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Hyperfunction Solutions to Fuchsian Hyperbolic Systems

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Abstract. To Fuchsian hyperbolic systems, the unique solvability theorem is proved in two cases: (1) hyperfunction (or microfunction) solutions with a real analytic parameter to Cauchy problem; (2) mild hyperfunction (or mild microfunction) solutions to boundary value problem. These results extend that of H. Tahara to general systems.

Introduction

In this paper, we shall show the unique solvability theorem for hyperfunction solutions with a real analytic parameter of Cauchy problem to Fuchsian hyperbolic systems.

Fuchsian partial differential operator was first defined by Baouendi-Goulaouic [B-G]. This includes non-characteristic type as a special case, and Cauchy-Kovalevskaja type theorem (namely, unique solvability for Cauchy problem) was proved in [B-G] under the conditions of characteristic exponents. After that, Tahara [T] treated a Fuchsian Volevič system and proved Cauchy-Kovalevskaja type theorem in the complex domain under the conditions of characteristic exponents. Further as an application, under the hyperbolicity condition he obtained a Cauchy-Kovalevskaja type theorem for this system in the framework of hyperfunctions. On the other hand, Laurent-Monteiro Fernandes [L-MF 1] extended the notion of Fuchsian type to a general system of differential equations; that is, coherent left $\mathscr{D}_X$-Module, here and in what follows, we shall write a *Module* with a capital letter, instead of a *sheaf of left modules*. Their notion includes Fuchsian Volevič system, and they proved a Cauchy-Kovalevskaja type theorem in the complex domain in general settings; that is, without conditions of characteristic exponents. As for the uniqueness of hyperfunction solutions for Cauchy problem, Oaku [O 1] and Oaku-Yamazaki [O-Y] extended

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the uniqueness result to Fuchsian systems. Hence in this paper, we shall prove the solvability theorem for general Fuchsian hyperbolic systems in the framework of hyperfunctions (that is, hyperfunctions with a real analytic parameter, or mild hyperfunctions) without the conditions of characteristic exponents. To this end, in addition to the Cauchy-Kovalevskaja type theorem due to Laurent-Monteiro Fernandes [L-MF 1], we use the theory of *microsupports* due to Kashiwara-Schapira (see [K-S]). This theory enables us to prove our desired result, in fact, our key theorem (Theorem 2.2) is only an exercise of this theory, and from this, we easily deduce the Cauchy-Kovalevskaja type theorem for general Fuchsian hyperbolic systems in the framework of hyperfunctions.

1. Preliminaries

In this section, we shall fix the notation and recall known results used in later sections. General references are made to Kashiwara-Schapira [K-S].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z};\, n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{>0} := \{r \in \mathbb{R};\, r > 0\}$.

In this paper, all the manifolds are assumed to be paracompact. If $\tau\colon E \to Z$ is a vector bundle over a manifold Z, then we set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. We denote by $f\colon N \to M$ the canonical embedding. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. We also denote by $f\colon Y \to X$ the canonical embedding with same notation f. By local coordinates $(z, \tau) = (x + \sqrt{-1}\, y, t + \sqrt{-1}\, s)$ of X around each point of N, we have locally the following relation:

$$
\begin{array}{ccccccc}
N & = & \mathbb{R}^n_x \times \{0\} & \overset{f}{\hookrightarrow} & M & = & \mathbb{R}^n_x \times \mathbb{R}_t \\
\downarrow & & & & \downarrow & & \\
Y & = & \mathbb{C}^n_z \times \{0\} & \overset{f}{\hookrightarrow} & X & = & \mathbb{C}^n_z \times \mathbb{C}_\tau
\end{array}
\tag{1.1}
$$

The embedding f induces a natural embedding $f'\colon T_N Y \hookrightarrow T_M X$ and by this mapping we regard $T_N Y$ as a closed submanifold of $T_M X$. Further,

f induces mappings:

$$
\begin{array}{ccccccc}
N & \overset{\pi}{\twoheadleftarrow} & \sqrt{-1}\,T_N^*M & \hookleftarrow & N & \overset{f}{\hookrightarrow} & M \\
\downarrow{\scriptstyle i_N} & \square & \downarrow{\scriptstyle i} & & \downarrow & \square & \downarrow{\scriptstyle i_M} \\
T_N^*Y & \overset{f_d}{\twoheadleftarrow} & N \underset{M}{\times} T_M^*X & = & N \underset{M}{\times} T_M^*X & \overset{f_\pi}{\hookrightarrow} & T_M^*X \\
\downarrow{\scriptstyle \pi_N} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi_M} \\
N & = & N & = & N & \overset{f}{\hookrightarrow} & M
\end{array}
\tag{1.2}
$$

Here π_N, π_M and π are canonical projections, i_N, i_M and i are zero-section embeddings, and $\square$ means that the square is *Cartesian*.

We write $M \setminus N = \Omega_+ \sqcup \Omega_-$, where each $\Omega_\pm$ is an open subset and $\partial\Omega_\pm = N$. We set $M_+ := \Omega_+ \cup N$. By local coordinates, we can write

$$\Omega_+ = \{(x,t) \in M;\ t > 0\} \subset M_+ = \{(x,t) \in M;\ t \geqslant 0\}.$$

We remark that a natural morphism $\mathbb{C}_{M_+} \to \mathbb{C}_N$ gives natural morphisms

$$\boldsymbol{R}\mathcal{H}om_{\mathbb{C}_M}(\mathbb{C}_N, \mathbb{C}_M) = \omega_{N/M} \to \boldsymbol{R}\mathcal{H}om_{\mathbb{C}_M}(\mathbb{C}_{M_+}, \mathbb{C}_M) = \mathbb{C}_{\Omega_+} \to \mathbb{C}_M\,.$$

Here $\omega_{N/M}$ denotes the *relative dualizing complex*. As usual, we denote by $\nu_\bullet(*)$ and $\mu_\bullet(*)$ the specialization and microlocalization functors respectively. Further, $\mu hom(*,*)$ denotes μhom bifunctor. We denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category of sheaves of $\mathbb{C}$-linear spaces with bounded cohomologies. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Then, by Kashiwara-Schapira [K-S, Chapter IV], we have

$$
\begin{aligned}
\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F)\otimes\omega_{N/M}) &\simeq \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\mathbb{C}_M, F)\otimes\omega_{N/M}) \\
&\to \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, F)\otimes\omega_{N/M}) \\
&\to \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\omega_{N/M}, F)\otimes\omega_{N/M}) \\
&= \boldsymbol{R}f_{d!}\, f_\pi^{-1}\mu_N(F) \to \mu_N(f^{-1}F\otimes\omega_{Y/X}).
\end{aligned}
$$

Therefore we obtain morphisms:

$$
\begin{aligned}
\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F)\otimes\omega_{N/M}) &\to \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, F)\otimes\omega_{N/M}) \\
&\to \mu_N(f^{-1}F\otimes\omega_{Y/X}).
\end{aligned}
\tag{1.3}
$$

