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Quasi-unipotent Logarithmic Riemann-Hilbert Correspondences

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Abstract. We generalize the logarithmic version of the Riemann-Hilbert correspondence defined in [KtNk] to local systems with quasi-unipotent local monodromies by working with a certain Grothendieck topology. We also discuss its behavior with respect to direct images and give applications to nearby cycles and the degeneration of relative log Hodge to log de Rham spectral sequences.

Introduction

Let X be a proper and smooth scheme over $\mathbb{C}$, and let $U \subset X$ be the complement of a divisor with normal crossings D on X. In its simplest form, the Riemann-Hilbert correspondence sets up an equivalence between the category of local systems of $\mathbb{C}$-vector spaces of finite dimension on the

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analytic space U_{an} associated to U and the category of vector bundles on U equipped with an integrable connection ∇ regular along D (i. e. extending to a vector bundle on X equipped with an integrable connection with log poles along D) (see Deligne ([D2], II 5.9) ; the (now common) terminology "Riemann-Hilbert correspondence", which does not appear in (loc. cit.), was introduced later). Under this equivalence unipotent local systems (in what follows we will say "local systems" for "local systems of $\mathbb{C}$-vector spaces of finite dimension") on U_{an} correspond to connections with log poles whose residues along the branches of D are nilpotent. M. Saito's theory of mixed Hodge modules ([Sai1], [Sai2]) provides far reaching amplifications and generalizations of this correspondence. On the other hand, in [KtNk] Kato and Nakayama presented a generalization of a more modest but rather different nature. They define an extension of the correspondence to certain log schemes over $\mathbb{C}$. For simplicity, consider a proper and log smooth log scheme X over $\mathbb{C}$ (a scheme X as above, with the log structure given by D, is an example). Let U be the largest open subset of X where the log structure is trivial (in the above example, this is $X - D$). Then, in [KtNk] there is constructed a "log Riemann-Hilbert" equivalence Φ between the category of unipotent local systems on U_{an} and the category of vector bundles on X, equipped with an integrable connection (in the log sense) with nilpotent residues. This equivalence involves passing through a certain topological space $X^{\log}$ mapping to X_{an} by a proper map τ, which is a homeomorphism over U_{an} (more generally with fiber $\tau^{-1}(x)$ homeomorphic to $(S^1)^r$ if r is the rank of the log structure at x), and such that the corresponding open inclusion of U_{an} into $X^{\log}$ is (locally at the boundary) a homotopy equivalence (in particular, local systems on U_{an} extend uniquely to local systems on $X^{\log}$). This space $X^{\log}$ is defined more generally for any fs log analytic space X and comes equipped with a sheaf of rings $\mathcal{O}_X^{\log}$, generated by analytic functions on X and logarithms of sections of M^{gp}. The purpose of the present paper is to address two questions left open in (loc. cit.) : (1) how does the log Riemann-Hilbert equivalence Φ extend to *quasi-unipotent* local systems ? (2) how does it behave with respect to maps $X \to Y$?

The answer to question (1) is rather easy : Φ extends to an equivalence between the category of quasi-unipotent local systems on U_{an} and the category of vector bundles on the "Kummer étale ringed site" X^{ket} of X endowed with an integrable connection satisfying a condition of nilpotence of the residues on this site ; the Kummer site is generated by Kummer log étale maps of target X (whose prototype is $\mathbb{A}^1 \to \mathbb{A}^1, t \mapsto t^n$) ; see Theorem 4.4 for a precise (and more general) statement (formulated in the analytic, rather than algebraic context). As a byproduct we obtain a (perhaps more canonical) description of the so-called "canonical extension" ([D2] II, 5.5). Let us also mention another generalization of Φ due to Ogus [O2], which uses a "Lorenzon algebra" instead of the Kummer site, and works for local systems with arbitrary (i. e. non necessarily quasi-unipotent) monodromies.

