**A Limit Theorem for Solutions of Some Functional Stochastic Difference Equations**

By Takashi Kato

**Abstract.** In this paper, we study a limit theorem for solutions of some functional stochastic difference equations under strong mixing conditions and some dimensional conditions. This work is an extension of the work of Hisao Watanabe.

1. Introduction and Main Results

Diffusion approximations for certain stochastic difference equations or stochastic ordinary differential equations have been discussed in several papers. [9] [15], [16] and [17] treated such problem and derived the weak limit of appropriately scaled and interpolated process, which was given by the solution of a stochastic difference equation as a diffusion process. Concerning this, [5], [6], [10], [11] and many other papers dealt with weak convergence of the solution of a stochastic ordinary differential equation.

In this paper, we study a limit theorem for stochastic processes $X^n_t$ given by the following functional stochastic difference equations

$$X^n_{(k+1)/n} - X^n_{k/n} = \frac{1}{\sqrt{n}} F^n_k(X^n, \omega) + \frac{1}{n} G^n_k(X^n, \omega)$$

and by linear interpolation as

$$X^n_t = (1 - nt + k)X^n_{k/n} + (nt - k)X^n_{(k+1)/n}$$

for $k/n < t < (k + 1)/n$, and

$$X^n_0 = x_0 \in \mathbb{R}^d.$$  

Here $F^n_k$ and $G^n_k$ are $d$ dimensional random functions on $C([0, \infty); \mathbb{R}^d)$, the space of continuous functions from $[0, \infty)$ to $\mathbb{R}^d$, such that $F^n_k$ has mean zero.

1991 *Mathematics Subject Classification.* Primary 60F05; Secondary 60B10, 39A12.
Under certain assumptions for $F^n_k$ and $G^n_k$, we show that the distribution of $X^n$ converges weakly to the solution of a martingale problem corresponding to functional coefficients.

The methods of the proof are based on [5] and [16]. However, we cannot use mixing inequalities in these papers, since the dimension of parameter space $C([0, \infty); \mathbb{R}^d)$ is infinite.

We show another version of mixing inequalities by assuming certain dimensional conditions for the set of random variables $F^n_k(w)$ and $G^n_k(w)$, which may look artificial but we give sufficient conditions for this assumption later.

The author thanks Professor Shigeo Kusuoka for a lot of precious advice and discussions.

Let $(\Omega^n, \mathcal{F}^n, P^n)$, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, be complete probability spaces. Let $F^n_k(w, \omega) = (F^n_{k,i}(w, \omega))_{i=1}^d$ and $G^n_k(w, \omega) = (G^n_{k,i}(w, \omega))_{i=1}^d : C([0, \infty); \mathbb{R}^d) \times \Omega^n \rightarrow \mathbb{R}^d$, $k \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, be random functions. Let $\mathcal{B}_t$ be the $\sigma$-algebra of $C([0, \infty); \mathbb{R}^d)$ given by $\mathcal{B}_t = \sigma(w(s) ; s \leq t)$.

We introduce the following conditions.

[A1] $F^n_{k,i}$ and $G^n_{k,i}$ are measurable with respect to $\mathcal{B}_{k/n} \otimes \mathcal{F}^n$.

By [A1], we can regard $F^n_{k,i}$ and $G^n_{k,i}$ as random functions defined on the Banach space $C([0, k/n]; \mathbb{R}^d)$.

[A2] $F^n_{k,i}(w, \omega)$ (respectively, $G^n_{k,i}(w, \omega)$) is twice (respectively, once) continuously Fréchet differentiable in $w$ for $P^n$-almost surely $\omega$.

We denote by $L^m_{T}$ the space of real valued continuous $m$-multilinear operators on $C([0, T]; \mathbb{R}^d)$ and denote by $| \cdot |_{L^m_{T}}$ its norm. Then the $m$-th Fréchet derivative $\nabla^m F^n_{k,i}(w) : (w_1, \ldots, w_m) \mapsto \nabla^m F^n_{k,i}(w; w_1, \ldots, w_m)$ is regarded as the element of $L^m_{k/n}$ for each $w$ (and so is $\nabla^m G^n_{k,i}(w)$). For $m = 0$, $L^0_{T} = \mathbb{R}$ and $\nabla^0 F^n_{k,i}(w) = F^n_{k,i}(w)$.

Let $p_0 > 3$ and $\gamma_0 > 0$. We assume the moment conditions with respect to $p_0$ and the dimensional conditions with respect to $\gamma_0$ as [A3] and [A4].
For each $M > 0$, there exists a constant $C(M) > 0$ such that

\[ \sum_{m=0}^{2} E^n \left[ \sup_{|w| \leq M} \left| \nabla^m F_{n,i}^k (w) \right|^{p_0}_{L_{k/n}^m} \right] \leq C(M) \]

and

\[ \sum_{m=0}^{1} E^n \left[ \sup_{|w| \leq M} \left| \nabla^m G_{n,i}^k (w) \right|^{p_0}_{L_{k/n}^m} \right] \leq C(M) \]

for any $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, where $E^n[\cdot]$ denotes the expectation under the probability measure $P^n$ and $|w|_{\infty} = \sup_{t \geq 0} |w(t)|$.

Let $C_M^d$ denote the set of $w \in C([0, \infty); \mathbb{R}^d)$ such that $|w|_{\infty} \leq M$. For a random function $U : C([0, \infty); \mathbb{R}^d) \times \Omega^n \rightarrow \mathbb{R}$ and $\varepsilon > 0$, let $N_n(\varepsilon, M; U)$ be the smallest integer $m$ such that there exist sets $S_1, \ldots, S_m$ which satisfy

\[ C_M^d = \bigcup_{i=1}^{m} S_i \]

and

\[ E^n \left[ \max_{i=1, \ldots, m} \sup_{x,y \in S_i} |U(x) - U(y)|^{p_0} \right]^{1/p_0} < \varepsilon. \]

\[ \sup_{n,k} \sup_{\varepsilon > 0} \varepsilon^{70} N_n(\varepsilon, M; F_{k}^{n,i}) < \infty, \]

\[ \sup_{n,k} \sup_{l \leq k} \sup_{\varepsilon > 0} \varepsilon^{70} N_n(\varepsilon, M; \nabla F_{k}^{n,i}(\cdot; I_i^n e_j)) < \infty, \]

\[ \sup_{n,k} \sup_{l,m \leq k} \sup_{\varepsilon > 0} \varepsilon^{70} N_n(\varepsilon, M; \nabla^2 F_{k}^{n,i}(\cdot; I_i^n e_j, I_m^n e_\nu)) < \infty, \]

\[ \sup_{n,k} \sup_{\varepsilon > 0} \varepsilon^{70} N_n(\varepsilon, M; G_{k}^{n,i}) < \infty \]

and

\[ \sup_{n,k} \sup_{l \leq k} \sup_{\varepsilon > 0} \varepsilon^{70} N_n(\varepsilon, M; \nabla G_{k}^{n,i}(\cdot; I_i^n e_j)) < \infty \]

for each $M > 0$ and $i,j,\nu = 1, \ldots, d$, where $e_j \in \mathbb{R}^d$ denotes the unit vector along the $j$-th axis, i.e. $e_j = (0, \ldots, 0, \overset{\_}{1}, 0, \ldots, 0)$, and the function $I_i^n :$
[0, ∞) → ℝ is given by

\[
I^n_l(t) = \begin{cases} 
0 & \text{if } 0 \leq t < \frac{l}{n} \\
n(t - l) & \text{if } \frac{l}{n} \leq t < \frac{l+1}{n} \\
1 & \text{if } t \geq \frac{l+1}{n}.
\end{cases}
\]

[A5] Let

\[
\mathcal{F}^n_{k,l} = \sigma \left( F^{n,i}_m(w), G^{n,i}_m(w) ; i = 1, \ldots, d, \ k \leq m \leq l, \ w \in C([0, \infty); \mathbb{R}^d) \right)
\]

and

\[
\alpha_k = \sup_n \sup_l \sup \{|P^n(A \cap B) - P^n(A)P^n(B)| ; A \in \mathcal{F}^n_{0,l}, B \in \mathcal{F}^n_{k+l,\infty}\}.
\]

Then

\[
\sum_{k=1}^{\infty} \alpha^g_0 < \infty,
\]

where \( g_0 = \frac{1}{2s_0 + 4\gamma_0} \) and \( s_0 = \frac{p_0}{p_0 - 3} \).

[A6] \( E^n[F^{n,i}_k(w)] = 0 \).

We denote by \( \mathcal{K}^d \) the family of a compact set \( K \) of \( C([0, \infty); \mathbb{R}^d) \) such that sup \( w \in K |w|_{\infty} < \infty \).

[A7] Let

\[
a^{n,i,j}_0(k, w) = E^n[F^{n,i}_k(w)F^{n,j}_k(w)],
\]

\[
b^{n,i}_0(k, w) = E^n[G^{n,i}_k(w)],
\]

\[
A^{n,i,j}(k, w) = \sum_{l=1}^{\infty} E^n \left[ F^{n,i}_{k+l}(w \left( \cdot \wedge \frac{k}{n} \right))F^{n,j}_k(w) \right],
\]

\[
B^{n,i,j}(k, w) = \sum_{l=1}^{\infty} E^n \left[ \nabla F^{n,i}_{k+l}(w \left( \cdot \wedge \frac{k}{n} \right); I^n_{k,j})F^{n,j}_k(w) \right]
\]
for \( k \in \mathbb{Z}_+ \) and \( w \in C([0, \infty); \mathbb{R}^d) \), where \( a \wedge b = \min\{a, b\} \). The following limits exist uniformly on any \( K \in \mathcal{K}^d \) for each \( t \geq 0 \):

\[
\begin{align*}
\text{(1.12)} & \quad a_{0}^{ij}(t, w) = \lim_{n \to \infty} a_{0}^{n,ij}([nt], w), \\
\text{(1.13)} & \quad b_{0}^{i}(t, w) = \lim_{n \to \infty} b_{0}^{n,i}([nt], w), \\
\text{(1.14)} & \quad A^{ij}(t, w) = \lim_{n \to \infty} A^{n,ij}([nt], w), \\
\text{(1.15)} & \quad B^{ij}(t, w) = \lim_{n \to \infty} B^{n,ij}([nt], w),
\end{align*}
\]

where \([x]\) denotes the greatest integer less than or equal to \( x \).

[A8] Define \( a(t, w) = (a^{ij}(t, w))_{i,j=1}^{d} \) and \( b(t, w) = (b^{i}(t, w))_{i=1}^{d} \) by

\[
a^{ij}(t, w) = a_{0}^{ij}(t, w) + A^{ij}(t, w) + A^{ji}(t, w)
\]

and

\[
b^{i}(t, w) = b_{0}^{i}(t, w) + \sum_{j=1}^{d} B^{ij}(t, w).
\]

For each \( T > 0 \), there exists a positive constant \( C(T) \) such that

\[
\text{(1.16)} \quad |a^{ij}(t, w)| \leq C(T) \left( 1 + \sup_{0 \leq s \leq t} |w(s)|^2 \right)
\]

and

\[
\text{(1.17)} \quad |b^{i}(t, w)| \leq C(T) \left( 1 + \sup_{0 \leq s \leq t} |w(s)| \right)
\]

for \( t \in [0, T] \) and \( w \in C([0, \infty); \mathbb{R}^d) \).

[A9] Let

\[
\mathcal{L} f(t, w) = \frac{1}{2} \sum_{i,j=1}^{d} a^{ij}(t, w) \frac{\partial^2}{\partial x^i \partial x^j} f(w(t)) + \sum_{i=1}^{d} b^{i}(t, w) \frac{\partial}{\partial x^i} f(w(t))
\]

for \( f \in C^2(\mathbb{R}^d) \). The martingale problem associated with the generator \( \mathcal{L} \) and initial value \( x_0 \) has a unique solution \( Q \) on \( C([0, \infty); \mathbb{R}^d) \).

We will introduce the sufficient conditions for [A4] and [A9] in Section 5.
Define the stochastic process \( X^n_t = (X^n_{t,i})_{i=1}^d \) by (1.1), (1.2) and (1.3). Let \( Q^n \) be the probability measure induced by \( X^n \) on \( C([0, \infty); \mathbb{R}^d) \).

**Theorem 1.** Assume \([A1] - [A9]\). Then \( Q^n \) converges weakly to \( Q \) on \( C([0, \infty); \mathbb{R}^d) \).

Let us give some remarks on Theorem 1.

(i) In fact, using the arguments in [16], we can prove Theorem 1 without assuming the condition (1.10).

(ii) We can replace the assumption \([A5]\) with \([A5']\). For each \( M > 0 \)

\[
\sum_{k=1}^{\infty} \alpha_k(M)^{\varphi_0} < \infty,
\]

where

\[
\mathcal{F}^n_{k,l}(M) = \sigma(F^n_{m,i}(w), G^n_{m,i}(w) ; i = 1, \ldots, d, \ k \leq m \leq l, \ |w|_\infty \leq M)
\]

and

\[
\alpha_k(M) = \sup_n \sup_l \sup\{ |P^n(A \cap B) - P^n(A)P^n(B)| ; A \in \mathcal{F}^n_{0,l}(M), B \in \mathcal{F}^n_{k+l,\infty}(M) \}.
\]

The proof needs no change.

(iii) Assuming the following uniform mixing condition \([A5'']\) instead of \([A5]\), we can remove the dimensional condition \([A4]\):

\([A5'']\) It holds that

\[
\sum_{k=1}^{\infty} \phi_k^{\varphi_2} < \infty,
\]

where \( \varphi_2 = \frac{p_0 - 2}{2p_0} \) and

\[
\phi_k = \sup_n \sup_l \sup\{ \left| \frac{P^n(A \cap B)}{P^n(A)} - P^n(B) \right| ; \ A \in \mathcal{F}^n_{0,l}, B \in \mathcal{F}^n_{k+l,\infty}, P^n(A) > 0 \}.
\]
Next we provide another version of Theorem 1. We introduce the following conditions.

- **[B4]** For some $\gamma_1 > 0$, (1.6) – (1.10) hold with $\log N_n$ instead of $N_n$.

- **[B5]** Let $\alpha_k$ be as in [A5]. Then there exists $\varrho_1 \in \left(0, \frac{1}{2\gamma_1}\right)$ such that

$$
\sum_{k=1}^{\infty} \left(\frac{1}{\log(1/\alpha_k)}\right)^{\frac{\varrho_1}{\gamma_1}} < \infty.
$$

\textbf{Theorem 2.} Assume [A1] – [A3], [B4], [B5] and [A6] – [A9]. Then $Q^n$ converges weakly to $Q$ on $C([0, \infty); \mathbb{R}^d)$.

\section{Mixing Inequalities}

In this section we prepare some inequalities for strong mixing coefficients. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset \mathcal{F}$ be sub $\sigma$-algebras. Define $\alpha(\mathcal{A}, \mathcal{B})$ by

$$
\alpha(\mathcal{A}, \mathcal{B}) = \sup\{ |P(A \cap B) - P(A)P(B)| ; A \in \mathcal{A}, B \in \mathcal{B} \}.
$$

The following lemma is shown in the proof of Theorem 17.2.2 in [4].

\textbf{Lemma 1.} Let $1 \leq p, q, r \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $X$ be an $\mathcal{A}$-measurable random variable and $Y$ be a $\mathcal{B}$-measurable random variable. Then

$$
\left| \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \right| \leq 8 \mathbb{E}[|X|^p]^{1/p} \mathbb{E}[|Y|^q]^{1/q} \alpha(\mathcal{A}, \mathcal{B})^{1/r}.
$$

Let $(S, d)$ be a metric space, $\varepsilon, p > 0$ and $U : S \times \Omega \to \mathbb{R}$ be a continuous random function. We say that a family of sets $(S_i)_{i=1}^{m}$ is an $(\varepsilon, p, U)$-net of $S$ if $S = \bigcup_{i=1}^{m} S_i$ and

$$
\mathbb{E} \left[ \max_{i=1,\ldots,m} \sup_{x,y \in S_i} |U(x) - U(y)|^p \right]^{1/p} < \varepsilon.
$$
We denote the minimum of cardinals of $(\varepsilon, p, U)$-nets by $N(\varepsilon, p; U)$.

**Lemma 2.** Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} < 1$ and $U : S \times \Omega \rightarrow \mathbb{R}$ be a continuous random function such that $U(x)$ is $\mathcal{A}$-measurable and $E[U(x)] = 0$ for each $x \in S$, and $X : \Omega \rightarrow S$, $V : \Omega \rightarrow \mathbb{R}$ be $\mathcal{B}$-measurable random variables. Then for any $\varepsilon > 0$

\[
\|E[U(X)V]\| \leq 8 \left( E[\sup_{x \in S} |U(x)|^p]^{1/p} + 1 \right) \times E[|V|^q]^{1/q} \left\{ \varepsilon + \varepsilon^{1-r} N(\varepsilon, p; U)\alpha(A, B) \right\},
\]

where $\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}$.