2. Near-Hyperbolicity Condition

Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. We denote by $\mathrm{SS}(F)$ the *microsupport* of F due to Kashiwara-Schapira (see [K-S]). $\mathrm{SS}(F)$ is a closed conic involutive subset of T^*X and described as follows: Let (w) be local coordinates of X and $(w_0;\zeta_0)$ a point of T^*X. Then $(w_0;\zeta_0)\notin \mathrm{SS}(F)$ if and only if the following condition holds: There exist an open neighborhood U of w_0 in X and a proper convex closed cone $\gamma\subset X$ satisfying $\zeta_0\in \operatorname{Int}\gamma^{\circ a}\cup\{0\}$ such that

$$\boldsymbol{R}\Gamma(H_\varepsilon\cap(x+\gamma);F)\xrightarrow{\sim}\boldsymbol{R}\Gamma(L_\varepsilon\cap(x+\gamma);F)$$

holds for any $w\in U$ and any sufficiently small $\varepsilon>0$. Here $\operatorname{Int}A$ denotes the *interior* of A, $\gamma^{\circ a}:=\bigcap_{\zeta\in\gamma}\{w\in X;\,\mathrm{Re}\langle w,\zeta\rangle\leqslant 0\}$ and

$$L_\varepsilon:=\{w\in X;\,\mathrm{Re}\langle w-w_0,\zeta_0\rangle=-\varepsilon\}\subset H_\varepsilon:=\{w\in X;\,\mathrm{Re}\langle w-w_0,\zeta_0\rangle\geqslant-\varepsilon\}.$$

Next, we shall recall the definition of the near-hyperbolicity condition due to Laurent-Monteiro Fernandes [L-MF 2, Definition 1.3.1]:

Definition 2.1. Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. We say F is *near-hyperbolic* at $x_0\in N$ in $\pm dt$-codirection if there exist positive constants C and ε_1 such that

$$\begin{aligned}&\mathrm{SS}(F)\cap\{(z,\tau;z^*,\tau^*)\in T^*X;\,|z-x_0|<\varepsilon_1,\,|\tau|<\varepsilon_1,\,t\neq0\}\\&\quad\subset\{(z,\tau;z^*,\tau^*)\in T^*X;\,|t^*|\leqslant C\big(|y^*|(|y|+|s|)+|x^*|\big)\}\end{aligned}$$

holds by local coordinates of X in (1.1) and the following associated coordinates of T^*X:

$$(z,\tau;z^*,\tau^*)=(x+\sqrt{-1}\,y,t+\sqrt{-1}\,s;x^*+\sqrt{-1}\,y^*,t^*+\sqrt{-1}\,s^*).$$

Our first main result is as follows:

Theorem 2.2. *Let F be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Assume that F is near-hyperbolic at $x_0\in N$ in $\pm dt$-codirection. Then, the morphisms in* (1.3) *induce isomorphisms for any* $p^* \in T^*_N Y \cap \pi_N^{-1}(x_0)$:

$$\begin{aligned}\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F) \otimes \omega_{N/M})_{p^*} &\simeq \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, F) \otimes \omega_{N/M})_{p^*} \\ &\simeq \mu_N(f^{-1}F \otimes \omega_{Y/X})_{p^*}\,.\end{aligned}$$

Proof. (a) First we shall prove that

$$\boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F) \otimes \omega_{N/M})_{p^*} \to \mu_N(f^{-1}F \otimes \omega_{Y/X})_{p^*}$$

is an isomorphism. Since the Fourier-Sato transformation gives an equivalence, we may prove that

$$\nu_M(F)\big|_{T_N Y \cap \tau_N^{-1}(x_0)} \to \nu_N(f^{-1}F)\big|_{T_N Y \cap \tau_N^{-1}(x_0)}$$

is an isomorphism. If $q = x_0 \in T_N N$, then

$$\begin{aligned}f'^{-1}\nu_M(F)_{x_0} = \nu_M(F)_{f(x_0)} &= F_{f(x_0)} \\ &\to \nu_N(f^{-1}F)_{x_0} = (f^{-1}F)_{x_0} = F_{f(x_0)}.\end{aligned}$$

Next consider in $v = (x_0, \eta_0) \in \dot{T}_N Y \cap \tau_N^{-1}(x_0)$. We may assume $x_0 = 0$ under local coordinates of X in (1.1). We set as in Bony-Schapira [B-S 2]

$$\begin{aligned}B(0,a) &:= \{(x,t) \in \mathbb{R}^{n+1};\ |x| + |t| < a\}, \\ B'(0,a) &:= \{x \in \mathbb{R}^n;\ |x| < a\}, \\ K(a,\delta) &:= \operatorname{Int}\gamma\big[B'(0,a) \cup \{(0, \pm a\delta)\}\big].\end{aligned}$$

Here $\gamma[\cdot]$ denotes the *convex hull.* For an open convex cone $\Gamma \subset \mathbb{R}^{n+1}$ (resp. $\Gamma' \subset \mathbb{R}^n$), we set $\Gamma_\varepsilon := \Gamma \cap B(0,\varepsilon)$ (resp. $\Gamma'_\varepsilon := \Gamma' \cap B'(0,\varepsilon)$). Then, for any $k \in \mathbb{Z}$ we have

$$\begin{aligned}\mathcal{H}^k\nu_M(F)_v = \varinjlim_{V(a,\delta;\Gamma_\varepsilon)} & H^k(V(a,\delta;\Gamma_\varepsilon); F) \\ &\to \mathcal{H}^k\nu_N(f^{-1}F)_v = \varinjlim_{U(a;\Gamma'_\varepsilon)} H^k(U(a;\Gamma'_\varepsilon); F).\end{aligned}$$

Here $\Gamma \subset \mathbb{R}^{n+1}$ (resp. $\Gamma' \subset \mathbb{R}^n$) ranges through the family of open conic neighborhoods of $(\eta_0, 0)$ (resp. η_0), and $V(a,\delta;\Gamma_\varepsilon)$ (resp. $U(a;\Gamma'_\varepsilon)$) ranges through the family of open neighborhoods of $K(a,\delta) + \sqrt{-1}\,\Gamma_\varepsilon$ (resp. $B'(0,a) + \sqrt{-1}\,\Gamma'_\varepsilon$). Then the proof of (a) is reduced to Proposition 2.3 below.

