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Log Crepant Birational Maps and Derived Categories

By Yujiro Kawamata

1. Introduction

The purpose of this paper is to extend the conjecture stated in the paper [5] to the logarithmic case and prove some supporting evidences. [5] Conjecture 1.2 predicts that birationally equivalent smooth projective varieties have equivalent derived categories if and only if they have equivalent canonical divisors.

According to the experience of the minimal model theory, one has to deal with singular varieties instead of only smooth varieties for the classification of algebraic varieties. Moreover, we should consider not only varieties but also pairs consisting of varieties and divisors on them. These pairs are expected to have some mild singularities, the log terminal singularities.

On the other hand, the theory of derived categories works well under the smoothness assumption of the variety. The reason is that the global dimension is finite only in the case of smooth varieties.

In this paper we shall consider pairs of varieties and $\mathbb{Q}$-divisors which have smooth local coverings (Definition 2.1), and conjecture that, if there is an equivalence of log canonical divisors between birationally equivalent pairs, then there is an equivalence of derived categories (Conjecture 2.2). We note that we need to consider the sheaves on the stacks associated to the pairs instead of the usual sheaves on the varieties in order to have equivalences of derived categories as already notices in [4]. This is a generalization of the conjecture in [5], and includes the case considered in [2]. We note that crepant resolutions in higher dimensions are rare but there are many log crepant partial resolutions of quotient singularities by finite subgroups of general linear groups. We note also that, even in the case in which there is a birational morphism between varieties, the direction of the inclusion of the category may be different from that of the morphism, but coincides with that of the inequality of the log canonical divisors.

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In §3, we consider the problem on recovering the variety from the category. We prove that some basic birational invariants related to the canonical divisors can be recovered from the derived category (Theorem 3.1). In particular, we prove the converse statement of the conjecture. On the other hand, we remark that the variety itself can be reconstructed from the category of coherent sheaves (Theorem 3.2).

In §4, we consider the toroidal varieties and prove that the conjecture holds in this case (Theorem 4.2). This is a generalization of [4] Theorem 5.2. We note that our result implies the McKay correspondence for abelian quotient singularities as a special case.

We conclude the paper with a remark on the relationship with the noncommutative geometry in Proposition 4.7. We maybe need the moduli theoretic interpretation of the log crepant maps in order to deal with the conjecture in the difficult general case.

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2. Derived Equivalence Conjecture

We shall consider pairs of varieties with $\mathbb{Q}$-divisors which have local coverings by smooth varieties:

Definition 2.1. Let X be a normal variety, and B an effective $\mathbb{Q}$-divisor on X whose coefficients belong to the standard set $\{1-1/n; n \in \mathbb{N}\}$. Assume the following condition.

(*) There exists a quasi-finite and surjective morphism $\pi : U \to X$ from a smooth variety, which may be reducible, such that $\pi^*(K_X + B) = K_U$.

Let $R = (U \times_X U)^\sim$ be the normalization of the fiber product. Then the projections $p_i : R \to U$ are étale for $i = 1, 2$, because there is no ramification divisor for p_i.

We define the associated Deligne-Mumford stack $\mathcal{X}$ as a 2-functor

$$\mathcal{X} : (Sch) \to (Groupoid)$$

which is defined in the following way, where (Sch) is the category of schemes and $(Groupoid)$ is the 2-category of groupoids, categories themselves whose morphisms are only isomorphisms. For any scheme B, an object of the category $\mathcal{X}(B)$ is an element of the set $U(B) = \mathrm{Hom}(B, U)$, and a morphism of the category $\mathcal{X}(B)$ is an element of the set $R(B) = \text{Hom}(B, R)$. For a morphism $f : B' \to B$, we have a functor $f^* : \mathcal{X}(B) \to \mathcal{X}(B')$ given by $f^* : U(B) \to U(B')$ and $f^* : R(B) \to R(B')$.

If we replace U by its étale covering, then the category $\mathcal{X}(B)$ is replaced by an equivalent category so that the stack $\mathcal{X}$ does not depend on the choice of the covering $\pi : U \to X$ but only on the pair (X, B).

CONJECTURE 2.2. *Let (X, B) and (Y, C) be pairs of quasi-projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there are proper birational morphisms $\mu : W \to X$ and $\nu : W \to Y$ from a third variety W such that $\mu^*(K_X + B) = \nu^*(K_Y + C)$. Then there exists an equivalence as triangulated categories $D^b(Coh(\mathcal{X})) \to D^b(Coh(\mathcal{Y}))$.*

The pairs considered in the conjecture are very special kind of log terminal pairs. But our assumption is sufficiently general in dimension 2:

PROPOSITION 2.3. *(1) Let (X, B) be a pair which satisfies the condition (*) in Definition 2.1. Then the pair (X, B) is log terminal.*

(2) Let X be a normal surface, and B an effective $\mathbb{Q}$-divisor on X whose coefficients belong to the standard set $\{1 - 1/n; n \in \mathbb{N}\}$. Assume that the pair (X, B) is log terminal. Then the pair satisfies the condition (*) in Definition 2.1.

PROOF. (1) Let $\mu : Y \to X$ be a proper birational morphism, and E an exceptional prime divisor of μ. Then there is a birational morphism $\nu : V \to U$ with a quasi-finite morphism $\rho : V \to Y$ and a exceptional prime divisor F of ρ which dominates E. We can write $\mu^*(K_X+B) = K_Y+aE+\dots$ and $\nu^* K_U = K_V + bF + \dots$. If e is the ramification index of the morphism ρ along F, then we have $b = ae + (e - 1)$. Since $b > 0$, we conclude that $a > -1$.

(2) We may always replace X by its open covering in the étale topology in the course of the proof. Fixing a point $x \in X$, we shall construct our covering in a neighborhood of x. Let $\{n_1, n_2, \dots\}$ be the set of all integers such that the numbers $1 - 1/n_i$ appear as coefficients of irreducible components of B which pass through x. We may assume that $n_1 > n_2 > \dots$.

By the classification of log terminal singularities on surfaces, X has only quotient singularities in the étale topology. Thus there is a quasi-finite and surjective morphism $\pi_1 : X_1 \to X$ from a smooth surface which is étale in codimension 1. Let $B_1 = \pi_1^* B$. By shrinking X_1 if necessary, we can take a cyclic Galois covering $\pi_2 : X_2 \to X_1$ of order n_1 which ramifies along the irreducible components of B_1 whose coefficients are equal to $1 - 1/n_1$. Let B_1' be the sum of all the other irreducible components. Then we can show that the pair (X_2, B_2) with $B_2 = \pi_2^* B_1'$ is again log terminal by a similar argument as in (1).

