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Gauge Theory and the A-Polynomial

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Abstract. In this article, we explain how to use instanton Floer homology of various Dehn surgeries along knots in integer homology spheres to prove that their A-polynomial is non-trivial. In particular, we show that all non-trivial knots in S^3 have non-trivial A-polynomial.

Not long ago, Kronheimer and Mrowka gave a gauge theoretic proof of Property P for knots in S^3. In fact, the proof found in [11] establishes much more: all $(1/n)$–surgeries along a non-trivial knot in S^3 admit a representation of their fundamental group in $SU(2)$ with non-abelian image. Their work has been used in [2] by Boyer and Zhang to show that the A–polynomial of any non-trivial knot in S^3 is non-trivial. In this short note we give a gauge-theoretic proof of this fact and various generalizations, the approach being closer in spirit to the results of Kronheimer and Mrowka contained in [11] and [12] since we use holonomy perturbations that naturally arise in gauge theory.

Let us begin by recalling a crucial step in the proof of Property P: one must establish that the 0–surgery of S^3 along K has non-vanishing Floer homology $HF_*(Y_0(K))$. These Floer groups are generated by non-degenerate flat connections on the $SO(3)$–bundle P over $Y_0(K)$ with non-trivial second Steifel-Whitney class. If the moduli space of flat connections $\mathcal{M}(Y_0(K))$ is degenerate, holonomy perturbations of the flatness equation are used to define the Floer chain groups. The explicit construction of Floer homology for $Y_0(K)$ is not important for our purpose, only matters the fact that if the moduli space $\mathcal{M}(Y_0(K))$ is empty, then $HF_*(Y_0(K))$ is trivial. The 0–surgery being obtained by Dehn filling the knot complement Y_K along the longitude λ_K, the moduli space $\mathcal{M}(Y_0(K))$ on P such that $w_2(P) \neq 0$ can be obtained from the moduli space of *irreducible* flat $SU(2)$–connections over the knot complement $\mathcal{M}^*(Y_K)$ by taking flat connections

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with holonomy $-I \in SU(2)$ along λ_K, see for example [3]. The result of Kronheimer and Mrowka implies the following:

Theorem 1. *Let K be a non-trivial knot in S^3. The moduli space of irreducible flat $SU(2)$–connections $\mathcal{M}^*(Y_K)$ is non-empty.*

We want to use this to say something about the A–polynomial of K. We first recall that $SU(2)$ being a subgroup of $SL(2,\mathbb{C})$, an $SU(2)$–representation naturally gives an $SL(2,\mathbb{C})$–representation. Moreover, it is well known that different $SU(2)$–characters yield different $SL(2,\mathbb{C})$–characters. For the construction of the A–polynomial, we refer the reader to [5]. The definition uses characters of $SL(2,\mathbb{C})$–representations of $\pi_1(Y_K)$ and a restriction map to $\mathbb{C}^* \times \mathbb{C}^*$ corresponding to characters of $SL(2,\mathbb{C})$–representations of $\pi_1(\partial Y_K)$. Showing that the A-polynomial is non-trivial amounts to showing that the image of the restriction map is of complex dimension 1 in $\mathbb{C}^* \times \mathbb{C}^*$ for some component in the character variety which contains an irreducible character, and that on such a curve the character of the knot longitude is not identically equal to 2. On the gauge-theoretic side, there is a corresponding restriction map for flat $SU(2)$–connections: $r\colon \mathcal{M}(Y_K) \to \mathcal{M}(\partial Y_K)$, commonly referred to as the pillow-case restriction. We prove the following about this restriction map:

Theorem 2. *Let K be a non-trivial knot in S^3. Then $\mathcal{M}^*(Y_K)$ contains an arc of irreducible connections $\{A_t\}$ that can be locally parametrized by the holonomy along the longitude λ_K.*

A short proof that the A-polynomial of K is non-trivial follows. The arc $\{A_t\} \subset \mathcal{M}^*(Y_K)$ of Theorem 2 yields, via the holonomy correspondance, an arc of irreducible $SU(2)$–characters with non-constant trace along the longitude of K. This arc therefore lies on a component X_0 of $SL(2,\mathbb{C})$–characters whose restriction to $\mathbb{C}^* \times \mathbb{C}^*$ is 1-dimensional and can be locally parametrized by the trace along the longitude. It follows directly that X_0 cannot contribute trivially to the A-polynomial of K.

