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On the Twisted de Rham Cohomology Group of the General Hypergeometric Integral of Type $(q+1, 1^{N-q})$

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Abstract. The twisted de Rham cohomology groups associated with the general hypergeometric integrals of type $(q+1, 1^{N-q})$ are computed. A vanishing of the cohomology groups is proved and an explicit basis of the top cohomology group is given.

1. Introduction

For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of a positive integer $N+1 (\geq 2)$, we consider the general hypergeometric integrals which have the form

$$\int_\gamma e^{g(u)} \prod_{k=1}^{\ell} f_k(u)^{\alpha_k}\, du_1 \wedge \cdots \wedge du_n, \tag{1.1}$$

where f_k are linear polynomials of the coodinates $u = (u_1, \ldots, u_n)$ of $\mathbb{C}^n$, α_k are complex constants satisfying $\sum_k \alpha_k = -n-1$, g is a rational function of u having poles of order $\lambda_k - 1$ $(k = 1, \ldots, \ell)$ along the hyperplanes $f_k = 0$ $(k = 1, \ldots, \ell)$, respectively. When one of the linear polynomials, say f_i, is a constant polynomial, then we understand that the hyperplane $f_i = 0$ is a hyperplane at infinity $\mathbb{P}^n \setminus \mathbb{C}^n$. For the precise definition of the general hypergeometric integrals, see Section 2 and also [17].

The objective of this paper is to compute explicitly the twisted rational de Rham cohomology groups (Definition 3.2) associated with the general hypergeometric integral (1.1) for the partition $(q+1, 1^{N-q})$.

We shall explain how the general hypergeometric integrals (1.1) enter into our sight and why we are interested in computing the twisted de Rham cohomology groups associated with the general hypergeometric integrals.

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It is well known that the Gauss hypergeometric function (Gauss HGF)

$$F(a,b,c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du \tag{1.2}$$

and the special functions of confluent type derived from Gauss HGF play important and interesting roles in various fields of mathematics and mathematical physics. So it is very reasonable that interesting and natural generalizations of Gauss HGF and its confluent family to several variables were expected and pursued by many mathematicians. We refer the reader to [4], [7] for the classical works. Among the works about this problem in this two decades, those of K. Aomoto and of I. M. Gelfand are important and worthy of notice.

K. Aomoto [1], noticing that the integrand of Gauss HGF (1.2) is a product of complex powers of linear polynomials $u, 1-u$ and $1-xu$, he defined a generalization of Gauss HGF to be an integral of a power product of N linear polynomials of variables of integration $u=(u_1,\dots,u_n)$. His integral has the form (1.1) with $g=0$ and $f_1=1$ and therefore it correspond to the partition $\lambda=(1,\dots,1)$ of $N+1$. The integral thus obtained gives a function of coefficients of the linear polynomials.

In 1986, I.M. Gelfand [9] reformulated the work of Aomoto from the point of view of integral geometry; let G be the Cartan subgroup of $\mathrm{GL}(N+1;\mathbb{C})$ consisting of diagonal matrices and let $\chi:\tilde{G}\to\mathbb{C}^\times$ be a character of the universal convering group $\tilde{G}$ of Cartan subgroup; then the Radon transform of the character χ gives the Aomoto's generalization of HGF. So this generalization is sometimes called Aomoto-Gelfand hypergeometric function.

Gelfand's formulation has merits in several respects. One of the merits is that his formulation opened a way to extend the classical HGFs of confluent type of single variable, such as Kummer's confluent hypergeometric function, Bessel function, Hermite-Weber function and Airy function, to those of several variables. The idea is to replace in the above construction the Cartan subgroup G by maximal abelian subgroups of $\mathrm{GL}(N+1;\mathbb{C})$ obtained as centralizers of regular elements. Here the partitions λ of $N+1$ appears as data for indicating the conjugacy classes of centralizers of regular elements. Then we get the functions defined by the integrals of the form (1.1) which we call the *general hypergeometric function of type* λ, see [16], [17] for the details. Note that Gauss hypergeometric function, Kummer's confluent hypergeometric function, Bessel function, Hermite-Weber function and Airy function are obtained as the general hypergeometric functions of type $\lambda = (1,1,1,1), (2,1,1), (2,2), (3,1)$ and (4), respectively when $n = 1$. So the general hypergeometric function of type $(q+1, 1^{N-q})$ treated in this paper covers generalizations of Kummer, Hermite-Weber and Airy. We remark also that the general hypergeometric function of type $\lambda = (N+1)$ is defined and studied by Gelfand et al. and is called the *generalized Airy function* (see [10]).

Another merit of Gelfand's approach is that it enables us to write down a system of differential equations which characterizes the functions in a unified way not only for the Aomoto-Gelfand's HGF but also for the general hypergeometric functions of confluent type. He used the theory of Radon transform to obtain such system.

For the purpose of characterizing his HGF, Aomoto [3] used the different approach from that of Gelfand, namely he used the twisted de Rham theory and Gauss-Manin systems. The use of Gauss-Manin system has a merit of allowing us the detailed study of the properties of the system, the structure of singularity of the system, for example. See [11], [18] for other merits of the approach of Gauss-Manin system and of the de Rham theory.

To derive the Gauss-Manin system for the general hypergeometric functions of type λ, it is needed to compute the twisted de Rham cohomology groups which are defined by using the integrand of the general hypergeometric integral (see Definition 3.3 for the case treated in this paper).

In this paper we compute the twisted de Rham cohomology groups for the general hypergeometric integral for the partition $\lambda = (q+1, 1^{N-q})$. More explicitly

- we prove the vanishing of all the twisted de Rham cohomology groups except for the n-th one (Theorem 3.3),
- we give explicitly a basis of the n-th cohomology group (Theorem 3.4).

So far, the above results about the twisted de Rham cohomology groups for the general hypergeometric integrals have been obtained for the case of one dimensional integral [13] or for the cases of multiple integrals with the partition λ of $N+1$ is either $(1,\dots,1)$ [2] or $(N+1)$ [14], namely, the case of Aomoto-Gelfand hypergeometric integral or the case of generalized Airy integral.

It should be mentioned that there is a work of Aomoto, Kita, Orlik and Terao [5] about the twisted de Rham cohomology groups for an integral of the form (1.1) where the function g is a polynomial in u. They showed the vanishing of the twisted de Rham cohomology and gave a basis of the top de Rham cohomology group under the condition that the homogeneous part g_0 of the highest degree of g has isolated critical point at $u = 0$. This condition is crucial in their argument and results. But this condition is not satisfied for the generalized hypergeometric integral of type $(q+1, 1^{N-q})$ which we consider in this paper and their results cannot be applied to our case.