Since the set of integers arising from the coefficients of new pair (X_2, B_2) is smaller, we obtain our assertion by induction. □

Example 2.4. (1) Let $X = \mathbb{C}^2/\mathbb{Z}_8(1,3)$, and $f : Y \to X$ the minimal resolution. Then $f^*K_X = K_Y + \frac{1}{2}(C_1 + C_2)$ for the exceptional divisors C_1 and C_2; f is log crepant as a morphism from $(Y, \frac{1}{2}(C_1 + C_2))$ to $(X, 0)$. Furthermore, let $g : Z \to Y$ be the blowing up at the point $y_0 = C_1 \cap C_2$. Then $g^*(K_Y + \frac{1}{2}(C_1 + C_2)) = K_Z + \frac{1}{2}(C_1' + C_2')$, where C_i' is the strict transform of C_i for $i = 1, 2$. Thus $f \circ g$ is log crepant as a morphism from $(Z, \frac{1}{2}(C_1' + C_2'))$ to $(X, 0)$.

(2) Let $X = \mathbb{C}^2$, D_i $(i = 1, 2)$ the coordinate curves, and $f : Y \to X$ the weighted blowing up at the origin $x_0 = D_1 \cap D_2$ with weight $(1, 2)$. Then Y has one ordinary double point, and $f^*(K_X + \frac{2}{3}(D_1 + D_2)) = K_Y + \frac{2}{3}(D_1' + D_2')$, where D_i' is the strict transform of D_i for $i = 1, 2$. Thus f is log crepant as a morphism from $(Y, \frac{2}{3}(D_1' + D_2'))$ to $(X, \frac{2}{3}(D_1 + D_2))$.

3. Recovery of Varieties from Categories

The variety cannot be recovered from the derived category, but the canonical divisor can because the Serre functor is categorical:

Theorem 3.1. *Let (X, B) and (Y, C) be pairs of projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there is an equivalence as triangulated categories:*

$$F : D^b(Coh(\mathcal{X})) \to D^b(Coh(\mathcal{Y})).$$

Then the following hold:

(1) $\dim X = \dim Y$.

(2) $K_X + B$ *(resp.* $-(K_X + B)$*) is nef if and only if* $K_Y + C$ *(resp.* $-(K_Y + C)$*) is nef. Moreover, if this is the case, then the numerical Kodaira dimensions are equal*

$$\nu(X, \pm(K_X + B)) = \nu(Y, \pm(K_Y + C)).$$

(3) If $K_X + B$ *or* $-(K_X + B)$ *is big, then there are birational morphisms* $\mu : W \to X$ *and* $\nu : W \to Y$ *from a third projective variety* W *such that* $\mu^*(K_X + B) = \nu^*(K_Y + C)$.

(4) There is an isomorphism of big canonical rings as graded $\mathbb{C}$*-algebras*

$$\bigoplus_{m \in \mathbb{Z}} H^0(X, \llcorner m(K_X + B) \lrcorner) \to \bigoplus_{m \in \mathbb{Z}} H^0(Y, \llcorner m(K_Y + C) \lrcorner).$$

Proof. By [6], there exists an object on the product stack $e \in D^b(\mathrm{Coh}(\mathcal{X} \times \mathcal{Y}))$ which is the kernel of the equivalence: F is isomorphic to the integral functor $\Phi^e_{\mathcal{X} \to \mathcal{Y}}$ defined by $\Phi^e_{\mathcal{X} \to \mathcal{Y}}(a) = p_{2*}(p_1^* a \otimes e)$. Then the proofs of the statements (1) through (3) are similar to those in [5].

(4) This is due to Bridgeland and Caldararu. Let $e' \in D^b(\mathrm{Coh}(\mathcal{X} \times \mathcal{Y}))$ be the kernel of the quasi-inverse of F. Let $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ and $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ be diagonal morphisms. Then $e \boxtimes e' \in D^b(\mathrm{Coh}(\mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y}))$ is the kernel of an equivalence

$$G : D^b(\mathrm{Coh}(\mathcal{X} \times \mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{Y} \times \mathcal{Y}))$$

which satisfies

$$G(\Delta_{\mathcal{X}*} \omega_{\mathcal{X}}^m) \cong \Delta_{\mathcal{Y}*} \omega_{\mathcal{Y}}^m$$

for any integer m by [3]. The conclusion follows from the following isomorphisms for $m, m' \in \mathbb{Z}$

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{X} \times \mathcal{X}}(\Delta_{\mathcal{X}*} \omega_{\mathcal{X}}^m, \Delta_{\mathcal{X}*} \omega_{\mathcal{X}}^{m'}) \cong \mathrm{Hom}_{\mathcal{X}}(\Delta_{\mathcal{X}}^* \Delta_{\mathcal{X}*} \omega_{\mathcal{X}}^m, \omega_{\mathcal{X}}^{m'}) \\ &\cong \mathrm{Hom}_{\mathcal{X}}(\omega_{\mathcal{X}}^m \otimes \bigoplus_{p=0}^{\dim X} \Omega_{\mathcal{X}}^p[p], \omega_{\mathcal{X}}^{m'}) \cong \mathrm{Hom}_{\mathcal{X}}(\omega_{\mathcal{X}}^m, \omega_{\mathcal{X}}^{m'}). \ \square \end{aligned}$$

The variety can be recovered from the category of sheaves:

Theorem 3.2. *Let (X, B) and (Y, C) be pairs of projective varieties with $\mathbb{Q}$-divisors which satisfy the condition (*) in Definition 2.1, and let $\mathcal{X}$ and $\mathcal{Y}$ be the associated stacks. Assume that there is an equivalence as abelian categories:*

$$F : Coh(\mathcal{X}) \to Coh(\mathcal{Y}).$$

Then there exists an isomorphism $f : X \to Y$ such that $f_*B = C$. Moreover, if X is smooth and $B = 0$, then there exists an invertible sheaf L on Y such that F is isomorphic to the functor f_*^L defined by $f_*^L(a) = f_*a \otimes L$.

Proof. The functor F is extendable to an equivalence as triangulated categories

$$\tilde{F} : D^b(\mathrm{Coh}(\mathcal{X})) \to D^b(\mathrm{Coh}(\mathcal{Y})).$$

By [6], there exists an object on the product stack $e \in D^b(\mathrm{Coh}(\mathcal{X} \times \mathcal{Y}))$ such that F is isomorphic to the integral functor $\Phi^e_{\mathcal{X}\to\mathcal{Y}}$. The object $\mathcal{O}_x \in \mathrm{Coh}(\mathcal{X})$ for $x \in X$ is simple, i.e, is non-zero and has no non-trivial subobject. Then the image $p_{2*}(p_1^*\mathcal{O}_x \otimes e)$ is also simple and supported at a point. Therefore, the support of e is a graph of a bijection $f : X \to Y$, which is an isomorphism by the Zariski main theorem. The coefficient of B at any irreducible component is determined by the number of non-isomorphic simple objects above its generic point. It follows that $C = f_*B$.