The proof of Theorem 2 relies on the use of holonomy perturbations. These were first used by Floer to define his invariant in [8] and, later on, to prove the existence of a surgery exact sequence in [9]. An extension to knot complements was defined by Herald, and this is the version we use here. We give a brief summary and refer to [10] and [4] for details. Take a collection $\{\gamma_i\colon S^1 \times D^2 \to Y_K\}_{1\leq i\leq n}$ of embeddings of solid tori in Y_K whose images are disjoint and away from the boundary torus of Y_K. Let $\eta\colon D^2 \to \mathbb{R}$ be a bump function on D^2 and define a function h on the moduli space of $SU(2)$–connections by

$$h(A) = \sum_{i=1}^{n} \int_{D^2} h_i(\operatorname{tr}\operatorname{hol}_A(\gamma_i(S^1 \times \{x\})))\eta(x)dx^2,$$

where $\{h_i\colon \mathbb{R} \to \mathbb{R}\}_{1\leq i\leq n}$ is a collection of smooth functions. The function h is called an admissible perturbation function. We shall make use of the perturbed moduli space $\mathcal{M}_h(Y_K)$ of flat $SU(2)$–connections satisfying the equation $*F_A + \nabla h(A) = 0$. Herald proved in [10] that a generic holonomy perturbation h makes $\mathcal{M}^*(Y_K)$ into a smooth 1-manifold $\mathcal{M}^*_h(Y_K)$. This allows us to explicitely construct a perturbed moduli space $\mathcal{M}_h(Y_0(K))$, by considering elements in $\mathcal{M}_h(Y_K)$ that have holonomy equal to $-I$ along λ_K. Alternatively, we could construct, as in [12], the space $\mathcal{M}_h(Y_0(K))$ by considering elements in $\mathcal{M}^*(Y_K)$ satisfying a perturbed holonomy condition along the longitude λ_K. The two approaches are equivalent. Also note that, while large scale perturbations are needed to obtain a result like the Floer surgery exact sequence, here we only need (small) holonomy perturbations that change the Floer chain groups but not the Floer homology groups.

Proof of Theorem 2. Consider $\mathcal{M}(Y_0(K))$ as a subset of $\mathcal{M}^*(Y_K)$. We first claim that some element in $\mathcal{M}(Y_0(K))$ lies on an arc $\{A_t\}$ in $\mathcal{M}^*(Y_K)$. Otherwise, since $\mathcal{M}^*(Y_K)$ is a compact real algebraic set, any element $A \in \mathcal{M}(Y_0(K)) \cap \mathcal{M}^*(Y_K)$ is isolated. Because A is isolated we can choose our generic holonomy perturbation h such that for any $A_h \in \mathcal{M}^*_h(Y_K)$, the holonomy of A_h along the longitude λ_K is different from $-I$. This means that we have a generic holonomy perturbation for the 3-manifold $Y_0(K)$ for which the perturbed Floer chain complex is empty, ie we have pushed off A from the top line in the pillow-case. It follows that $HF_*(Y_0(K))$ is trivial, since it is invariant under admissible perturbations, which contradicts Kronheimer and Mrowka [11]. The rest of the proof is very similar to the above. Suppose now that no arc $\{A_t\}$ constructed above in $\mathcal{M}^*(Y_K)$ admits a local parametrization by holonomy along λ_K. In particular, this means that $\{A_t\} \subset \mathcal{M}(Y_0(K))$. Now perturb exactly as above to exhibit an empty perturbed Floer chain complex, giving $HF_*(Y_0(K)) = 0$, again a contradiction. □

The method readily generalizes beyond the case of knots in S^3. Indeed, for knots in integer homology spheres, the Floer homology of the 0-surgery is defined and we obtain a criterion for the non-triviality of the A-polynomial.

THEOREM 3. *Let K be a knot in an integer homology sphere Y^3. Suppose that the Floer homology of the 0-surgery of Y^3 along K is non-vanishing. Then the A-polynomial of K is non-trivial.*

In our construction, there is nothing special about the longitudinal Dehn filling other than the fact that we know (for knots in S^3) the Floer homology of this 3-manifold to be non-trivial. We can use holonomy perturbations for any other filling, and this gives:

THEOREM 4. *Let K be a knot in a homology sphere Y^3 such that for some $r \in \{\infty, 0\} \cup \{1/k \mid k \in \mathbb{Z}^*\}$ the Dehn filling $Y_r(K)$ has non-trivial Floer homology. Then there exists an arc $\{A_t\} \subset \mathcal{M}^*(Y_K)$ locally parametrized by the holonomy along a peripheral element in $\pi_1(Y_K)$.*

Theorem 4 does not quite imply that the A-polynomial is non-trivial. We obtain existence of deformations of irreducible $SL(2,\mathbb{C})$–characters into a 1-dimensional family of irreducibles, but the A-polynomial could still be trivial. This would happen if all the irreducible characters for the knot complement Y_K send the longitude λ_K to $I \in SL(2,\mathbb{C})$, as happens in the example below. From the gauge theory side, this situation illustrates how much information can be lost by restriction to the pillow-case.