Proposition 2.3. *Let $\Gamma' \subset \mathbb{R}^n$ be a conic neighborhood of η_0. If a and ε are sufficiently small positive constants, then for any $k \in \mathbb{Z}$ there exist ε', $\delta > 0$ and a conic neighborhood $\Gamma \subset \mathbb{R}^{n+1}$ of $(\eta_0, 0)$ such that*

$$\varinjlim_{V(a,\delta;\Gamma_\varepsilon)} H^k(V(a,\delta;\Gamma_\varepsilon);F) \simeq \varinjlim_{U(a;\Gamma'_\varepsilon)} H^k(U(a;\Gamma'_\varepsilon);F).$$

Proof. The proof is same as [B-S 2, Lemme 3.2]. We use the following lemma instead of [B-S 2, Théorème 1.1]:

Lemma 2.4 (cf. [B-S 1, Théorème 2.1]). *Let $Z \subset \Omega \subset X$ be convex sets such that Ω is an open set and Z is closed in Ω. Let G be an object of $\mathbf{D}^{\mathrm{b}}(X)$. Set*

$$A := \{\zeta;\ (w;\zeta) \in \mathrm{SS}(G) \text{ for some } w \in \Omega\}.$$

Suppose that if a hyperplane with normal vector in A crosses Ω, then this hyperplane always crosses Z. Then it follows that

$$\boldsymbol{R}\Gamma(\Omega;G) \xrightarrow{\sim} \boldsymbol{R}\Gamma(Z;G).$$

Proof of Lemma. This lemma is proved in Yamazaki [Y]. For the sake of completeness of this paper, we reproduce the proof. Fix any $k \in \mathbb{Z}$. Then it is known that

$$\varinjlim_{V} H^k(V;G) \xrightarrow{\sim} H^k(Z;G),$$

where V ranges through the family of open neighborhoods of Z in Ω (cf. [K-S, Proposition 2.6.9]). Take every $u \in H^k(Z;G)$. Then there exists a neighborhood $\Omega' \subset \Omega$ of Z such that u is regarded as a section of $H^k(\Omega';G)$. Take any $p \in \Omega$. Set $U(p,\delta) := \{q \in \mathbb{R}^{n+1};\ |p-q| < \delta\}$. Then there exists a compact convex set $K \subset Z$ such that if a hyperplane with normal vector in A crosses the closure $\mathrm{Cl}\, U(p,\delta)$, then this hyperplane always crosses K. Take $p_0 \in K$ and choose sufficiently small $0<\varepsilon<\delta$. We set

$$\begin{aligned} p_t &:= (1-t)p_0 + tp, \\ K_\varepsilon &:= \{q \in \mathbb{R}^{n+1};\ \mathrm{dist}(q,K)<\varepsilon\} \subset \Omega', \\ S_t(p) &:= \mathrm{Int}\,\gamma[U(p_t,\varepsilon)\cup K_\varepsilon] \subset \Omega. \end{aligned}$$

We remark that $S_0(p) \subset \Omega'$. Further for any $q \in Z$, there exists a $p \in \Omega \setminus Z$ such that $q \in S_t(p)$ for any $0 \leqslant t \leqslant 1$ (we may choose that $p = q + c\zeta$ for suitable $c \in \mathbb{R}$ and $\zeta \in X \setminus Z$). Hence there exists a subset $\{p_i\}_{i\in I_0} \subset \Omega$ such that $\{S_t(p_i)\}_{i\in I_0}$ is an open covering of Z in Ω for any $0 \leqslant t \leqslant 1$. Now, instead of Zerner's theorem, we can use the theory of microsupports to prove:

$$\boldsymbol{R}\Gamma(S_1(p_i);G) \xrightarrow{\sim} \boldsymbol{R}\Gamma(S_0(p_i);G)$$

(see the proof of [B-S 1, Théorème 2.1] and [K-S, Lemma 5.2.2]). We refer this as a *sweeping out* method. Hence there exists a unique $u_i \in H^k(S_1(p_i);G)$ such that $u_i|_{S_0(p_i)} = u|_{S_0(p_i)}$. We shall prove

$$u_i|_{S_1(p_i)\cap S_1(p_j)} = u_j|_{S_1(p_i)\cap S_1(p_j)}.$$

We may show $u_i = u_j$ at any $p \in S_1(p_i)\cap S_1(p_j)$.

(1) Assume that $p \in S_0(p_i)\cap S_0(p_j)$. Then we see $u_i = u_j$ at p since

$$u_i|_{S_0(p_i)\cap S_0(p_j)} = u|_{S_0(p_i)\cap S_0(p_j)} = u_j|_{S_0(p_i)\cap S_0(p_j)}.$$

(2) Assume that $p \in S_0(p_i)\cap (S_1(p_j)\setminus S_0(p_j)) \subset \Omega'$. Then we can construct $S_t(p)$ such that

$$\begin{cases} S_t(p) \subset S_1(p_j)\cap \Omega' \text{ for } 0 \leqslant t \leqslant 1; \\ S_0(p) \subset S_0(p_j); \\ \qquad p \in S_1(p)\cap S_0(p_i). \end{cases}$$

Since $u_i|_{S_0(p_i)} = u|_{S_0(p_i)}$, we have $u_i|_{S_1(p)\cap S_0(p_i)} = u|_{S_1(p)\cap S_0(p_i)}$. On the other hand, we can find a unique $v \in H^k(S_1(p);G)$ such that $v|_{S_0(p)} = u|_{S_0(p)}$ by the sweeping out method. Using the uniqueness twice, we have $u_j|_{S_1(p)} = v = u|_{S_1(p)}$. Therefore we have $u_i = u = u_j$ at p.

(3) Assume that $p \in (S_1(p_i) \setminus S_0(p_i)) \cap (S_1(p_j) \setminus S_0(p_j))$. Then we can construct $S_t(p)$ as $S_t(p) \subset S_1(p_i) \cap S_1(p_j)$ for any $0 \leqslant t \leqslant 1$. By the uniqueness, we have $u_i\big|_{S_1(p)} = u_j\big|_{S_1(p)}$. Hence $u_i = u_j$ at p.

Thus there exists a unique $u' \in H^k(\bigcup_{i\in I_0} S_1(p_i); G)$ such that $u'\big|_{S_1(p_i)} = u_i$. In particular, $u' = u$ holds in $\bigcup_{p\in I_0} S_0(p_i)$. We set:

$$\begin{aligned}\Phi := \{ \bigcup_{j\in J} S_1(p_j);\ p_j \in \Omega,\ Z \subset \bigcup_{j\in J} S_1(p_j),\ &\text{there exists a unique} \\ v \in H^k(\bigcup_{j\in J} S_1(p_j); G) \text{ such that } v = u \text{ in } &\bigcup_{j\in J} S_0(p_j) \cap \Omega' \}.\end{aligned}$$

Then $\bigcup_{i\in I_0} S_1(p_i) \in \Phi$. Let $\{V_\alpha\}_{\alpha\in A}$ be a totally ordered set in Φ. There exists a unique $v_\alpha \in H^k(V_\alpha; G)$ such that $v\big|_{V_\alpha\cap\Omega'} = u\big|_{V_\alpha\cap\Omega'}$. Set $V := \bigcup_{\alpha\in A} V_\alpha$. Then we can write $V = \bigcup_{j\in J} S_1(p_j)$ for some $\{p_j\}_{j\in J} \subset \Omega$. Then by the uniqueness we can define a unique $v \in H^k(V; G)$ such that $v\big|_{V_\alpha} = v_\alpha$, hence $v = u$ holds in $\bigcup_{j\in J} S_0(p_j)$. Thus $V \in \Phi$.