Under the additional assumption, we have

$$p_{2*}(p_1^*\mathcal{O}_x \otimes e) \cong \mathcal{O}_{f(x)}$$

for any point $x \in X$. Therefore, e is an invertible sheaf on the graph of f. □

4. Toroidal Case

We confirm our conjecture 2.2 in the case of toroidal varieties.

Definition 4.1. Let X be a normal variety and $\bar{B}$ a reduced divisor. The pair $(X, \bar{B})$ is said to be *toroidal* if, for each $x \in X$, there exists a toric variety (P_x, Q_x) with a point t, called a *toric chart*, such that the completion

$(\hat{X}_x, \hat{\bar{B}}_x)$ at x is isomorphic to the completion $((\hat{P}_x)_t, (\hat{Q}_x)_t)$ at t. A toroidal pair is said to be *quasi-smooth* if it has only quotient singularities. Such a pair always satisfies the condition (*) in Definition 2.1.

A morphism $f : (X, \bar{B}) \to (Y, \bar{C})$ of toroidal pairs is said to be *toroidal* if, for any point $x \in X$ and any toric chart $(P'_y, Q'_y; t')$ at $y = f(x) \in Y$, there exists a toric chart $(P_x, Q_x; t)$ at x and a toric morphism $g_x : (P_x, Q_x) \to (P'_y, Q'_y)$ such that $g_x(t) = t'$ and the completion $\hat{f}_x : (\hat{X}_x, \hat{\bar{B}}_x) \to (\hat{Y}_y, \hat{\bar{C}}_y)$ is isomorphic to the completion $\hat{(g_x)}_t : ((\hat{P}_x)_t, (\hat{Q}_x)_t) \to ((\hat{P}'_y)_{t'}, (\hat{Q}'_y)_{t'})$ under the toric chart isomorphisms.

A toroidal morphism $f : (X, B) \to (Y, C)$ is said to be a *divisorial contraction* if it is a projective birational morphism and such that the exceptional locus is a prime divisor. A pair of toroidal morphisms $\phi : (X, B) \to (Z, D)$ and $\psi : (Y, C) \to (Z, D)$ is said to be a *flip* if they are projective birational morphisms such that the codimensions of their exceptional loci are at least 2, the composite map $(\psi)^{-1} \circ \phi$ is not an isomorphism, and that the relative Picard numbers $\rho(X/Z)$ and $\rho(Y/Z)$ are equal to 1.

THEOREM 4.2. *Let X and Y be quasi-smooth toroidal varieties, B and C be effective toroidal $\mathbb{Q}$-divisors on X and Y, respectively, whose coefficients are contained in the set $\{1 - 1/r | r \in \mathbb{Z}_{>0}\}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne-Mumford stacks associated to the pairs (X, B) and (Y, C), respectively. Assume that one of the following holds.*

(1) $X = Y$ and $K_X + B \geq K_Y + C$.

(2) There is a toroidal divisorial contraction $f : X \to Y$ such that $C = f_*B$ and

$$K_X + B \geq f^*(K_Y + C).$$

(3) There is a toroidal flip $f : X \to Z \leftarrow Y$ such that $C = f_*B$ and

$$\mu^*(K_X + B) \geq \nu^*(K_Y + C)$$

for common toroidal resolutions $\mu : W \to X$ and $\nu : W \to Y$ with $f = \nu \circ \mu^{-1}$.

(4) There is a toroidal divisorial contraction $f : Y \to X$ such that $B = f_*C$ and

$$f^*(K_X + B) \geq K_Y + C.$$

Setting $Z = Y$ in the cases (1) and (2) and $Z = X$ in the case (4), let $\mathcal{W} = (\mathcal{X} \times_Z \mathcal{Y})\tilde{}$ be the normalization of the fiber product with the natural morphisms $\tilde{\mu} : \mathcal{W} \to \mathcal{X}$ and $\tilde{\nu} : \mathcal{W} \to \mathcal{Y}$. Then the functor

$$F = \tilde{\mu}_* \tilde{\nu}^* : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X})).$$

is fully faithful. Moreover, if the inequality between the log canonical divisors become an equality, then F is an equivalence of triangulated categories.

Proof. Since the set of all the point sheaves span the derived categories, our assertion can be proved analytic locally. Hence we may assume that X, Y, Z are toric varieties, Z is affine, and that f is a toric map. We note that the toroidal structure is only necessary near the exceptional locus of the given birational maps for the application of the theorem.

It is sufficient to prove the first assertion. Indeed, if the pull backs of $K_X + B$ and $K_Y + C$ are equal, then F is automatically an equivalence, because it commutes with the Serre functors ([1]).

Let N be a lattice in $\mathbb{R}^n$ which contains $\mathbb{Z}^n$, M its dual lattice. Assume that the standard vectors $v_1 = (1, 0, \dots, 0)$, $\dots$, $v_n = (0, \dots, 0, 1)$ are primitive in N. Let $\{v_i^*\}_{1 \le i \le n}$ be the dual basis of $\{v_i\}_{1 \le i \le n}$.

(1) Let $X = Y = Z$ be the affine toric variety corresponding to the cone $\sigma \subset N_{\mathbb{R}}$ generated by $v_1, \dots, v_n$. Let D_i be the prime divisors on X corresponding to the v_i. Let r_i and s_i be positive integers attached to the divisors D_i for $1 \le i \le n$ such that we have $B = \sum_i (1 - \frac{1}{r_i}) D_i$ and $C = \sum_i (1 - \frac{1}{s_i}) D_i$. By assumption, we have $r_i \ge s_i$ for all i.

The stacks $\mathcal{X}$ and $\mathcal{Y}$ are defined by the coverings $\pi_X : \tilde{X} \to X$ and $\pi_Y : \tilde{Y} \to Y$ which correspond to the sublattices N_X and N_Y of N generated by the $r_i v_i$ and $s_i v_i$ for $1 \le i \le n$, respectively. The stack $\mathcal{W}$ is defined by the covering $\pi_W : \tilde{W} \to X$ which correspond to the sublattice $N_W = N_X \cap N_Y \subset N$. Let M_X, M_Y and M_W be the dual lattices of N_X, N_Y and N_W, respectively. The morphisms $\tilde{\mu} : \mathcal{W} \to \mathcal{X}$ and $\tilde{\nu} : \mathcal{W} \to \mathcal{Y}$ are given by inclusions $M_X \subset M_W$ and $M_Y \subset M_W$, respectively. M_X and M_Y are generated by the $\frac{1}{r_i} v_i^*$ and $\frac{1}{s_i} v_i^*$, respectively. The dual lattice M_W is equal to the sum $M_X + M_Y \subset M_{\mathbb{Q}}$ generated by the $\frac{1}{t_i} v_i^*$ with $t_i = \mathrm{LCM}(r_i, s_i)$.