Example 1. Take any integer homology sphere Y^3 with $HF_*(Y)$ non-trivial, and consider an unknotted curve K contained in a small 3-ball in Y^3. By construction, $\mathcal{M}^*(Y)$ is non-empty and the knot complement Y_K has fundamental group $\pi_1(Y_K) = \pi_1(Y) * \mathbb{Z}$. It is then clear that $\mathcal{M}^*(Y_K)$ contains at least one arc of irreducible flat connection parametrized by the holonomy along the meridian μ_K. The 0-surgery will be $Y \sharp S^1 \times S^2$ and hence $HF_*(Y_0(K))$ is trivial. Because the longitude λ_K is trivial in $\pi_1(Y_K)$, it follows directly that the A-polynomial is trivial.

In the case of a knot K in S^3, the non-triviality of $HF_*(Y_0(K))$ is equivalent to the non-triviality of the A-polynomial of K. For knots in arbitrary integer homology spheres, we can use Theorem 4 to see that the non-vanishing of $HF_*(Y_0(K))$ is not a necessary condition for the A-polynomial of K to be non-trivial. Various examples can be constructed using generalized Mazur manifolds. These are contractible 4-manifolds $W^{\pm}(l,k)$ for $k,l \in \mathbb{Z}$ whose boundaries are integer homology spheres $\partial W^{\pm}(l,k)$. We refer the reader to [1] for the construction, as we give below two examples of knots in Mazur homology spheres whose A-polynomial is non-trivial but for which $HF_*(Y_0(K)) = 0$.

Example 2. Consider Mazur's original manifold $W^+(0,0)$ whose boundary $\partial W^+(0,0)$ is the Brieskorn homology sphere $\Sigma(2,3,7)$. The Floer homology of Brieskorn manifolds was explicitely computed in [7], but all we need here is that $HF_*(\Sigma(2,3,7))$ is non-vanishing. Let K be a knot in $\partial W^+(0,0)$ given as a small linking circle about the 1-handle used in the construction of $W^+(0,0)$. Performing a 0-surgery along K clearly gives $S^1 \times S^2$, a 3-manifold over which there are no irreducible $SO(3)$–connections, so that $HF_*(Y_0(K)) = 0$. Now apply Theorem 4 to the filling $Y_\infty(K) = \Sigma(2,3,7)$ to construct an arc $\{A_t\} \subset \mathcal{M}^*(Y_K)$. To conclude that the A-polynomial of K is non-trivial, we simply need to show, moreover, that $\{A_t\}$ satisfies $\mathrm{hol}_{A_t}(\lambda_K) \neq I$. But this is immediate as otherwise the arc $\{A_t\}$ would provide irreducible flat $SO(3)$–connections over Y_K which extend to $Y_0(K) = S^1 \times S^2$, a contradiction.

Example 3. In $\partial W^+(2,0)$ let K be the knot given by a linking circle along which the $(+1)$-surgery corresponds to a crossing change between the two links in the Kirby diagram of $W^+(2,0)$. This $(+1)$-surgery along K is obtained by blowing down K and the 1-handle, therefore $Y_{+1}(K)$ can also be seen as the result of a $(+1)$-surgery along some non-trivial knot in S^3. By [11] we know that $HF_*(Y_{+1}(K))$ is non-vanishing. Also it is an easy exercice to see that $Y_0(K)$ is again $S^1 \times S^2$. As in Example 2, Theorem 4 therefore enables us to conclude that the A-polynomial of K is non-trivial although $HF_*(Y_0(K)) = 0$.

The following seems likely to be a difficult question: is it possible to find knots in integer homology spheres whose A-polynomial is non-trivial but such that the Floer homology groups $HF_*(Y_r(K))$ vanish for all $r \in \{\infty, 0\} \cup \{1/k \mid k \in \mathbb{Z}^*\}$?

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