the translation law induced from the right multiplication by $\rho(\gamma_1)$ as

$$\begin{aligned}&\Gamma_N(g(x)\rho(\gamma_1),A,B,C)\\ (4.14)\quad &:=\Gamma_N\left(\otimes_{j=1}^{n}\begin{pmatrix}x_{11}^{(j)} & x_{11}^{(j)}a_{1j}+x_{12}^{(j)} & x_{11}^{(j)}c_{1j}+x_{12}^{(j)}b_{1j}+x_{13}^{(j)}\\ x_{21}^{(j)} & x_{21}^{(j)}a_{1j}+x_{22}^{(j)} & x_{21}^{(j)}c_{1j}+x_{22}^{(j)}b_{1j}+x_{23}^{(j)}\\ x_{31}^{(j)} & x_{31}^{(j)}a_{1j}+x_{32}^{(j)} & x_{31}^{(j)}c_{1j}+x_{32}^{(j)}b_{1j}+x_{33}^{(j)}\end{pmatrix},A,B,C\right)\\ &=\Gamma_{N-1}(g(x),\check{A}_1,\check{B}_1,\check{C}_1)^{-1}\,\Gamma_N(g(x),A,B,C),\end{aligned}$$

where the $(N-1)\times n$ matrices $\check{A}_1,\check{B}_1$ and $\check{C}_1$ are defined to be the same way as in Corollary 3.3. The translation law relative to the left multiplication by $\rho(\gamma_N)$ is similarly obtained, while it becomes a much complicated one when $1<k<N$ because of the non-commutativity of those γ_j's. Moreover, using this gamma function, we introduce two (multiple) sine functions. Actually, the non-commutativity of the group G^n naturally leads us to define both a left sine function and a right sine function respectively as

$$S_N^L(g(x),A,B,C):=\Gamma_N(g(x),A,B,C)^{-1}\Gamma_N(\rho(\underline{\gamma})g(x)^{-1},A,B,C)^{(-1)^N},$$
$$S_N^R(g(x),A,B,C):=\Gamma_N(g(x),A,B,C)^{-1}\Gamma_N(g(x)^{-1}\rho(\underline{\gamma}),A,B,C)^{(-1)^N}.$$

In fact, since $\rho(\underline{\gamma})\rho(\underline{\gamma}^{\underline{\mathbf{m}}})\neq\rho(\underline{\gamma}^{\underline{\mathbf{m}}})\rho(\underline{\gamma})$ in general for $\underline{\mathbf{m}}\in\mathbb{Z}_{>0}^N$, the left sine function is not necessary equal to the right sine function. Although the periodicity of these sine functions is quite restrictive, under some conditions, the periodicity follows from the translation law of the gamma function $\Gamma_N(g(x),A,B,C)$. For instance, we now assume that the commutator $\gamma_1\gamma_j\gamma_1^{-1}\gamma_j^{-1}$ is always in $\ker\rho$ for $2\leq j\leq N$. Then, since $\rho(\underline{\gamma}^{\underline{\mathbf{m}}})\rho(\gamma_1)^{-1}=\rho(\gamma_1)^{-1}\rho(\underline{\gamma}^{\underline{\mathbf{m}}})$ for any $\underline{\mathbf{m}}\in\mathbb{Z}_{\geq0}^N$, with the help of the general property

$\mathop{\prod\!\!\!\!\!\!\bullet}_n \operatorname{tr}(A_nB_n) = \mathop{\prod\!\!\!\!\!\!\bullet}_n \operatorname{tr}(B_nA_n)$ (this follows from the uniqueness of the analytic continuation of the Dirichlet series $\sum_n \operatorname{tr}(A_nB_n)^{-s}$), we see by (4.14) that