Let $m = \sum_{i=1}^n \frac{m_i}{s_i} v_i^* \in M_Y$ for $m_i \in \mathbb{Z}$ be a monomial which generates an invertible sheaf $L = \mathcal{O}_{\mathcal{Y}}(-\sum_{i=1}^n \frac{m_i}{s_i} D_i)$ on $\mathcal{Y}$. Then the invertible sheaf

$\tilde{\nu}^*L$ on $\mathcal{W}$ is generated by the same monomial m, hence $F(L)$ on $\mathcal{X}$ is by the generator

$$F(m) = \sum_{i=1}^{n} \frac{\ulcorner \frac{m_i r_i}{s_i} \urcorner}{r_i} v_i^*$$

of $(m + \check{\sigma}) \cap M_X$. Let L' be another invertible sheaf with the corresponding monomials

$$m' = \sum_{i=1}^{n} \frac{m_i'}{s_i} v_i^*, \quad F(m') = \sum_{i=1}^{n} \frac{\ulcorner \frac{m_i' r_i}{s_i} \urcorner}{r_i} v_i^*.$$

We have

$$\begin{aligned} &\mathrm{Hom}(L', L) = \mathbb{C}^{(m - m' + \check{\sigma}) \cap M} \\ &\mathrm{Hom}(F(L'), F(L)) = \mathbb{C}^{(F(m) - F(m') + \check{\sigma}) \cap M}. \end{aligned}$$

We calculate

$$0 \le \ulcorner \frac{m_i - m_i'}{s_i} \urcorner r_i + \ulcorner \frac{m_i' r_i}{s_i} \urcorner - \frac{m_i r_i}{s_i} \le (1 - \frac{1}{s_i}) r_i + (1 - \frac{1}{s_i}) < r_i$$

because $s_i \le r_i$. Thus

$$0 \le \ulcorner \frac{m_i - m_i'}{s_i} \urcorner r_i + \ulcorner \frac{m_i' r_i}{s_i} \urcorner - \ulcorner \frac{m_i r_i}{s_i} \urcorner < r_i$$

hence

$$\ulcorner \frac{m_i - m_i'}{s_i} \urcorner = \ulcorner \frac{\ulcorner \frac{m_i r_i}{s_i} \urcorner - \ulcorner \frac{m_i' r_i}{s_i} \urcorner}{r_i} \urcorner.$$

Therefore, the homomorphism

$$\mathrm{Hom}(L', L) \to \mathrm{Hom}(F(L'), F(L))$$

is bijective.

(2) Let $Y = Z$ be the affine toric variety corresponding to the cone $\sigma \subset N_{\mathbb{R}}$ generated by $v_1, \dots, v_n$ as in (1). Let $v_{n+1} = (a_1, \dots, a_n) \in N$ be a

primitive vector such that $a_i > 0$ for $1 \le i \le n'$ and $a_i = 0$ for $n' < i \le n$. We assume that $n' \ge 2$. If we put $a_{n+1} = -1$, then we have

$$\sum_{i=1}^{n+1} a_i v_i = 0.$$

Let σ_{i_0} $(1 \le i_0 \le n')$ be the subcones of σ generated by the v_i for $1 \le i \le n$ with $i \ne i_0$. Then there is a decomposition $\sigma = \bigcup_{i=1}^{n'} \sigma_i$. Let $f : X \to Y$ be the corresponding projective birational toric morphism, let D_i be the prime divisors on X corresponding to the v_i for $1 \le i \le n+1$, and D'_i their strict transforms on Y. D_{n+1} is the exceptional divisor of f, and we put $D'_{n+1} = 0$. Then we have $f^*D'_i = D_i + a_i D_{n+1}$, hence $D_i \equiv -a_i D_{n+1}$. The divisors D_i for $1 \le i \le n'$ are positive for f.

Let r_i be positive integers attached to the divisors D_i for $1 \le i \le n+1$ such that we have $B = \sum_i (1 - \frac{1}{r_i}) D_i$. Our condition

$$K_X + \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D_i \ge f^*(K_Y + \sum_{i=1}^{n} \frac{r_i - 1}{r_i} D'_i).$$

is equivalent to

$$\sum_{i=1}^{n+1} \frac{a_i}{r_i} \ge 0$$

because $K_X + \sum_{i=1}^{n+1} D_i = f^*(K_Y + \sum_{i=1}^{n} D'_i)$.

The toric variety X is covered by the affine toric varieties $X_{\sigma_{i_0}}$ for $1 \le i_0 \le n'$. The stacks $\mathcal{X}$ and $\mathcal{Y}$ are defined by the coverings $\pi_{X,i_0} : \tilde{X}_{\sigma_{i_0}} \to X_{\sigma_{i_0}}$ and $\pi_Y : \tilde{Y} \to Y$ which correspond to the sublattices N_{i_0} of N generated by the $r_i v_i$ for $1 \le i \le n+1$ with $i \ne i_0$ and N_{n+1} generated by the $r_i v_i$ for $1 \le i \le n$, respectively. The stack $\mathcal{W}$ above X is defined by the coverings $\pi_{W,i_0} : \tilde{W}_{\sigma_{i_0}} \to X_{\sigma_{i_0}}$ which correspond to the sublattices $N_{i_0,n+1} = N_{i_0} \cap N_{n+1} \subset N$. We note that the morphism of stacks $\tilde{\mu} : \mathcal{W} \to \mathcal{X}$ is not necessarily an isomorphism. If $1 \le i_0 \le n$, then the dual lattice M_{i_0} of N_{i_0} is generated by $\frac{1}{r_i} v_i^* - \frac{a_i}{a_{i_0} r_i} v_{i_0}^*$ for $1 \le i \le n$ with $i \ne i_0$ and $\frac{1}{a_{i_0} r_{n+1}} v_{i_0}^*$. The dual lattice M_{n+1} of N_{n+1} is generated by $\frac{1}{r_i} v_i^*$ for $1 \le i \le n$. The dual lattice $M_{i_0,n+1}$ of $N_{i_0,n+1}$ is equal to the sum $M_{i_0} + M_{n+1} \subset M_{\mathbb{Q}}$.