By Zorn's lemma, there exists a maximal element $V = \bigcup_{i\in I} S_1(p_i) \in \Phi$. Thus there exists a unique $v \in H^k(V; G)$ such that $v = u$ holds in $\bigcup_{i\in I} S_0(p_i)$. Suppose that there exists a $p \in \Omega \setminus V$. By the sweeping out method, we can find a unique $u' \in H^k(S_1(p); G)$ such that $u'\big|_{S_0(p)} = u\big|_{S_0(p)}$. By the preceding argument, we have $u'\big|_{S_1(p_i)\cap S_1(p)} = v\big|_{S_1(p_i)\cap S_1(p)}$ for any $i \in I$. Therefore there exists a unique $v' \in H^k(V \cup S_1(p); G)$ such that $v'\big|_V = v$ and $v'\big|_{S_1(p)} = u'$. Therefore $V \subsetneq V \cup S_1(p) \in \Phi$, which is a contradiction. Thus $\Omega \in \Phi$. Hence there exists a unique $v \in H^k(\Omega; G)$ such that $v\big|_Z = u \in H^k(Z; G)$. Summing up, we obtain an isomorphism

$$H^k(\Omega; G) \simeq H^k(Z; G). \ \square$$

We shall prove Proposition 2.3. We have only to follow the argument in the proof of [B-S 1, Lemme 3.2] (cf. Tahara [T, Lemmata 2.1.1 and 2.1.2]) to obtain

$$H^k(M_{\eta,\varepsilon};F) \simeq \varinjlim_{U(a;\Gamma'_\varepsilon)} H^k(U(a;\Gamma'_\varepsilon);F).$$

Here

$$M_{\eta,\varepsilon} := \operatorname{Int}\gamma\big[(B'(0,a)+\sqrt{-1}\,\Gamma'_{\varepsilon/2})\cup\{(0,\pm\alpha\delta)+\sqrt{-1}\,\eta\}\big]$$

for an $\eta\in\Gamma'_{\varepsilon/4}$ and an independent constant $\alpha>0$. By the same argument as in the proof of Lemma 2.4, we have

$$H^k(\bigcup_{\eta\in\Gamma'_{\varepsilon/4}} M_{\eta,\varepsilon};F) \simeq \varinjlim_{U(a;\Gamma'_\varepsilon)} H^k(U(a;\Gamma'_\varepsilon);F).$$

We can find ε', $\delta>0$ and a conic neighborhood $\Gamma\subset\mathbb{R}^{n+1}$ of $(y_0,0)$ such that

$$K(a,\delta')+\sqrt{-1}\,\Gamma_{\varepsilon'}\subset\bigcup_{\eta\in\Gamma'_{\varepsilon/4}} M_{\eta,\varepsilon}\,.$$

The proof is complete. □

(b) Next, we shall prove that

$$\boldsymbol{R}f_{d!}\,f_\pi^{-1}\mu_M(F)_{p^*}\to\boldsymbol{R}f_{d!}\,f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+},F)_{p^*}$$

is an isomorphism.

Let $\nu_{\Omega_+}(F)$ be the inverse Fourier-Sato transform of $\mu hom(\mathbb{C}_{\Omega_+},F)$. By the same argument as in the proof of (a), we have only to prove that a natural morphism

$$\nu_M(F)\big|_{T_NY\cap\tau_N^{-1}(x_0)}\to\nu_{\Omega_+}(F)\big|_{T_NY\cap\tau_N^{-1}(x_0)}$$

is an isomorphism. To this end, for any $v\in TX$ and $k\in\mathbb{Z}$, we calculate

$$\mathcal{H}^k\nu_{\Omega_+}(F)_v=\varinjlim_V H^k(V;\nu_{\Omega_+}(F)),$$

where V ranges through the family of open conic neighborhoods of v such that $V^{\circ\circ}=\operatorname{Int}V$. Here V° denotes the dual cone. By Kashiwara-Schapira [K-S, Proposition 3.7.12], we have

$$H^k(V;\nu_{\Omega_+}(F))=H^{k+n+1}_{V^{\circ a}}(T^*X;\mu hom(\mathbb{C}_{\Omega_+},F)\otimes or_{M/X}).$$

Here $V^{\circ a} := \{\xi \in T^*X;\ -\xi \in V^\circ\}$, and $or_{M/X}$ denotes the *relative orientation sheaf.* By Schapira-Zampieri [Sc-Z, Theorem 1.1], it follows that

$$H^k(V;\nu_{\Omega_+}(F)) = \varinjlim_{U} H^k(U;F),$$

where U ranges through the family of open subsets of X such that

$$C(X \setminus U, M_+) \cap V = \emptyset.$$

Here $C(*,*)$ denotes the *normal cone* (see [K-S]). Therefore we have

$$H^k\nu_{\Omega_+}(F)_v = \varinjlim_{W} H^k(W;F),$$

where W ranges through the family of open subsets of X such that $v \notin C(X \setminus W, M_+)$. Take $v = (0,\eta_0) \in T_N Y \subset T_M X \subset TX$ arbitrary. Hence we have

$$\mathcal{H}^k\nu_{\Omega_+}(F)_v = \varinjlim_{a,\delta,\Gamma'_\varepsilon} H^k(D_+(a,\delta;\Gamma'_\varepsilon);F).$$

Here $\Gamma' \subset \mathbb{R}^n$ ranges through the family of open conic neighborhoods of η_0, and

$$D_+(a,\delta;\Gamma'_\varepsilon) := \{(z,\tau);\ (x,t) \in K(a,\delta),\ y \in \Gamma'_\varepsilon,\ \max\{-t,0\} + |s| < \varepsilon|y|\}.$$

For the same reasoning as in the proof of (a), we have

$$\mathcal{H}^k\nu_M(F)_v \simeq \mathcal{H}^k\nu_{\Omega_+}(F)_v\,. \ \square$$

3. Microfunction with a Real Analytic Parameter

Recall the diagram (1.2). As usual, we denote by $\mathcal{O}_X$, $\mathcal{A}_M := \mathcal{O}_X\big|_M$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaves of *holomorphic functions* on X, of *real analytic functions* on M, of *hyperfunctions* on M and of *microfunctions* on T^*_MX respectively. Although it is well-known that $f_{d!}\,f_\pi^{-1}\mathcal{C}_M$ is the sheaf of *microfunction with a real analytic parameter* t, and $f_{d!}\,f_\pi^{-1}\mathcal{C}_M\big|_N$ is the sheaf of *hyperfunction with a real analytic parameter* t (see Sato [S] and Sato-Kawai-Kashiwara [S-K-K]), we give a detailed exposition about these sheaves for the convenience of the reader.