2. Hypergeometric Integral

Let $V = \mathbb{C}^n$ be the complex n-space with the affine coordinates $(u_1, \ldots, u_n)$ and let N be a positive integer such that $n < N$. For a partition $(q+1, 1^{N-q})$ of $N+1$, the general hypergeometric integral of type $(q+1, 1^{N-q})$ on the space V is defined as follows. Consider the $(N+1)$-dimensional maximal abelian subgroup G of $\mathrm{GL}(N+1, \mathbb{C})$ of the form

$$G = J(q+1) \times (\mathbb{C}^\times)^{N-q},$$

where

$$J(q+1) = \left\{ \sum_{i=0}^{q} h_i \Lambda^i \mid h_0 \neq 0, h_i \in \mathbb{C} \right\} \subset \mathrm{GL}(q+1; \mathbb{C})$$

is the Jordan group of size $q+1$,

$$\Lambda = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \in M(q+1; \mathbb{C})$$

being the shift matrix. Let $\chi : \tilde{G} \to \mathbb{C}^\times$ be a character of the universal covering group $\tilde{G}$ of G, namely a complex analytic homomorphism from the Lie group $\tilde{G}$ to the complex torus $\mathbb{C}^\times$. The characters can be described as follows. Let $\theta_i(h)$ be the functions defined by

$$\log(h_0 I + x_1\Lambda + \cdots + h_q\Lambda^q) = \sum_{i=0}^{q} \theta_i(h)\Lambda^i. \tag{2.1}$$

The explicit form of θ_i can be obtained by expanding the left-hand side of (2.3). In fact we have $\theta_0 = \log h_0$ and

$$\begin{aligned}\theta_m(h) = &\sum_{\substack{\lambda_1+2\lambda_2+\cdots+m\lambda_m=m\\ \lambda_1,\dots,\lambda_m\geq 0}} (-1)^{\lambda_1+\cdots+\lambda_m-1}\frac{(\lambda_1+\cdots+\lambda_m-1)!}{\lambda_1!\cdots\lambda_m!}\\ &\times\left(\frac{h_1}{h_0}\right)^{\lambda_1}\cdots\left(\frac{h_m}{h_0}\right)^{\lambda_m}.\end{aligned} \tag{2.2}$$

First few of them are

$$\begin{aligned}
\theta_0(h) &= \log h_0,\\
\theta_1(h) &= \frac{h_1}{h_0},\\
\theta_2(h) &= \frac{h_2}{h_0} - \frac{1}{2}\left(\frac{h_1}{h_0}\right)^2,\\
\theta_3(h) &= \frac{h_3}{h_0} - \left(\frac{h_1}{h_0}\right)\left(\frac{h_2}{h_0}\right) + \frac{1}{3}\left(\frac{h_1}{h_0}\right)^3,\\
\theta_4(h) &= \frac{h_4}{h_0} - \frac{1}{2}\left(\frac{h_2}{h_0}\right)^2 - \left(\frac{h_1}{h_0}\right)\left(\frac{h_3}{h_0}\right) + \left(\frac{h_1}{h_0}\right)^2\left(\frac{h_2}{h_0}\right) - \frac{1}{4}\left(\frac{h_1}{h_0}\right)^4.
\end{aligned}$$

Then the map $J(q+1) \to \mathbb{C}^\times \times \mathbb{C}^q$ defined by $\sum_{i=0}^q h_i\Lambda^i \mapsto (h_0, \theta_1(h), \dots, \theta_q(h))$ gives an isomorphism and it follows from this fact that any character χ is described as

$$\chi(h,\alpha) = \exp\left(\sum_{k=0}^{q} \alpha_k\theta_k(h_0,\dots,h_k)\right) h_{q+1}^{\alpha_{q+1}}\cdots h_N^{\alpha_N} \tag{2.3}$$

with some complex constants $\alpha = (\alpha_0, \dots, \alpha_N)$.

Let us define the hypergeometric integral as a Radon transform of a character χ. To this end we let $Z_{n+1,N+1}$ be the set of $(n+1)\times(N+1)$ complex matrices $z = (z_0, \dots, z_N)$ satisfying the conditions:

(1) $z_0 = {}^t(1, 0, \dots, 0)$,

(2) *for any integer* $0 \le k \le q$ *and any indices* $(i_1, \dots, i_{n-k})$ *such that* $q + 1 \le i_1 < \cdots < i_{n-k} \le N$, *we have*

$$\det(z_0, z_1, \dots, z_k, z_{i_1}, \dots, z_{i_{n-k}}) \ne 0.$$

For $z \in Z_{n+1,N+1}$, we define $N + 1$ linear polynomials of $u = (u_1, \dots, u_n)$ by

$$f_i = f_i(u, z) = z_{0,i} + z_{1,i}u_1 + \cdots + z_{n,i}u_n, \quad i = 0, \dots, N,$$

and a polynomial

$$g(u, z) = \sum_{k=1}^{q} \alpha_k \theta_k(f_0, \dots, f_q).$$

Since $f_0 = 1$ by the condition (1) for z, the explicit form (2.2) of the functions $\theta_m(h)$ says that $g(u, z)$ is a polynomial of degree q in u.

DEFINITION 2.1. Suppose that the parameters α of a character χ, given in (2.3), satisfy

$$\alpha_0 + \alpha_{q+1} + \cdots + \alpha_N = -n - 1 \quad \text{and} \quad \alpha_q \ne 0. \tag{2.4}$$

Then for $z \in Z_{n+1,N+1}$ we define the hypergeometric integral of type $(q + 1, 1^{N-q})$ by

$$F(z, \alpha) = \int_c \chi(\vec{u}z, \alpha) du = \int_c e^{g(u,z)} \prod_{i=q+1}^{N} f_i^{\alpha_i} du, \tag{2.5}$$

where $\vec{u} = (1, u_1, \dots, u_n)$, $du = du_1 \wedge \cdots \wedge du_n$ and c is an n-cycle of the homology group with coefficients in the local system on $\mathbb{C}^n \setminus \cup_{j=q+1}^{N} \{f_j = 0\}$ determined by the power product $\prod_{i=q+1}^{N} f_i^{\alpha_i}$ and with the family of supports determined by g. We used in (2.5) the convention of identifying $\vec{u}z$ with a group element $(\sum_{i=0}^{q} \vec{u}z_i\Lambda^i, \vec{u}z_{q+1}, \dots, \vec{u}z_N) \in G$.

REMARK 2.2. The condition (2) for $z \in Z_{n+1,N+1}$ assure that the system of differential equations, which characterizes the general HGF defined above, has no singularity on $Z_{n+1,N+1}$. This can be seen by calculating the characteristic variety for the system. The condition (1) is added for the simplicity of description as is seen from the discussion of the begining of Section 4.

Since we are concerned only with the cohomology group, detailed account of the homology group is omitted in this paper.

3. Theorems

When a matrix $z \in Z_{n+1,N+1}$ is given, we have an arrangement $\mathcal{A} = \{H_{q+1}, \ldots, H_N\}$ of hyperplanes in V defined by $H_i = \ker(f_i), (i = q+1, \ldots, N)$. Let $|\mathcal{A}|$ denote the number of hyperplanes in $\mathcal{A}$ and let $N(\mathcal{A})$ denotes the union of hyperplanes H_i in $\mathcal{A}$:

$$N(\mathcal{A}) = \bigcup_{i=q+1}^{N} H_i.$$

Note that the hyperplanes in the arrangement $\mathcal{A}$ are in general position by virtue of the condition (2) for $z \in Z_{n+1,N+1}$ and that the integrand of the hypergeometric integral for z is a multi-valued holomorphic function on $V \setminus N(\mathcal{A})$.

Let $\Omega^p[V]$ be the set of polynomial p-forms on V and let $\Omega^p(*\mathcal{A})$ be the set of rational p-forms having poles at most on $N(\mathcal{A})$.

Definition 3.1. A form $\eta \in \Omega^p(*\mathcal{A})$ is called a logarithmic p-form if only if

$$Q\eta \in \Omega^p[V] \quad \text{and} \quad Qd\eta \in \Omega^{p+1}[V],$$

where $Q = \prod_{i=q+1}^{N} f_i(u)$. The set of logarithmic p-forms is denoted by $\Omega^p(\mathcal{A})$.