**Proof.** We may assume that the right-hand side of (2.2) is finite and $\alpha(A, B) > 0$. Set $N_\varepsilon = N(\varepsilon, p; U)$ and $U^* = \sup_{x \in S} |U(x)|$. Let $\delta = p/r$, $\bar{\delta} = q/r$,

\[
I = E[|U^*|^p]^{1/p} \varepsilon^{-1/\delta}, \quad J = E[|V|^q]^{1/q} \varepsilon^{-1/\bar{\delta}}
\]

and

\[
U_I(x) = U(x)1_{\{|U^*| \leq I\}}, \quad V_J = V1_{\{|V| \leq J\}}.
\]

Then we have

\[
\frac{1}{\delta} + \frac{1}{\bar{\delta}} = r - 1.
\]

Let $(S_i)_{i=1}^{N_\varepsilon}$ be an $(\varepsilon, p, U)$-net. We may assume that all $S_i$ are disjoint and not empty. Take any $x_i \in S_i$, and define the random variable $\tilde{X} : \Omega \rightarrow S$ by

\[
\tilde{X}(\omega) = \sum_{i=1}^{N_\varepsilon} x_i 1_{\Omega_i}(\omega),
\]

where $\Omega_i = \{X \in S_i\}$. Then it follows that

\[
|E[U(X)V]| \leq |E[(U(X) - U(\tilde{X}))V]| + |E[(U(\tilde{X}) - U_I(\tilde{X}))V]| + |E[U_I(\tilde{X})(V - V_J)]| + |E[U_I(\tilde{X})V_J]|.
\]
By the definition of $\tilde{X}$, we have

$$
\begin{align*}
|E[(U(X) - U(\tilde{X}))V]| & \leq E \left[ \max_{i=1, \ldots, N_\varepsilon} \sup_{x, y \in S_i} |U(x) - U(y)| \cdot |V| \right] \\
& \leq E \left[ \max_{i=1, \ldots, N_\varepsilon} \sup_{x, y \in S_i} |U(x) - U(y)|^p \right]^{1/p} E[|V|^q]^{1/q} \\
& \leq \varepsilon E[|V|^q]^{1/q}.
\end{align*}
$$

By the Chebyshev inequality and the Hölder inequality, we have

$$
\begin{align*}
|E[(U(\tilde{X}) - U_I(\tilde{X}))V]| & \leq \frac{1}{I^\delta} E \left[ |U^*|^{1+\delta} |V| \right] \\
& \leq \frac{1}{I^\delta} E[|U^*|^p]^{(1+\delta)/p} E[|V|^q]^{1/q} = E[|U^*|^p]^{1/p} E[|V|^q]^{1/q} \varepsilon.
\end{align*}
$$

Similarly we obtain

$$
\begin{align*}
|E[U_I(\tilde{X}')(V - V_J)]| & \leq E[|U^*|^p]^{1/p} E[|V|^q]^{1/q} \varepsilon.
\end{align*}
$$

Set $\bar{U}_I(x) = E[U_I(x)]$ and $\tilde{U}_I(x) = U_I(x) - \bar{U}_I(x)$. Then it follows that

$$
\begin{align*}
|E[U_I(\tilde{X})V_J]| & \leq |E[\bar{U}_I(\tilde{X})V_J]| + |E[\tilde{U}_I(\tilde{X})V_J]| \\
& \leq \sup_{x \in S} |\tilde{U}_I(x)| E[|V|^q]^{1/q} + \sum_{i=1}^{N_\varepsilon} |E[\tilde{U}_I(x_i)V_J 1_{\Omega_i}]|.
\end{align*}
$$

Since $E[U(x)] = 0$, we have

$$
\begin{align*}
|\bar{U}_I(x)| & = \left| E[U_I(x) - U(x)] \right| \leq \frac{1}{I^\delta} E[|U^*|^{1+\delta}] = E[|U^*|^p]^{1/p} \varepsilon.
\end{align*}
$$

By Lemma 1 and (2.3), we get

$$
\begin{align*}
\sum_{i=1}^{N_\varepsilon} |E[\tilde{U}_I(x_i)V_J 1_{\Omega_i}]| & \leq 8N_\varepsilon I J \alpha(\mathcal{A}, \mathcal{B}) \\
& = 8 E[|U^*|^p]^{1/p} E[|V|^q]^{1/q} \varepsilon^{1-r} N_\varepsilon \alpha(\mathcal{A}, \mathcal{B}).
\end{align*}
$$

By (2.4)-(2.10), we obtain the assertion. □
Lemma 3. Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} < 1$ and $U : S \times \Omega \rightarrow \mathbb{R}$ be a continuous random function such that $U(x)$ is $\mathcal{A}$-measurable and $\mathbb{E}[U(x)] = 0$ for each $x \in S$, and $X : \Omega \rightarrow S$, $V : \Omega \rightarrow \mathbb{R}$ be $\mathcal{B}$-measurable random variables. Suppose that there exist positive constants $C_0$ and $\gamma$ such that

$$\sup_{\varepsilon > 0} \varepsilon^{\gamma} N(\varepsilon, p; U) \leq C_0. \tag{2.11}$$

Then it holds that

$$|\mathbb{E}[U(X)V]| \leq 16(C_0 + 1)(\mathbb{E}[\sup_{x \in S} |U(x)|^p]^{1/p} + 1) \times \mathbb{E}[|V|^q]^{1/q} \alpha(\mathcal{A}, \mathcal{B})^p, \tag{2.12}$$

where $\rho = \frac{1}{r + \gamma}$ and $\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}$.

Proof. By Lemma 2, we get

$$|\mathbb{E}[U(X)V]| \leq 8(C_0 + 1)(\mathbb{E}[\sup_{x \in S} |U(x)|^p]^{1/p} + 1) \times \mathbb{E}[|V|^q]^{1/q}\{\varepsilon + \varepsilon^{1-r-\gamma}\alpha(\mathcal{A}, \mathcal{B})\}. \tag{2.12}$$

The assertion now follows by taking $\varepsilon = \alpha(\mathcal{A}, \mathcal{B})^6$. \(\square\)

We denote by $\mathcal{A} \vee \mathcal{B}$ the smallest $\sigma$-algebra which includes both $\mathcal{A}$ and $\mathcal{B}$. The following lemma is obtained by Lemma 3 and the arguments in the proof of Lemma 2 in [5].

Lemma 4. Let $1 < p, q, r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let $U, V : S \times \Omega \rightarrow \mathbb{R}$ be continuous random functions such that $U(x)$ and $V(x)$ are $\mathcal{A}$ and $\mathcal{B}$-measurable respectively and $\mathbb{E}[U(x)] = 0$ for each $x \in S$, and $X : \Omega \rightarrow S$, $Z : \Omega \rightarrow \mathbb{R}$ be $\mathcal{C}$-measurable random variables. Suppose that there exist positive constants $C_0, u^*, v^*$ and $\gamma$ such that

$$\sup_{\varepsilon > 0} \varepsilon^{\gamma}\{N(\varepsilon, p; U) + N(\varepsilon, q; V)\} \leq C_0, \tag{2.13}$$

$$\mathbb{E}[\sup_{x \in S} |U(x)|^p]^{1/p} \leq u^* \tag{2.14}$$
and

\[
E[\max_{x \in S} |V(x)|^q]^{1/q} \leq v^*.
\]

Then there exists a constant \( C > 0 \) depending only on \( C_0, u^*, v^* \) and \( \gamma \) such that

\[
E[\Xi(X)Z] \leq C E[Z]^{1/r}\alpha(A \lor B, C)^{\phi'} \alpha(A, B \lor C)^{\phi'},
\]

where \( \Xi(x) = U(x)V(x) - E[U(x)V(x)] \), \( \phi' = \frac{1}{2s + 4\gamma} \) and \( \frac{1}{s} = 1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} \).

**Proof.** Set \( \tilde{\varepsilon} = \frac{\varepsilon}{2(u^* + v^*)} \). Let \( t \geq 1 \) be such that \( \frac{1}{t} = \frac{1}{p} + \frac{1}{q} \). Then we have

\[
N(\varepsilon, t; \Xi) \leq N(\tilde{\varepsilon}, p; U)N(\tilde{\varepsilon}, q; V).
\]

Indeed, if we let \( (S_i)_{i=1}^{N(\tilde{\varepsilon}, p; U)} \) and \( (\tilde{S}_j)_{j=1}^{N(\tilde{\varepsilon}, q; V)} \) be \((\tilde{\varepsilon}, p, U)\)-net and \((\tilde{\varepsilon}, p, U)\)-net respectively, then the Hölder inequality implies

\[
E \left[ \max_{i,j} \sup_{x,y \in S_i \cap \tilde{S}_j} |\Xi(x) - \Xi(y)|^t \right]^{1/t} \\
\leq 2 \left\{ E \left[ \max_{i,j} \sup_{x \in S_i} |U(x)|^t \right] \max_{y \in \tilde{S}_j} \sup_{x \in S_i} |V(x) - V(y)|^t \right]^{1/t} \\
+ E \left[ \max_{i,j} \sup_{x,y \in \tilde{S}_j} |U(x) - U(y)|^t \sup_{x \in S_i} |V(x)|^t \right]^{1/t} \\
\leq 2 \left\{ u^* E \left[ \max_{i,j} \sup_{x,y \in \tilde{S}_j} |V(x) - V(y)|^q \right]^{1/q} \\
+ E \left[ \max_{i,j} \sup_{x,y \in \tilde{S}_j} |U(x) - U(y)|^p \right]^{1/p} v^* \right\} \\
\leq 2(u^* + v^*) \tilde{\varepsilon} = \varepsilon.
\]

Thus \( (S_i \cap \tilde{S}_j)_{i=1,\ldots,N(\tilde{\varepsilon}, p; U), j=1,\ldots,N(\tilde{\varepsilon}, q; V)} \) is an \((\varepsilon, t, \Xi)\)-net. This implies (2.17).

So we get

\[
N(\varepsilon, t; \Xi) \leq 2^{2\gamma}(u^* + v^*)^{2\gamma} C_0^2 \varepsilon^{-2\gamma}.
\]
Then, using Lemma 3 with $\Xi$ substituted for $U$, we have

\begin{equation}
\left| E[\Xi(X)Z] \right| \leq C_1 \left( E[\sup_{x \in S} |\Xi(x)|^t]^{1/t} + 1 \right) E[|Z|^r]^{1/r} \alpha(\mathcal{A} \lor \mathcal{B}, \mathcal{C})^{\varrho''}
\end{equation}

for some $C_1 > 0$ depending only on $C_0, u^*, v^*$ and $\gamma > 0$.

On the other hand, using Lemma 3 with $V(X)Z$ substituted for $V$, we have

\begin{equation}
\left| E[U(X)\Xi(X)Z] \right| \leq C_2 (u^* + 1) E[|V(X)Z|^{t'/t'}]^{1/t'} \alpha(\mathcal{A}, \mathcal{B} \lor \mathcal{C})^{\varrho''}
\end{equation}

for some $C_2 > 0$ depending only on $C_0$ and $\gamma > 0$, where $\frac{1}{t'} = \frac{1}{q} + \frac{1}{r}$ and $\varrho'' = \frac{1}{s + \gamma}$.

Set $W(x) = E[U(x)V(x)]$. By Lemma 1, we see

$$|W(x)| \leq 8u^*v^* \alpha(\mathcal{A}, \mathcal{B})^{1-1/t} \leq 8u^*v^* \alpha(\mathcal{A}, \mathcal{B} \lor \mathcal{C})^{2\varrho'}$$

for each $x \in S$. Thus

\begin{equation}
\left| E[W(X)Z] \right| \leq 8u^*v^* E[|Z|^r]^{1/r} \alpha(\mathcal{A}, \mathcal{B} \lor \mathcal{C})^{2\varrho'}.
\end{equation}

By (2.19), (2.20) and (2.21), it follows that

$$\left| E[\Xi(X)Z] \right| = \left| E[\Xi(X)Z] \right|^{1/2} \left| E[\Xi(X)Z] \right|^{1/2} \leq C_3 E[|Z|^r]^{1/r} \alpha(\mathcal{A} \lor \mathcal{B}, \mathcal{C})^{\varrho'} \alpha(\mathcal{A}, \mathcal{B} \lor \mathcal{C})^{\varrho'}$$

for some $C_3 > 0$ depending only on $C_0, u^*, v^*$ and $\gamma > 0$. This implies the assertion. \(\square\)

3. Proof of Theorem 1

Let $\varphi_M \in C^\infty(\mathbb{R}^d; \mathbb{R})$ be such that $0 \leq \varphi_M \leq 1$,

$$\varphi_M(x) = \begin{cases} 
1 & \text{if } |x| \leq M/2 \\
0 & \text{if } |x| \geq M,
\end{cases}$$
The gradient of $\varphi_M(x)$ is bounded uniformly in $x \in \mathbb{R}^d$ and $M \geq 1$. Define the truncated functions $F_k^n(w) = \left( F_k^{n,M,i}(w) \right)_{i=1}^d$ and $G_k^n(w) = \left( G_k^{n,M,i}(w) \right)_{i=1}^d$ by

\[
F_k^n(w) = \varphi_M(w(k/n))F_k^n(w), \quad G_k^n(w) = \varphi_M(w(k/n))G_k^n(w).
\]

We also define the stochastic process $X_t^{n,M} = (X_t^{n,M,i})_{i=1}^d$ by (1.1) and (1.2) for which $F_k^n$ and $G_k^n$ are replaced by $F_k^{n,M}$ and $G_k^{n,M}$.

Let $H_k^{n,M,i}(w) = F_k^{n,M,i}(w) + \frac{1}{\sqrt{n}}G_k^{n,M,i}(w)$. Then $X_t^{n,M,i}$ satisfies the following equation

\[
X_{(k+1)/n}^{n,M,i} - X_{k/n}^{n,M,i} = \frac{1}{\sqrt{n}}H_k^{n,M}(X_t^{n,M}).
\]

**Proposition 1.** For each $\omega \in \Omega^n$, if $|X_t^{n,M}(w)| \leq M$, then $|X^{n,M}_s(\omega)| \leq M$ for any $s \in [0,t]$.

**Proof.** We prove the contraposition of the assertion. Suppose that $|X^{n,M}_s| > M$ holds for some $s \in [0,t]$. Let $k = \lfloor ns \rfloor$. If $|X_{k/n}^{n,M}| > M$, we have $|X_{(k+1)/n}^{n,M}| = |X^{n,M}_s| > M$, obviously, so we may suppose $|X_{k/n}^{n,M}| \leq M$.

Then we see $|X_{(k+1)/n}^{n,M}| \geq M$. Indeed, if $|X_{(k+1)/n}^{n,M}| \leq M$, then $|X^{n,M}_s| \leq M$ holds by the convexity of the set $\{x \in \mathbb{R}^d, |x| \leq M\}$, that contradicts the supposition. So $X^{n,M}_t$ is in $\{uX^{n,M}_s + (1 - u)X_{(k+1)/n}; 0 \leq u \leq 1\} \subset \{uX^{n,M}_s + (1 - u)X^{n,M}_{k/n}; u \geq 1\}$, since $|X_k^{n,M}| \leq M$ and $|X^{n,M}_s| > M$ hold, we have $|uX^{n,M}_s + (1 - u)X^{n,M}_{k/n}| > M$ for each $u \geq 1$. Thus $|X^{n,M}_t| > M$ holds and we obtain the assertion. $\Box$

By Proposition 1, the assumption $[A3]$ and the definition of $X_t^{n,M}$, we see that $X_t^{n,M}$ is $\mathcal{F}_{0,[nt]}$-measurable and that there exists a constant $C(M) > 0$ such that

\[
\sum_{m=0}^{2} E^n \left[ \left| \nabla^m F_k^{n,M,i}(X^{n,M}) \right|_{L^p_{k/n}}^{p_0} \right] \leq C(M)
\]
and

$$\sum_{m=0}^{1} E_n \left[ |\nabla^m G_k^{n,M,i}(X_{t}^{n,M})|_{L_{k/n}^p}^{p_0} \right] \leq C(M)$$

for $n \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Let

$$Y_{k}^{n,M}(u,t) = X_{t\wedge (k/n)}^{n,M} + u \left( X_{t\wedge ((k+1)/n)}^{n,M} - X_{t\wedge (k/n)}^{n,M} \right), \quad u \in [0,1].$$

Easily we have

$$Y_{k}^{n,M}(u,t) = \begin{cases} 
X_{t}^{n,M} & \text{if } t \leq \frac{k}{n} \\
X_{k/n+u(t-k/n)}^{n,M} & \text{if } \frac{k}{n} < t \leq \frac{k+1}{n} \\
X_{(k+u)/n}^{n,M} & \text{if } \frac{k+1}{n} < t.
\end{cases}$$

By Lemma 3 and Lemma 4, we obtain the following two propositions.

**Proposition 2.** Let $1 < q < \infty$ be such that $\frac{1}{q} \leq \frac{1}{2} \left( 1 + \frac{1}{p_0} \right)$, and let $U : C([0,\infty);\mathbb{R}^d) \times \Omega^n \rightarrow \mathbb{R}$ be such that $U(w)$ is $F_{k,\infty}^n$-measurable and $E^n[U(w)] = 0$ for each $w \in C_M^d$, and $V : \Omega^n \rightarrow \mathbb{R}$ be an $F_{0,1}^n$-measurable random variable. Suppose that there exists a constant $C_0 = C_0(M) > 0$ such that

$$\sup_{\varepsilon > 0} \varepsilon^\gamma N_n(\varepsilon, M; U) \leq C_0.$$

Then there exists a constant $C > 0$ depending only on $M$ and $C_0$ such that for all $l \leq k$, $u \in [0,1]$ and $\beta = (\beta^1, \ldots, \beta^d) \in \mathbb{Z}_+^d$ with $|\beta| = \beta^1 + \cdots + \beta^d \leq 2$

$$|E^n[U_{\beta}^M(Y_{l}^{n,M}(u,\cdot))V]| \leq C(E^n[\sup_{|w|_{\infty} \leq M} |U(w)|_{p_0}^{p_0} + 1] E^n[|V|^1/q] \alpha_{k-l}^{p_0},$$

where $U_{\beta}^M(w) = D^\beta \varphi_M(w(k/n))U(w)$ and $D^\beta = \frac{\partial^{\beta}}{\partial x^{\beta^1} \cdots \partial x^{\beta^d}}$. 