Let $m = \sum_{i=1}^n \frac{k_i}{r_i} v_i^* \in M_{n+1}$ with $k_i \in \mathbb{Z}$ be a monomial which corresponds to an invertible sheaf L on $\mathcal{Y}$. Then the invertible sheaf $F(L)$ on $\mathcal{X}$ is given by the generators m_{i_0} of $(m + \check{\sigma}_{i_0}) \cap M_{i_0}$ given by

$$m_{i_0} = \sum_{1 \le i \le n, i \ne i_0} \frac{k_i}{r_i} v_i^* + (\frac{k_{n+1}}{a_{i_0} r_{n+1}} - \sum_{1 \le i \le n, i \ne i_0} \frac{a_i k_i}{a_{i_0} r_i}) v_{i_0}^*$$

where k_{n+1} is a smallest integer such that

$$\frac{k_{n+1}}{a_{i_0} r_{n+1}} - \sum_{1 \le i \le n, i \ne i_0} \frac{a_i k_i}{a_{i_0} r_i} \ge \frac{k_{i_0}}{r_{i_0}}$$

that is

$$k_{n+1} = \ulcorner r_{n+1} \sum_{i=1}^n \frac{a_i k_i}{r_i} \urcorner.$$

We note that k_{n+1} is independent of i_0. Let L' be another invertible sheaf corresponding to the monomial $m' = \sum_{i=1}^n \frac{k'_i}{r_i} v_i^*$, and let

$$m'_{i_0} = \sum_{1 \le i \le n, i \ne i_0} \frac{k'_i}{r_i} v_i^* - \sum_{1 \le i \le n+1, i \ne i_0} \frac{a_i k'_i}{a_{i_0} r_i} v_{i_0}^*, \quad k'_{n+1} = \ulcorner r_{n+1} \sum_{i=1}^n \frac{a_i k'_i}{r_i} \urcorner$$

where we note that $a_{n+1} = -1$.

Since f is birational and $\mathcal{H}om(L', L)$ is torsion free, the natural homomorphism

$$\mathrm{Hom}(L', L) \to \mathrm{Hom}(F(L'), F(L))$$

is injective. It is also surjective because the complement of the indeterminacy locus of f^{-1} in Y has codimension at least 2. Since $\mathrm{Hom}^p(L', L) = 0$ for $p > 0$, it is sufficient to prove that $\mathrm{Hom}^p(F(L'), F(L)) = 0$ for $p > 0$ in order to prove that F is fully faithful.

The invertible sheaf $\mathcal{H}om(F(L'), F(L))$ is given by the monomials

$$m_{i_0} - m'_{i_0} = \sum_{1 \le i \le n, i \ne i_0} \frac{k_i - k'_i}{r_i} v_i^* - \sum_{1 \le i \le n+1, i \ne i_0} \frac{a_i (k_i - k'_i)}{a_{i_0} r_i} v_{i_0}^*.$$

We consider the divisorial reflexive sheaf

$$\mathcal{H}om(F(L'), F(L))_X = \mathcal{O}_X(-\sum_{i=1}^{n+1} \ulcorner \frac{k_i - k'_i}{r_i} \urcorner D_i)$$

on X. Since $f^*D'_i \equiv D_i + a_i D_{n+1}$ for $1 \le i \le n$, we have

$$-\sum_{i=1}^{n+1} \ulcorner \frac{k_i - k'_i}{r_i} \urcorner D_i \equiv \sum_{i=1}^{n+1} a_i \ulcorner \frac{k_i - k'_i}{r_i} \urcorner D_{n+1}.$$

We calculate

$$\begin{aligned}&\sum_{i=1}^{n+1} \frac{a_i(k_i - k'_i)}{r_i} \\ &= \sum_{i=1}^{n} \frac{a_i(k_i - k'_i)}{r_i} - \frac{\ulcorner r_{n+1} \sum_{i=1}^{n} \frac{a_i k_i}{r_i} \urcorner}{r_{n+1}} + \frac{\ulcorner r_{n+1} \sum_{i=1}^{n} \frac{a_i k'_i}{r_i} \urcorner}{r_{n+1}} \\ &< \frac{1}{r_{n+1}} \le \sum_{i=1}^{n} \frac{a_i}{r_i}.\end{aligned}$$

Therefore we have

$$\sum_{i=1}^{n+1} a_i \ulcorner \frac{k_i - k'_i}{r_i} \urcorner \le \sum_{i=1}^{n+1} \frac{a_i(k_i - k'_i)}{r_i} + \sum_{i=1}^{n} \frac{a_i(r_i - 1)}{r_i} < \sum_{i=1}^{n} a_i.$$

On the other hand, we have $K_X + D_{n+1} \equiv (\sum_{i=1}^{n} a_i) D_{n+1}$. Since $-D_{n+1}$ is an f-ample $\mathbb{Q}$-divisor, we conclude by the vanishing theorem ([7] Theorem 1.2.5) that

$$\mathrm{Hom}^p(F(L'), F(L)) \cong H^p(\mathcal{H}om(F(L'), F(L))_X) = 0$$

for $p > 0$.

(3) Let $v_{n+1} = (a_1, \ldots, a_n) \in N$ be a primitive vector such that $a_i > 0$ for $1 \le i \le n'$, $a_i = 0$ for $n' < i \le n''$ and $a_i < 0$ for $n'' < i \le n$. We assume that $2 \le n'$ and $n'' < n$. If we put $a_{n+1} = -1$, then we have

$$\sum_{i=1}^{n+1} a_i v_i = 0.$$

Let $\langle v_1, \ldots, v_{n+1} \rangle$ be the convex cone generated by the v_i for $1 \leq i \leq n+1$, and let σ_{i_0} $(1 \leq i_0 \leq n+1)$ be the subcones generated by the v_i for $1 \leq i \leq n+1$ with $i \neq i_0$. Then there are two decompositions:

$$\langle v_1, \ldots, v_{n+1} \rangle = \bigcup_{i=1}^{n'} \sigma_i = \bigcup_{i=n''+1}^{n+1} \sigma_i.$$

Let Z be an affine toric variety corresponding to the lattive N and the cone $\langle v_1, \ldots, v_{n+1} \rangle$. Let $g : X \to Z$ and $h : Y \to Z$ be projective birational toric morphisms corresponding to these subdivisions, and let $f = h \circ g^{-1} : X - \to Y$ be the composite proper birational map. Let D_i be the prime divisors on X corresponding to the v_i for $1 \leq i \leq n+1$, and D'_i their strict transforms on Y. Then we have $D_i \equiv -a_i D_{n+1}, \quad D'_i \equiv -a_i D'_{n+1}$. The divisors D_i $(1 \leq i \leq n')$ are positive for g, and the D_i $(n'' < i \leq n+1)$ negative.