Definition 3.1. We set

$$\mathscr{C}^A_{N|M} := \boldsymbol{R} f_{d!}\, f_\pi^{-1} \mathscr{C}_M\,, \quad \mathscr{B}^A_{N|M} := i_N^{-1} \mathscr{C}^A_{N|M} = \boldsymbol{R}\pi_{N*} \mathscr{C}^A_{N|M}\,.$$

Proposition 3.2 (cf. [S-K-K, Chapter I, Theorem 2.2.6]).

(1) $\mathscr{C}^A_{N|M}$ *is concentrated in degree zero and conically soft*; *that is, the direct image of* $\mathscr{C}^A_{N|M}\big|_{\dot{T}^*_N Y}$ *on* $\dot{T}^*_N Y/\mathbb{R}_{>0}$ *is a soft sheaf.*

(2) *There exists the following exact sequence*:

$$0 \to \mathscr{A}_M\big|_N \to \mathscr{B}^A_{N|M} \to \dot{\pi}_{N*}\mathscr{C}^A_{N|M} \to 0.$$

(3) *There exists the following exact sequence*:

$$0 \to \mathscr{B}^A_{N|M} \to \mathscr{B}_M\big|_N \to \dot{\pi}_*(\mathscr{C}_M\big|_{\sqrt{-1}\,\dot{T}^*_N M}) \to 0.$$

Proof. (1) Consider a distinguished triangle:

$$\boldsymbol{R}\Gamma_M(\mathscr{C}_M) \to \mathscr{C}_M \to \boldsymbol{R}\Gamma_{\dot{T}^*_M X}(\mathscr{C}_M) \xrightarrow{+1} \tag{3.1}$$

and apply the functor $\boldsymbol{R} f_{d!}\, f_\pi^{-1}$. Since $i_M \colon M \to T^*_M X$ and $i_N \colon N \to T^*_N Y$ are closed embeddings, we have

$$\begin{aligned} \boldsymbol{R} f_{d!}\, f_\pi^{-1} \boldsymbol{R}\Gamma_M(\mathscr{C}_M) &= \boldsymbol{R} f_{d!}\, f_\pi^{-1} \boldsymbol{R} i_{M!}\, \mathscr{A}_M \simeq \boldsymbol{R} i_{N!}\, f^{-1} \mathscr{A}_M \\ &= \boldsymbol{R} i_{N*}\, f^{-1}\mathscr{A}_M = i_{N*}\, f^{-1}\mathscr{A}_M\,. \end{aligned} \tag{3.2}$$

On the other hand, since $\boldsymbol{R}\Gamma_{\dot{T}^*_M X}(\mathscr{C}_M) = \Gamma_{\dot{T}^*_M X}(\mathscr{C}_M)$ is a conically flabby sheaf, $\boldsymbol{R} f_{d!}\, f_\pi^{-1} \boldsymbol{R}\Gamma_{\dot{T}^*_M X}(\mathscr{C}_M) = f_{d!}\, \dot{f}_\pi^{-1} \mathscr{C}_M$ and $f_{d!}\, \dot{f}_\pi^{-1} \mathscr{C}_M\big|_{\dot{T}^*_N Y}$ is conically soft. Here we set $\dot{f}_\pi \colon N \underset{M}{\times} \dot{T}^*_M X \to \dot{T}^*_M X$. Summing up, we obtain a distinguished triangle:

$$i_{N*}\, f^{-1}\mathscr{A}_M \to \mathscr{C}^A_{N|M} \to f_{d!}\, \dot{f}_\pi^{-1} \mathscr{C}_M \xrightarrow{+1} .$$

Hence $\mathscr{C}^A_{N|M}$ is concentrated by degree zero, and

$$\mathscr{C}^A_{N|M}\big|_{\dot{T}^*_N Y} = f_{d!}\, \dot{f}_\pi^{-1} \mathscr{C}_M\big|_{\dot{T}^*_N Y}$$

is conically soft.

(2) Consider Sato's distinguished triangle:

$$\boldsymbol{R}\pi_{N!}\mathscr{C}^A_{N|M} \to \boldsymbol{R}\pi_{N*}\mathscr{C}^A_{N|M} \to \boldsymbol{R}\dot{\pi}_{N*}\mathscr{C}^A_{N|M} \xrightarrow{+1} .$$

Then we have $\boldsymbol{R}\pi_{N*}\mathscr{C}^A_{N|M} = \mathscr{C}^A_{N|M}\big|_N = \mathscr{B}^A_{N|M}$, $\boldsymbol{R}\dot{\pi}_{N*}\mathscr{C}^A_{N|M} = \dot{\pi}_{N*}\mathscr{C}^A_{N|M}$ and

$$\begin{aligned}\boldsymbol{R}\pi_{N!}\mathscr{C}^A_{N|M} &= \boldsymbol{R}\pi_{N!}\,\boldsymbol{R}f_{d!}\,f_\pi^{-1}\mathscr{C}_M \simeq \boldsymbol{R}\pi_!\,f_\pi^{-1}\mathscr{C}_M \\ &\simeq f^{-1}\boldsymbol{R}\pi_{M!}\mathscr{C}_M \simeq \mathscr{A}_M\big|_N .\end{aligned}$$

(3) Set $V := N \underset{M}{\times} \dot{T}^*_M X \setminus \sqrt{-1}\,\dot{T}^*_N M$ for short. Consider a distinguished triangle:

$$(\dot{f}_\pi^{-1}\mathscr{C}_M)_V \to \dot{f}_\pi^{-1}\mathscr{C}_M \to (\dot{f}_\pi^{-1}\mathscr{C}_M)_{\sqrt{-1}\,\dot{T}^*_N M} \xrightarrow{+1} . \tag{3.3}$$

Applying the functor $\boldsymbol{R}f_{d!}$ to (3.3) we have

$$\boldsymbol{R}f_{d!}((\dot{f}_\pi^{-1}\mathscr{C}_M)_V) \to \boldsymbol{R}f_{d!}\,\dot{f}_\pi^{-1}\mathscr{C}_M \to \boldsymbol{R}f_{d!}\,(\dot{f}_\pi^{-1}\mathscr{C}_M)_{\sqrt{-1}\,\dot{T}^*_N M} \xrightarrow{+1} .$$

Since

$$\boldsymbol{R}f_{d!}\,(\dot{f}_\pi^{-1}\mathscr{C}_M)_{\sqrt{-1}\,\dot{T}^*_N M}\big|_{\dot{T}^*_N Y} = \boldsymbol{R}i_{N!}\boldsymbol{R}\dot{\pi}_!\,i^{-1}\dot{f}_\pi^{-1}\mathscr{C}_M\big|_{\dot{T}^*_N Y} = 0,$$

we have

$$\boldsymbol{R}f_{d!}((\dot{f}_\pi^{-1}\mathscr{C}_M)_V)\big|_{\dot{T}^*_N Y} \xrightarrow{\sim} \boldsymbol{R}f_{d!}\,\dot{f}_\pi^{-1}\mathscr{C}_M\big|_{\dot{T}^*_N Y} \simeq \mathscr{C}^A_{N|M}\big|_{\dot{T}^*_N Y} . \tag{3.4}$$