Proof. Define $\hat{Y}_{l}^{n,M}(u,t)$ and $\hat{V}$ by

\[
(3.7) \quad \hat{Y}_{l}^{n,M}(u,t) = \begin{cases} 
Y_{l}^{n,M}(u,t) & \text{if } |X_{(l+u)/n}^{n,M}| \leq M \\
0 & \text{otherwise} 
\end{cases}
\]

and

\[
\hat{V} = \begin{cases} 
V & \text{if } |X_{(l+u)/n}^{n,M}| \leq M \\
0 & \text{otherwise} 
\end{cases}.
\]

By (3.4) and Proposition 1, we see that $|\hat{Y}_{l}^{n,M}(u,t)| \leq M$ for all $t \geq 0$ almost surely and

\[
(3.8) \quad E^{n}[U^{M}_{\beta}(\hat{Y}_{l}^{n,M}(u,\cdot))V] = E^{n}[U(\hat{Y}_{l}^{n,M}(u,\cdot))D^{\beta}\varphi_{M}(X_{(l+u)/n}^{n,M})\hat{V}].
\]

Using Lemma 3, we see that

\[
\left|E^{n}[U(\hat{Y}_{l}^{n,M}(u,\cdot))D^{\beta}\varphi_{M}(X_{(l+u)/n}^{n,M})\hat{V}]\right| 
\leq 16(C_{0} + 1)\left(E^{n}\left[\sup_{|w|_{\infty} \leq M} |U(w)|^{p_{0}}\right]^{1/p_{0}} + 1\right) \times E^{n}[|D^{\beta}\varphi_{M}(X_{(l+u)/n}^{n,M})\hat{V}|^{q}]^{1/q} \alpha'_{k-l},
\]

where $\alpha'_{0} = \frac{1}{s'_{0} + \gamma}$ and $\frac{1}{s'_{0}} = 1 - \frac{1}{p_{0}} - \frac{1}{q}$. Since $s'_{0} \leq 2s_{0}$ holds, which implies $\alpha'_{0} \geq 2\alpha_{0}$, and $D^{\beta}\varphi_{M}$ is bounded uniformly in $x$, we have our assertion. $\square$

Proposition 3. Let $U, V : C([0, \infty); \mathbb{R}^{d}) \times \Omega^{n} \rightarrow \mathbb{R}$ be such that $U(w)$ and $V(w)$ are $\mathcal{F}_{k,k}^{n}$ and $\mathcal{F}_{l,l}^{n}$-measurable respectively and $E^{n}[U(w)] = 0$ for each $w \in C_{M}^{d}$, and $Z : \Omega^{n} \rightarrow \mathbb{R}$ be an $\mathcal{F}_{0,m}^{n}$-measurable random variable. Suppose that there exists $C_{0} = C_{0}(M) > 0$ such that

\[
(3.9) \quad \sup_{\varepsilon > 0} \varepsilon^{\gamma}\{N_{n}(\varepsilon, M; U) + \varepsilon^{\gamma}N_{n}(\varepsilon, M; V)\} \leq C_{0},
\]

\[
(3.10) \quad E^{n}\left[\sup_{|w|_{\infty} \leq M} |U(w)|^{p_{0}}\right]^{1/p_{0}} \leq C_{0}
\]

and

\[
(3.11) \quad E^{n}\left[\sup_{|w|_{\infty} \leq M} |V(w)|^{p_{0}}\right]^{1/p_{0}} \leq C_{0}.
\]
Then there exists a constant $C > 0$ depending only on $M$ and $C_0$ such that for all $m \leq l \leq k$, $u \in [0, 1]$ and $\beta, \beta' \in \mathbb{Z}^d_+ \text{ with } |\beta| + |\beta'| \leq 2$

$$|E^n[\Xi_{\beta, \beta'}^M(Y_{m,M}^n(u, \cdot))Z]| \leq C E^n[|Z|^{p_0}]^{1/p_0} \alpha_{k-l}^{\alpha_{l-m}};$$

where $\Xi_{\beta, \beta'}^M(w) = D^\beta \varphi_M(w(k/n)) D^{\beta'} \varphi_M(w(l/n)) \Xi(w)$, $\Xi(w) = U(w)V(w) - E^n[U(w)V(w)]$.

**Proof.** Define $\hat{Z}$ by

$$\hat{Z} = \begin{cases} Z & \text{if } |X_{(m+u)/n}| \leq M \\ 0 & \text{otherwise}. \end{cases}$$

Then we have

$$(3.12) \quad E^n[\Xi^M(Y_{m,M}^n(u, \cdot))Z] = E^n[\Xi(Y_{m,M}^n(u, \cdot)) D^\beta \varphi_M(X_{(m+u)/n}) D^{\beta'} \varphi_M(X_{(m+u)/n}) \hat{Z}],$$

where $Y_{m,M}^n(u, t)$ is given by (3.7). Using Lemma 4, we see that there exists $C_1 > 0$ depending only on $M$ and $C_0$ such that

$$|E^n[\Xi(Y_{m,M}^n(u, \cdot)) \varphi_M(X_{(m+u)/n})^2 \hat{Z}]| \leq C_1 E^n[|D^\beta \varphi_M(X_{(m+u)/n}) D^{\beta'} \varphi_M(X_{(m+u)/n}) \hat{Z}|^{p_0}]^{1/p_0} \alpha_{k-l}^{\alpha_{l-m}}.$$

Then we have our assertion. \( \square \)

Let $Q_{n,M}$ be the probability measure induced by $X_{n,M}$ on $C([0, \infty); \mathbb{R}^d)$.

**Proposition 4.** The family of measures $(Q_{n,M})_n$ is tight for each fixed $M > |x_0|$.

**Proof.** Take any $T > 0$. Let $0 \leq s < t < u \leq T$, $0 < \delta_0 < \frac{p_0 - 3}{2}$ and set

$$J^n_0 = E^n[|X_{u,M,i} - X_{t,M,i}|^2 |X_{t,M,i} - X_{s,M,i}|^{1+\delta_0}].$$

By the argument in [1], [5] and [16], it suffices to show that there exists a constant $C_0 = C_0(M, T) > 0$ which is independent of $s, t, u$ and $n$ such that

$$(3.13) \quad J^n_0 \leq C_0 |u - s|^{1+1/q_0},$$
where \( q_0 = \frac{p_0}{1 + \delta_0} \).

First we consider the case of \( u - s < 1/n \). In this case, it follows that \([ns] + 1 = [nt] = [nu]\) or \([ns] = [nt] = [nu] - 1\).

If \([ns] + 1 = [nt] = [nu]\), by assumption [A3] and Proposition 1, we have

\[
J_n^0 = E^n \left[ \left( \frac{1}{\sqrt{n}} (nt - [nt]) H_{[nt]}^{n,M} (X^{n,M}) \right)^2 \right]
\]

\[
\times \left( \left( \frac{1}{\sqrt{n}} (1 - ns + [ns]) H_{[ns]}^{n,M} (X^{n,M}) \right)^{1+\delta_0} \right)
\]

\[
= \left( \frac{1}{\sqrt{n}} \right)^{1-\delta_0} |u - s|^2 E^n \left[ \left| \mathcal{H}_{[nt]}^{n,M,i} (X^{n,M}) \right|^2 \right]
\]

\[
\times \left\{ \left( nt - [nt] \right) H_{[nt]}^{n,M,i} (X^{n,M}) - (1 - ns + [ns]) H_{[ns]}^{n,M,i} (X^{n,M}) \right\}^2
\]

\[
\leq \left( \frac{1}{\sqrt{n}} \right)^{1-\delta_0} |u - s|^2 \left\{ E^n \left[ \left| \mathcal{H}_{[nt]}^{n,M,i} (X^{n,M}) \right|^2 \right] \right. \\
\quad + E^n \left[ \left| \mathcal{H}_{[nt]}^{n,M,i} (X^{n,M}) \right|^{p_0(3+\delta_0)/p_0} \right]^{2/p_0} \\
\quad \times E^n \left[ \left| \mathcal{H}_{[ns]}^{n,M,i} (X^{n,M}) \right|^{p_0(1+\delta_0)/p_0} \right]
\]

\[
\leq C_1 \left( \frac{1}{\sqrt{n}} \right)^{1-\delta_0} |u - s|^2 \leq C_1 |u - s|^{3+\delta_0}/2 \leq C_2 |u - s|^{1+1/q_0}
\]

for some \( C_1 = C_1(M) > 0 \) and \( C_2 = C_2(M,T) > 0 \).

If \([ns] = [nt] = [nu] - 1\), the similar calculation gives us the following estimation

\[
J_n^0 \leq C_3 |u - s|^{1+1/q_0}
\]

for some \( C_3 = C_3(M,T) > 0 \). So the inequality (3.13) holds when \( u - s < 1/n \).

Next we consider the case of \( u - s \geq 1/n \). We will show that there exists a constant \( C_4 = C_4(M,T) > 0 \) such that

\[
E^n [\|X_v^{n,M,i} - X_r^{n,M,i}\|^2 \Phi] \leq C_4 |u - s| E^n [\Phi^q_0]^{1/q_0}
\]

for each \( r,v \in [s,u] \) with \( r \leq v \) and each \( \mathcal{F}_{0,([nr] - 1)\vee 0} \)-measurable non-negative random variable \( \Phi \).
Since we have

\[
|X^n_{v,M,i} - X^n_{r,M,i}|^2 \leq 3 \left\{ |X^n_{((nv)+1)/n} - X^n_{v,M,i}|^2 + |X^n_{r,M,i} - X^n_{[nr]/n}|^2 \right. \\
+ \left. \left| \sum_{k=[nr]}^{[nv]} (X^n_{(k+1)/n} - X^n_{k/n}) \right|^2 \right\}
\]

and the following equality

\[
(\sum_{l=1}^{k} x_l)^2 = \sum_{l=1}^{k} x_l^2 + 2 \sum_{l=1}^{k} x_l(x_1 + \cdots + x_l), \quad x_1, \ldots, x_k \in \mathbb{R},
\]

it follows that

\[
E^n |X^n_{v,M,i} - X^n_{r,M,i}|^2 \Phi \leq 6(J^n_1 + J^n_2 + J^n_3 + J^n_4 + J^n_5),
\]

where

\[
J^n_1 = E^n |X^n_{((nv)+1)/n} - X^n_{v,M,i}|^2 \Phi,
\]

\[
J^n_2 = E^n |X^n_{r,M,i} - X^n_{[nr]/n}|^2 \Phi,
\]

\[
J^n_3 = \frac{1}{n} \sum_{k=[nr]}^{[nv]} E^n \left[ |H^n_{k,M,i}(X^n_{v,M})|^2 \Phi \right],
\]

\[
J^n_4 = \frac{1}{\sqrt{n}} \sum_{k=[nr]}^{[nv]} |E^n [F^n_{k,M,i}(X^n_{v,M})(X^n_{k/n} - X^n_{[nr]/n})\Phi]|
\]

\[
J^n_5 = \frac{1}{n} \sum_{k=[nr]}^{[nv]} |E^n [G^n_{k,M,i}(X^n_{v,M})(X^n_{k/n} - X^n_{[nr]/n})\Phi]|
\]

Since \(\frac{2}{p_0} + \frac{1}{q_0} < 1\), we have

\[
J^n_1 \leq \frac{1}{n} ([nv] + 1 - v)^2 E^n [H^n_{(nv)}(X^n_{v,M})]^{p_0}^{2/p_0} E^n [\Phi^{q_0}]^{1/q_0}
\]

\[
J^n_5 \leq C_5 \times \frac{1}{n} E^n [\Phi^{q_0}]^{1/q_0} \leq C_5 |u - s| E^n [\Phi^{q_0}]^{1/q_0}
\]
for some $C_5 = C_5(M) > 0$. Similarly we have

\begin{equation}
J_n^2 \leq C_6 |u - s| E^n[\Phi^q_0]^{1/q_0}
\end{equation}

for some $C_6 = C_6(M) > 0$. We also have

\begin{equation}
J_n^3 \leq C_7 \left( |v - r| + \frac{2}{n} \right) E^n[\Phi^q_0]^{1/q_0} \leq 3C_7 |u - s| E^n[\Phi^q_0]^{1/q_0}
\end{equation}

for some $C_7 = C_7(M) > 0$.

To estimate $J_n^4$, using Taylor’s theorem (Theorem 1.43 in [12]), we have

\begin{equation}
E^n[F_{n,M,i}^k(X_{n,M}^l)(X_{n,M}^l/n - X_{l/n})\Phi]
= \sum_{l=\lfloor nr \rfloor}^{k-1} \left\{ E^n[F_{n,M,i}^k(X_{\lceil (l+1)/n \rceil})(X_{(l+1)/n} - X_{l/n})\Phi]
+ E^n[(F_{n,M,i}^k(X_{\lceil (l+1)/n \rceil})
- F_{n,M,i}^k(X_{\lceil l/n \rceil}))(X_{l/n}^n - X_{\lfloor nr/n \rfloor}^n)\Phi]\right\}
= \frac{1}{\sqrt{n}} \sum_{l=\lfloor nr \rfloor}^{k-1} \left\{ \Lambda_{n,1}^{k,l} + \Lambda_{n,2}^{k,l} + \Lambda_{n,3}^{k,l} \right\},
\end{equation}

where

\begin{align*}
\Lambda_{n,1}^{k,l} &= E^n[\varphi_M(X_{(l+1)/n}^n)F^n_{n,i}^k(X_{\lceil (l+1)/n \rceil}^n)H^n_{l}^{n,M,i}M(X_{n,M}^n)\Phi], \\
\Lambda_{n,2}^{k,l} &= \sum_{j=1}^{d} \int_0^1 E^n[\frac{\partial}{\partial x} \varphi_M(Y^M_{l}^n(u,k/n))F^n_{k}^i(Y^M_{l}^n(u,\cdot))
\times H^n_{l}^{n,M,j}(X_{n,M}^n)(X_{l/n}^n - X_{\lfloor nr/n \rfloor}^n)\Phi] du, \\
\Lambda_{n,3}^{k,l} &= \sum_{j=1}^{d} \int_0^1 E^n[\varphi_M(Y^M_{l}^n(u,k/n))\nabla F^n_{k}^i(Y^M_{l}^n(u,\cdot); I^n_{l}^e e_j)
\times H^n_{l}^{n,M,j}(X_{n,M}^n)(X_{l/n}^n - X_{\lfloor nr/n \rfloor}^n)\Phi] du.
\end{align*}
Let \( r_0 \) be such that \( \frac{1}{r_0} = \frac{1}{p_0} + \frac{1}{q_0} \). Since

\[
\frac{1}{2} \left( 1 + \frac{1}{p_0} \right) - \frac{1}{r_0} = \frac{p_0 - 3 - 2\delta_0}{2p_0} > 0, \tag{3.21}
\]

using Proposition 2 with \( U = F_k^{n,i} \), \( V = H_l^{n,M,i}(X^{n,M}) \) and \( u = 1 \), we have

\[
|\Lambda_{k,l}^{n,(1)}| \leq C_8 \left( E^n [ \sup_{|w| \leq M} |F_k^{n,i}(w)|]^{1/p_0} + 1 \right) \times E^n [H_l^{n,M,i}(X^{n,M})\Phi]^{1/r_0} \alpha_{k-l}^{\frac{q_0}{r_0}} \leq C_9 E^n [\Phi]^{1/q_0} \alpha_{k-l}^{\frac{q_0}{r_0}}. \tag{3.22}
\]

for some \( C_8, C_9 > 0 \) depending only on \( M \).

Also we see

\[
E^n [H_l^{n,M,j}(X^{n,M})(X_{l/n}^{n,M,i} - X_{l/n}^{n,M,i})\Phi]^{1/r_0} \leq M E^n [\varphi_M(X_{l/n}^{n,M})H_l^{n,M}(X^{n,M})\Phi]^{1/r_0} \leq M E^n [\varphi_M(X_{l/n}^{n,M})H_l^{n,M}(X^{n,M})\Phi]^{1/r_0} \leq M E^n [\Phi]^{1/q_0}. \tag{3.23}
\]

Then, using Proposition 2 again, we have

\[
|\Lambda_{k,l}^{n,(2)}|, |\Lambda_{k,l}^{n,(3)}| \leq C_{10} E^n [\Phi]^{1/q_0} \alpha_{k-l}^{\frac{q_0}{r_0}} \tag{3.24}
\]

for some \( C_{10} = C_{10}(M) > 0 \). Thus

\[
J_4^n \leq C_{11} \times \frac{1}{n} \sum_{k=[nr]} \sum_{l=[nr]} E^n [\Phi]^{1/q_0} \alpha_{k-l}^{\frac{q_0}{r_0}} \leq 3C_{11} \left( \sum_{k=1}^{\infty} \alpha_k^{\frac{q_0}{r_0}} \right) |u - s| E^n [\Phi]^{1/q_0} \tag{3.25}
\]

for some \( C_{11} = C_{11}(M) > 0 \).

By the similar calculation of (3.23), we have

\[
J_5^n \leq C_{12} |u - s| E^n [\Phi]^{1/q_0} \tag{3.26}
\]
for some \( C_{12} = C_{12}(M) > 0 \). Then the inequality (3.15) holds.