Let $v_{n+2} = \sum_{i=1}^{n'} \lambda a_i v_i = \sum_{i=n''+1}^{n+1} \lambda(-a_i) v_i$, where λ is the smallest positive number such that v_{n+2} becomes a primitive vector in N. We define the subcones $\sigma_{i_0 i_1}$ for $1 \leq i_0 \leq n'$ and $n'' < i_1 \leq n+1$ to be the ones generated by the v_i for $1 \leq i \leq n+1$ with $i \neq i_0, i_1$ and v_{n+2}. Then we have two decompositions

$$\sigma_{i_1} = \bigcup_{i=1}^{n'} \sigma_{i i_1}, \quad \sigma_{i_0} = \bigcup_{i=n''+1}^{n+1} \sigma_{i_0 i}$$

so that

$$\langle v_1, \ldots, v_{n+1} \rangle = \bigcup_{1 \leq i \leq n', n''+1 \leq j \leq n+1} \sigma_{ij}.$$

Let W be the toric variety corresponding to this subdivision with natural projective birational morphisms $\mu : W \to X$ and $\nu : W \to Y$:

$$\begin{array}{ccc} W & \xrightarrow{\nu} & Y \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow[g]{} & Z. \end{array}$$

Let D''_i be the prime divisors on W corresponding to the v_i for $1 \le i \le n+2$. We have

$$\mu^* D_i = \begin{cases} D''_i & \text{if } 1 \le i \le n' \\ D''_i + \lambda(-a_i) D''_{n+2} & \text{if } n'' < i \le n+1 \end{cases}$$
$$\nu^* D'_i = \begin{cases} D''_i + \lambda a_i D''_{n+2} & \text{if } 1 \le i \le n' \\ D''_i & \text{if } n'' < i \le n+1. \end{cases}$$

If $r_1, \dots, r_{n+1}$ are the positive integers attached to the divisors D_i, then our condition

$$\mu^*(K_X + \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D_i) \ge \nu^*(K_Y + \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D'_i).$$

is equivalent to

$$\sum_{i=1}^{n+1} \frac{a_i}{r_i} \ge 0$$

because $\mu^*(K_X + \sum_{i=1}^{n+1} D_i) = \nu^*(K_Y + \sum_{i=1}^{n+1} D'_i) = K_W + \sum_{i=1}^{n+2} D''_i$.

The toric varieties X and Y are covered by the affine toric varieties $X_{\sigma_{i_0}}$ for $1 \le i_0 \le n'$ and $Y_{\sigma_{i_0}}$ for $n'' < i_0 \le n+1$, respectively. The stacks $\mathcal{X}$ and $\mathcal{Y}$ are defined by the coverings $\pi_{X,i_0} : \tilde{X}_{\sigma_{i_0}} \to X_{\sigma_{i_0}}$ and $\pi_{Y,i_0} : \tilde{Y}_{\sigma_{i_0}} \to Y_{\sigma_{i_0}}$ which correspond to the sublattices N_{i_0} of N generated by the $r_i v_i$ for $1 \le i \le n+1$ with $i \ne i_0$. W is covered by $W_{\sigma_{i_0 i_1}}$ for $1 \le i_0 \le n'$ and $n'' < i_1 \le n+1$, and the stack $\mathcal{W}$ is defined by the coverings $\pi_{W,i_0 i_1} : \tilde{W}_{\sigma_{i_0 i_1}} \to W_{\sigma_{i_0 i_1}}$ which correspond to the sublattices $N_{i_0 i_1} = N_{i_0} \cap N_{i_1} \subset N$. The morphisms $\mu : W \to X$ and $\nu : W \to Y$ are covered by morphisms $\tilde{\mu} : \mathcal{W} \to \mathcal{X}$ and $\tilde{\nu} : \mathcal{W} \to \mathcal{Y}$ of stacks. If $1 \le i_0 \le n$, then the dual lattice M_{i_0} of N_{i_0} is generated by $\frac{1}{r_i} v_i^* - \frac{a_i}{a_{i_0} r_i} v_{i_0}^*$ for $1 \le i \le n$ with $i \ne i_0$ and $\frac{1}{a_{i_0} r_{n+1}} v_{i_0}^*$. The dual lattice M_{n+1} of N_{n+1} is generated by $\frac{1}{r_i} v_i^*$ for $1 \le i \le n$. The dual lattice $M_{i_0 i_1}$ of $N_{i_0 i_1}$ is equal to the sum $M_{i_0} + M_{i_1} \subset M_{\mathbb{Q}}$.

Let $L = \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D'_i)$ be an invertible sheaf on $\mathcal{Y}$, where k_i are integers. We have

$$\begin{aligned}\tilde{\nu}^* L &= \mathcal{O}_{\mathcal{W}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D''_i + \sum_{i=1}^{n'} \frac{\lambda a_i k_i}{r_i} D''_{n+2}) \\ &= \tilde{\mu}^* \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D_i) \otimes \mathcal{O}_{\mathcal{W}}(\sum_{i=1}^{n+1} \frac{\lambda a_i k_i}{r_i} D''_{n+2}).\end{aligned}$$

Since

$$K_W + \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D''_i + D''_{n+2} = \mu^*(K_X + \sum_{i=1}^{n+1} \frac{r_i - 1}{r_i} D_i) - \sum_{i=n''+1}^{n+1} \frac{\lambda a_i}{r_i} D''_{n+2}$$

we have $R^p \tilde{\mu}_* \tilde{\nu}^* L = 0$ for $p > 0$, if $\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < - \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i}$ by [7] Theorem 1.2.5. Thus if

$$0 \le \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < - \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i} \tag{4.1}$$

then we have $F(L) = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D_i)$.

Let $L' = \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{n+1} \frac{k'_i}{r_i} D'_i)$ be another invertible sheaf. We assume that both L and L' are in the range (4.1). Since f is isomorphic in codimension 1, the natural homomorphism

$$\text{Hom}(L', L) \to \text{Hom}(F(L'), F(L))$$

is bijective. We shall prove that their higher Hom's vanish on both X and Y.

We have $\mathcal{H}om(L', L)_Y = \mathcal{O}_Y(\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D'_i)$ and

$$\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D'_i \equiv -(\sum_{i=1}^{n+1} a_i \llcorner \frac{k_i - k'_i}{r_i} \lrcorner) D'_{n+1}.$$

Since $- \sum_{i=1}^{n+1} a_i \frac{k_i - k'_i}{r_i} > \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i}$, we have

$$- \sum_{i=1}^{n+1} a_i \llcorner \frac{(k_i - k'_i)}{r_i} \lrcorner > \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i} + \sum_{i=n''+1}^{n+1} a_i \frac{r_i - 1}{r_i} = \sum_{i=n''+1}^{n+1} a_i.$$

Since $K_Y + \sum_{i=1}^{n'} D'_i \equiv -\sum_{i=n''+1}^{n+1} D'_i \equiv (\sum_{i=n''+1}^{n+1} a_i)D'_{n+1}$ and D'_{n+1} is ample, we conclude that

$$\mathrm{Hom}^p(L', L) = H^p(Y, \mathcal{H}om(L', L)_Y) = 0$$

for $p > 0$ by [7] Theorem 1.2.5.