Question (2) is more delicate, and we have only partial answers. Our main results (6.2, 6.3, 6.4) concern a proper, separated, log smooth morphism $f : X \to Y$ of fs log analytic spaces. We have to assume in addition that f is exact (roughly speaking, does not involve any log blow-up) or that Y is log smooth. For simplicity, let us consider here this second case (so that X is also log smooth). Let L be a quasi-unipotent local system on the open subset of triviality U of the log structure of X (which is contained in but can be strictly smaller than the inverse image by f of the open subset of triviality V of the log structure of Y) and let E be the vector bundle with connection associated with it via the log Riemann-Hilbert correspondence. Then, for all q, $R^q f_* L$ is a quasi-unipotent local system on V, and the vector bundle associated with it via the log Riemann-Hilbert correspondence is $R^q f_*^{\text{ket}} \omega^{\cdot,\text{ket}}_{X/Y}(E)$ endowed with the Gauss-Manin connection (here $f^{\text{ket}} : X^{\text{ket}} \to Y^{\text{ket}}$ is the extension of f to the Kummer étale sites, and $\omega^{\cdot,\text{ket}}_{X/Y}(E)$ the corresponding relative de Rham complex of E).

Let us now briefly describe the contents of the paper. In Section 1 we review the definition of $X^{\log}$ and the log Riemann-Hilbert correspondence of [KtNk]. The Kummer étale (ket, for short) sites and the extended ring $\mathcal{O}_X^{\text{klog}}$ on $X^{\log}$ are defined in sections 2 and 3. The log Riemann-Hilbert correspondence in the quasi-unipotent case is constructed in Section 4, in a way quite analogous to the unipotent case of (loc. cit.). The main tool for the functoriality results sketched above is a "log Poincaré lemma" for log smooth maps in the ket context, established in Section 5. It generalizes results of F. Kato [KtF], Matsubara [M] and Ogus [O1], [O2]. The functoriality results are then stated and proven in Section 6, using the log Poincaré lemma and a key result on higher direct images of local systems, due to Kajiwara-Nakayama [KjNk].

Sections 7 and 8 present some natural complements to the previous results. Section 7 is of a more algebraic flavor. For a proper and exact log smooth map $f : X \to Y$ of fs log schemes of finite type over $\mathbb{C}$, we know, by a special case of the main theorems of Section 6 (for L the constant sheaf $\mathbb{C}$), that the higher direct images $R^n f_*^{\mathrm{ket}} \omega_{X/Y}^{\cdot,\mathrm{ket}}$ are locally free of finite type on Y^{ket}. We show that the relative Hodge to de Rham spectral sequence

$$E_1^{pq} = R^q f_*^{\mathrm{ket}} \omega_{X/Y}^{p,\mathrm{ket}} \Rightarrow R^{p+q} f_*^{\mathrm{ket}} \omega_{X/Y}^{\cdot,\mathrm{ket}}$$

degenerates at E_1 and that its E_1 terms consists of vector bundles on Y^{ket} as well. This generalizes results of Deligne [D1], Steenbrink [Ste2], Illusie [I1], Cailotto [C], and Fujisawa [Fjs1]. The proof combines some results of Section 6 (to reduce to the case where the base Y is a reduced point) with the standard mod p^2 techniques of Deligne-Illusie [DI]. One basic theorem of Steenbrink [Ste1] is that for a map $X \to Y$, where Y is the unit disc, which is semistable and smooth outside the origin, the relative (log) de Rham complex $\omega_{X_0}^{\cdot}$ (of the log space X_0 over the log point) calculates the complex of nearby cycles $R\Psi(\mathbb{C})$. In Section 8 we generalize this to higher dimensional bases. It is known that, because of the possible presence of blow-ups, there is no "good" theory of nearby cycles on bases of dimension > 1 ("good" meaning in particular preserving constructibility in a suitable way). Some definitions and results, however, were sketched in [L]. Here we present a different approach (whose relation with that of [L] remains to be investigated). For a map $f : X \to Y$ of fs log analytic spaces and a bounded below complex L of abelian sheaves on the space $X^{\log}$ (which can be thought of some substitute for the generic fiber), we define a complex of nearby cycles $R\Psi^{\log}L$ on a certain space $X'^{\log}$ playing the role of the special fiber (see 8.1 for a precise definition). When the base is a disc and L is a locally constant sheaf, it is easy to recover the classical $R\Psi L$ from this more sophisticated object. In general, for f log smooth and exact and L a quasi-unipotent local system, we obtain a comparison theorem à la Steenbrink between the (ket) relative de Rham complex associated to L by the log Riemann-Hilbert correspondence and the complex of nearby cycles $R\Psi^{\log}L$ (8.6). This yields an alternate proof of one of the main results of Section 6. Finally, in the appendix we collect some technical results (used in sections 6 and 7) inspired by Tsuji's theory of "saturated maps" [T1]. For example, we show how to render an exact log smooth map saturated by Kummer extensions of the base.