Using (3.15) with \( v = u, r = t \) and \( \Phi = |X_t^{n,M,i} - X_s^{n,M,i}|^{1+\delta_0}1_{\{|X_s^{n,M} - X_t^{n,M,i}| \leq M\}} \), we get

\[
J_0^n \leq C_4 |u - s| E^n(|X_t^{n,M,i} - X_s^{n,M,i}|^{p_0}1_{\{|X_s^{n,M} - X_t^{n,M,i}| \leq M\}})^{1/p_0}.
\]

Using (3.15) again with \( v = [nt]/n, r = s \) and \( \Phi = 1 \), we get

\[
E^n(|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^2) \leq C_4 |u - s|.
\]

Thus

\[
E^n(|X_t^{n,M,i} - X_s^{n,M,i}|^{p_0}1_{\{|X_s^{n,M} - X_t^{n,M,i}| \leq M\}})
\]

\[
\leq C_{13} \left\{ E^n(|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^{p_0}1_{\{|X_s^{n,M} - X_{[nt]/n}^{n,M,i}| \leq M\}})
\right.
\]

\[
+ E^n(|X_{t}^{n,M,i} - X_{[nt]/n}^{n,M,i}|^{p_0}1_{\{|X_s^{n,M} - X_{[nt]/n}^{n,M,i}| \leq M\}})\}
\]

\[
\leq C_{14} \left\{ M^{p_0-2} E^n(|X_{[nt]/n}^{n,M,i} - X_s^{n,M,i}|^2)
\right.
\]

\[
+ \frac{1}{(\sqrt{n})^{p_0}} (nt - [nt]) E^n([H_{[nt]}^{n,M,i}(X_s^{n,M})]^{p_0}) \}
\]

\[
\leq C_{15} \left(|u - s| + \frac{1}{(\sqrt{n})^{p_0}} \right) \leq 2C_{15}|u - s|
\]

for some \( C_{13}, C_{14}, C_{15} > 0 \) depending only on \( M \). Thus the inequality (3.13) holds also when \( u - s \geq 1/n \). This completes the proof of Proposition 4. \( \Box \)

By Proposition 4, for any subsequence \((n_k)_k\), there is a further subsequence \((n_{k_l})_l\) such that \( Q^{n_{k_l},M} \) converges weakly to some probability measure \( Q^M \) on \( C([0, \infty); \mathbb{R}^d) \) as \( l \to \infty \) for each fixed \( M > 1 + |x_0| \).

**Proposition 5.** \( Q^M (c_M^d) = 1 \).

**Proof.** For each \( T > 0 \), it follows that

\[
Q^M(\sup_{0 \leq t \leq T} |w(t)| > M) = \lim_{\varepsilon \searrow 0} Q^M(\sup_{0 \leq t \leq T} |w(t)| > M + \varepsilon)
\]

\[
\leq \lim_{\varepsilon \searrow 0} \liminf_{n \to \infty} P^{n_{k_j}}(\sup_{0 \leq t \leq T} |X_t^{n,M} > M + \varepsilon|).
\]
Here we see
\[
P^n(\sup_{0 \leq t \leq T} |X^n_{t}^n| > M + \varepsilon) \\
\leq P^n(|X^n_{k/n}| \leq M, |X^n_{k/n}| + \frac{1}{\sqrt{n}}|H^M_k(X^n,M)| > M + \varepsilon \\
\text{for some } k = 0, \ldots, [nT])
\]
\[
\leq \sum_{k=0}^{[nT]} P^n(|H^M_k(X^n,M)| \geq \varepsilon \sqrt{n}) \leq C_0 \times \frac{1}{\varepsilon^3 \sqrt{n}}
\]
for some \( C_0 = C_0(M, T) > 0 \). Thus
\[(3.30) \quad Q^n_M(\sup_{0 \leq t \leq T} |w(t)| > M) = 0, \ T > 0.\]
This implies the assertion. \( \square \)

Next we define functions \( a^{M,ij}(t, w) \) and \( b^{M,i}(t, w) \) by
\[
a^{M,ij}(t, w) = \varphi_M(w(t))^2 a^{ij}(t, w)
\]
\[
b^{M,i}(t, w) = \varphi_M(w(t)) b_0^i(t, w) + \sum_{j=1}^{d} \left\{ \varphi_M(w(t))^2 B^{ij}(t, w) + \varphi_M(w(t)) \frac{\partial}{\partial x^j} \varphi_M(w(t)) A^{ij}(t, w) \right\}
\]
and let
\[
\mathcal{L}^M f(t, w) = \frac{1}{2} \sum_{i,j=1}^{d} a^{M,ij}(t, w) \frac{\partial^2}{\partial x^i \partial x^j} f(w(t)) + \sum_{i=1}^{d} b^{M,i}(t, w) \frac{\partial}{\partial x^i} f(w(t))
\]
for \( f \in C^2(\mathbb{R}^d) \).

**Proposition 6.** \( Q^n_M \) is a solution of the martingale problem associated with the generator \( \mathcal{L}^M \) and starting at \( x_0 \).

By Proposition 5, in order to prove Proposition 6, it suffices to show that
\[(3.31) \quad E^{Q^n_M}[(f(w(t)) - f(w(s)))\Phi(w(s_1), \ldots, w(s_N))] = E^{Q^n_M}\int_{s}^{t} \mathcal{L}^M f(u, w)dw\Phi(w(s_1), \ldots, w(s_N))\]
for any $C^\infty$ function $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, integer $N$, real numbers $0 \leq s_1 < \ldots < s_N \leq s < t$ and bounded continuous function $\Phi : (\mathbb{R}^N)^m \to \mathbb{R}$. Until Proposition 14, we omit the $M$ in $X^n_{t,M}$ and $Y^n_{k,M}(u,t)$ as long as there is no misunderstanding, and simply denote $(n_{k_l})$ by $(n)$.

Since $f$ and $\Phi$ are bounded, it follows that

$$E^{Q_n,M}_{\text{X},t}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\ldots,w(s_N))]$$

$$\to E^{Q,M}_{\text{X},t}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\ldots,w(s_N))].$$

On the other hand, Taylor’s theorem implies

$$E^{Q_n,M}_{\text{X},t}[(f(w(t)) - f(w(s)))\Phi(w(s_1),\ldots,w(s_N))]$$

$$= K^n_1 + K^n_2 + K^n_3 + K^n_4 + \frac{1}{2} K^n_5 + K^n_6 + \frac{1}{2} K^n_7 + \frac{1}{2} K^n_8,$$

where

$$K^n_1 = E^n[(f(X^n_t) - f(X^n_{[nt]/n}))\Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$

$$K^n_2 = E^n[(f(X^n_{[ns]/n}) - f(X^n_s))\Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$

$$K^n_3 = \frac{1}{\sqrt{n}} \sum_{i=1}^d \sum_{k=[ns]}^{[nt]-1} E^n[\frac{\partial}{\partial x^i} f(X^n_{k/n}) F^n_{k,M,i}(X^n) \Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$

$$K^n_4 = \frac{1}{n} \sum_{i=1}^d \sum_{k=[ns]}^{[nt]-1} E^n[\frac{\partial}{\partial x^i} f(X^n_{k/n}) G^n_{k,M,i}(X^n) \Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$

$$K^n_5 = \frac{1}{n} \sum_{i,j=1}^d \sum_{k=[ns]}^{[nt]-1} E^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X^n_{k/n})$$

$$\times F^n_{k,M,i}(X^n) F^n_{k,M,j}(X^n) \Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$

$$K^n_6 = \frac{1}{n \sqrt{n}} \sum_{i,j=1}^d \sum_{k=[ns]}^{[nt]-1} E^n[\frac{\partial^2}{\partial x^i \partial x^j} f(X^n_{k/n})$$

$$\times F^n_{k,M,i}(X^n) G^n_{k,M,j}(X^n) \Phi(X^n_{s_1},\ldots,X^n_{s_N})],$$
\[ K^n_7 = \frac{1}{n^2} \sum_{i,j=1}^d \sum_{k=[ns]}^{[nt]-1} E^n \left[ \frac{\partial^2}{\partial x^i \partial x^j} f(X^n_k) \right] \times G^n_{k,i} (X^n) G^n_{k,j} (X^n) \Phi(X^n_s, \ldots, X^n_s) \],
\[ K^n_8 = \frac{1}{n \sqrt{n}} \sum_{i,j,\nu=1}^d \sum_{k=[ns]}^{[nt]-1} \int_0^1 (1-u)^2 E^n \left[ \frac{\partial^3}{\partial x^i \partial x^j \partial x^\nu} f(Y^n_u) \right] \times H^n_{k,i} (X^n) H^n_{k,j} (X^n) H^n_{k,\nu} (X^n) \Phi(X^n_s, \ldots, X^n_s) \] du.

Proposition 7. \( K^n_j \longrightarrow 0 \) as \( n \to \infty, \ j = 1, 2, 6, 7, 8. \)

Proof. By (3.2) and (3.3), we have
\[ |K^n_6| \leq \frac{1}{n \sqrt{n}} \sum_{k=[ns]}^{[nt]-1} C(M, f, \Phi) \longrightarrow 0 \]
for some constant \( C(M, f, \Phi) > 0. \) Similarly we get \( K^n_7 \longrightarrow 0 \) and \( K^n_8 \longrightarrow 0. \)

Taylor’s theorem implies
\[ |K^n_1| \leq \frac{1}{\sqrt{n}} \sum_{i=1}^d \int_0^1 E^n \left[ \left| \frac{\partial}{\partial x^i} f(Y^n_{nt}(u, t)) (nt - [nt]) H^n_{[nt]i} (X^n) \Phi \right| \right] du \leq \text{const.} \times \frac{1}{\sqrt{n}} \longrightarrow 0. \]

Similar arguments give us \( K^n_2 \longrightarrow 0. \) Then we obtain the assertion. \( \Box \)

To treat the convergent of \( K^n_3, K^n_4 \) and \( K^n_5, \) we will show the following three propositions.

Proposition 8. Let \( U^n_k : C([0, \infty); \mathbb{R}^d) \times \Omega^n \longrightarrow \mathbb{R} \) be a continuously Fréchet differentiable random function such that \( U^n_k(w) \) is \( \mathcal{F}^{n}_{k,\infty} \)-measurable and \( E^n[U^n_k(w)] = 0 \) for each \( w \in C^n_M, \) and \( V^n : \Omega^n \longrightarrow \mathbb{R} \) be an \( \mathcal{F}^{n}_{0,[ns]} \)-measurable random variable. Suppose that there exists a constant \( C_0 = \)
\( C_0(M) > 0 \) such that

\[
\begin{align*}
(3.34) \quad & \sup_{\varepsilon > 0} \varepsilon^\gamma N_n(\varepsilon, M; U^n_k) \leq C_0, \\
& \sup_{t \leq k} \sup_{\varepsilon > 0} \varepsilon^\gamma N_n(\varepsilon, M; \nabla U^n_k(\cdot; I^n_k e_j)) \leq C_0,
\end{align*}
\]

\[
(3.35) \quad \sum_{m=0}^{1} \mathbb{E}^n \left[ \sup_{|w|_{\infty} \leq M} |\nabla^m U^n_k(w)|_{p_0}^{p_0}_{L^m_{k/n}} \right] \leq C_0,
\]

and

\[
(3.36) \quad \mathbb{E}^n[|V^n|^{p_0/2}] \leq C_0
\]

for any \( j = 1, \ldots, d, n \in \mathbb{N} \) and \( k \in \mathbb{Z}_+ \). Then it holds that

\[
(3.37) \quad \frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor - 1} \mathbb{E}^n[D^\beta \varphi_M(X^n_k(X^n)V^n)] \rightarrow 0, \quad n \to \infty
\]

for \( \beta \in \mathbb{Z}^d_+ \) with \(|\beta| \leq 1\).

**Proof.** By Taylor’s theorem, we have

\[
\begin{align*}
\mathbb{E}^n[D^\beta \varphi_M(X^n_k(X^n)V^n)] &= \frac{1}{n} \sum_{i=1}^{d} \sum_{l=\lfloor ns \rfloor}^{k-1} \int_0^1 \left\{ \mathbb{E}^n \left[ \frac{\partial}{\partial x^i} D^\beta \varphi_M(Y^n_{l,i}(u, k/n)) \right] \\
& \times U^n_k(Y^n_{l,i}(u, \cdot)) H^n_{l,i}(X^n) V^n \right\} du \\
& + \mathbb{E}^n[D^\beta \varphi_M(Y^n_{l,i}(u, k/n)) \\
& \times \nabla U^n_k(Y^n_{l,i}(u, \cdot); I^n_k e_i) H^n_{l,i}(X^n) V^n] \right\} du \\
& + \mathbb{E}^n[D^\beta \varphi_M(X^n_{\lfloor ns \rfloor/n}) U^n_k(X^n_{\lfloor ns \rfloor/n}) V^n].
\end{align*}
\]
Then we obtain the assertion.

\[ (3.38) \quad |E^n [\frac{\partial}{\partial x} D^\beta \varphi_\mathcal{M}(Y_t^{n,\mathcal{M}}(u, k/n))U_k^n (Y_t^{n,\mathcal{M}}(u, \cdot)) H_t^{n,\mathcal{M},i}(X^n)V^n]| \]

\[ \leq C_1 \alpha_{k-l}^0 \]

\[ (3.39) \quad |E^n [D^\beta \varphi_\mathcal{M}(Y_t^{n,\mathcal{M}}(u, k/n)) \nabla U_k^n (Y_t^{n,\mathcal{M}}(u, \cdot); I_t^n e_i) H_t^{n,\mathcal{M},i}(X^n)V^n]| \]

\[ \leq C_1 \alpha_{k-l}^0 \]

and

\[ (3.40) \quad |E^n [D^\beta \varphi_\mathcal{M}(X^n_{[ns]/n}) U_k^n (X^n_{\Lambda([ns]/n)}) V^n]| \leq C_1 \alpha_{k-[ns]}^0 \]

for some \( C_1 > 0 \) depending only on \( M \) and \( C_0 \). Thus

\[ \frac{1}{n} \sum_{k=[ns]}^{[nt]-1} |E^n [D^\beta \varphi_\mathcal{M}(X^n_{k/n}) U_k^n (X^n)V^n]| \]

\[ \leq 2C_1 d \times \frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \left\{ \sum_{l=[ns]}^{k-1} \frac{1}{\sqrt{n}} \alpha_{k-l}^0 + \alpha_{k-[ns]}^0 \right\} \]

\[ \leq 2C_1 d \left( \sum_{k=1}^{\infty} \alpha_k^0 \right) (t + 1) \times \frac{1}{\sqrt{n}} \to 0, \quad n \to \infty. \]

Then we obtain the assertion. \( \Box \)

**Proposition 9.** Let \( U_k^n, V_k^n : C([0, \infty); \mathbb{R}^d) \times \Omega^n \to \mathbb{R} \) be such that \( U_k^n(w) \) and \( V_k^n(w) \) are \( \mathcal{F}_{k,k} \)-measurable and continuously Fréchet differentiable random functions such that \( E^n[U_k^n(w)] = 0 \) for each \( w \in C^d_M \), and \( Z^n : \Omega^n \to \mathbb{R} \) be an \( \mathcal{F}_{0,[ns]} \)-measurable random variable. Suppose that there exists a constant \( C_0 = C_0(M) > 0 \) such that

\[ (3.41) \quad \sup_{\varepsilon > 0} \{ N_n(\varepsilon, M; U_k^n) + N_n(\varepsilon, M; V_k^n) \} \leq C_0, \]

\[ (3.42) \quad \sup_{l \leq k} \sup_{\varepsilon > 0} \{ N_n(\varepsilon, M; \nabla U_k^n(\cdot; I_t^n e_j)) + N_n(\varepsilon, M; \nabla V_k^n(\cdot; I_t^n e_j)) \} \leq C_0, \]

\[ (3.43) \quad \sum_{m=0}^{1} E^n \left[ \sup_{|w|_\infty \leq M} |\nabla^m U_k^n(w)|_{L^p_{k/n}}^{p_0} \right] \leq C_0 \]

or

\[ \sum_{m=0}^{1} E^n \left[ \sup_{|w|_\infty \leq M} |\nabla^m V_k^n(w)|_{L^p_{k/n}}^{p_0} \right] \leq C_0. \]
and

\[(3.44) \quad E^n[|Z^n|^p] \leq C_0\]

for any \(j = 1, \ldots, d, n \in \mathbb{N}\) and \(k \in \mathbb{Z}_+\). Then it holds that

\[(3.45) \quad (i) \quad \frac{1}{n} \sum_{k=1}^{[nt]-1} E^n[D^\beta \varphi_M(X^n_{k/n})D^\beta' \varphi_M(X^n_{k/n})\Xi_{kl}^n(X^n)Z^n] \to 0,\]

\[(3.46) \quad (ii) \quad \frac{1}{n} \sum_{k=1}^{[nt]-1} \sum_{l=1}^{k-1} E^n[D^\beta \varphi_M(X^n_{l/n})\times D^\beta' \varphi_M(X^n_{l/n})\Xi_{kl}^n(X^n_{\wedge(l/n)})Z^n] \to 0\]

as \(n \to \infty\) for \(\beta, \beta' \in \mathbb{Z}_d^+\) with \(|\beta| + |\beta'| \leq 1\), where \(\Xi_{kl}^n(w) = U^n_k(w)V^n_l(w) - E^n[U^n_k(w)V^n_l(w)].\)