Similarly, we have $\mathcal{H}om(F(L'), F(L))_X = \mathcal{O}_X(\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D_i)$ and

$$\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D_i \equiv -(\sum_{i=1}^{n+1} a_i \llcorner \frac{(k_i - k'_i)}{r_i} \lrcorner) D_{n+1}.$$

Since $\sum_{i=1}^{n+1} \frac{a_i(k_i - k'_i)}{r_i} > \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i}$, we have

$$\begin{aligned} \sum_{i=1}^{n+1} a_i \llcorner \frac{k_i - k'_i}{r_i} \lrcorner &> \sum_{i=n''+1}^{n+1} \frac{a_i}{r_i} - \sum_{i=1}^{n'} a_i \frac{r_i - 1}{r_i} \\ &\geq -\sum_{i=1}^{n'} \frac{a_i}{r_i} - \sum_{i=1}^{n'} a_i \frac{r_i - 1}{r_i} = -\sum_{i=1}^{n'} a_i. \end{aligned}$$

Since $K_X + \sum_{i=n''+1}^{n+1} D_i \equiv -\sum_{i=1}^{n'} D_i \equiv (\sum_{i=1}^{n'} a_i) D_{n+1}$ and $-D_{n+1}$ is ample, we conclude that

$$\mathrm{Hom}^p(F(L'), F(L)) = H^p(X, \mathcal{H}om(F(L'), F(L))_X) = 0$$

for $p > 0$ by [7] Theorem 1.2.5. Therefore, the proof of the fully faithfulness of F is reduced to the following Lemma 4.3 by using [1].

(4) We interchange X and Y, and use the notation of (2). The inequality $K_X + B \leq f^*(K_Y + C)$ implies that

$$\sum_{i=1}^{n} \frac{a_i}{r_i} \leq \frac{1}{r_n}.$$

Let $L = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D_i)$ be an invertible sheaf on $\mathcal{X}$, where k_i are integers. Since $f^* D'_i = D_i + a_i D_{n+1}$, we have

$$\tilde{\mu}^* L = \mathcal{O}_{\mathcal{W}}(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D''_i) = \tilde{\nu}^* \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^{n} \frac{k_i}{r_i} D'_i) \otimes \mathcal{O}_{\mathcal{W}}(-\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} D''_{n+1})$$

where we note that $a_{n+1} = -1$. Since

$$K_X + \sum_{i=1}^n \frac{r_i - 1}{r_i} D_i + D_{n+1} = f^*(K_Y + \sum_{i=1}^n \frac{r_i - 1}{r_i} D'_i) + \sum_{i=1}^n \frac{a_i}{r_i} D_{n+1}$$

we have $R^p \tilde{\nu}_* \tilde{\mu}^* L = 0$ for $p > 0$ if $-\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=1}^n \frac{a_i}{r_i}$. Thus if

$$0 \le -\sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} < \sum_{i=1}^n \frac{a_i}{r_i} \tag{4.2}$$

then we have $F(L) = \mathcal{O}_{\mathcal{Y}}(\sum_{i=1}^n \frac{k_i}{r_i} D'_i)$.

Let $L' = \mathcal{O}_{\mathcal{X}}(\sum_{i=1}^{n+1} \frac{k'_i}{r_i} D_i)$ be another invertible sheaf, so that we have $\mathcal{H}om(L', L)_X = \mathcal{O}_X(\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D_i)$. We assume that both L and L' are in the range (4.2). Since we have $\sum_{i=1}^n \frac{a_i}{r_i} \le \frac{1}{r_n}$, the homomorphism

$$\mathrm{Hom}(L', L) \to \mathrm{Hom}(F(L'), F(L))$$

is bijective.

We have

$$\sum_{i=1}^{n+1} \llcorner \frac{k_i - k'_i}{r_i} \lrcorner D_i \equiv -(\sum_{i=1}^{n+1} a_i \llcorner \frac{k_i - k'_i}{r_i} \lrcorner) D_{n+1}.$$

Since $-\sum_{i=1}^{n+1} a_i \frac{k_i - k'_i}{r_i} < \sum_{i=1}^n \frac{a_i}{r_i}$, we have

$$-\sum_{i=1}^{n+1} a_i \llcorner \frac{(k_i - k'_i)}{r_i} \lrcorner < \sum_{i=1}^n \frac{a_i}{r_i} + \sum_{i=1}^n a_i \frac{r_i - 1}{r_i} = \sum_{i=1}^n a_i.$$

Since $K_X + D_{n+1} \equiv -\sum_{i=1}^n D_i \equiv (\sum_{i=1}^n a_i) D_{n+1}$ and D_{n+1} is ample, we conclude that

$$\mathrm{Hom}^p(L', L) = H^p(X, \mathcal{H}om(L', L)_X) = 0$$

for $p > 0$. Therefore, F is fully faithful by the following Lemma 4.4. □

LEMMA 4.3. *The set of all the invertible sheaves on $\mathcal{Y}$ in the range (4.1) is a spanning class of the category $D^b(Coh(\mathcal{Y}))$.*

Proof. Let Ω be the set of all such invertible sheaves. We shall prove that the full subcategory $\mathcal{D}$ spanned by Ω coincides with $D^b(\mathrm{Coh}(\mathcal{Y}))$. Let i_p $(1 \le p \le t)$ be integers such that $n'' < i_1 < \cdots < i_t \le n+1$ and $t < n - n''$. Let i_0 be an integer such that $n'' < i_0 \le n+1$ and $i_0 \ne i_p$ for any p. Let S be the intersection of the Cartier divisors $\frac{1}{r_{i_p}} D'_{i_p}$ for $1 \le p \le t$ on $\mathcal{Y}$. We call S a stratum. We have an exact Koszul complex

$$0 \to \mathcal{O}_{\mathcal{Y}}(-\sum_{p=1}^{t} \frac{1}{r_{i_p}} D'_{i_p}) \to \cdots \to \bigoplus_{p=1}^{t} \mathcal{O}_{\mathcal{Y}}(-\frac{1}{r_{i_p}} D'_{i_p}) \to \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_S \to 0.$$

Let k_i $(1 \le i \le n+1,\ i \ne i_0)$ be any choice of integers, and let k_{i_0} be an integer such that $-\frac{a_{i_0}}{r_{i_0}} > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} \ge 0$. Then we have

$$-\sum_{i=n''+1}^{n+1} \frac{a_i}{r_i} > \sum_{i=1}^{n+1} \frac{a_i k_i}{r_i} - \sum_{p=1}^{t} \frac{a_{i_p} \epsilon_p}{r_{i_p}} \ge 0$$

if $\epsilon_p = 0, 1$. Therefore, $\mathcal{D}$ does not change if we add all the sheaves of the form $\mathcal{O}_S(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D'_i)$ to the set Ω.

Assume that $t = n - n'' - 1$. Then S is affine and $\mathcal{O}_S(\frac{1}{r_{i_0}} D'_{i_0}) \cong \mathcal{O}_S$. Therefore, $\mathcal{D}$ does not change if we add the invertible sheaves $\mathcal{O}_S(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D'_i)$ for all integers k_i to the set Ω. Hence any coherent sheaf whose support is contained in S belongs to $\mathcal{D}$.