Definition 3.2. Let $\nabla_g : \Omega^p(*\mathcal{A}) \to \Omega^{p+1}(*\mathcal{A})$ be the differential defined by

$$\nabla_g(\eta) = d\eta + \left(dg + \sum_{i=q+1}^{N} \alpha_i \frac{df_i}{f_i} \right) \wedge \eta. \tag{3.1}$$

Since ${\nabla_g}^2 = 0$, we get a complex $(\Omega^\bullet(*\mathcal{A}), \nabla_g)$. The cohomology groups of this complex are called the twisted rational de Rham cohomology groups for the hypergeometric integral (2.5).

The main results of this paper are the following two theorems.

THEOREM 3.3. *Suppose the condition*

$$\alpha_q \neq 0 \quad \textit{and} \quad \alpha_i \notin \mathbb{Z}, \quad i = q+1, \dots N \tag{3.2}$$

holds. Then

(1) the complex $(\Omega^\bullet(*\mathcal{A}), \nabla_g)$ *is pure, namely,*

$$H^p(\Omega^\bullet(*\mathcal{A}), \nabla_g) = 0 \quad \textit{for } p \neq n,$$

(2)

$$\dim_{\mathbb{C}} H^n(\Omega^\bullet(*\mathcal{A}), \nabla_g) = \binom{N-1}{n}.$$

Next theorem gives a basis of the n-th cohomology group $H^n(\Omega^\bullet(*\mathcal{A}), \nabla_g)$. To state the theorem we prepare some notation. Let $\mathcal{Y}(k, l)$ denote the set of Young diagrams contained in the $k \times l$ rectangular diagram, namely, a diagram λ belongs to $\mathcal{Y}(k, l)$ if and only if the length $|\lambda|$ of λ is not greater than k and the size of each row is not greater than l. Take an integer $0 \leq k \leq n$. For any pair (μ, I) of a diagram $\mu \in \mathcal{Y}(k, q-1-k)$ and indices $I = (i_1, \dots, i_{n-k})$ satisfying $q+1 \leq i_1 < \dots < i_{n-k} \leq N$, put

$$\omega_{\mu,I} := S_\mu(f_1, \dots, f_k, 0, \dots, 0) df_1 \wedge \dots \wedge df_k \wedge \frac{df_{i_1}}{f_{i_1}} \wedge \dots \wedge \frac{df_{i_{n-k}}}{f_{i_{n-k}}} \in \Omega^n(\mathcal{A}),$$

where $S_\mu(x)$ is a polynomial in $x = (x_1, \dots, x_n)$ such that $S_\mu(e(y))$ is the Schur polynomial indexed by the Young diagram μ, $e(y) = (e_1(y), \dots, e_n(y))$ being the elementary symmetric polynomials in $y = (y_1, \dots, y_n)$.

THEOREM 3.4. *The set of logarithmic n-forms*

$$\bigcup_{k=0}^{n} \left\{ \omega_{\mu,I} \;\middle|\; \begin{array}{l} \mu \in \mathcal{Y}(k, q-1-k), \\ I = (i_1, \dots, i_{n-k}) \textit{ such that } q+1 \leq i_1 < \dots < i_{n-k} \leq N \end{array} \right\}$$

provides a basis of $H^n(\Omega^\bullet(*\mathcal{A}), \nabla_g)$.

As is pointed out in [5] pp120, it is not hard to see that $\nabla_g(\Omega^p(\mathcal{A})) \subset \Omega^{p+1}(\mathcal{A})$. Therefore we can define the complex $(\Omega^\bullet(\mathcal{A}), \nabla_g)$ using logarithmic forms.

Proposition 3.5. *Let* $(\Omega^\bullet(*\mathcal{A}), \nabla_g)$ *and* $(\Omega^\bullet(\mathcal{A}), \nabla_g)$ *be as above and suppose that the condition (3.2) holds. Then the inclusion* $(\Omega^\bullet(\mathcal{A}), \nabla_g) \subset (\Omega^\bullet(*\mathcal{A}), \nabla_g)$ *is a quasi-isomorphism:*

$$H^p(\Omega^\bullet(\mathcal{A}), \nabla_g) \simeq H^p(\Omega^\bullet(*\mathcal{A}), \nabla_g), \quad (\forall p \geq 0) \tag{3.3}$$

Proof. Since we can prove the proposition in the same manner as that of Theorem 3.1 of [5], the detail is omitted. □

Thus, to compute the twisted rational de Rham cohomology groups, it is sufficient to compute the cohomology groups of the complex $C_{\nabla_g}(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), \nabla_g)$.

4. Filtration

Let K be the subgroup of $\mathrm{GL}(n+1) \times G$ consisting of the elements (a, g) such that it satisfies $azg \in Z_{n+1,N+1}$ for any element $z \in Z_{n+1,N+1}$. Thus the group K acts on $Z_{n+1,N+1}$. For $z \in Z_{n+1,N+1}$, we take $(a, 1_{N+1}) \in K$ such that

$$\tilde{z} = a^{-1}z = (1_{n+1}, \tilde{z}'').$$

Using this a, we define the bijective affine map $\rho_a : V \to V,\ u \mapsto \rho_a(u)$ by

$$(1, \rho_a(u)) = (1, u_1, \ldots, u_n)a.$$

Then the map ρ_a sends the polynomial g to the polynomial $\tilde{g}$ on the target space and the arrangement $\mathcal{A}$ to the arrangement $\tilde{\mathcal{A}}$ on the target space. The polynomial $\tilde{g}$ is determined by the first $q+1$ columns of $\tilde{z}$ and $\tilde{\mathcal{A}}$ is determined by the last $N-q$ columns of $\tilde{z}$. Since we have an isomorphism

$$\rho_a^* : H^p(\Omega^\bullet(\tilde{\mathcal{A}}), \nabla_{\tilde{g}}) \to H^p(\Omega^\bullet(\mathcal{A}), \nabla_g),$$

if we can show a property for the cohomology groups $H^p(\Omega^\bullet(\tilde{\mathcal{A}}), \nabla_{\tilde{g}})$, we can obtain the corresponding property for $H^p(\Omega^\bullet(\mathcal{A}), \nabla_g)$ via the isomorphism

ρ_a^*. Thus we may suppose, in what follows, that a matrix $z \in Z_{n+1,N+1}$ has the form $z = (1_{n+1}, z'')$. Note that, in this case, the polynomial g has the form

$$g(u) = \sum_{i=1}^{q} \alpha_i \theta_i(1, f_1(u), \dots, f_i(u))$$

with $f_i(u) = u_i$ for $1 \le i \le n$. Put also

$$\bar{g}(u) = \begin{cases} \alpha_q \theta_q(1, u_1, u_2, \dots u_n, 0, \dots 0), & q > n \\ \alpha_q \theta_q(1, u_1, u_2, \dots u_q), & q \le n \end{cases}$$

In addition to $C_{\nabla_g}(\mathcal{A})$, we also consider the Koszul complexes $C_g(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), dg\wedge)$ and $C_{\bar{g}}(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), d\bar{g}\wedge)$. The point in proving the theorems is to show that these three complexes are quasi-isomorphic:

$$H^p(C_{\nabla_g}(\mathcal{A})) \simeq H^p(C_g(\mathcal{A})) \simeq H^p(C_{\bar{g}}(\mathcal{A}))$$

and to reduce the proof to that for the Koszul complex $C_{\bar{g}}(\mathcal{A})$. We will prove these isomorphisms by introducing a filtration on $\Omega^\bullet(\mathcal{A})$ and by using the vanishing of cohomology and E_1-degeneracy of the spectral sequence of the introduced filtered complexes. See Section 6 for the detailed discussion.