**Proof.** By Taylor’s theorem, we have

\[
E^n[D^\beta \varphi_M(X^n_{l/n})D^\beta \varphi_M(X^n_{l/n})\Xi_{kl}^n(X^n_{\wedge(l/n)})Z^n]
\]

\[
= \sum_{m=1}^{l-1} E^n[\{D^\beta \varphi_M(X^n_{(m+1)/n})D^\beta \varphi_M(X^n_{(m+1)/n})\Xi_{kl}^n(X^n_{\wedge((m+1)/n)})
\]

\[-D^\beta \varphi_M(X^n_{m/n})D^\beta' \varphi_M(X^n_{m/n})\Xi_{kl}^n(X^n_{\wedge(m/n)})\}Z^n]
\]

\[
+ E^n[D^\beta \varphi_M(X^n_{[ns]/n})D^\beta' \varphi_M(X^n_{[ns]/n})\Xi_{kl}^n(X^n_{\wedge([ns]/n)})Z^n]
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^d \sum_{m=1}^{l-1} \int_0^1 \left\{ E^n \left[ \left\{ \frac{\partial}{\partial x^i} D^\beta \varphi_M D^\beta' \varphi_M \right\}(Y^n_{m,M}(u, l/n))
\right.
\]

\[
+ D^\beta \varphi_M \frac{\partial}{\partial x^i} D^\beta' \varphi_M \left( Y^n_{m,M}(u, l/n) \right)
\]

\[
\times \Xi_{kl}^n(Y^n_{m,M}(u, \cdot))H_{m,M;i}^n(X^n)Z^n
\]

\[
+ E^n \left[ D^\beta \varphi_M(Y^n_{m,M}(u, l/n))D^\beta' \varphi_M(Y^n_{m,M}(u, l/n))
\right.
\]

\[
\times \nabla \Xi_{kl}^n(Y^n_{m,M}(u, \cdot); I_{m,M;i}^n)H_{m,M;i}^n(X^n)Z^n
\]

\[
\left. + E^n[D^\beta \varphi_M(X^n_{[ns]/n})D^\beta' \varphi_M(X^n_{[ns]/n})\Xi_{kl}^n(X^n_{\wedge([ns]/n)})Z^n] \right\} du
\]

\[
+ E^n[D^\beta \varphi_M(X^n_{[ns]/n})D^\beta' \varphi_M(X^n_{[ns]/n})\Xi_{kl}^n(X^n_{\wedge([ns]/n)})Z^n].
\]
Since
\[\nabla \Xi_{kl}^n (w; I_m^i e_i) = \nabla U_k^n (w; I_m^i e_i) V_l^n (w) - E^n [\nabla U_k^n (w; I_m^i e_i) V_l^n (w)] + U_k^n (w) \nabla V_l^n (w; I_m^i e_i) - E^n [U_k^n (w) \nabla V_l^n (w; I_m^i e_i)]\]
holds, using Proposition 3, we get
\[\left| E^n [D^\beta \varphi_M (X^n_{l/n}) D^{\beta'} \varphi_M (X^n_{l/n}) \Xi_{kl}^n (X^n_{\wedge (l/n)}) Z^n] \right| \leq C_1 \left\{ \frac{1}{\sqrt{n}} \sum_{m=\lfloor ns \rfloor}^{l-1} \alpha_{k-l}^0 \alpha_{l-m}^0 + \alpha_{k-l}^0 \alpha_{l-[ns]}^0 \right\} \]
for some $C_1 > 0$ depending only on $M$ and $C_0$. In particular it follows that
\[\left| E^n [D^\beta \varphi_M (X^n_{k/n}) D^{\beta'} \varphi_M (X^n_{k/n}) \Xi_{kk}^n (X^n) Z^n] \right| \leq C_1 \left\{ \frac{1}{\sqrt{n}} \sum_{m=\lfloor ns \rfloor}^{k-1} \alpha_{k-m}^0 + \alpha_{k-[ns]}^0 \right\}.\]

Thus we have
\[
\frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{[nt]-1} \left| E^n [D^\beta \varphi_M (X^n_{l/n}) D^{\beta'} \varphi_M (X^n_{l/n}) \Xi_{kl}^n (X^n) Z^n] \right| \leq 2C_1 \left( \sum_{k=1}^{\infty} \alpha_k^0 \right) (t + 1) \times \frac{1}{\sqrt{n}} \rightarrow 0, \quad n \rightarrow \infty
\]
and
\[
\frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{[nt]-1} \sum_{l=\lfloor ns \rfloor}^{k-1} \left| E^n [D^\beta \varphi_M (X^n_{l/n}) D^{\beta'} \varphi_M (X^n_{l/n}) \Xi_{kl}^n (X^n) Z^n] \right| \leq 2C_1 \left( \sum_{k=1}^{\infty} \alpha_k^0 \right)^2 (t + 1) \times \frac{1}{\sqrt{n}} \rightarrow 0, \quad n \rightarrow \infty.
\]

Then we obtain the assertion. □

**Proposition 10.** Let $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuously differentiable function such that $\psi(x) = 0$ for any $x \in \mathbb{R}^d$ with $|x| > M$ and $g^n :$
\[ Z_+ \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R}, \quad g : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R} \] be functionals. Suppose that \( g^n(k, \cdot) \) is \( B_{k/n} \)-measurable and continuous, and that there exists a constant \( C_0 = C_0(M) > 0 \) such that

\[
\sup_{|w|_\infty \leq M} |g^n(k, w)| \leq C_0 \tag{3.50}
\]

for each \( n \in \mathbb{N} \) and \( k \in \mathbb{Z}_+ \). Moreover suppose

\[
\sup_{w \in K} |g^n([nt], w) - g(t, w)| \rightarrow 0, \quad n \rightarrow \infty \tag{3.51}
\]

for each \( K \in \mathcal{K}^d \) and \( t \geq 0 \). Then it holds that

\[
\frac{1}{n} \sum_{k=\lfloor ns \rfloor}^{[nt]-1} E^n[\psi(X^n_{k/n})g^n(k, X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})] \rightarrow \int_s^t E^Q \left[ \psi\left( w(u) \right) g(u, w) \Phi\left( w(s_1), \ldots, w(s_N) \right) \right] du, \quad n \rightarrow \infty \tag{3.52}
\]

**Proof.** Denote the left-hand side of (3.52) by \( K^n \). Define \( L^n \) and \( S^n \) by

\[
L^n = \int_s^t E^n[\psi(X^n_{k/n})g^n([nu], X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})] du
\]

and

\[
S^n = \int_s^t E^n[\psi(X^n_{u})g(u, X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})] du.
\]

Then we have

\[
|K^n - L^n| \leq C_0 \int_s^t E^n[|\psi(X^n_{u}) - \psi(X^n_{[nu]/n})| \cdot |\Phi|] du
\]

\[
\leq \text{const.} \times \frac{1}{\sqrt{n}} \sum_{i=1}^d \int_s^t \int_0^{1} E^n \left[ \left| \frac{\partial}{\partial x_i} \psi(Y^n_{[nu]}(v, u)) \right| \right. \\
\left. \times (nu - [nu]) H^n_{[nu]}(X^n) \right] dv du
\]

\[
\leq \text{const.} \times \frac{1}{\sqrt{n}} \rightarrow 0.
\]
Next we will show

$$L^n - S^n \longrightarrow 0.$$  (3.53)

Take any $\varepsilon > 0$. Then, by Proposition 4, there exists a compact set $K \subset C([0, \infty); \mathbb{R}^d)$ such that

$$\inf_n Q^{n,M}(K) > 1 - \varepsilon.$$  \hspace{1cm} (3.54)

Set $K_M = K \cap C^d_M$. Then, by Proposition 1, we have

$$\left| E^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\right| \\
\leq \text{const.} \times \left\{ \sup_{w \in K_M} |g^n([nu], w) - g(u, w)| \\
+ \left| E^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n)); X^n \notin K]\right| \right\} \\
\leq \text{const.} \times \left\{ \sup_{w \in K_M} |g^n([nu], w) - g(u, w)| \\
+ \sup_{|w|_{\infty} \leq M} \{|g^n([nu], w)| + |g(u, w)|\} \varepsilon \right\}.$$

for each $u \in [s, t]$. Since $K_M \in K^d$ holds, by (3.50), we have

$$\limsup_{n \to \infty} \left| E^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\right| \leq \text{const.} \times \varepsilon.$$  (3.55)

Thus

$$\lim_{n \to \infty} \left| E^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\right| = 0$$  \hspace{1cm} (3.56)

for each $u \in [s, t]$. By (3.50) again and the bounded convergence theorem, we get

$$|L^n - S^n| \\
\leq \int_s^t \left| E^n[\psi(X^n_u)(g^n([nu], X^n) - g(u, X^n))\Phi]\right| du \longrightarrow 0.$$  \hspace{1cm} (3.57)

Since

$$F(w) = \int_s^t \psi(w(u))g(u, w)\Phi(w(s_1), \ldots, w(s_N))du$$
is continuous and Proposition 1 implies
\begin{equation}
Q^n,M(|F(w)| \leq C_1) = 1
\end{equation}
for each \( n \in \mathbb{N} \), where
\[
C_1 = C_0 |t - s| \sup_{|x| \leq M} |\psi(x)| \sup_{y_1, \ldots, y_N \in \mathbb{R}^d} |\Phi(y_1, \ldots, y_N)|,
\]
using the continuous mapping theorem, we get
\[
S^n \rightarrow \int_s^t E Q^M \left[ \psi(w(u)) g(u, w) \Phi(w(s_1), \ldots, w(s_N)) \right] du.
\]
This completes the proof of Proposition 10. □

By Proposition 8, 9(i) and 10, we have the following.

**Proposition 11.**

(i) \( K^n_1 \rightarrow \sum_{i=1}^d \int_s^t E Q^M \left[ \frac{\partial}{\partial x^i} f(w(u)) \varphi_M(w(u)) \right] b^i_0(u, w) \Phi(w(s_1), \ldots, w(s_N)) du, \)

(ii) \( K^n_3 \rightarrow \sum_{i,j=1}^d \int_s^t E Q^M \left[ \frac{\partial^2}{\partial x^i \partial x^j} f(w(u)) \varphi_M(w(u))^2 \right] a^{ij}_0(u, w) \Phi(w(s_1), \ldots, w(s_N)) du \)
as \( n \to \infty \).

Next we calculate the limit of \( K^n_3 \). Using Taylor’s theorem, we have
\[
K^n_3 = K^n_{3,1} + K^n_{3,2} + K^n_{3,3} + K^n_{3,4} + K^n_{3,5} + K^n_{3,6} + K^n_{3,7} + K^n_{3,8},
\]
where
\[
K^n_{3,1} = \frac{1}{\sqrt{n}} \sum_{i=1}^d \sum_{k=[ns]}^{[nt]-1} E^n \left[ \frac{\partial}{\partial x^i} f(X_{[ns]/n}^{n}) \varphi_M(X_{[ns]/n}^{n}) F_{k}^{n,i}(X_{(\leftarrow [ns]/n)}^{n}) \Phi \right],
\]
\[
K^n_{3,2} = \frac{1}{n} \sum_{i,j=1}^d \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{[nt]-1} E^n \left[ \frac{\partial^2}{\partial x^i \partial x^j} f(X_{l/n}^{n}) \varphi_M(X_{l/n}^{n})^2 \right] F_{k}^{n,i}(X_{\leftarrow (l/n)}^{n}) F_{l}^{n,j}(X^{n}) \Phi,
\]
\[
K_{3,3}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} E^n [\frac{\partial^2}{\partial x^i \partial x^j} f(X^i_{l/n}) \varphi_M(X^n_{l/n})^2]
\times F^{n,i}_k(X^n_{\wedge(l/n)})G^{n,j}_l(X^n)\Phi,
\]

\[
K_{3,4}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} E^n [\frac{\partial}{\partial x^i} f(X^i_{l/n}) \varphi_M(X^n_{l/n})]
\times \frac{\partial}{\partial x^j} \varphi_M(X^n_{l/n}) F^{n,i}_k(X^n_{\wedge(l/n)}) F^{n,j}_l(X^n)\Phi,
\]

\[
K_{3,5}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} E^n [\frac{\partial}{\partial x^i} f(X^i_{l/n}) \varphi_M(X^n_{l/n})]
\times \frac{\partial}{\partial x^j} \varphi_M(X^n_{l/n}) F^{n,i}_k(X^n_{\wedge(l/n)}) G^{n,j}_l(X^n)\Phi,
\]

\[
K_{3,6}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} E^n [\frac{\partial}{\partial x^i} f(X^i_{l/n}) \varphi_M(X^n_{l/n})]
\times \nabla F^{n,i}_k(X^n_{\wedge(l/n)}; I^n_{l} e_j) F^{n,j}_l(X^n)\Phi,
\]

\[
K_{3,7}^n = \frac{1}{n\sqrt{n}} \sum_{i,j=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} E^n [\frac{\partial}{\partial x^i} f(X^i_{l/n}) \varphi_M(X^n_{l/n})]
\times \nabla F^{n,i}_k(X^n_{\wedge(l/n)}; I^n_{l} e_j) G^{n,j}_l(X^n)\Phi,
\]

\[
K_{3,8}^n = \frac{1}{n\sqrt{n}} \sum_{i,j,\nu=1}^{[nt]-1} \sum_{k=1}^{k-1} \sum_{l=1}^{[ns]} \int_0^1 (1 - u) E^n [\eta_{kl}^{n,M,ij\nu}(Y^n_{l}(u, \cdot))]
\times H^{n,M,j}_l(X^n) H^{n,M,\nu}_l(X^n)\Phi du
\]

and

\[
\eta_{kl}^{n,M,ij\nu}(w) = \frac{\partial^3}{\partial x^i \partial x^j \partial x^\nu} f(w(l/n)) F^{n,i}_k(w)
\]

\[
+ \frac{\partial^2}{\partial x^i \partial x^j} f(w(l/n)) \nabla F^{n,i}_k(w; I^n_{l} e_\nu)
\]

\[
+ \frac{\partial^2}{\partial x^i \partial x^\nu} f(w(l/n)) \nabla F^{n,i}_k(w; I^n_{l} e_j)
\]

\[
+ \frac{\partial}{\partial x^i} f(w(l/n)) \nabla^2 F^{n,i}_k(w; I^n_{l} e_j, I^n_{l} e_\nu).
\]
Proposition 12. $K_{3,j}^n \to 0$ as $n \to \infty$, $j = 1, 3, 5, 7, 8$.

Proof. Applying Proposition 2 with $U = F_{n,i}^m$ and $V = \frac{\partial}{\partial x^i} f(X_{[ns]/n})\Phi$, we have

$$|K_{3,1}^n| \leq \text{const} \cdot \frac{1}{\sqrt{n}} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor-1} \alpha_{k-[ns]}^{\phi_0} \leq \text{const} \cdot \left( \sum_{k=0}^{\infty} \alpha_k^{\phi_0} \right) \frac{1}{\sqrt{n}} \to 0.$$

Applying Proposition 2 again with $U = F_{n,i}^m$ and $V = \frac{\partial^2}{\partial x^i \partial x^j} f(X_{l/n})\varphi_M(Y_{l/n})C_{l,j}(X^n)\Phi$, we have

$$|K_{3,3}^n| \leq \text{const} \cdot \frac{1}{n\sqrt{n}} \sum_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor-1} \sum_{l=\lfloor ns \rfloor}^{\lfloor nt \rfloor-1} \alpha_{k-l}^{\phi_0} \leq \text{const} \cdot \left( \sum_{k=0}^{\infty} \alpha_k^{\phi_0} \right) \frac{1}{\sqrt{n}} \to 0.$$

Similarly we have $K_{3,5}^n \to 0$ and $K_{3,7}^n \to 0$. Since $\eta_{ki}^{n,M,ij\nu}(w)$ is the finite sum of the following terms

$$D^\beta f(w(l/n))D^{\beta'} \varphi_M(w(k/n))U(w)$$

with $\beta, \beta' \in \mathbb{Z}_d^+$ and $U(w) = F_{n,i}^m(w), \nabla F_{n,i}^m(w; I_l^e e_j)$ or $\nabla^2 F_{n,i}^m(w; I_l^e e_j, I_l^e e_\nu)$, by Proposition 2, it follows that $K_{3,8}^n \to 0$. Then we obtain the assertion. $\Box$

For $K_{3,2}^n, K_{3,4}$ and $K_{3,6}^n$, we will show the following proposition.