We wish to thank A. Ogus for useful discussions on these topics.

Notation. For a log analytic space X, we denote by $\overset{\circ}{X}$ its underlying analytic space. By abuse of notation, we denote by $\overset{\circ}{X}$ (sometimes by X) the log analytic space $\overset{\circ}{X}$ endowed with the trivial log structure.

1. Log Riemann-Hilbert Correspondences (Review)

We review the results in [KtNk] on log Riemann-Hilbert correspondences.

Let X be an fs log analytic space with log structure M_X.

(1.1). In (1.1)–(1.3), we review the ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ over the ringed space $(X, \mathcal{O}_X)$, defined in [KtNk].

It is constructed as follows.

Any fs log analytic space is covered by open fs log analytic subspaces U_λ having the following property: There exists an fs monoid $\mathcal{S}$ such that U_λ is isomorphic to a locally closed analytic subspace of $\operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ endowed with the log structure associated to $\mathcal{S} \to \mathcal{O}_{U_\lambda}$. Here $\mathbf{C}[\mathcal{S}]$ denotes the semi-group ring.

Step 1. First assume $X = \operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ with the log structure associated to $\mathcal{S} \to \mathcal{O}_X$. As a topological space, X is identified with $\operatorname{Hom}(\mathcal{S}, \mathbf{C})$, where $\mathbf{C}$ is regarded here as a multiplicative monoid. In this case,

$$X^{\log} = \operatorname{Hom}(\mathcal{S}, \mathbf{R}_{\geq 0} \times \mathbf{S}^1)$$

($\mathbf{R}_{\geq 0} = \{r \in \mathbf{R}; r \geq 0\}$ regarded as a multiplicative monoid here, and $\mathbf{S}^1 = \{z \in \mathbf{C}; |z| = 1\}$ regarded as a multiplicative group) with the natural topology. We have a canonical continuous map $\tau : X^{\log} \to X$ induced by the homomorphism $\mathbf{R}_{\geq 0} \times \mathbf{S}^1 \to \mathbf{C}; (r, u) \mapsto ru$. The sheaf of rings $\mathcal{O}_X^{\log}$ is defined as follows. Let $U = \operatorname{Spec}(\mathbf{C}[\mathcal{S}^{\mathrm{gp}}])_{\mathrm{an}} = \operatorname{Hom}(\mathcal{S}^{\mathrm{gp}}, \mathbf{C}^\times)$ regarded as an open fs log analytic subspace of X. Then the log structure of U is trivial and U is non-singular. The inclusion map $j : U \to X$ factors as $\tau \circ j^{\log}$ where $j^{\log} : U \to X^{\log}$ is induced by $\mathbf{C}^\times \to \mathbf{R}_{\geq 0} \times \mathbf{S}^1; ru \mapsto (r, u)$ ($r \in \mathbf{R}_{>0}, u \in \mathbf{S}^1$). Since $\mathcal{S}^{\mathrm{gp}} \subset \Gamma(U, \mathcal{O}_U^\times) = \Gamma(X^{\log}, j_*^{\log}(\mathcal{O}_U^\times))$, the constant sheaf $\mathcal{S}^{\mathrm{gp}}|_{X^{\log}}$ on $X^{\log}$ can be viewed as a subsheaf of $j_*^{\log}(\mathcal{O}_U^\times)$. As a subsheaf of rings of $j_*^{\log}(\mathcal{O}_U)$, $\mathcal{O}_X^{\log}$ is generated by $\tau^{-1}(\mathcal{O}_X)$ and

$\log(\mathcal{S}^{\mathrm{gp}}|_{X^{\log}})$, where $\log(\mathcal{S}^{\mathrm{gp}}|_{X^{\log}})$ denotes the inverse image of $\mathcal{S}^{\mathrm{gp}}|_{X^{\log}}$ under $\exp\colon j_*^{\log}(\mathcal{O}_U) \to j_*^{\log}(\mathcal{O}_U^\times)$.

Step 2. Next assume that X is a locally closed analytic subspace of $Y = \mathrm{Spec}\,(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ endowed with the log structure associated to $\mathcal{S} \to \mathcal{O}_X$. Then

$$X^{\log} = X \times_Y Y^{\log}, \quad \mathcal{O}_X^{\log} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^{\log}.$$

Step 3. Finally any fs log analytic space X is covered by open fs log analytic subspaces which are as in Step 2. The ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ is constructed by glueing those constructed in Step 2. The reason why we can glue is seen by the categorical definition of $(X^{\log}, \mathcal{O}_X^{\log})$ given in (1.3) below.