Define the weights of the variables $u_1, \dots, u_n$ by $\mathrm{wt}(u_i) = i$. For a monomial p-form

$$\eta = u_1^{m_1} \cdots u_n^{m_n} du_{i_1} \wedge \cdots \wedge du_{i_p} \in \Omega^p[V]$$

we define the weight $\mathrm{wt}(\eta)$ of η by

$$\mathrm{wt}(\eta) = m_1 + 2m_2 + \cdots + nm_n + i_1 + \cdots + i_p.$$

For an element $\eta \in \Omega^p[V]$ not necessarily monomial form, we say $\mathrm{wt}(\eta) \le k$ if any monomial p-forms consisting of η are of weight less than or equal to k. We see that $\bar{g}(u)$ is the weighted homogeneous polynomial of weight q and

$$g(u) = \bar{g}(u) + (\text{monomials of weight less than } q). \tag{4.1}$$

Definition 4.1. For an element $\omega \in \Omega^p(\mathcal{A})$, we write $\omega = \eta/Q$ with $\eta \in \Omega^p[V]$ and $Q = \prod_{i=q+1}^{N} f_j$. We say

$$\mathrm{wt}(\omega) \leq r \quad \text{if} \quad \mathrm{wt}(\eta) \leq r + M,$$

where

$$M = \begin{cases} n(N-q) & q \geq n \\ (q+1) + \cdots + (n-1) + n(N-n+1) & q < n. \end{cases}$$

Put

$$\Omega^p(\mathcal{A})_{\leq r} = \{\omega \in \Omega^p(\mathcal{A}) \mid \mathrm{wt}(\omega) \leq r\}.$$

It will be better to notice that the coefficient of u_n in the linear polynomial $f_j(u)$ is not zero for $\max\{q, n\} + 1 \leq j \leq N$ by virtue of the condition (2) of $z \in Z_{n+1,N+1}$ and that $f_j(u) = u_j$ for $q+1 \leq j \leq n$ when $q < n$. So the polynomial $Q(u)$ has the highest weight term u_n^{N-q} of weight $n(N-q)$ in the case $q \geq n$ or the highest weight term $u_{q+1} \cdots u_{n-1} u_n^{N-n+1}$ in the case $q < n$.

Definition 4.2. Define an increasing filtration for the complex $C_{\nabla_g}(\mathcal{A})$ by

$$F_r C_{\nabla_g}(\mathcal{A}): \; 0 \to \Omega^0(\mathcal{A})_{\leq r-nq} \xrightarrow{\nabla_g} \cdots \xrightarrow{\nabla_g} \Omega^{n-1}(\mathcal{A})_{\leq r-q} \xrightarrow{\nabla_g} \Omega^n(\mathcal{A})_{\leq r} \to 0$$

and put

$$\begin{aligned} &\mathrm{Gr}_r C_{\nabla_g}(\mathcal{A}) = F_r C_{\nabla_g}(\mathcal{A}) / F_{r-1} C_{\nabla_g}(\mathcal{A}) \quad \text{and} \\ &\mathrm{Gr} C_{\nabla_g}(\mathcal{A}) = \oplus_{r \in \mathbb{Z}} \mathrm{Gr}_r C_{\nabla_g}(\mathcal{A}). \end{aligned}$$

Similarly we define

$$F_r C_{\bar{g}}(\mathcal{A}): \; 0 \to \Omega^0(\mathcal{A})_{\leq r-nq} \xrightarrow{d\bar{g}\wedge} \cdots \xrightarrow{d\bar{g}\wedge} \Omega^{n-1}(\mathcal{A})_{\leq r-q} \xrightarrow{d\bar{g}\wedge} \Omega^n(\mathcal{A})_{\leq r} \to 0.$$

and put

$$\mathrm{Gr}_r C_{\bar{g}}(\mathcal{A}) = F_r C_{\bar{g}}(\mathcal{A}) / F_{r-1} C_{\bar{g}}(\mathcal{A}) \quad \text{and} \quad \mathrm{Gr} C_{\bar{g}}(\mathcal{A}) = \oplus_{r \in \mathbb{Z}} \mathrm{Gr}_r C_{\bar{g}}(\mathcal{A}).$$

Lemma 4.3. *We have*

$$\mathrm{Gr}_r C_{\nabla_g}(\mathcal{A}) = \mathrm{Gr}_r C_{\bar{g}}(\mathcal{A}).$$

Proof. In the expression (3.1) of ∇_g, $\eta \mapsto d\eta$ and $\eta \mapsto \left(\sum_i \alpha_i \frac{df_i}{f_i}\right) \wedge \eta$ preserve the weight. Thus the map $\eta \mapsto \nabla_g(\eta)$ induces the map $\eta \mapsto d\bar{g} \wedge \eta$ on the quotient space $\Omega^p(\mathcal{A})_{\leq r-(n-p)q}/\Omega^p(\mathcal{A})_{\leq r-(n-p)q-1}$. This proves the lemma. □

Corollary 4.4. *We have*

$$H^p(\mathrm{Gr} C_{\nabla_g}(\mathcal{A})) \simeq H^p(\mathrm{Gr} C_{\bar{g}}(\mathcal{A})).$$

Remark 4.5. We will show that the spectral sequence for the filtered complex $F_r C_{\bar{g}}(\mathcal{A})$ degenerates at E_1-term. It follows from the generality of spectral sequence that we have

$$\mathrm{Gr} H^p(C_{\nabla_g}(\mathcal{A})) \simeq H^p(\mathrm{Gr} C_{\nabla_g}(\mathcal{A})) = H^p(\mathrm{Gr} C_{\bar{g}}(\mathcal{A})) \simeq \mathrm{Gr} H^p(C_{\bar{g}}(\mathcal{A})).$$

Thus the computation of the cohomology groups $H^p(C_{\nabla_g}(\mathcal{A}))$ is reduced to that for the Koszul complex $C_{\bar{g}}(\mathcal{A})$.

5. Residue Map

To compute the cohomology groups for the Koszul complex $C_{\bar{g}}(\mathcal{A})$, we use the induction on the number of hyperplanes $|\mathcal{A}|$ and on the dimension n of the space on which the hypergeometric integral is defined. The induction process will be enabled by the deletion-restriction method for arrangements, which we will explain in the following.

For the n-arrangement $\mathcal{A} = \{H_{q+1}, \dots, H_N\}$ with $H_i = \ker f_i$, we put

$$\mathcal{A}' = \{H_{q+1}, \dots, H_{N-1}\}, \quad \mathcal{A}'' = \mathcal{A}'|_{H_N} = \{H''_{q+1}, \dots, H''_{N-1}\}$$

where H''_i is the hyperplane in H_N obtained by restricting H_i to H_N. We will explain more in detail for the $(n-1)$-arrangement $\mathcal{A}''$. The hyperplanes H_j of the n-arrangement $\mathcal{A}$ are defined by the linear polynomial

$$f_j(u) = \vec{u} z_j = z_{0,j} + z_{1,j} u_1 + \cdots + z_{n,j} u_n.$$

As noted in the paragraph just after Definition 4.1, the coefficient $z_{n,j}$ of u_n in f_j is not zero for $\max\{q,n\}+1 \le i \le N$. In particular the coefficent $z_{n,N}$ of u_n of f_N is nonzero, so we can take $(u_1,\dots,u_{n-1})$ as the coordinates of H_N. Let f_j'' be the linear polynomials of $(u_1,\dots,u_{n-1})$ obtained from f_j by eliminating u_n by the condition $f_N=0$. Then $H_j''=\ker(f_j'')$. The coefficients of f_j'' $(0\le j\le N-1)$ are given by the $n\times N$ matrix

$$z''=\begin{pmatrix}1 & & & z''_{0,n} & \cdots & z''_{0,N-1}\\ & \ddots & & \vdots & & \vdots\\ & & 1 & z''_{n-1,n} & \cdots & z''_{n-1,N-1}\end{pmatrix}$$

where

$$z''_{i,j}=z_{i,j}-\frac{z_{n,j}}{z_{n,N}}z_{i,N}. \tag{5.1}$$

We will show that the matrix z'', which define the $(n-1)$-arrangement $\mathcal{A}''$, enjoys a similar condition as z.