$$\begin{aligned}
&S_N^R(g(x)\rho(\gamma_1), A, B, C)\\
=&\Gamma_N(g(x)\rho(\gamma_1), A, B, C)^{-1}\,\Gamma_N(\rho(\gamma_1)^{-1}g(x)^{-1}\rho(\underline{\gamma}), A, B, C)^{(-1)^N}\\
=&\Big\{\Gamma_{N-1}(g(x), \check{A}_1, \check{B}_1, \check{C}_1)^{-1}\Gamma_N(g(x), A, B, C)\Big\}^{-1}\\
&\quad\times \Gamma_N(g(x)^{-1}\rho(\underline{\gamma})\rho(\gamma_1)^{-1}, A, B, C)^{(-1)^N}\\
=&\Gamma_{N-1}(g(x), \check{A}_1, \check{B}_1, \check{C}_1)\,\Gamma_N(g(x), A, B, C)^{-1}\\
&\quad\times \Big\{\Gamma_{N-1}(g(x)^{-1}\rho(\underline{\gamma})\rho(\gamma_1)^{-1}, \check{A}_1, \check{B}_1, \check{C}_1)\Gamma_N(g(x)^{-1}\rho(\underline{\gamma}), A, B, C)\Big\}^{(-1)^N}\\
=&\Big\{\Gamma_{N-1}(g(x), \check{A}_1, \check{B}_1, \check{C}_1)^{-1}\\
&\quad\times \Gamma_{N-1}(g(x)^{-1}\rho(\gamma_2\cdots\gamma_N)^{-1}, \check{A}_1, \check{B}_1, \check{C}_1)^{(-1)^{N-1}}\Big\}^{-1} S_N^R(g(x), A, B, C)\\
=&S_{N-1}^R(g(x), \check{A}_1, \check{B}_1, \check{C}_1)^{-1}S_N^R(g(x), A, B, C).
\end{aligned}$$

This indeed shows that the periodicity, or rather, the difference equation satisfied by the right sine function $S_N^R(g(x), A, B, C)$ with respect to the translation $g(x) \mapsto g(x)\rho(\gamma_1) \in GL(\operatorname{End}(V))$. We remark that this difference equation is a generalization of (3.5) for the Shintani sine function $S_S^{(N,n)}(\underline{\omega}, A)$. Beside the discussion concerning such difference equations, it is also interesting to seek an integral representation of the gamma function and an algebraic differential equation satisfied by these sine functions (cf. [KW6]). We will discuss these points in the future.

5. Closing Remarks

One of the most important general question is to ask a meromorphy of the zeta function, i.e. when the zeta function can be analytically extended to a region containing the origin $s = 0$ or further to the entire plane $\mathbb{C}$ as a meromorphic function. When it is true and if the coefficient of the linear term of the Laurent expansion of the zeta function at $s = 0$ is not a constant, we obtain a "non-trivial" gamma function. Here, apart from this problem, we define generalized versions of the present zeta function and give brief comments. Our main concern is to construct functions having a certain invariance and a given set of zeros. We, however, restrict ourselves to developing the detailed study here. We will leave it in another occasion.

• We define a zeta generating function $\zeta^{(r)}_{G,\rho}(s,g;\underline{\gamma},t)$ for the data $(G,\rho,\underline{\gamma},r)$. Put $\Psi(t,A)=\det(tI-A)$. Then

$$\zeta^{(r)}_{G,\rho}(s,g;\underline{\gamma},t):=\sum_{0\le n_i<N_i\ (i=1,\dots,r)}\Psi(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})g))^{-s}. \tag{5.1}$$

It is immediate to see that $\frac{\partial^j}{\partial t^j}\zeta^{(r)}_{G,\rho}(s,g;\underline{\gamma},0)=(s)_j\,\zeta^{(r)}_{G,\rho,n-j}(s,g;\underline{\gamma})$. Here $(s)_n=s(s+1)\cdots(s+n-1)=\Gamma(s+n)/\Gamma(s)$ denotes the Pochhammer symbol. Moreover, since $\{\mathrm{tr}(A^k)\}_{k=1,\dots,n}$ forms another basis of the ring of invariants, we may also define zeta functions attached to the same data by $\sum_{n_1,\dots,n_r}L_1\big((\rho(\gamma_1^{n_1}\cdot\gamma_2^{n_2}\cdots\gamma_r^{n_r})g)^k\big)^{-s}$. Then, one can ask whether there is any relation (like the Newton formula) or not between the gamma functions attached to the L_j's and the ones attached to the powers of the trace function.