Next, applying the functor $\boldsymbol{R}f_{d*}$ to (3.3) we have

(3.5)
$$\boldsymbol{R}f_{d*}((\dot{f}_\pi^{-1}\mathscr{C}_M)_V) \to \boldsymbol{R}f_{d*}\,\dot{f}_\pi^{-1}\mathscr{C}_M \to \boldsymbol{R}i_{N*}\boldsymbol{R}\dot{\pi}_*(\mathscr{C}_M\big|_{\sqrt{-1}\,\dot{T}^*_N M}) \xrightarrow{+1} .$$

Since $f_d\colon V \to \dot{T}^*_N Y$ is proper, by (3.4) we have

$$\boldsymbol{R}f_{d*}((\dot{f}_\pi^{-1}\mathscr{C}_M)_V)\big|_{\dot{T}^*_N Y} \simeq \boldsymbol{R}f_{d!}((\dot{f}_\pi^{-1}\mathscr{C}_M)_V)\big|_{\dot{T}^*_N Y} \simeq \mathscr{C}^A_{N|M}\big|_{\dot{T}^*_N Y} .$$

On the other hand, from (3.1), we have

$$\begin{array}{ccc}
f^{-1}\boldsymbol{R}\pi_{M*}\boldsymbol{R}\Gamma_M(\mathscr{C}_M) & \longrightarrow & \boldsymbol{R}\pi_{N*}\boldsymbol{R}f_{d*}\,f_\pi^{-1}\boldsymbol{R}\Gamma_M(\mathscr{C}_M) \\
\downarrow & & \downarrow \\
f^{-1}\boldsymbol{R}\pi_{M*}\mathscr{C}_M & \longrightarrow & \boldsymbol{R}\pi_{N*}\boldsymbol{R}f_{d*}\,f_\pi^{-1}\mathscr{C}_M \\
\downarrow & & \downarrow \\
f^{-1}\boldsymbol{R}\pi_{M*}\boldsymbol{R}\Gamma_{\dot{T}^*_M X}(\mathscr{C}_M) & \longrightarrow & \boldsymbol{R}\pi_{N*}\boldsymbol{R}f_{d*}\,\dot{f}_\pi^{-1}\boldsymbol{R}\Gamma_{\dot{T}^*_M X}(\mathscr{C}_M) \\
\downarrow{\scriptstyle +1} & & \downarrow{\scriptstyle +1}
\end{array}$$

As in (3.2) we have

$$\boldsymbol{R}\pi_{N*}\boldsymbol{R}f_{d*}\,f_\pi^{-1}\boldsymbol{R}\Gamma_M(\mathscr{C}_M) = \mathscr{A}_M\big|_N$$

and by [K-S, Proposition 3.7.5]

$$\boldsymbol{R}\pi_{N*}\boldsymbol{R}f_{d*}\,f_\pi^{-1}\mathscr{C}_M = \boldsymbol{R}\pi_*\,f_\pi^{-1}\mathscr{C}_M = f_\pi^{-1}\mathscr{C}_M\big|_N = \mathscr{B}_M\big|_N\,.$$

Hence we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{A}_M\big|_N & \longrightarrow & \mathscr{B}_M\big|_N & \longrightarrow & \dot{\pi}_{M*}\mathscr{C}_M\big|_N & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathscr{A}_M\big|_N & \longrightarrow & \mathscr{B}_M\big|_N & \longrightarrow & \dot{\pi}_*\,\dot{f}_\pi^{-1}\mathscr{C}_M & \longrightarrow & 0
\end{array}$$

thus $\dot{\pi}_{M*}\mathscr{C}_M\big|_N = \dot{\pi}_*\,\dot{f}_\pi^{-1}\mathscr{C}_M$. Applying the functor $\boldsymbol{R}\dot{\pi}_{N*}$ to (3.5) we have

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathscr{A}_M\big|_N & \longrightarrow & \mathscr{B}^A_{N|M} & \longrightarrow & \dot{\pi}_{N*}\mathscr{C}^A_{N|M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{A}_M\big|_N & \longrightarrow & \mathscr{B}_M\big|_N & \longrightarrow & \dot{\pi}_*\,\dot{f}_\pi^{-1}\mathscr{C}_M & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & \dot{\pi}_*(\mathscr{C}_M\big|_{\sqrt{-1}\,\dot{T}^*_N M}) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Thus by Snake Lemma we obtain

$$0 \to \mathscr{B}^{A}_{N|M} \to \mathscr{B}_M\big|_N \to \dot{\pi}_*(\mathscr{C}_M\big|_{\sqrt{-1}\,\dot{T}^*_N M}) \to 0. \ \square$$

We recall that the morphism in (1.3) induces *restriction morphisms*:

$$\mathscr{C}^{A}_{N|M} \to \mathscr{C}_N\,, \quad \mathscr{B}^{A}_{N|M} \to \mathscr{B}_N\,.$$

In order to study microlocal boundary value problems, Kataoka [Kt] defined the sheaf $\overset{\circ}{\mathscr{C}}_{N|M_+}$ of *mild microfunctions* on $T^*_N Y$, and $\overset{\circ}{\mathscr{B}}_{N|M_+} := \overset{\circ}{\mathscr{C}}_{N|M_+}\big|_N$ is called the sheaf of *mild hyperfunctions.* Note that

$$\overset{\circ}{\mathscr{C}}_{N|M_+} = \boldsymbol{R}f_{d!}\, f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, \mathscr{O}_X) \otimes or_{M/X}[n+1]$$

holds by Schapira-Zampieri [Sc-Z]. $\overset{\circ}{\mathscr{C}}_{N|M_+}$ is conically soft, and there exists an exact sequence:

$$0 \to \mathscr{A}_M\big|_N \to \overset{\circ}{\mathscr{B}}_{N|M_+} \to \dot{\pi}_{N*}\overset{\circ}{\mathscr{C}}_{N|M_+} \to 0.$$

Further by (1.3), the restriction morphism $\mathscr{C}^{A}_{N|M} \to \mathscr{C}_N$ factorizes through the *boundary value morphism* $\overset{\circ}{\mathscr{C}}_{N|M_+} \to \mathscr{C}_N$:

$$\begin{array}{ccc} \mathscr{C}^{A}_{N|M} & \longrightarrow & \mathscr{C}_N \\ & \searrow \quad \nearrow & \\ & \overset{\circ}{\mathscr{C}}_{N|M_+} & \end{array}$$

4. Cauchy and Boundary Value Problems for Fuchsian Hyperbolic Systems

First, we recall the definition of Fuchsian differential operators in the sense of Baouendi-Goulaouic [B-G].