LEMMA 5.1. *The matrix z'' defined above belongs to $Z_{n,N}$.*

PROOF. Take a pair (k,J) of an integer $1\le k\le n-1$ and indices $J=(j_1,\dots,j_k)$ of length k satisfying $q+1\le j_1<\dots<j_k\le N-1$. Then by using the relation (5.1), we have

$$\begin{aligned}&\det\begin{pmatrix}1 & & & z''_{0,j_1} & \cdots & z''_{n-k,j_k}\\ & \ddots & & \vdots & & \vdots\\ & & 1 & z''_{n-k-1,j_1} & \cdots & z''_{n-k-1,j_k}\\ & & & z''_{n-k,j_1} & \cdots & z''_{n-k,j_k}\\ & & & \vdots & & \vdots\\ & & & z''_{n-1,j_1} & \cdots & z''_{n-1,j_k}\end{pmatrix}\\
&=\det\begin{pmatrix}z''_{n-k,j_1} & \cdots & z''_{n-k,j_k}\\ \vdots & & \vdots\\ z''_{n-1,j_1} & \cdots & z''_{n-1,j_k}\end{pmatrix}\\
&=\det\begin{pmatrix}z_{n-k,j_1} & \cdots & z_{n-k,j_k}\\ \vdots & & \vdots\\ z_{n-1,j_1} & \cdots & z_{n-1,j_k}\end{pmatrix}\end{aligned}$$

$$+\sum_{p=1}^{k} -\frac{z_{n,j_p}}{z_{n,N}} \det \begin{pmatrix} z_{n-k,j_1} & \cdots & z_{n-k,N} & \cdots & z_{n-k,j_k} \\ \vdots & & \vdots & & \vdots \\ z_{n-1,j_1} & \cdots & z_{n-1,N} & \cdots & z_{n-1,j_k} \end{pmatrix}$$

$$= \frac{1}{z_{n,N}} \left\{ \sum_{p=1}^{k} (-1)^{k+1+p} z_{n,j_p} \right.$$

$$\times \det \begin{pmatrix} z_{n-k,j_1} & \cdots & z_{n-k,j_{p-1}} & z_{n-k,j_{p+1}} & \cdots & z_{n-k,j_k} & z_{n-k,N} \\ \vdots & & \vdots & \vdots & & \vdots & \\ z_{n-1,j_1} & \cdots & z_{n-1,j_{p-1}} & z_{n-1,j_{p+1}} & \cdots & z_{n-1,j_k} & z_{n-1,N} \end{pmatrix}$$

$$\left. +(-1)^{k+1+k+1} z_{n,N} \det \begin{pmatrix} z_{n-k,j_1} & \cdots & z_{n-k,j_k} \\ \vdots & & \vdots \\ z_{n-1,j_1} & \cdots & z_{n-1,j_k} \end{pmatrix} \right\}$$

$$= \frac{1}{z_{n,N}} \det \begin{pmatrix} z_{n-k,j_1} & \cdots & z_{n-k,j_k} & z_{n-k,N} \\ \vdots & & \vdots & \vdots \\ z_{n,j_1} & \cdots & z_{n,j_k} & z_{n,N} \end{pmatrix} \neq 0.$$

Nonvanishing of the last determinant is assured by the condition (2) for $z \in Z_{n+1,N+1}$. This proves the lemma. □

We recall the definition of the residue map

$$\mathrm{res} : \Omega^p(\mathcal{A}) \to \Omega^{p-1}(\mathcal{A}'').$$

Lemma 5.2 ([21]). *For $\omega \in \Omega^p(\mathcal{A})$, there exist $\omega' \in \Omega^{p-1}(*\mathcal{A})$ and $\omega'' \in \Omega^p(*\mathcal{A})$ such that*

(1) $\omega = \omega' \wedge \dfrac{df_N}{f_N} + \omega''$,

(2) neither ω' nor ω'' has a pole along H_N,

(3) $\omega'|_{H_N}$ belongs to $\Omega^{p-1}(\mathcal{A}'')$ and depends only on ω and H_N.

Definition 5.3. For $\omega \in \Omega^p(\mathcal{A})$, the $(p-1)$-form $\omega'|_{H_N} \in \Omega^{p-1}(\mathcal{A}'')$ is called the residue of ω along H_N and is denoted by $\mathrm{res}(\omega)$.

LEMMA 5.4 ([5] Corollary 6.7). *Let $n \geq 2$. Then the sequence*

$$0 \longrightarrow \Omega^p(\mathcal{A}') \xrightarrow{i} \Omega^p(\mathcal{A}) \xrightarrow{\mathrm{res}} \Omega^{p-1}(\mathcal{A}'') \to 0$$

is exact for $1 \leq p \leq n$, where $i : \Omega^p(\mathcal{A}') \to \Omega^p(\mathcal{A})$ is an inclusion.

LEMMA 5.5. *The following sequence is exact.*

$$0 \to \Omega^p(\mathcal{A}')_{\leq r} \xrightarrow{i} \Omega^p(\mathcal{A})_{\leq r} \xrightarrow{\mathrm{res}} \Omega^{p-1}(\mathcal{A}'')_{\leq r} \to 0.$$

PROOF. It is obvious that the map i is injective. We show that res is surjective. Take $\eta'' \in \Omega^{p-1}(\mathcal{A}'')_{\leq r}$. Since the arrangement $\mathcal{A}''$ is in general position, we can express η'' as

$$\eta'' = \sum \frac{df''_{i_1}}{f''_{i_1}} \wedge \cdots \wedge \frac{df''_{i_s}}{f''_{i_s}} \wedge \varphi_{i_1 \cdots i_s},$$

where $\varphi_{i_1 \cdots i_s}$ are polynomial $(p-s-1)$-forms in $(u_1, \ldots, u_{n-1})$ such that $\mathrm{wt}(\varphi_{i_1 \cdots i_s}) \leq r$. Put

$$\eta = \sum (-1)^{p-s-1} \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_s}}{f_{i_s}} \wedge \frac{df_N}{f_N} \wedge \varphi_{i_1 \cdots i_s}.$$

It is easy to see that $\eta \in \Omega^p(\mathcal{A})$ and $\mathrm{res}(\eta) = \eta''$. The exactness of the sequence at $\Omega^p(\mathcal{A})_{\leq r}$ is assured by Lemma 5.4. □

From Lemma 5.5, we have the following.