Proposition 13. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function such that $\psi(x) = 0$ for any $x \in \mathbb{R}^d$ with $|x| > M$, and $\xi_{k,l}^n : C([0, \infty); \mathbb{R}^d) \to \mathbb{R}, k, l \in \mathbb{Z}_+ \times \mathbb{Z}_+ : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}$ be functionals. Suppose that $\xi_{k,l}^n$ is $\mathcal{B}_{l/n}$-measurable and continuous, and that there exists a constant $C_0 = C_0(M) > 0$ such that

$$\sum_{k=1}^{\infty} \sup_{l \in \mathbb{Z}_+} \sup_{|w|_\infty \leq M} |\xi_{k,l}^n(w)| \leq C_0$$

(3.59)
for each $n \in \mathbb{N}$. Moreover suppose
\\[
\sup_{w \in K} \left| \sum_{k=1}^{\infty} \xi_{k,[nt]}^n(w) - \Xi(t,w) \right| \to 0, \quad n \to \infty
\]
for each $K \in \mathcal{K}^d$ and $t \geq 0$. Then it holds that
\\[
\frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{[nt]-1} E^n[\psi(X^n_{l/n})\xi_{k,l}^n(X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})]
\to \int_s^t E^Q F^{M}[\psi(w(u))\Xi(u,w)\Phi(w(s_1), \ldots, w(s_N))] du, \quad n \to \infty.
\]

PROOF. Denote the left-hand side of (3.61) by $U^n$ and set
\[
V^n = \frac{1}{n} \sum_{k=[ns]}^{[nt]-1} \sum_{l=[ns]}^{[nt]-1} E^n[\psi(X^n_{l/n})\xi_{k,l}^n(X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})].
\]
Since Fubini’s theorem implies
\\[
U^n = \frac{1}{n} \sum_{l=[ns]}^{[nt]-1} \sum_{k=1}^{[nt]-2} E^n[\psi(X^n_{l/n})\xi_{k,l}^n(X^n)\Phi(X^n_{s_1}, \ldots, X^n_{s_N})],
\]
we have
\\[
|U^n - V^n| \leq C_1(M, \psi, \Phi) \left\{ \frac{1}{n} + \int_s^t \sup_{l \in \mathbb{Z}_+} \sup_{|w| \leq M} |\xi_{k,l}^n(w)| du \right\}
\]
for some $C_1(M, \psi, \Phi) > 0$. By (3.59), the integrand in the right-hand side of (3.63) is bounded and converges to zero as $n \to \infty$ for $u \in [s,t)$. Thus, using the bounded convergence theorem, we have
\\[
U^n - V^n \to 0.
\]
Since Proposition 10 implies
\\[
V^n \to \int_s^t E^Q F^{M}[\psi(w(u))\Xi(u,w)\Phi(w(s_1), \ldots, w(s_N))] du,
\]
we have our assertion. □

**Proposition 14.**

(i) \( K^n_{3,2} \rightarrow \sum_{i,j=1}^{d} \int_{t}^{s} E^{Q^{M}} \left[ \frac{\partial^2}{\partial x_i \partial x_j} f(w(u)) \varphi_M(w(u))^2 \right] \times A^{ij}(u,w) \Phi(w(s_1), \ldots, w(s_N)) \text{d}u, \)

(ii) \( K^n_{3,4} \rightarrow \sum_{i,j=1}^{d} \int_{t}^{s} E^{Q^{M}} \left[ \frac{\partial}{\partial x_i} f(w(u)) \varphi_M(w(u)) \frac{\partial}{\partial x_j} \varphi_M(w(u)) \right] \times A^{ij}(u,w) \Phi(w(s_1), \ldots, w(s_N)) \text{d}u, \)

(iii) \( K^n_{3,6} \rightarrow \sum_{i,j=1}^{d} \int_{t}^{s} E^{Q^{M}} \left[ \frac{\partial}{\partial x_i} f(w(u)) \varphi_M(w(u))^2 \varphi_M(w(u)) \right] \times B^{ij}(u,w) \Phi(w(s_1), \ldots, w(s_N)) \text{d}u \)

as \( n \to \infty. \)

**Proof.** Define \( \xi_{k,l}^{n,ij} \) by

\[
\xi_{k,l}^{n,ij} = E^n \left[ F_{n,i}^{k,+l}(w) \right] F_{n,j}^{l}(w).
\]

By assumption [A7], we have

\[
\sup_{w \in K} \left| \sum_{k=1}^{\infty} \xi_{k,[nt]}^{n,ij}(w) - A^{n,ij}(t,w) \right| \to 0, \quad n \to \infty
\]

for any \( K \in K^d \) and \( t \geq 0. \)

By Proposition 9, it follows that

\[
K^n_{3,2} - K^n_{3,2,1} \to 0, \quad n \to \infty
\]

where

\[
K^n_{3,2,1} = \frac{1}{n} \sum_{i,j=1}^{d} \sum_{k=1}^{[nt]-1} \sum_{l=1}^{k-1} E^n \left[ \frac{\partial^2}{\partial x_i \partial x_j} f(X_{l/n}^n) \varphi_M(X_{l/n}^n)^2 \right] \times \xi_{k-l,l}^{n,ij}(X^n) \Phi(X_{s_1}^n, \ldots, X_{s_N}^n).
\]

Since Lemma 1 implies

\[
|\xi_{k,l}^{n,ij}(w)| \leq 8 E^n[|F_{k+l}(w)|^{3}]^{1/3} E^n[|F_{l}^{n,j}(w)|^{3}]^{1/3} \alpha_k^{1/3},
\]
we have
\begin{equation}
\sum_{k=1}^{\infty} \sup_{t \in Z^+ \cap |w|_\infty \leq M} |\xi_{k,t}^{n,ij}(w)| \leq C_0 \sum_{k=1}^{\infty} \alpha_k^{1/3}
\end{equation}
for some $C_0 = C_0(M) > 0$. Then, applying Proposition 13, we get
\begin{equation}
K_{3,2,1} \rightarrow \sum_{i,j=1}^{d} \int_{s}^{t} E^{Q^{M}} \left[ \sum_{k=1}^{\infty} \sup_{t \in Z^+ \cap |w|_\infty \leq M} |\xi_{k,t}^{n,ij}(w)| \leq C_0 \sum_{k=1}^{\infty} \alpha_k^{1/3}ight]
\end{equation}
\begin{equation}
\times A^{ij}(u,w)\Phi(w(s_1), \ldots, w(s_N))du.
\end{equation}
Then we obtain the assertion (i).

The assertions (ii) and (iii) follow by the same way. □

By Proposition 7, 11, 12 and 14, it follows that
\begin{equation}
E^{Q^{n,M}}[(f(w(t)) - f(w(s)))\Phi(w(s_1), \ldots, w(s_N))]
\end{equation}
\begin{equation}
\rightarrow E^{Q^{M}}[\int_{s}^{t} E^{M} f(u,w)du\Phi(w(s_1), \ldots, w(s_N))].
\end{equation}
The equality (3.31) now follows by (3.32) and (3.70). This completes the proof of Proposition 6.

**Proposition 15.** The family of measures $(Q^{M})_{M>1+|x_0|}$ is tight on $C([0, \infty); \mathbb{R}^d)$.

**Proof.** We define the matrix $\sigma^{M}(t,w) = (\sigma^{M,ij}(t,w))_{i,j=1}^{d}$ by
\begin{equation}
\sigma^{M}(t,w) = \varphi_M(w(t))a^{1/2}(t,w),
\end{equation}
where $a^{1/2}(t,w)$ is the square root matrix of $a(t,w)$. By Proposition 6, there exists the weak solution $(\Omega^{M}, \mathcal{F}^{M}, (\mathcal{F}^{M}_t)_t, P^{M}, (B^{M}_t)_t, (X^{M}_t)_t)$ of the following stochastic differential equation
\begin{equation}
\begin{cases}
dX^{M}_t = \sigma^{M}(t,X^{M})dB^{M}_t + b^{M}(t,X^{M})dt \\
X^{M}_0 = x_0
\end{cases}
\end{equation}
such that the distribution of $X^{M}$ under $P^{M}$ is equal to $Q^{M}$.

Let $T > 0$. We will show that there exists a constant $C_0(T) > 0$ such that
\begin{equation}
E^{M}[\sup_{0 \leq t \leq T} |X^{M}_t|^4] \leq C_0(T)
\end{equation}
Fix any $R > 0$ and define the stopping time $\tau_R$ and the function $m_R(t)$ by
\[
\tau_R = \inf\{ t \in \mathbb{R}_+ ; |X^M_t| \geq R \},
\]
and
\[
m_R(t) = E^M[\sup_{0 \leq s \leq t} |X^M_{s \wedge \tau_R}|^4],
\]
where $E^M$ denotes the expectation under $P^M$.

By the continuity of $X^M$, we see that $\tau_R \to \infty$ as $R \to \infty$ almost surely under $P^M$. By the assumption [A8], the Hölder inequality and the Burkholder-Davis-Gundy inequality, we have
\[
m_R(t) \leq C_1 \left\{ E^M \left[ \sup_{0 \leq s \leq t} \int_0^{s \wedge \tau_R} \sigma^M(u, X^M_s) dB^M_u \right]^4 \right\}
+ E^M \left[ \sup_{0 \leq s \leq t} \int_0^{s \wedge \tau_R} b^M(u, X^M_s) du \right]^4 \right\}
\leq C_1 \left\{ t E^M \left[ \int_0^t 1_{\{s \leq \tau_R\}} |\sigma^M(s, X^M)|^4 ds \right]
+ t^3 E^M \left[ \int_0^t 1_{\{s \leq \tau_R\}} |b^M(s, X^M)|^4 ds \right]\right\}
\leq C_2(T) E^M \left[ \int_0^t 1_{\{s \leq \tau_R\}} (1 + \sup_{0 \leq u \leq s} |X^M_u|)^4 ds \right]
\leq C_3(T) \left\{ 1 + \int_0^t m_R(s) ds \right\}
\]
for each $t \leq T$ and for some constants $C_1, C_2(T), C_3(T) > 0$. Applying the Gronwall inequality, we see
\[
(3.73) \quad \sup_{0 \leq t \leq T} m_R(t) \leq C_4(T)
\]
for some $C_4(T) > 0$. Letting $R \to \infty$, we get (3.72) by Fatou’s lemma.

Then, using the Hölder inequality and the Burkholder-Davis-Gundy inequality again, we have
\[
E^{P^M}[|X^M_t - X^M_s|^4]
\leq C_1 \left\{ E^M \left[ \int_0^t 1_{\{u \geq s\}} \sigma^M(u, X^M_u) dB^M_u \right]^4 \right\}
+ E^M \left[ \int_s^t b^M(u, X^M_u) du \right]^4 \right\}
\]
\[
\leq C_1 \left\{ |t - s| \mathbb{E}^M \left[ \int_s^t |\sigma^M(u, X^M)|^4 du \right] \\
+ |t - s|^3 \mathbb{E}^M \left[ \int_s^t |b^M(u, X^M)|^4 du \right] \right\}
\leq C_5(T)|t - s| \int_s^t \left( 1 + \mathbb{E}^M \left[ \sup_{0 \leq v \leq u} |X^M_v|^4 \right] \right) du \leq C_0(T)C_5(T)|t - s|^2
\]
for some $C_5(T) > 0$. Obviously $Q^M(w \in C([0, \infty); \mathbb{R}^d); w(0) = x_0) = 1$ holds for all $M$. Then, using theorem 2.3 in [13], we obtain the tightness of $(Q^M)_{M > 1 + |x_0|}$. □

**Proof of Theorem 1.** Proposition 15 implies that for any subsequence $(M_k)_k$, there exists a further subsequence $(M_{k_l})_l$ such that $Q^{M_{k_l}}$ converges to some probability measure $Q^*$ on $C([0, \infty); \mathbb{R}^d)$.

Take $M_0$ large enough so that the support of $f$ is contained in $\{ x \in \mathbb{R}^d; |x| \leq M_0/2 \}$. Since $\mathcal{L}^M f = \mathcal{L} f$ holds for $M > M_0$, by (3.31), it follows that

\begin{equation}
(3.74) \quad E^{Q^{M_{k_l}}}[ (f(w(t)) - f(w(s))) \Phi(w(s_1), \ldots, w(s_N))] \\
= E^{Q^{M_{k_l}}}[ \int_s^t \mathcal{L} f(u, w) du \Phi(w(s_1), \ldots, w(s_N))]
\end{equation}

for $M_{k_l} > M_0$. Letting $l \to \infty$, we see that $Q^*$ is a solution of the martingale problem associated with the generator $\mathcal{L}$. Moreover, by the assumption [A10], $Q^*$ equals to $Q$ and is independent of a subsequence $(M_{k_l})_l$. Then it follows that $Q^M$ converges weakly to $Q$ on $C([0, \infty); \mathbb{R}^d)$ as $M \to \infty$.

Finally, repeating the arguments in [5] p.119-120, we show that $Q^n$ converges weakly to $Q$ on $C([0, \infty); \mathbb{R}^d)$. This completes the proof of Theorem 1. □

4. **Proof of Theorem 2**

To prove Theorem 2, we will show two lemmas below. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, d)$ be a metric space.

**Lemma 5.** Let $1 < p, q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} < 1$ and $U : S \times \Omega \to \mathbb{R}$ be a continuous random function such that $U(x)$ is $\mathcal{A}$-measurable and
E[U(x)] = 0 for each x ∈ S, and X : Ω → S, V : Ω → ℝ be B-measurable random variables. Suppose that there exist positive constants C_0 and γ such that

$$\sup_{\varepsilon > 0} \varepsilon^\gamma \log N(\varepsilon, p; U) \leq C_0. \tag{4.1}$$

Then for each ϱ ∈ (0, 1/γ) there exists a constant C > 0 depending only on p, q, γ, ϱ and C_0 such that

$$\left| \mathbb{E}[U(X)V] \right| \leq C \left( \mathbb{E}\left[ \sup_{x \in S} |U(x)|^p \right]^{1/p} + 1 \right) \mathbb{E}[|V|^q]^{1/q} \left( \frac{1}{\log(1/\alpha(A,B))} \right)^\varrho. \tag{4.2}$$

**Proof.** We may assume that the right-hand side of (4.2) is finite. Set $\xi = \frac{1}{\log(1/\alpha(A,B))}$. Using Lemma 2 with $\varepsilon = \xi^\varrho$, we have

$$\left| \mathbb{E}[U(X)V] \right| \leq 8 \left( \mathbb{E}\left[ \sup_{x \in S} |U(x)|^p \right]^{1/p} + 1 \right) \mathbb{E}[|V|^q]^{1/q} \left( \frac{1}{\log(1/\alpha(A,B))} \right)^\varrho \times \mathbb{E}[V]^q \left( \xi^\varrho + \xi^{(1-r)\varrho} \exp(C_0\xi^{-\varrho} - \xi^{-1}) \right), \tag{4.3}$$

where $\frac{1}{r} = 1 - \frac{1}{p} - \frac{1}{q}$. Since $\varrho\gamma \in (0, 1)$ and $\xi \in (0, 1)$, there is a constant $C_1 > 0$ which depends only on $p, q, \gamma, \varrho$ and $C_0$ such that

$$\xi^{(1-r)\varrho} \exp(C_0\xi^{-\varrho} - \xi^{-1}) \leq C_1 \xi^\varrho. \tag{4.4}$$

By (4.3) and (4.4), we obtain our assertion. □

**Lemma 6.** Let 1 < p, q, r < ∞ be such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Let $U, V : S \times \Omega \rightarrow \mathbb{R}$ be continuous random functions such that $U(x)$ and $V(x)$ are $A$ and $B$-measurable respectively and $\mathbb{E}[U(x)] = 0$ for each $x \in S$, and $X : \Omega \rightarrow S$, $Z : \Omega \rightarrow \mathbb{R}$ be $C$-measurable random variables. Suppose that there exist positive constants $C_0, u^*, v^* > 0$ and $\gamma$ such that

$$\sup_{\varepsilon > 0} \varepsilon^\gamma \left\{ \log N(\varepsilon, p; U) + \log N(\varepsilon, q; V) \right\} \leq C_0. \tag{4.5}$$
(4.6) \[ E\left[ \sup_{x \in S} |U(x)|^p \right]^{1/p} \leq u^* \]

and

(4.7) \[ E\left[ \sup_{x \in S} |V(x)|^q \right]^{1/q} \leq v^*. \]

Then for each \( \varrho' \in \left( 0, \frac{1}{2\gamma} \right) \) there exists a constant \( C > 0 \) depending only on \( p, q, r, \gamma, \varrho', u^*, v^* \) and \( C_0 \) such that

(4.8) \[ \left| E[\Xi(X)Z] \right| \leq C E[|Z|^r]^{1/r} \left( \frac{1}{\log(1/\alpha(A \vee B, C))} \right)^{\varrho'} \left( \frac{1}{\log(1/\alpha(A, B \vee C))} \right)^{\varrho'}, \]

where \( \Xi(x) = U(x)V(x) - E[U(x)V(x)] \).

**Proof.** By (2.17), we have

(4.9) \[ \sup_{\varepsilon > 0} \varepsilon^\gamma \log N(\varepsilon, p; \Xi) \leq 2^{\gamma+1} C_0 (u^* + v^*)^\gamma. \]

Then, by Lemma 5, we see that

(4.10) \[ \left| E[\Xi(X)Z] \right| \leq C_1 E[|Z|^r]^{1/r} \left( \frac{1}{\log(1/\alpha(A \vee B, C))} \right)^{2\varrho'} \]

for some \( C_1 = C_1(p, q, r, \gamma, \varrho', u^*, v^*, C_0) > 0 \). By Lemma 1 and Lemma 5, we have

(4.11) \[ \left| E[\Xi(X)Z] \right| \leq C_2 E[|Z|^r]^{1/r} \left\{ \alpha(A, B \vee C)^{1-1/p-1/q} + \left( \frac{1}{\log(1/\alpha(A, B \vee C))} \right)^{2\varrho'} \right\} \]

for some \( C_2 = C_2(p, q, r, \gamma, \varrho', u^*, v^*, C_0) > 0 \). Since there is \( C_3 = C_3(p, q, \varrho') > 0 \) such that

(4.12) \[ t^{1-1/p-1/q} \leq C_3 \left( \frac{1}{\log(1/t)} \right)^{2\varrho'} \]

for all \( t \in (0, 1/4] \), we get

(4.13) \[ \left| E[\Xi(X)Z] \right| \leq C_2 (C_3 + 1) E[|Z|^r]^{1/r} \left( \frac{1}{\log(1/\alpha(A, B \vee C))} \right)^{2\varrho'}. \]
By (4.10) and (4.13), we obtain the assertion. □

By Lemma 5, Lemma 6 and the same arguments in the proof of Theorem 1, we obtain Theorem 2.

5. Appendix

5.1. Sufficient conditions for [A9]

Let \( a(t, w) = (a_{ij}(t, w))_{i,j=1}^d \) and \( b(t, w) = (b_i(t, w))_{i=1}^d \) be as in [A8], and let \( \sigma(t, w) = (\sigma_{ij}(t, w))_{i,j=1}^d = a^{1/2}(t, w) \). It is well-known that if we assume the Lipschitz condition of \( \sigma_{ij}(t, w) \) and \( b_i(t, w) \), then the condition [A9] holds. In fact, the local Lipschitz continuity of \( b_i(t, w) \) is obtained by [A3] and [A5]. In this section we introduce the sufficient condition under which \( \sigma_{ij}(t, w) \) is Lipschitz continuous.