We claim that, for any staratum S, $\mathcal{D}$ does not change if we add the invertible sheaves $\mathcal{O}_S(\sum_{i=1}^{n+1} \frac{k_i}{r_i} D'_i)$ for all integers k_i to the set Ω. Indeed, we proceed by the descending induction on t. Let $S' = S \cap \frac{1}{r_{i_0}} D'_{i_0}$. Then our claim follows from the following exact sequence

$$0 \to \mathcal{O}_S(-\frac{1}{r_{i_0}} D'_{i_0}) \to \mathcal{O}_S \to \mathcal{O}_{S'} \to 0.$$

Hence any coherent sheaf belongs to $\mathcal{D}$. □

Lemma 4.4. *The set of all the invertible sheaves on $\mathcal{X}$ in the range (4.2) is a spanning class of $D^b(Coh(\mathcal{X}))$.*

Proof. The proof is similar to that of Lemma 4.3. □

Corollary 4.5. *Let X and Y be quasi-smooth projective toric (not only toroidal) varieties, let B and C be effective toric $\mathbb{Q}$-divisors on X and*

Y, respectively, whose coefficients are contained in the set $\{1-1/r | r \in \mathbb{Z}_{>0}\}$, and let $\mathcal{X}$ and $\mathcal{Y}$ be smooth Deligne-Mumford stacks attached to the pairs (X, B) and (Y, C), respectively. Let $f : X - \to Y$ be a toric proper birational map which is log crepant in the sense that

$$g^*(K_X + B) = h^*(K_Y + C)$$

for toric proper birational morphisms from a common toric variety $g : Z \to X$ and $h : Z \to Y$ such that $f = h \circ g^{-1}$. Then there is an equivalence of triangulated categories

$$F : D^b(Coh(\mathcal{Y})) \to D^b(Coh(\mathcal{X})).$$

PROOF. By [8] with some additional argument due to Matsuki, f is shown to be decomposed into a sequence of toric divisorial contractions and flips which are log crepant. □

REMARK 4.6. (1) We note that the Fourier-Mukai functors for both divisorial contractions and flips are of the same type consisting of pull-backs and push-downs. Indeed, divisorial contractions and flips are of the same kind of operations from the view point of the Minimal Model Program though they look very different geometrically.

(2) In the situation of Theorem 4.2 (1), suppose that there is a third $\mathbb{Q}$-divisor D on X with standard coefficients such that $B \geq C \geq D$. Let $\mathcal{Z}$ be the stack corresponding to the pair (X, D). Then there are fully faithful functors $F_1 : D^b(\mathrm{Coh}(\mathcal{Z})) \to D^b(\mathrm{Coh}(\mathcal{Y}))$, $F_2 : D^b(\mathrm{Coh}(\mathcal{Y})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ and $F_3 : D^b(\mathrm{Coh}(\mathcal{Z})) \to D^b(\mathrm{Coh}(\mathcal{X}))$ as proved there. But we have $F_3 \not\cong F_2 \circ F_1$ in general, e.g., in the case $B = \frac{3}{4}D_0$, $C = \frac{2}{3}D_0$ and $D = \frac{1}{2}D_0$.

(3) Let $X = \mathbb{C}^2$, $\bar{B} = B_1 + B_2$ the union of the two coordinate lines, $f : Y \to X$ the blowing up at the singular point $B_1 \cap B_2$ of $\bar{B}$, and $\bar{C} = B_1' + B_2' + C_3$ the union of the strict transforms and the exceptional divisor. Let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be the stacks associated to the pairs $(X, \frac{1}{2n}(B_1 + B_2))$ and $(Y, \frac{1}{2n}(B_1' + B_2') + \frac{1}{n}C_3)$, respectively. Since $f^*(K_X + \frac{1}{2n}(B_1 + B_2)) = K_Y + \frac{1}{2n}(B_1' + B_2') + \frac{1}{n}C_3$, we have equivalences $\Phi_n : D^b(\mathrm{Coh}(\mathcal{X}_n)) \to D^b(\mathrm{Coh}(\mathcal{Y}_n))$ for each n by Theorem 4.2 (2). On the other hand, we have fully faithful functors $\Psi_{X,nn'} : D^b(\mathrm{Coh}(\mathcal{X}_n)) \to D^b(\mathrm{Coh}(\mathcal{X}_{n'}))$ and $\Psi_{Y,nn'} : D^b(\mathrm{Coh}(\mathcal{Y}_n)) \to D^b(\mathrm{Coh}(\mathcal{Y}_{n'}))$

for $n < n'$ by Theorem 4.2 (1). We have to be careful about the non-commutativity of the following diagram

$$\begin{array}{ccc} D^b(\mathrm{Coh}(\mathcal{X}_n)) & \xrightarrow{\Phi_n} & D^b(\mathrm{Coh}(\mathcal{Y}_n)) \\ {\scriptstyle \Psi_{X,nn'}}\downarrow & & \downarrow{\scriptstyle \Psi_{Y,nn'}} \\ D^b(\mathrm{Coh}(\mathcal{X}_{n'})) & \xrightarrow{\Phi_{n'}} & D^b(\mathrm{Coh}(\mathcal{Y}_{n'})). \end{array}$$

The reason is that the Serre functors, defined by using different invertible sheaves, are not compatible.

We conclude this paper with a remark on the non-commutative geometry as in [9]. We consider the situation of Theorem 4.2 (3) under the additional assumption that $\mu^*(K_X + B) = \nu^*(K_Y + C)$.

Although the set of all invertible sheaves on $\mathcal{Y}$ in the range (4.1) is infinite, there are only finitely many ismomorphism classes. Let P_Y be the direct sum of these representatives, and let $A_Y = \mathrm{Hom}(P_Y, P_Y)$ be the non-commutative ring of endomorphisms. We denote by $\mathrm{Mod}(A_Y)$ the abelian category of finitely generated right A_Y-modules.

PROPOSITION 4.7. *There is an equivalence of triangulated categories:*

$$D^b(Coh(\mathcal{Y})) \cong D^b(Mod(A_Y)).$$

PROOF. By the proof of Theorem 4.2 (3), we have $\mathrm{Hom}^p(P_Y, P_Y) = 0$ for $p > 0$ and the set $\{P_Y\}$ is spanning. We claim that the functors

$$G : D^b(\mathrm{Coh}(\mathcal{Y})) \to D^b(\mathrm{Mod}(A_Y))$$
$$H : D^b(\mathrm{Mod}(A_Y)) \to D^b(\mathrm{Coh}(\mathcal{Y}))$$

given by $G(a) = R\mathrm{Hom}(P_Y, a)$ and $H(m) = m \otimes^L_{A_Y} P_Y$ are quasi-inverses each other. Indeed, we have $G(P_Y) \cong A_Y$ and $H(A_Y) \cong P_Y$. Since P_Y and A_Y respectively span $D^b(\mathrm{Coh}(\mathcal{Y}))$ and $D^b(\mathrm{Mod}(A_Y))$, it follows that G and H are fully faithful by [1], hence quasi-inverses each other. □

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Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro
Tokyo 153-8914, Japan
E-mail: kawamata@ms.u-tokyo.ac.jp