Definition 4.1. Let us take local coordinates in (1.1). Then we say that P is a *Fuchsian differential operator* of weight (k, m) in the sense of Baouendi-Goulaouic [B-G] if P can be written in the following form

$$P(z,\tau,\partial_z,\partial_\tau) = \tau^k\partial_\tau^m + \sum_{j=1}^{k} P_j(z,\tau,\partial_z)\,\tau^{k-j}\partial_\tau^{m-j} + \sum_{j=k+1}^{m} P_j(z,\tau,\partial_z)\,\partial_\tau^{m-j}.$$

Here $\operatorname{ord} P_j \leqslant j$ $(0 \leqslant j \leqslant m)$, and $P_j(z, 0, \partial_z) \in \mathcal{O}_Y$ $(1 \leqslant j \leqslant k)$.

Note that a Fuchsian differential operator of weight (m, m) is nothing but *an operator with regular singularity along Y in a weak sense* due to Kashiwara-Oshima [K-O].

Let $\mathcal{M}$ be a $\mathcal{D}_X$-Module. The *inverse image* in the sense of $\mathcal{D}$-Module is defined by

$$\boldsymbol{D}f^*\mathcal{M} := \mathcal{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{f^{-1}\mathcal{D}_X}{\otimes}} f^{-1}\mathcal{M} \in \operatorname{Ob} \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y).$$

Here $\mathcal{D}_{Y\to X} := \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{D}_X$ is the *transfer bi-Module* as usual. Further we set $\mathcal{M}_Y := \mathcal{H}^0 \boldsymbol{D}f^*\mathcal{M} = \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}\mathcal{M}$.

Next, let $\mathcal{M}$ be a *Fuchsian system along Y* in the sense of Laurent-Monteiro Fernandes [L-MF 1]. Since the precise definition of Fuchsian system is complicated, we do not recall it here. We remark that $\mathcal{M}$ is Fuchsian along Y if and only if there exists locally an epimorphism $\bigoplus_{i=1}^{m} \mathcal{D}_X / \mathcal{D}_X P_i \twoheadrightarrow \mathcal{M}$, where each differential operator P_i is an operator with regular singularity along Y in a weak sense.

Remark 4.2. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module for which Y is non-characteristic. Then $\mathcal{M}$ is Fuchsian. More generally, any regular-specializable system is Fuchsian.

(2) Let $\mathcal{M}$ be a Fuchsian system along Y. Then:

(i) By Laurent-Schapira [L-S, Théorème 3.3], all the cohomologies of $\boldsymbol{D}f^*\mathcal{M}$ are coherent $\mathcal{D}_Y$-Modules.

(ii) Laurent-Monteiro Fernandes [L-MF 1, Théorème 3.2.2] proved that there exists the following isomorphism (that is, Cauchy-Kovalevskaja type theorem):

$$f^{-1}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^*\mathcal{M}, \mathcal{O}_Y).$$

Definition 4.3. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$-Module. Then we say $\mathcal{M}$ is *near-hyperbolic* at $x_0 \in N$ in $\pm dt$ codirection if $\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is near-hyperbolic in the sense of Definition 2.1. We remark that

$$\operatorname{SS}\big(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\big) = \operatorname{char}(\mathcal{M}).$$

Here char($\mathcal{M}$) is the characteristic variety of $\mathcal{M}$.

Example 4.4. (1) Let P be a Fuchsian differential operator of weight (k, m) in the sense of Baouendi-Goulaouic [B-G]. Then $\mathcal{D}_X/\mathcal{D}_X P$ is Fuchsian along Y. Moreover, assume that P is Fuchsian hyperbolic in the sense of Tahara [T]; that is, the principal symbol is written as $\sigma_m(P)(z, \tau; z^*, \tau) = \tau^k p(z, \tau; z^*, \tau^*)$. Here $p(z, \tau; z^*, \tau^*)$ satisfies the following condition:

$$\text{(4.1)} \qquad \begin{cases} \text{If } (x, t; x^*) \text{ are real, then all the roots } \tau_j^*(x, t; x^*) \text{ of the} \\ \text{equation } p(x, t; x^*, \tau^*) = 0 \text{ with respect to } \tau^* \text{ are real.} \end{cases}$$

Then $\mathcal{D}_X/\mathcal{D}_X P$ is near-hyperbolic (see [L-MF 2, Lemma 1.3.2]).

(2) Let $P = \vartheta - A(z, \tau, \partial_z)$ be a *Fuchsian Volevič system* of size m due to Tahara [T]; that is,

(i) $A(z, \tau, \partial_z) = (A_{ij}(z, \tau, \partial_z))_{i,j=1}^m$ is a matrix of size m whose components are in $\mathcal{D}_X$ with $[A_{ij}, \tau] = 0$;

(ii) There exists $\{n_i\}_{i=1}^m \subset \mathbb{Z}$ such that $\operatorname{ord} A_{ij}(z, \tau, \partial_z) \leqslant n_i - n_j + 1$ and $A_{ij}(z, 0, \partial_z) \in \mathcal{O}_Y$ for any $1 \leqslant i, j \leqslant m$.

Set $\sigma(A)(z, \tau; z^*) := (\sigma_{n_i - n_j + 1}(A_{ij})(z, \tau; z^*))_{i,j=1}^m$. Then

$$\operatorname{char}(\mathcal{D}_X^m/\mathcal{D}_X^m P) = \{(z, \tau; z^*, \tau^*) \in T^*X;\ \det(\tau\tau^* - \sigma(A)(z, \tau; z^*)) = 0\},$$

and we can prove that $\mathcal{D}_X^m/\mathcal{D}_X^m P$ is Fuchsian along Y. Moreover assume that P is Fuchsian hyperbolic in the sense of Tahara [T]; that is,

$$\det(\tau\tau^* - \sigma(A)(z, \tau; z^*)) = \tau^m p(z, \tau; z^*, \tau^*),$$

and $p(z, \tau; z^*, \tau^*)$ satisfies the condition (4.1). Then $\mathcal{D}_X^m/\mathcal{D}_X^m P$ satisfies the near-hyperbolicity condition.