• We consider the following zeta function which has some invariance. Let K be a compact subgroup of $GL(V)$, that is, $K\subset SU(V)$. We then define a zeta function attached to the data $(G,\rho,j,K,r,\underline{\gamma},g)$ which is right K-invariant as

$$\zeta^{(r,K)}_{G,\rho,j}(s,g;\underline{\gamma}):=\sum_{0\le n_i<N_i\ (i=1,\dots,r)}\int_K L_j(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})gk))^{-s}dk, \tag{5.2}$$

where dk denotes the normalized Haar measure on K. Of course, if $K=\{1_V\}$, 1_V being the identity operator in $\mathrm{End}(V)$, we have $\zeta^{(r,\{1\})}_{G,\rho,j}(s,g;\underline{\gamma})=\zeta^{(r)}_{G,\rho,j}(s,g;\underline{\gamma})$. Also, if K is a finite group then the integral should be considered as a sum over the elements of K divided by the order of K. Thus, it is important to study the integral $\int_K L_j(\rho(\gamma)gk))^{-s}dk$ as a function of $(s,g)\in\mathbb{C}\times\mathrm{End}(V)^K$. We refer to, e.g., [Su] for the study of harmonic analysis on Lie groups. Notice also that the zeta function $\zeta^{(r,\{1\})}_{G,\rho,j}(s,g;\underline{\gamma})$ still has the ladder relation (1.3), whence the corresponding gamma function has a translation law. As we saw in the previous section, if we want to describe such a law explicitly, we need to handle a large number of variables. Thus, if we take a average on K, it is possible to reduce a necessary number of variables for the discription of the translation law.

• Another candidate for defining invariant zeta function is of the form

$$(5.3)\quad \zeta^{(K_1,\dots,K_M)}_{G,\rho,j}(s,g_1,\dots,g_M;\underline{\gamma}_1,\dots,\underline{\gamma}_M) := \sum_{\underline{\mathbf{n}}_1\in\mathbb{Z}^{r_1}_{\geq 0}}\cdots\sum_{\underline{\mathbf{n}}_M\in\mathbb{Z}^{r_M}_{\geq 0}} \times\int_{K_1}\cdots\int_{K_M} L_j(\rho(\underline{\gamma}_1^{\underline{\mathbf{n}}_1})g_1k_1\cdots\rho(\underline{\gamma}_1^{\underline{\mathbf{n}}_M})g_Mk_M)^{-s}dk_1\cdots dk_M,$$

where $\underline{\gamma}^{\underline{\mathbf{n}}}=\gamma_1^{n_1}\cdots\gamma_r^{n_r}$, when $\underline{\gamma}=(\gamma_1,\dots,\gamma_r)$ and $\underline{\mathbf{n}}=(n_1,\dots,n_r)$, and $K_1,\dots,K_M$ being subgroups of $GL(V)$. Obviously, this zeta function has also ladder structure, whence the attached gamma function has translation laws.

• It is also interesting to observe the following situation. Let S be a group and (π,V) be a representation of S. Let (G,K) be a dual pair of subgroups of S (see, e.g., [Ho]). By definition, $\pi(G)$ and $\pi(K)$ commute with each other. Then, as a function of g, the integral $f_j(s,g;h) := \int_K L_j(\pi(h)g\pi(k))^{-s}dk$ defines a K-bi-invariant function on $\mathrm{End}(V)$, that is, $f_j(s,g;h)\in C(K\backslash \mathrm{End}(V)/K)$.

• For a given function ϕ, we can define the zeta function of the following type.

$$(5.4)\quad \zeta^{(r)}_{G,\rho,j}(s,g;\underline{\gamma};\phi) := \sum_{0\leq n_i<N_i\,(i=1,\dots,r)} \phi\big(L_j(\rho(\gamma_1^{n_1}\gamma_2^{n_2}\cdots\gamma_r^{n_r})g))\big)^{-s}.$$

Then, like the case where $\phi(t)=\sinh(t)$ treated in [KiW], the zeta function $\zeta^{(r)}_{G,\rho,j}(s,g;\underline{\gamma};\phi)$ may have a log-branch at $s=0$.

Note added in proof. After this paper was completed, the author knew the work of E. Friedman and S. Ruijsenaars: "Shintani-Barnes zeta and gamma functions," *Advances in Math.*, **187** (2004), 362-395. The definition of our $\zeta(s,A,\underline{\omega})$ in §3 is identical with their Shintani-Barnes zeta function.

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