LEMMA 5.6. *The following sequence is exact.*

$$0 \to \mathrm{Gr}_r\Omega^p(\mathcal{A}') \xrightarrow{i} \mathrm{Gr}_r\Omega^p(\mathcal{A}) \xrightarrow{\mathrm{res}} \mathrm{Gr}_r\Omega^{p-1}(\mathcal{A}'') \to 0.$$

Using the above exact sequence, we construct an exact sequence of the Koszul complexes:

$$0 \longrightarrow \mathrm{Gr}C_{\bar{g}}(\mathcal{A}') \xrightarrow{i} \mathrm{Gr}C_{\bar{g}}(\mathcal{A}) \xrightarrow{j} \mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'') \longrightarrow 0. \tag{5.2}$$

For the complexes $C_{\bar{g}}(\mathcal{A}') = (\Omega^\bullet(\mathcal{A}'), d\bar{g}\wedge)$ and $C_{\bar{g}}(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), d\bar{g}\wedge)$, it is obvious that the diagram

$$\begin{array}{ccc}
\mathrm{Gr}_r\Omega^p(\mathcal{A}') & \xrightarrow{i} & \mathrm{Gr}_r\Omega^p(\mathcal{A}) \\
{\scriptstyle d\bar{g}\wedge}\Big\downarrow & & \Big\downarrow{\scriptstyle d\bar{g}\wedge} \\
\mathrm{Gr}_{r+q}\Omega^{p+1}(\mathcal{A}') & \xrightarrow{i} & \mathrm{Gr}_{r+q}\Omega^{p+1}(\mathcal{A})
\end{array}$$

is commutative. Thus the inclusion i defines a chain map $i : \mathrm{Gr}C_{\bar{g}}(\mathcal{A}') \longrightarrow \mathrm{Gr}C_{\bar{g}}(\mathcal{A})$. Since the coefficient of u_n in f_N is not zero, as is already mentioned, $(u_1, \dots, u_{n-1})$ can be taken as the coordinates of H_N. Put

$$\bar{g}'' = \begin{cases} \alpha_q\theta_q(1, u_1, \dots, u_{n-1}, 0, \dots, 0) & q > n-1, \\ \alpha_q\theta_q(1, u_1, \dots, u_q) & q \leq n-1. \end{cases}$$

Then if we restrict the polynomial function $\bar{g}$ to H_N, we have

$$\bar{g}|_{H_N} = \bar{g}'' + (\text{terms of lower weight}).$$

Put

$$j = (-1)^p \mathrm{res} : \mathrm{Gr}_r\Omega^p(\mathcal{A}) \to \mathrm{Gr}_r\Omega^{p-1}(\mathcal{A}'').$$

Then we have the commutative diagram:

$$\begin{array}{ccc}
\mathrm{Gr}_r\Omega^p(\mathcal{A}) & \xrightarrow{j} & \mathrm{Gr}_r\Omega^{p-1}(\mathcal{A}'') \\
{\scriptstyle d\bar{g}\wedge}\Big\downarrow & & \Big\downarrow{\scriptstyle d\bar{g}''\wedge} \\
\mathrm{Gr}_{r+q}\Omega^{p+1}(\mathcal{A}) & \xrightarrow{j} & \mathrm{Gr}_{r+q}\Omega^{p}(\mathcal{A}'').
\end{array}$$

Thus the map j defines a chain map $j : \mathrm{Gr}C_{\bar{g}}(\mathcal{A}) \longrightarrow \mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'')$. Now the Lemma 5.6 implies that the sequence (5.2) is exact.

6. Proof of the Theorems

6.1. Lemmas

We prove the Theorems 3.3 and 3.4 by induction on the number of hyperplanes $|\mathcal{A}|$ and the dimension n of the space on which the integrals are considered.

LEMMA 6.1. *The Milnor number $\mu_{\bar{g}}$ of the polynomial $\bar{g}(u)$ is*

$$\mu_{\bar{g}} = \begin{cases} \binom{q-1}{n}, & q \geq n+1, \\ 0 & q < n+1. \end{cases}$$

PROOF. Since $\bar{g}(u)$ is a weighted homogeneous polynomial of weight q with the weights $\mathrm{wt}(u_i) = i$. When $q \geq n+1$, it has the isolated singular point at $u = 0$ and using Cor.1 of [6], Chap. II, §12, we see that the Milnor number $\mu_{\bar{g}}$ is given by

$$\mu_{\bar{g}} = \prod_{i=1}^{n} \left(\frac{q}{i} - 1 \right) = \binom{q-1}{n}.$$

When $q < n+1$, the polynomial $\bar{g}(u)$ has no critical point. □

LEMMA 6.2. *When $|\mathcal{A}| = 0$, the complexes $C_{\bar{g}}(\mathcal{A})$ and $\mathrm{Gr}C_{\bar{g}}(\mathcal{A})$ are pure and*

$$\dim_{\mathbb{C}} H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) = \binom{N-1}{n}.$$

A basis of $H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A}))$ is given by

$$S_\mu(u_1, \ldots, u_n) du_1 \wedge \cdots \wedge du_n, \qquad (\mu \in \mathcal{Y}(n, N-n-1))$$

These forms provide also a basis of $H^n(C_{\bar{g}}(\mathcal{A}))$.

PROOF. This is already proved in [14] with the result on the cohomology ring of Grassmannian manifold ([8] pp27). See also [12]. □

The following lemma is proved in [13].

LEMMA 6.3. *When $n = 1$, the complex $C_{\bar{g}}(\mathcal{A})$ is pure and a basis of $H^1(C_{\bar{g}}(\mathcal{A}))$ is given by*

$$S_\mu(u) du, \ \frac{df_{q+1}}{f_{q+1}}, \ldots, \frac{df_N}{f_N}, \qquad (\mu \in \mathcal{Y}(1, q-2)).$$

LEMMA 6.4. *In the case $n = N$, we have*

$$H^p(C_{\bar{g}}(\mathcal{A})) = 0$$

for any $p \geq 0$.

PROOF. We may assume that $z \in Z_{n+1,n+1}$ has the form $z = 1_{n+1}$ and therefore

$$\bar{g}(u) = \alpha_q \theta_q(1, u_1, \dots, u_q)$$

and the n-arrangement $\mathcal{A} = \{H_{q+1}, \dots, H_n\}$ is given by

$$H_i = \ker(f_i), \quad f_i = u_i.$$

In this case the space V is written as $V = V_1 \times V_2$, where $V_1 = \mathbb{C}^q$ with the coordinates $(u_1, \dots, u_q)$ and $V_2 = \mathbb{C}^{n-q}$ with the coordinates $(u_{q+1}, \dots, u_n)$. We have the empty arrangement Φ in V_1 and an $(n-q)$-arrangement $\mathcal{A}_2 = \{H'_{q+1}, \dots, H'_n\}$ in V_2 defined by $H'_i = \{u_i = 0\} \subset V_2$, and $\mathcal{A} = \Phi \times \mathcal{A}_2$. Thus from [OT, Prop 4.8.4] we see that

$$\Omega^p(\mathcal{A}) \simeq \bigoplus_{k+l=p} \Omega^k[V_1] \otimes_{\mathbb{C}} \Omega^l(\mathcal{A}_2)$$

Therefore the complex $C_{\bar{g}}(\mathcal{A})$ becomes $(\Omega^\bullet[V_1] \otimes \Omega^\bullet(\mathcal{A}_2), (d\bar{g}\wedge) \otimes \mathrm{id}.)$, where $\bar{g}$ is considered as a polynomial on V_1. Since $\bar{g}$ has no critical point on V_1 by Lemma 6.1, the cohomology of $(\Omega^\bullet[V_1], d\bar{g}\wedge)$ is trivial by Lemma 6.2. Using the Künneth formula, we see that

$$H^p(C_{\bar{g}}(\mathcal{A})) = \sum_{k+l=p} H^k(\Omega^\bullet[V_1], d\bar{g}\wedge) \otimes \Omega^l(\mathcal{A}_2) = 0.$$