Our main Theorem is:

Theorem 4.5. *Let $\mathcal{M}$ be a Fuchsian system along Y. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$ in $\pm dt$-codirection. Then for any $p^* \in$*

$T^*_N Y \cap \pi_N^{-1}(x_0)$, *the morphisms in* (1.3) *induce isomorphisms*

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^A_{N|M})_{p^*} &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{C}}_{N|M_+})_{p^*} \\ &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^*\mathcal{M}, \mathcal{C}_N)_{p^*}\,.\end{aligned}$$

In particular, the morphisms in (1.3) *induce isomorphisms*

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}^A_{N|M})_{x_0} &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{B}}_{N|M_+})_{x_0} \\ &\simeq \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^*\mathcal{M}, \mathcal{B}_N)_{x_0}\,.\end{aligned}$$

Proof. Set $F := \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ for short. Then, we have

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^A_{N|M}) &= \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu_M(F) \otimes or_{M/X})[n+1]\,, \\ \boldsymbol{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{C}}_{N|M_+}) &= \boldsymbol{R}f_{d!}(f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, F) \otimes or_{M/X})[n+1]\,,\end{aligned}$$

and (see Remark 4.2):

$$\begin{aligned}\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^*\mathcal{M}, \mathcal{C}_N) &= \mu_N(\boldsymbol{R}\mathcal{H}om_{\mathcal{D}_Y}(\boldsymbol{D}f^*\mathcal{M}, \mathcal{O}_Y) \otimes or_{N/Y})[n] \\ &= \mu_N(f^{-1}F \otimes or_{N/Y})[n]\,.\end{aligned}$$

Therefore by Theorem 2.2, we obtain the theorem. □

Remark 4.6. Oaku-Yamazaki [O-Y] showed that for any Fuchsian system $\mathcal{M}$ along Y, two morphisms

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^A_{N|M})_{p^*} \rightarrowtail \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathring{\mathcal{C}}_{N|M_+})_{p^*} \rightarrowtail \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)_{p^*}$$

are always injective without the near-hyperbolicity condition. Precisely speaking, we always assumed that $\operatorname{codim}_M N \geqslant 2$ in [O-Y]. However, the same proof also works even in the case where $\operatorname{codim}_M N = 1$; Oaku ([O 1], [O 2]) defined the sheaf $\mathcal{C}^F_{N|M_+}$ of *F-mild microfunctions* on $T^*_N Y$, and $\mathcal{B}^F_{N|M_+} := \mathcal{C}^F_{N|M_+}|_N$ is called the sheaf of *F-mild hyperfunctions* ([O 1], [O 2], cf. [O-Y]). As is mentioned above, we can apply the methods in [O-Y] of the higher-codimensional case to the one-codimensional case to prove the following: there exist natural morphisms

$$\mathscr{C}^A_{N|M} \rightarrowtail \mathring{\mathscr{C}}_{N|M_+} \rightarrowtail \mathscr{C}^F_{N|M_+} \to \mathscr{C}_N$$

such that the composite coincides with $\mathscr{C}^A_{N|M} \to \mathscr{C}_N$, and that these induce monomorphisms:

$$\begin{array}{ccc} \mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{C}^A_{N|M}) & \rightarrowtail & \mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathring{\mathscr{C}}_{N|M_+}) \\ \downarrow & \circlearrowleft & \downarrow \\ \mathcal{H}om_{\mathscr{D}_Y}(\mathcal{M}_Y, \mathscr{C}_N) & \leftarrowtail & \mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{C}^F_{N|M_+}) \end{array}$$

for any Fuchsian system $\mathcal{M}$ along Y. Hence, under the near-hyperbolic condition, we obtain isomorphisms:

$$\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{C}^F_{N|M_+}) \simeq \mathcal{H}om_{\mathscr{D}_Y}(\mathcal{M}_Y, \mathscr{C}_N),$$
$$\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{B}^F_{N|M_+}) \simeq \mathcal{H}om_{\mathscr{D}_Y}(\mathcal{M}_Y, \mathscr{B}_N).$$

Our conjecture is: if $\mathcal{M}$ is a Fuchsian system along Y and satisfies near-hyperbolicity condition, then the following holds:

$$\boldsymbol{R}\mathcal{H}om_{\mathscr{D}_X}(\mathcal{M}, \mathscr{C}^F_{N|M_+}) \simeq \boldsymbol{R}\mathcal{H}om_{\mathscr{D}_Y}(\mathcal{M}_Y, \mathscr{C}_N).$$

References

[B-G] Baouendi, M. S. and C. Goulaouic, Cauchy problems with characteristic initial hypersurface, Comm. Pure Appl. Math. **26** (1973), 455–475.

[B-S 1] Bony, J.-M. et P. Schapira, Existence et prolongement des solutions holomorphes des équations aux dérivées partielles, Invent. Math. **17** (1972), 95–105.

[B-S 2] Bony, J.-M. et P. Schapira, Solutions hyperfonctions du problème de Cauchy, Hyperfunctions and Pseudo-Differential Equations, Proceedings Katata 1971 (H. Komatsu, ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 82–98.

[K-O] Kashiwara, M. and T. Oshima, Systems of differential equations with regular singularities and their boundary value problems, Ann. of Math. (2) **106** (1977), 145–200.

[K-S] Kashiwara, M. and P. Schapira, *Sheaves on Manifolds*, Grundlehren Math. Wiss. **292**, Springer, Berlin–Heidelberg–New York, 1990.

[Kt] Kataoka, K., Micro-local theory of boundary value problems, I–II, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **27** (1980), 355–399; *ibid.* **28** (1981), 31–56.

[L-MF 1] Laurent, Y. et T. Monteiro Fernandes, Systèmes différentiels fuchsiens le long d'une sous-variété, Publ. Res. Inst. Math. Sci. **24** (1981), 397–431.

[L-MF 2] Laurent, Y. et T. Monteiro Fernandes, Topological boundary values and regular $\mathcal{D}$-modules, Duke Math. J. **93** (1998), 207–230.

[L-S] Laurent, Y. et P. Schapira, Images inverse des modules différentiels, Compositio Math. **61** (1987), 229–251.

[O 1] Oaku, T., F-mild hyperfunctions and Fuchsian partial differential equations, *Group Representation and Systems of Differential Equations*, Proceedings Tokyo 1982 (K. Okamoto, ed.), Advanced Studies in Pure Math. **4**, Kinokuniya, Tokyo; North-Holland, Amsterdam–New York–Oxford, 1984, pp. 223–242.

[O 2] Oaku, T., Microlocal boundary value problem for Fuchsian operators, I, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **32** (1985), 287–317.

[O-Y] Oaku, T. and S. Yamazaki, Higher-codimensional boundary value problems and F-mild microfunctions, Publ. Res. Inst. Math. Sci. **34** (1998), 383–437.

[S] Sato, M., Theory of hyperfunctions. I–II, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **8** (1959–1960), 139–193 and 387–436.

[S-K-K] Sato, M., Kawai, T. and M. Kashiwara, Microfunctions and pseudo-differential equations, Hyperfunctions and Pseudo-Differential Equations, Proceedings Katata 1971 (H. Komatsu, ed.), Lecture Notes in Math. **287**, Springer, Berlin–Heidelberg–New York, 1973, pp. 265–529.

[Sc-Z] Schapira, P. and G. Zampieri, Microfunctions at the boundary and mild microfunctions, Publ. Res. Inst. Math. Sci. **24** (1988), 495–503.

[T] Tahara, H., Fuchsian type equations and Fuchsian hyperbolic equations, Japan J. Math. (N.S.) **5** (1979), 245–347.

[Y] Yamazaki, S., Microlocal boundary value problem for regular-specializable systems, J. Math. Soc. Japan **56**-4 (2004), 1109–1129.

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