\([A10]\) \( a_{ij}(t, w) \) is twice continuously Fréchet differentiable in \( w \) for each \( t \geq 0 \), and for each \( T > 0 \) there exists a positive constant \( C(T) > 0 \) such that

\[
\left| \nabla^2_w a_{ij}(t, w) \right|_{L^2_t} \leq C(T) \tag{5.1}
\]

for each \( t \in [0, T] \) and \( w \in C([0, \infty); \mathbb{R}^d) \), where \( \nabla^2_w a_{ij}(t, w) \) denotes the second Fréchet derivative of \( a_{ij}(t, w) \) with respect to \( w \).

Here we remark that since \( a_{ij}(t, \cdot) \) is measurable with respect to \( \mathcal{B}_t \), we can regard \( \nabla^2_w a_{ij}(t, w) \) as the element of \( L^2_t \) for each fixed \( t \geq 0 \).

**Theorem 3.** Assume [A1] – [A8] and [A10]. Then the conclusion of Theorem 1 holds.

**Proof.** Let \( \sigma(t, w) = a^{1/2}(t, w) \). To check the condition [A9], it suffices to show that for each \( M > 0 \) and \( T > 0 \) there exists a constant \( C_0 = C_0(M, T) > 0 \) such that

\[
\left| \sigma_{ij}(t, w) - \sigma_{ij}(t, w') \right| \leq C_0 \sup_{0 \leq s \leq t} |w(s) - w'(s)|, \tag{5.2}
\]

\[
\left| b_i(t, w) - b_i(t, w') \right| \leq C_0 \sup_{0 \leq s \leq t} |w(s) - w'(s)| \tag{5.3}
\]
for any \( t \in [0, T] \) and \( w, w' \in \mathcal{C}_M^d \).

By [A3], we have
\[
\| \nabla b_0^{n,i}(k, w) \|_{L_{k/n}^1} \leq E_n[\| \nabla G_k^{n,i}(w) \|_{L_{k/n}^1}] \leq C_1, \quad k \in \mathbb{Z}_+, \ w \in \mathcal{C}_M^d
\]
for some \( C_1 = C_1(M) > 0 \). Moreover, by [A3], [A5] and Lemma 1, we have
\[
\begin{align*}
\| \nabla b_0^{n,i}(k, w) \|_{L_{k/n}^1} & \leq C_2 \sup_{0 \leq s \leq t} \| w(s) - w'(s) \|^2, \quad t \in [0, T], \ x \in \mathbb{R} \\
& \leq C_2 \sum_{l=1}^{\infty} \alpha_l^{1/3}, \ k \in \mathbb{Z}_+, \ w \in \mathcal{C}_M^d
\end{align*}
\]
for some \( C_2 = C_2(M) > 0 \). By (5.4) and (5.5), we get (5.3).

To see (5.2), we introduce the following theorem (Theorem 5.2.3 in [14]).

**THEOREM 4.** Let \( f(t, x) = (f_{ij}(t, x))_{i,j=1}^d : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) be a symmetric non-negative definite matrix-valued function. Suppose that \( f_{ij}(t, x) \) is twice continuously differentiable in \( x \) for each \( t \geq 0 \) and that there is a positive constant \( C(T) \) such that
\[
\left| \frac{\partial^2}{\partial x^2} f_{ij}(t, x) \right| \leq C(T)
\]
for each \( t \in [0, T] \), \( x \in \mathbb{R} \) and \( i, j = 1, \ldots, d \). Then it holds that
\[
|g_{ij}(t, x) - g_{ij}(t, y)| \leq d \sqrt{2C(T)}|x - y|
\]
for each \( t \in [0, T] \) and \( x, y \in \mathbb{R} \), where \( g(t, x) = f^{1/2}(t, x) \).

For each fixed \( T > 0 \) and \( w, w' \in C([0, \infty); \mathbb{R}^d) \), define the functions \( f, g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) by \( f(t, x) = a(t, w' + x(w - w')) \) and \( g(t, x) = f^{1/2}(t, x) \). By [A10], \( f(t, x) \) is twice continuously differentiable in \( x \) for each \( t \) and
\[
\left| \frac{d^2}{dx^2} f_{ij}(t, x) \right| = \left| \nabla_w a_{ij}(t, w' + x(w - w')); w - w', w - w' \right|
\]
\[
\leq C_4 \sup_{0 \leq s \leq t} |w(s) - w'(s)|^2, \quad t \in [0, T], \ x \in \mathbb{R}
\]
for some $C_4(T) > 0$. Then Theorem 4 implies

$$|\sigma^{ij}(t, w) - \sigma^{ij}(t, w')| = |g^{ij}(t, 1) - g^{ij}(t, 0)| \leq d\sqrt{2C_4} \sup_{0 \leq s \leq t} |w(s) - w'(s)|.$$  

This implies (5.2). Then the condition [A9] holds and we obtain the conclusion. □

5.2. Sufficient conditions for [A4] and [B4]

In this section we provide sufficient conditions under which [A4] and [B4] are filled.

Let $\varepsilon > 0$, $(S, d)$ be a metric space and $A$ be a totally bounded subset of $S$. We say that a family of sets $(A_i)_{i=1}^m$ is an $\varepsilon$-net of $A$ if $A \subset \bigcup_{i=1}^m A_i$ and $\sup_{x, y \in A_i} d(x, y) < \varepsilon$ for each $i = 1, \ldots, m$. We denote by $\hat{N}(\varepsilon; A, d)$ the minimum of cardinals of $\varepsilon$-nets of $A$ in the metric $d$.

**Theorem 5.** Let $(\Omega, \mathcal{F}, P)$ be a probability space, $p \geq 1$, $(S, d)$ be a metric space, $(B, ||\cdot||_B)$ be a Banach space and $A$ be a totally bounded subset of $B$. Let $f : B \times \Omega \longrightarrow \mathbb{R}$ be a continuously Fréchet differentiable random function and $u : S \longrightarrow B$ be a continuous function such that $u(x) \in A$ for any $x \in S$. Suppose that there exists a positive constant $C_0$ such that

$$E[\sup_{y \in \tilde{A}} ||\nabla f(y)||_{B^*}^p]^{1/p} \leq C_0, \quad (5.9)$$

where $\tilde{A}$ is a convex hull of $A$ and

$$||\nabla f(y)||_{B^*} = \sup_{z \in B, z \neq 0} \frac{||\nabla f(y; z)||}{||z||_B}, \quad y \in B.$$  

Then for any $\varepsilon > 0$

$$N(\varepsilon, p; U) \leq \hat{N}(\varepsilon/C_0; A, d_B), \quad (5.10)$$

where $U(x, \omega) = f(u(x), \omega)$ and $d_B(y, y') = ||y - y'||_B$, $y, y' \in B$.

**Proof.** Let $(A_i)_{i=1}^m$ be an $\varepsilon$-net of $A$. We define $S_i \subset S$ by

$$S_i = \{x \in S \mid u(x) \in A_i\}.$$
Then we have

\[ S = \bigcup_{i=1}^{m} S_i \]

and for each \( x, x' \in S_i \)

\[
|U(x) - U(x')| \leq \int_{0}^{1} ||\nabla f(tu(x) + (1 - t)u(x'))||_{B^*} dt ||u(x) - u(x')||_B
\]

\[
\leq \sup_{y \in \tilde{A}} ||\nabla f(y)||_{B^*} \times \varepsilon.
\]

Then we have

\[ \mathbb{E}[\max_{i=1,\ldots,m} \sup_{x,x' \in S_i} |U(x) - U(x')|^p]^{1/p} \leq C_0 \varepsilon. \] (5.12)

By (5.11) and (5.12), we see that \((S_i)_{i=1}^{m}\) is an \((C_0 \varepsilon, p, U)\)-net of \(S\). Then we obtain the assertion. \( \square \)

Let \( B \) be a Banach space and \( \mathcal{B}(B) \) be a Borel field of \( B \). By Theorem 5, under suitable conditions, we can check conditions \([A4]\) and \([B4]\) when \( F_k^{n,i} \) and \( G_k^{n,i} \) are represented in the following form

\[ F_k^{n,i}(w, \omega) = f_k^{n,i}(u(k/n, w), \omega), \quad G_k^{n,i}(w, \omega) = g_k^{n,i}(v(k/n, w), \omega), \] (5.13)

where \( f_k^{n,i}(x, \omega), g_k^{n,i}(x, \omega) : B \times \Omega \to \mathbb{R} \) be \( \mathcal{B}(B) \otimes \mathcal{F}^n \)-measurable random functions and \( u(t, w), v(t, w) : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to B \) be \( (\mathcal{B}_t)_{t} \)-adapted (i.e. \( u(t, \cdot) \) and \( v(t, \cdot) \) are \( \mathcal{B}_t \)-measurable for each \( t \geq 0 \)) deterministic functions.

We also have the condition \([A4]\) when the image spaces of \( F_k^{n,i} \) and \( G_k^{n,i} \) are finite dimensional in \( L^p_0(\Omega^n) \). Let \( p \geq 1 \), \((\Omega, \mathcal{F}, P)\) be a probability space, \((S, d)\) be a metric space and \( U : S \times \Omega \to \mathbb{R} \) be a continuous random function which satisfies \( \mathbb{E}[|U(x)|^p] < \infty \) for any \( x \in S \). We define the metric space \((\mathcal{S}_p(U), d_p)\) by

\[ \mathcal{S}_p(U) = \{U(x) \in L^p(\Omega) : x \in S\} \]

and \( d_p(X, Y) = \mathbb{E}[|X - Y|^p]^{1/p} \).
THEOREM 6. Suppose that there are constants $\gamma \in (0, p/2)$, $C_0 > 0$ and $C_1 > 0$ such that

\begin{equation}
\sup_{\epsilon > 0} \epsilon^\gamma \hat{N}(\epsilon; S_p(U), d_p) \leq C_0 \tag{5.14}
\end{equation}

and

\begin{equation}
\mathbb{E}[\sup_{x \in S} |U(x)|^p] \leq C_1. \tag{5.15}
\end{equation}

Then for each $\lambda \in \left(0, \frac{p - 2\gamma}{p}\right)$ there exists a constant $C > 0$ which depends only on $p, \gamma, \lambda, C_0$ and $C_1$ such that

\begin{equation}
\sup_{\epsilon > 0} \epsilon^{\gamma/\lambda} N(\epsilon, p; U) \leq C. \tag{5.16}
\end{equation}

PROOF. Define $F : S_p(U) \times \Omega \rightarrow \mathbb{R}$ by $F(X, \omega) = X(\omega)$. Then we have

\begin{equation}
\mathbb{E}[|F(X) - F(Y)|^p] = \mathbb{E}[|X - Y|^p] = d_p(X, Y)^p \tag{5.17}
\end{equation}

for any $X, Y \in S_p(U)$. By (5.14), (5.17) and the similar arguments in the proof of Theorem 1.4.1 in [7], we see that there exist a continuous modification $\tilde{F}$ of $F$ and a constant $C_2 > 0$ depending only on $p, \gamma, \lambda$ and $C_0$ such that

\begin{equation}
\mathbb{E} \left[ \sup_{X, Y \in S_p(U), 0 < d_p(X, Y) < 1} \left| \frac{\tilde{F}(X) - \tilde{F}(Y)}{d_p(X, Y)^\lambda} \right|^p \right] \leq C_2. \tag{5.18}
\end{equation}

Define the random variable $K$ by

\[K = \sup_{X, Y \in S_p(U), X \neq Y} \frac{|\tilde{F}(X) - \tilde{F}(Y)|}{d_p(X, Y)^\lambda}.\]

Then it holds that

\begin{equation}
\mathbb{E}[|K|^p] \leq 2^{p-1} C_1 + C_2. \tag{5.19}
\end{equation}
Thus, for each subsets $S_1, \ldots, S_m \subset \mathcal{S}_p(U)$, we have

$$
E[\max_{i=1,\ldots,m} \sup_{x,y \in S_i} |U(x) - U(y)|^p]^{1/p} = E[\max_{i=1,\ldots,m} \sup_{x,y \in S_i} |\tilde{F}(U(x)) - \tilde{F}(U(y))|^p]^{1/p} \\
\leq E[K^p]^{1/p} \max_{i=1,\ldots,m} \sup_{x,y \in S_i} d_p(U(x), U(y))^\lambda \\
\leq C_3 \max_{i=1,\ldots,m} \sup_{x,y \in S_i} E[|U(x) - U(y)|^\lambda]^{p\lambda/p},
$$

where $C_3 = (2^{p-1}C_1 + C_2)^{1/p}$. So we get

$$N(\varepsilon, p; U) \leq \tilde{N}(\varepsilon^{1/\lambda}/C_3; \mathcal{S}_p(U), d_p)$$

for any $\varepsilon > 0$. Then we have

$$\sup_{\varepsilon > 0} \varepsilon^{\gamma/\lambda} N(\varepsilon, p; U) \leq C_3^{\gamma} \sup_{\varepsilon > 0} \varepsilon^\gamma \tilde{N}(\varepsilon; \mathcal{S}_p(U), d_p) \leq C_3^{\gamma} C_0.$$

This implies our assertion. ∎

By Theorem 6, we can check $[A4]$ under the following condition $[A4']$.

$[A4']$ For some $\gamma_2 \in (0, p_0/2)$, (1.6)–(1.10) hold with $\gamma_2$ and $\tilde{N}_n(\varepsilon, M; U)$ instead of $\gamma_0$ and $N_n(\varepsilon, M; U)$, where $\tilde{N}_n(\varepsilon, M; U)$ is the smallest integer $m$ such that there exist sets $S_1, \ldots, S_m$ which satisfy $\mathcal{C}_M^d = \bigcup_{i=1}^m S_i$ and

$$\sup_{x,y \in S_i} E^n[|U(x) - U(y)|^{p_0}]^{1/p_0} < \varepsilon$$

for each $i = 1, \ldots, m$.

### 5.3. Examples

In this section, we give two examples of Theorem 2. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $\xi_k = (\xi^i_k)_{i=1}^{m_1}, k \in \mathbb{Z}_+$, be an $m_1$-dimensional stationary Gaussian process.

(a.) Let $f(x) = (f^i(x))_{i=1}^{d} : \mathbb{R}^{m_2} \to \mathbb{R}^d, u(t, x, y) = (u^i(t, x, y))_{i=1}^{m_2}$:
$[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^{m_3} \rightarrow \mathbb{R}^{m_2}$ and $\psi(x) = (\psi^i(x))_{i=1}^{m_3} : \mathbb{R}^{m_1} \rightarrow \mathbb{R}^{m_3}$ be Borel measurable functions. Let $\Psi(t, w, y) = (\Psi^i(t, w, y))_{i=1}^{m_2}$ and $h(t, w, y) = (h^i(t, w, y))_{i=1}^{d}$ be such that

$$\Psi^i(t, w, y) = \int_0^t u^i(s, w(t-s), \psi(y)) \, ds$$

and

$$h^i(t, w, y) = f^i(\Psi(t, w, y)).$$

We define $F^n_{k,i}(w)$ and $G^n_{k,i}(w)$ by

$$G^n_{k,i}(w) = \mathbb{E}[h^i(k/n, w, \xi_k)]$$

and

$$F^n_{k,i}(w) = h^i(k/n, w, \xi_k) - G^n_{k,i}(w).$$

We introduce the following conditions.

[C1] $f^i(x)$ is three times continuously differentiable in $x$. Moreover $u(t, x, y)$ is three times continuously differentiable in $x$ and $y$, and all derivatives are continuous in $t$.

[C2] It holds that

$$\sum_{|\beta| \leq 3} \sup_{x \in \mathbb{R}^{m_2}} |D^\beta f^i(x)| < \infty,$$

$$\sum_{|\beta| + |\beta'| \leq 2} \int_0^\infty \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}^{m_3}} |D_x^\beta D_y^{\beta'} u^j(t, x, y)| \, dt < \infty$$

and

$$\sup_{x \in \mathbb{R}^{m_1}} |\psi^{\nu'}(x)| < \infty$$

for each $i = 1, \ldots, d$, $j = 1, \ldots, m_2$ and $\nu = 1, \ldots, m_3$.

[C3] Let $G_{k,l} = \sigma(\xi^i_\nu; \ i = 1, \ldots, d, \ k \leq \nu \leq l)$ and

$$\beta_k = \sup_l \sup \{|P(A \cap B) - P(A)P(B)|; \ A \in G_{0,l}, \ B \in G_{k+l,\infty}\}.$$
Then for some $\varrho_4 \in (0, 1/2)$

$$
\sum_{k=1}^{\infty} \left( \frac{1}{\log(1/\beta_k)} \right)^{\varrho_4} < \infty.
$$

(5.27)

Define $\hat{b}^i(t, w)$ and $\eta_k^{ij}(t, w)$ by

$$
\hat{b}^i(t, w) = \mathbb{E}[h^i(t, w, \xi_0)]
$$

(5.28)

and

$$
\eta_k^{ij}(t, w) = \mathbb{E}[h^i(t, w, \xi_k)h^j(t, w, \xi_0)] - \hat{b}^i(t, w)\hat{b}^j(t, w),
$$

(5.29)

and $\hat{a}^{ij}(t, w)$ by

$$
\hat{a}^{ij}(t, w) = \eta_0^{ij}(t, w) + \sum_{k=1}^{\infty} \{ \eta_k^{ij}(t, w) + \eta_k^{ji}(t, w) \}. 
$$

(5.30)

Let

$$
\hat{\mathcal{L}} f(t, w) = \frac{1}{2} \sum_{i,j=1}^{d} \hat{a}^{ij}(t, w) \frac{\partial^2}{\partial x^i \partial x^j} f(w(t)) + \sum_{i=1}^{d} \hat{b}^i(t, w) \frac{\partial}{\partial x^i} f(w(t))
$$

for $f \in C^2(\mathbb{R}^d)$.