This completes the proof. □

6.2. Case $q > n$

LEMMA 6.5. *Let $q > n$. Then the complex $\mathrm{Gr} C_{\bar{g}}(\mathcal{A})$ is pure and*

$$\dim_{\mathbb{C}} H^n(\mathrm{Gr} C_{\bar{g}}(\mathcal{A})) = \binom{N-1}{n}.$$

Proof. We prove the lemma using the induction on the number $|\mathcal{A}|$ of hyperplanes of the arrangement $\mathcal{A}$. The case $|\mathcal{A}| = 0$ is already proved by Lemmas 6.2. We assume as the hypothesis of induction that the theorem is proved for the cases $|\mathcal{A}| < N - q$ and we shall prove the result for the case $|\mathcal{A}| = N - q$. From the exact sequence (5.2) of the complexes, we have the long exact sequence

$$\begin{aligned} &\to H^p(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A}')) \to H^p(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A})) \to H^{p-1}(\mathrm{Gr}\, C_{\bar{g}''}(\mathcal{A}'')) \to \\ &\to H^{p+1}(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A}')) \to \end{aligned} \tag{6.1}$$

By the induction hypothesis we have

$$\begin{aligned} H^p(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A}')) &= 0 \quad \text{for } p \neq n, \\ H^p(\mathrm{Gr}\, C_{\bar{g}''}(\mathcal{A}'')) &= 0 \quad \text{for } p \neq n-1. \end{aligned}$$

Thus $H^p(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A})) = 0$ for $p \neq n$ and therefore the complex $\mathrm{Gr} C_{\bar{g}}(\mathcal{A})$ is pure. The above exact sequence (6.1) for $p = n$ gives

$$0 \to H^n(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A}')) \to H^n(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A})) \to H^{n-1}(\mathrm{Gr}\, C_{\bar{g}''}(\mathcal{A}'')) \to 0. \tag{6.2}$$

Now from the induction hypothesis, we know that

$$\begin{aligned} \dim_{\mathbb{C}} H^n(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A})) &= \dim_{\mathbb{C}} H^n(\mathrm{Gr}\, C_{\bar{g}}(\mathcal{A}')) + \dim_{\mathbb{C}} H^{n-1}(\mathrm{Gr}\, C_{\bar{g}''}(\mathcal{A}'')) \\ &= \binom{N-2}{n} + \binom{N-2}{n-1} = \binom{N-1}{n}. \ \square \end{aligned}$$

Proof of Theorem 3.3. The purity of $C_{\bar{g}}(\mathcal{A})$ is proved as follows. Let $E_t^{r,s}$ be the spectral sequence associated with the filtered complex defined in Definition 4.2 which we denote by $(C_{\bar{g}}(\mathcal{A}), F)$. Then the cohomology groups $H^p(\mathrm{Gr}_r C_{\bar{g}}(\mathcal{A}))$ can be regarded as the E_1-terms of the spectral sequence:

$$E_1^{r,s}(C_{\bar{g}}(\mathcal{A}), F) = H^{r+s}(\mathrm{Gr}_{-r} C_{\bar{g}}(\mathcal{A})). \tag{6.3}$$

The filtration F on the complex $C_{\bar{g}}(\mathcal{A})$ induces an increasing filtration on the cohomology group $H^p(C_{\bar{g}}(\mathcal{A}))$ which is also denoted by F. Put

$$\mathrm{Gr}_r H^p(C_{\bar{g}}(\mathcal{A})) = F_r H^p(C_{\bar{g}}(\mathcal{A})) / F_{r-1} H^p(C_{\bar{g}}(\mathcal{A})).$$

This is equal to $E_\infty^{-r,p+r}$. Since $E_1^{r,s}=0$ for $r+s\neq n$ by Lemma 6.5 and by (6.3), the spectral sequence $E_t^{r,s}$ degenerates at E_1-terms. It follows that $E_\infty^{r,s}\simeq E_1^{r,s}$ for any $r,s\in\mathbb{Z}$ and

$$\mathrm{Gr}_r H^p(C_{\bar g}(\mathcal{A}))=E_\infty^{-r,p+r}\simeq E_1^{-r,p+r}=H^p(\mathrm{Gr}_r C_{\bar g}(\mathcal{A})).$$

Thus $\mathrm{Gr}H^p(C_{\bar g}(\mathcal{A}))=0$ for $p\neq n$ and

$$\mathrm{Gr}_r H^n(C_{\bar g}(\mathcal{A}))\simeq H^n(\mathrm{Gr}_r C_{\bar g}(\mathcal{A})). \tag{6.4}$$

This completes the proof of Theorem 3.3. □

PROOF OF THEOREM 3.4. We look at the exact sequence (6.2). From the induction hypothesis, we can take a basis of the group $H^n(\mathrm{Gr}C_{\bar g}(\mathcal{A}'))$ as

$$\bigcup_{k=0}^{n}\left\{\omega_{\mu,I}\ \middle|\ \begin{array}{l}\mu\in\mathcal{Y}(k,q-1-k),\\ I=(i_1,\dots,i_{n-k})\text{ such that }q+1\le i_1<\cdots<i_{n-k}\le N-1\end{array}\right\}. \tag{6.5}$$

By the same reason, we can take a basis of the group $H^{n-1}(\mathrm{Gr}C_{\bar g''}(\mathcal{A}''))$ consisting of the forms

(6.6)

$$\bigcup_{k=0}^{n-1}\left\{\omega''_{\nu,J}\ \middle|\ \begin{array}{l}\nu\in\mathcal{Y}(k,q-1-k),\\ J=(j_1,\dots,j_{n-k-1})\text{ such that }q+1\le j_1<\cdots<j_{n-k-1}\le N-1\end{array}\right\},$$

where

$$\omega''_{\nu,J}=S_\nu(u_1,\dots,u_k,0,\dots,0)du_1\wedge\cdots\wedge du_k\wedge\frac{df''_{j_1}}{f''_{j_1}}\wedge\cdots\wedge\frac{df''_{j_{n-k-1}}}{f''_{j_{n-k-1}}}$$

with f''_j being the restriction of f_j to the hyperplane H_N. Looking at the exact sequence (6.2), we see that the form $\omega''_{\nu,J}$ corresponds to $\omega_{\nu,(J,N)}\in\Omega^n(\mathcal{A})$ by the map j as

$$j(\omega_{\nu,(J,N)})=(-1)^n\omega''_{\nu,J}.$$

From the exact sequence (6.2), we conclude that the forms (6.5) and

$$\bigcup_{k=0}^{n-1}\left\{\omega_{\nu,(J,N)}\ \middle|\ \begin{array}{l}\nu\in\mathcal{Y}(k,q-1-k),\\ J=(j_1,\dots,j_{n-k-1})\text{ such that }q+1\le j_1<\cdots<j_{n-k-1}\le N-1\end{array}\right\},$$

provide a basis of the cohomology group $H^n(\mathrm{Gr}C_{\bar g}(\mathcal{A}))$. Then the theorem is derived by the isomorphism (6.4). This complete the proof of Theorem 3.4 in the case $q>n$. □