**Theorem 7.** Assume $[C1] - [C3]$. Then the conclusion of Theorem 1 holds replacing $\mathcal{L}$ with $\hat{\mathcal{L}}$.

**Proof.** We will check that $F_k'^{n,i}$ and $G_k'^{n,i}$ satisfy the assumptions of Theorem 2. $[A1] - [A3], [B5]$ and $[A6]$ are obvious.

**Proposition 16.** The condition $[B4]$ holds with $\gamma_1 = 1$.

**Proof.** Let $U(w, \omega) = h^i(t, w, \xi_k(\omega))$. We define $g(v, \omega) : \hat{\mathcal{C}}_R \times \Omega \rightarrow \mathbb{R}$ by $g(v, \omega) = f^i(v(\psi(\xi_k(\omega))))$, where $\hat{\mathcal{C}}_R = C(K_R; \mathbb{R}^{m_1})$, $K_R = \{ x \in \mathbb{R}^m_3 ; |x| \leq R \}$ and $R = \sum_{i=1}^{m_3} \sup_{x \in \mathbb{R}^m_1} |\psi^i(x)|$. We also define $\hat{\Psi}(t, w, y) = (\hat{\Psi}^j(t, w, y))_{j=1}^{m_2} : [0, \infty) \times \mathcal{C}^d_M \times K_R \rightarrow \mathbb{R}^{m_2}$ by

$$
\hat{\Psi}^j(t, w, y) = \int_0^t \omega^j(s, w(t - s), y)ds.
$$
Then it follows that
\begin{equation}
U(w, \omega) = g(\tilde{\Psi}(t, w, \cdot), \omega). \tag{5.32}
\end{equation}

By \([C2]\), we see that there is a constant \(C_0 > 0\) such that
\begin{equation}
\sum_{j=1}^{m_2} \sum_{|\beta| \leq 1} |D^{\beta}_y \tilde{\Psi}(t, w, y)| \leq C_0, \quad w \in C^d_M, \ y \in K_R. \tag{5.33}
\end{equation}

Then we have
\begin{equation}
\tilde{\Psi}(t, w, \cdot) \in A_R, \quad w \in C^d_M, \tag{5.34}
\end{equation}
where
\[ A_R = \left\{ v \in \hat{C}_R : v \text{ is continuously differentiable and } \sum_{j=1}^{m_2} \sum_{|\beta| \leq 1} \sup_{|y| \leq R} |D^{\beta}v^j(y)| \leq C_0 \right\}. \]

\([C2]\) also implies
\begin{equation}
|\nabla g(v, \omega; \tilde{v})| \leq C_1 \sum_{j=1}^{m_2} \sup_{|y| \leq R} \tilde{v}^j(y), \quad v, \tilde{v} \in A_R, \ \omega \in \Omega \tag{5.35}
\end{equation}
for some \(C_1 > 0\). Then, by Theorem 5, we get
\begin{equation}
N(\varepsilon, p, M; U) \leq \hat{N}(\varepsilon/C_1; A_R, d_\infty) \tag{5.36}
\end{equation}
for each \(M > 0\) and \(p \geq 1\), where \(d_\infty(v, v') = \sup_{y \in K_R} |v(y) - v'(y)|\) and \(N(\varepsilon, p, M; U)\) is the minimum of cardinals of \((\varepsilon, p, U)\)-nets of \(C^d_M\).

Moreover, by Theorem XIII in [8], we have
\begin{equation}
\log \hat{N}(\varepsilon/C_1; A_R, d_\infty) \leq C_1C_2\varepsilon^{-1} \tag{5.37}
\end{equation}
for some \(C_2 > 0\) depending only on \(R\) and \(C_0\). Then we get
\begin{equation}
\log N(\varepsilon, p, M; U) \leq C_3\varepsilon^{-1} \tag{5.38}
\end{equation}
for some \(C_3 > 0\) with \(U(w, \omega) = h^i(t, w, \xi_k(\omega))\).
Similarly we see that (5.38) holds with $U(w, \omega) = \nabla w h^i(t, w, \xi_k(\omega); I^n e_j)$ and $U(w, \omega) = \nabla w^i h^i(t, w, \xi_k(\omega); I^n e_j, I^n e_r)$. Then we obtain the assertion. □

To check the condition [A7], we will show the following proposition.

**Proposition 17.** For each $K \in \mathcal{K}^d$, $t \geq 0$ and $k \in \mathbb{Z}_+$, it holds that

$$\sup_{w \in K, y \in \mathbb{R}^{m_1}} |\Psi^i\left(\frac{[nt] + k}{n}, w \left(\cdot \wedge \frac{[nt]}{n}\right), y \right) - \Psi^i(t, w, y)| \to 0, \; n \to \infty. \tag{5.39}$$

**Proof.** Let

$$\delta_T(s; w) = \sup\{|w(r) - w(r')| ; 0 \leq r, r' \leq T, \; |r - r'| \leq s\},$$

$$s, T > 0, \; w \in C([0, \infty); \mathbb{R}).$$

Then we have

$$\sup_{w \in K, y \in \mathbb{R}^{m_1}} \left|\Psi^i\left(\frac{[nt] + k}{n}, w \left(\cdot \wedge \frac{[nt]}{n}\right), y \right) - \Psi^i(t, w, y)\right|$$

$$\leq \int_t^{([nt] + k)/n} \sup_{x, y} |u^i(s, x, y)| ds$$

$$+ \sum_{j=1}^d \int_0^t \sup_{x, y} \left|\frac{\partial}{\partial x^j} u^i(s, x, y)\right| ds$$

$$\times \sup_{w \in K} \left|w^j\left(\frac{[nt] + k}{n} - s \right) \wedge \frac{[nt]}{n} - w^j(t - s)\right| ds$$

$$\leq \int_t^{([nt] + k)/n} \sup_{x, y} |u^i(s, x, y)| ds$$

$$+ \sum_{j=1}^d \int_0^t \sup_{x, y} \left|\frac{\partial}{\partial x^j} u^i(s, x, y)\right| ds \sup_{w \in K} \delta_t\left(\frac{k + 1}{n}; w^j\right).$$

Since $K$ is compact, we see that

$$\sup_{w \in K} \delta_t\left(\frac{k + 1}{n}; w^j\right) \to 0, \; n \to \infty, \; k \in \mathbb{Z}_+. \tag{5.40}$$
Then we have the assertion. □

Define $a_{0}^{n,ij}(k, w), b_{0}^{n, i}(k, w), A^{n, ij}(k, w)$ and $B^{n, ij}(k, w)$ as in [A7].

**Proposition 18.** It holds that

(i) $\sup_{w \in K} |a_{0}^{n, ij}(\lfloor nt \rfloor, w) - \eta_{0}^{ij}(t, w)| \to 0$,

(ii) $\sup_{w \in K} |b_{0}^{n, i}(\lfloor nt \rfloor, w) - \hat{b}^{i}(t, w)| \to 0$,

(iii) $\sup_{w \in K} |A^{n, ij}(\lfloor nt \rfloor, w) - \hat{A}^{ij}(t, w)| \to 0$,

(iv) $\sup_{w \in K} |B^{n, ij}(\lfloor nt \rfloor, w)| \to 0$

for each $t \geq 0$ and $K \in \mathcal{K}^{d}$, where $\hat{A}^{ij}(t, w) = \sum_{k=1}^{\infty} \hat{\eta}_{k}^{ij}(t, w)$.

**Proof.** By Proposition 17, we get

\[
E[\sup_{w \in K} |h^{i}(\lfloor nt \rfloor/n, w, \xi_{k}) - h^{i}(t, w, \xi_{k})|] \\
\leq \sum_{j=1}^{m_{2}} \sup_{x} \left| \frac{\partial}{\partial x^{j}} f^{i}(x) \right| \\
\times E \left[ \sup_{w \in K, y \in \mathbb{R}^{m_{1}}} \left| \Psi^{j} \left( \frac{\lfloor nt \rfloor}{n}, w \left( \cdot \wedge \frac{\lfloor nt \rfloor}{n} \right), y \right) - \Psi^{j}(t, w, y) \right| \right] \to 0
\]

as $n \to \infty$. Then we have the assertion (ii). Moreover this implies

\[
\sup_{w \in K} |a_{0}^{n, ij}(\lfloor nt \rfloor, w) - \eta_{0}^{ij}(t, w)| \\
\leq 2 \left\{ \sup_{x} |f^{i}(x)| E[\sup_{w \in K} |h^{i}(\lfloor nt \rfloor/n, w, \xi_{k}) - h^{i}(t, w, \xi_{k})|] \\
+ \sup_{x} |f^{j}(x)| E[\sup_{w \in K} |h^{i}(\lfloor nt \rfloor/n, w, \xi_{k}) - h^{i}(t, w, \xi_{k})|] \right\} \to 0, \quad n \to \infty.
\]

Then the assertion (i) holds.

Since $\xi_{k}$ is stationary, we have

\[
A^{n, ij}(\lfloor nt \rfloor, w) = \sum_{l=1}^{\infty} \hat{\eta}_{l}^{n, ij}(\lfloor nt \rfloor, w),
\]

(5.41)
where
\[ \hat{\eta}^{n,ij}(k, w) = \mathbb{E} \left[ h^i \left( \frac{k + l}{n}, w \left( \cdot \wedge \frac{k}{n} \right), \xi_l \right) h^j \left( \frac{k}{n}, w, \xi_0 \right) \right] - \mathbb{E} \left[ h^i \left( \frac{k + l}{n}, w \left( \cdot \wedge \frac{k}{n} \right), \xi_l \right) \mathbb{E} \left[ h^j \left( \frac{k}{n}, w, \xi_0 \right) \right] \right]. \]

By Proposition 17, we have
\[ \sup_{w \in K} |\hat{\eta}^{n,ij}_k([nt], w) - \eta^{ij}_k(t, w)| \leq 2 \sum_{\nu=1}^{m^2} \sup_x \left| \frac{\partial}{\partial x^\nu} f^i(x) \right| \sup_x |f^j(x)| \times \sup_{w \in K, y \in \mathbb{R}^{m^2}} \left| \Psi^\nu \left( \frac{[nt] + k}{n}, w \left( \cdot \wedge \frac{[nt]}{n} \right), y \right) - \Psi^\nu(t, w, y) \right| + \sup_x |f^i(x)| E \left[ \sup_{w \in K} |h^j([nt]/n, w, \xi_0) - h^j(t, w, \xi_0)| \right] \rightarrow 0, \quad n \rightarrow \infty \]
for each \( k \in \mathbb{Z}_+ \) and \( t \geq 0 \). Moreover, using Lemma 1, we have
\[ (5.42) \sup_{w \in K} |\hat{\eta}^{n,ij}_k([nt], w) - \eta^{ij}_k(t, w)| \leq 16 \sup_x |f^i(x)| \sup_x |f^j(x)| \beta_k, \]
and [C3] implies
\[ (5.43) \sum_{k=1}^{\infty} \beta_k < \infty. \]

Thus the dominated convergence theorem implies
\[ (5.44) \sup_{w \in K} |A^{n,ij}([nt], w) - \hat{A}^{ij}(t, w)| \leq \sum_{k=1}^{\infty} \sup_{w \in K} |\hat{\eta}^{n,ij}_k([nt], w) - \eta^{ij}_k(t, w)| \rightarrow 0, \quad n \rightarrow \infty. \]

This implies the assertion (iii).

Since
\[ \nabla_w h^i \left( \frac{[nt] + k}{n}, w \left( \cdot \wedge \frac{[nt]}{n} \right), y; \eta^*_{[nt]} e_j \right) = \sum_{\nu=1}^{m^2} \frac{\partial}{\partial x^\nu} f^i \left( \Psi \left( \frac{[nt] + k}{n}, w \left( \cdot \wedge \frac{[nt]}{n} \right), y \right) \right) \]
× \int_0^{k/n} \frac{\partial}{\partial x^j} u^\nu \left( \frac{[nt] + k}{n}, w\left( \left( \frac{[nt] + k}{n} - s \right) \wedge \frac{[nt]}{n} \right), y \right) I^\nu_{[nt]}
× \left( \frac{[nt] + k}{n} - s \right) ds,

we have

\sup_{w \in K} |B^{n,ij}([nt], w)| \leq 8 \sum_{\nu = 1}^{m_2} \sup_x |\frac{\partial}{\partial x^\nu} f^i(x)| \sup_x |f^j(x)|
\times \int_0^{k/n} \sup_{x,y} \left| \frac{\partial}{\partial x^j} u^\nu(s, x, y) \right| ds \beta_k.

(5.45)

Then, \[C2\], (5.43) and the dominated convergence theorem imply the assertion (iv). □

By Proposition 18, we see that \[A7\] holds. Obviously \(\hat{a}^{ij}\) and \(\hat{b}^i\) satisfies the condition \[A8\] and \[A10\]. Then, using Theorem 3, we obtain Theorem 7. □

(b.) Let \(f(x) = (f^i(x))_{i=1}^d : \mathbb{R}^{m_2} \rightarrow \mathbb{R}^d, u(t, x, y) = (u^i(t, x, y))_{i=1}^{m_2} : [0, \infty) \times \mathbb{R}^{m_3} \times \mathbb{R}^{m_1} \rightarrow \mathbb{R}^{m_2} , \) and \(\psi(t, x) = (\psi^i(t, x))_{i=1}^{m_3} : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{m_3}\) be Borel measurable functions. Let \(\Psi(t, w, y) = (\Psi^i(t, w, y))_{i=1}^{m_2}\) and \(h(t, w, y) = (h^i(t, w, y))_{i=1}^d\) be such that

\[\Psi^i(t, w, y) = \int_0^t u^i(s, \int_s^t \psi(r, w(r)) dr, y) ds\]
and

\[h^i(t, w, y) = f^i(\Psi(t, w, y)).\]

We define \(F^{n,i}_k(w)\) and \(G^{n,i}_k(w)\) by (5.22) and (5.23). We introduce the following conditions.

[D1] \(f^i(x)\) is three times continuously differentiable in \(x\). Moreover \(u(t, x, y)\) (respectively, \(\psi^i(t, x)\)) is three times (respectively, twice) continuously differentiable in \(x\), and all derivatives are continuous in \(t\).
It holds that

\[ \sum_{|\beta| \leq 3} \sup_{x \in \mathbb{R}^{m_2}} |D^\beta f(x)| < \infty, \]  

\[ \sum_{|\beta| \leq 2} \int_0^\infty \sup_{x \in \mathbb{R}^{m_3}, y \in \mathbb{R}^{m_1}} |D_x^\beta u^j(t, x, y)| dt < \infty \]  

and

\[ \sum_{|\beta| \leq 2} \int_0^\infty \sup_{x \in \mathbb{R}^d} |D_x^\beta \psi^\nu(t, x)| dt < \infty \]

for each \( i = 1, \ldots, d \), \( j = 1, \ldots, m_2 \) and \( \nu = 1, \ldots, m_3 \).

**Theorem 8.** Assume \([D1],[D2]\) and \([C3]\). Then the conclusion of Theorem 1 holds replacing \( \mathcal{L} \) with \( \hat{\mathcal{L}} \) which is defined by (5.28)-(5.31).

Theorem 8 is obtained by the similar arguments in the proof of Theorem 7. So we will check only the condition \([B4]\).

**Proposition 19.** The condition \([B4]\) holds with \( \gamma_1 = 1 \).

**Proof.** Let \( U(w, \omega) = h^i(t, w, \xi_k(\omega)) \) and \( \tilde{C}_t = C([0, t]; \mathbb{R}^{m_3}) \). We define \( \varphi(w) = (\varphi^j(w))_{j=1}^{m_3} : C([0, \infty); \mathbb{R}^d) \rightarrow \tilde{C}_t \) and \( g(v, \omega) : \tilde{C}_t \times \Omega \rightarrow \mathbb{R} \) by

\[ (\varphi^j(w))(s) = \int_s^t \psi^j(r, w(r)) dr \]

and

\[ g(v, \omega) = f^i\left( \int_0^t u(s, v(s), \xi_k(\omega)) ds \right). \]

Then it follows that

\[ U(w, \omega) = g(\varphi(w), \omega). \]

Set

\[ C_0 = \sum_{j=1}^{m_3} \sum_{|\beta| \leq 1} \int_0^\infty \sup_{x \in \mathbb{R}^d} |D_x^\beta \psi^j(s, x)| ds. \]
By \([D2]\), we see that \(C_0\) is finite and

\[
\varphi(w) \in \tilde{A}_t, \quad w \in C([0, \infty); \mathbb{R}^d),
\]

where

\[
\tilde{A}_t = \left\{ v \in \tilde{C}_t ; v \text{ is continuously differentiable and}
\right. \\
\left. \sum_{j=1}^{m_3} \left( \sup_{0 \leq s \leq t} |v^j(s)| + \sup_{0 \leq s \leq t} \left| \frac{d}{ds} v^j(s) \right| \right) \leq C_0 \right\}.
\]

Moreover we have

\[
|\nabla g(v, \omega; \tilde{v})| \leq C_1 \sum_{j=1}^{m_3} \sup_{0 \leq s \leq t} |\tilde{v}^j(s)|, \quad v, \tilde{v} \in \tilde{C}_t, \quad \omega \in \Omega
\]

for some \(C_1 > 0\). Then we have the assertion by the same arguments in the proof of Proposition 16. \(\square\)

References


(Received February 14, 2005)

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