6.3. Case $q \le n$

By the same argument as in the begining of Section 4, we may assume that the matrix z has the form

$$
z = \left(\begin{array}{ccc|ccc|ccc}
1 & & & & & & z_{0,n+1} & \cdots & z_{0,N} \\
& \ddots & & & & & \vdots & & \vdots \\
& & 1 & & & & z_{q,n+1} & \cdots & z_{q,N} \\
\hline
& & & 1 & & & z_{q+1,n+1} & \cdots & z_{q+1,N} \\
& & & & \ddots & & \vdots & & \vdots \\
& & & & & 1 & z_{n,n+1} & \cdots & z_{n,N}
\end{array}\right) \tag{6.7}
$$

The condition (2) for $z \in Z_{n+1,N+1}$ reads that for any integer $0 \le k \le q$, any $(n-k)$-minor of

$$
\left(\begin{array}{ccc|ccc}
& & & z_{k+1,n+1} & \cdots & z_{k+1,N} \\
& & & \vdots & & \vdots \\
& & & z_{q,n+1} & \cdots & z_{q,N} \\
\hline
1 & & & z_{q+1,n+1} & \cdots & z_{q+1,N} \\
& \ddots & & \vdots & & \vdots \\
& & 1 & z_{n,n+1} & \cdots & z_{n,N}
\end{array}\right)
$$

does not vanish.

By using the induction argument as in the case $q > n$, we reduce the proof of the theorems to either of the cases:

(1) $n = 1$,

(2) $n \ge 2$ and $q > n$,

(3) $n \ge 2$ and $q \le n, n+1 = N$.

So the first step of the induction process is to prove the theorems in the above cases. Since the case (1) is O.K by Lemma 6.3 and the case (2) is already treated in Subsection 6.2, we will establish the result for the third case.

LEMMA 6.6. *In the case $n \ge 2, q \le n$ and $N = n+1$, the complexes $C_{\bar{g}}(\mathcal{A})$ and $\mathrm{Gr}C_{\bar{g}}(\mathcal{A})$ are pure and*

$$\dim_{\mathbb{C}} H^n(C_{\bar{g}}(\mathcal{A})) = \dim_{\mathbb{C}} H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) = 1.$$

A basis of the cohomology group $H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A}))$ *is given by*

$$du_1 \wedge \cdots \wedge du_{q-1} \wedge \frac{df_{q+1}}{f_{q+1}} \wedge \cdots \wedge \frac{df_N}{f_N}. \tag{6.8}$$

This form also provides a basis of $H^n(C_{\bar{g}}(\mathcal{A}))$.

PROOF. We may assume that z has the form

$$\begin{pmatrix} 1 & & & z_{0,n+1} \\ & \ddots & & \vdots \\ & & 1 & z_{n,n+1} \end{pmatrix} \in Z_{n+1,n+2}.$$

Note that $|\mathcal{A}| = n - q + 1$. For $0 \le p \le n$, we consider the p-th cohomology group. In the exact sequence (6.1) we have $H^k(\mathrm{Gr}C_{\bar{g}}(\mathcal{A}')) = 0$ $(k \in \mathbb{Z})$ by virtue of Lemma 6.4. It follows from (6.1) that

$$H^p(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) \simeq H^{p-1}(\mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'')). \tag{6.9}$$

Here the complex $\mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'')$ is determined by the matrix of the form

$$z'' = \left(\begin{array}{ccc|ccc|c} 1 & & & & & & z''_{0,n} \\ & \ddots & & & & & \vdots \\ & & 1 & & & & z''_{q,n} \\ \hline & & & 1 & & & z''_{q+1,n} \\ & & & & \ddots & & \vdots \\ & & & & & 1 & z''_{n-1,n} \end{array}\right) \in Z_{n,n+1}, \qquad z''_{i,n} = -\frac{z_{i,n+1}}{z_{n,n+1}}$$

in which the numbers of rows and columns are smaller than that of z by 1 and $|\mathcal{A}''| = n - q$. First we consider the case $p \le n - q$. If $p = 1$ we are done since the right hand side of (6.9), $H^0(\mathrm{Gr}C_{\bar{g}''}(\mathcal{A}''))$, vanishes from the definition of the cohomology group. If $p \ge 2$, we proceed in a similar way as above to get

$$H^{p-1}(\mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'')) \simeq H^{p-2}(\mathrm{Gr}C_{\bar{g}^{(3)}}(\mathcal{A}^{(3)})),$$

where $\mathcal{A}^{(3)}$ and $\bar{g}^{(3)}$ are respectively the $(n-2)$-arrangement and the polynomial of $(u_1, \ldots, u_{n-2})$ obtained from $\mathcal{A}''$ and $\bar{g}''$ by restricting them to the last hyperplane of $\mathcal{A}''$. Since we assumed $p \leq n-q$, we can proceed successively until we arrive at the situation where

$$H^p(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) \simeq H^{p-1}(\mathrm{Gr}C_{\bar{g}''}(\mathcal{A}'')) \simeq \cdots \simeq H^0(\mathrm{Gr}C_{\bar{g}^{(p+1)}}(\mathcal{A}^{(p+1)})) = 0.$$

Next we consider the case $n-q < p \leq n$. We apply the deletion-restriction method $n-q+1$ times and arrive at the situation

$$H^p(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) \simeq H^{p-(n-q+1)}(\mathrm{Gr}C_{\bar{g}^{(n-q+2)}}(\mathcal{A}^{(n-q+2)})) \tag{6.10}$$

where the complex $\mathrm{Gr}C_{\bar{g}^{(n-q+2)}}(\mathcal{A}^{(n-q+2)})$ is defined on $\mathbb{C}^{q-1}$ and is determined by the matrix

$$\tilde{z} = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{pmatrix} \in M(q, q+1; \mathbb{C}),$$

namely $\mathcal{A}^{(n-q+2)}$ is the empty arrangement and

$$\bar{g}^{(n-q+2)} = \alpha_q \theta_q(1, u_1, \ldots, u_{q-1}, 0).$$

So we can apply Lemma 6.2, to $\mathrm{Gr}C_{\bar{g}^{(n-q+2)}}(\mathcal{A}^{(n-q+2)})$. If $p < n$, then the right hand side of (6.10) vanishes since $p-(n-q+1) \leq q-1$. Thus we have established the purity of the complex $C_{\bar{g}}(\mathcal{A})$. If $p = n$, then

$$\dim_{\mathbb{C}} H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A})) = \dim_{\mathbb{C}} H^{q-1}(\mathrm{Gr}C_{\bar{g}^{(n-q+2)}}(\mathcal{A}^{(n-q+2)})) = 1,$$

since the Milnor number of the polynomial $\bar{g}^{(n-q+2)}(u_1, \ldots, u_{q-1})$ is equal to 1 by Lemma 6.1. Noting that the cohomology class of $H^{q-1}(\mathrm{Gr}C_{\bar{g}^{(n-q+2)}}(\mathcal{A}^{(n-q+2)}))$ is given by

$$du_1 \wedge \cdots \wedge du_{q-1}$$

and tracing back the isomorphism (6.10) for $p = n$, we get a basis of $H^n(\mathrm{Gr}C_{\bar{g}}(\mathcal{A}))$ given by the n-form (6.8). □

PROOF OF THEOREMS 3.3, 3.4 ($q \leq n$). We can prove the theorem in a similar way as in the case $q > n$ by induction on the number of hyperplanes in the arrangement and on the dimension of the space V. The details are omitted. □

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