(1.2). We give a categorical definition of $X^{\log}$. Let F be the functor from the category of topological spaces over X to the category of sets defined as follows. For a topological space T over X, $F(T)$ is defined to be the set of all homomorphisms $c : M_X^{\mathrm{gp}}|_T \to \mathrm{Cont}_T(-, \mathbf{S}^1)$ such that $c(f) = f/|f|$ for $f \in \mathcal{O}_X^\times|_T$. Here $|_T$ means the inverse image on T of a sheaf on X, and $\mathrm{Cont}_T(-, \mathbf{S}^1)$ is the sheaf on T of continuous maps into $\mathbf{S}^1$. This functor F is represented by the topological space $X^{\log}$ over X. The canonical map $X^{\log} \to X$ is denoted by τ.

For $\mathcal{S}$ and X as in (1.1) Step 1, M_X (resp. M_X^{gp}) is the subsheaf of $j_*(\mathcal{O}_U^\times)$ generated, as a sheaf of monoids (resp. groups), by $\mathcal{O}_X^\times$ and $\mathcal{S}$ (resp. $\mathcal{S}^{\mathrm{gp}}$). The map c for $T = X^{\log} = \mathrm{Hom}\,(\mathcal{S}, \mathbf{R}_{\geq 0} \times \mathbf{S}^1)$ in (1.1) Step 1 is induced from the evident homomorphism $\mathcal{S}^{\mathrm{gp}} \to \mathrm{Cont}_T(-, \mathbf{S}^1)$.

An explicit construction of $X^{\log}$ is given in [KtNk] as follows. As a set,

$$X^{\log} = \{(x, h) | x \in X, h \in \mathrm{Hom}\,(M_{X,x}^{\mathrm{gp}}, \mathbf{S}^1), \\ h(f) = \frac{f(x)}{|f(x)|} \text{ for any } f \in \mathcal{O}_{X,x}^\times\},$$

and $\tau : X^{\log} \to X$ is the projection $(x, h) \mapsto x$. The topology of $X^{\log}$ is described as follows locally on X. Locally X has a chart. If X has a chart $\mathcal{S} \to M_X$ with $\mathcal{S}$ an fs monoid, the topological space $X^{\log}$ is embedded in $X \times \mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbf{S}^1)$ as a closed subset, by $(x, h) \mapsto (x, h_{\mathcal{S}})$ where $h_{\mathcal{S}}$ is the composition $\mathcal{S}^{\mathrm{gp}} \to M_{X,x}^{\mathrm{gp}} \xrightarrow{h} \mathbf{S}^1$. From this, we see that $\tau : X^{\log} \to X$ is a proper map.

Proposition (1.2.1). *Let X be a log smooth fs log analytic space over $\mathbf{C}$. Then any point of $X^{\log}$ has a basis of neighborhoods whose intersection with X_{triv} is contractible.*

Proof. By [O2] Theorem 3.1. □

Remark (1.2.2). In the above proposition, $X^{\log}$ is actually a topological manifold with the boundary $X^{\log} - X_{\mathrm{triv}}$. This is proved in [KjNk] Lemma 1.2. Cf. [KtNk] (1.5.1).

(1.3). We give a categorical definition of the ringed space $(X^{\log}, \mathcal{O}_X^{\log})$ over $(X, \mathcal{O}_X)$. Let G be the functor from the category of ringed spaces $(T, \mathcal{O}_T)$ over $(X, \mathcal{O}_X)$ such that all stalks of $\mathcal{O}_T$ are non-zero rings, to the category of sets, defined as follows: $G(T, \mathcal{O}_T)$ is the set of all triples $(\log, c, \iota)$ where log is a homomorphism $M_X^{\mathrm{gp}}|_T \to \mathcal{O}_T/\mathbf{Z}(1)$ ($\mathbf{Z}(1)$ denotes $\mathbf{Z}\cdot 2\pi i$) such that the composition $\log \circ \exp : \mathcal{O}_X|_T \xrightarrow{\exp} M_X^{\mathrm{gp}}|_T \xrightarrow{\log} \mathcal{O}_T/\mathbf{Z}(1)$ coincides with the evident map, c is a homomorphism $M_X^{\mathrm{gp}}|_T \to \mathrm{Cont}_T(-, \mathbf{S}^1)$ such that $c(f) = f/|f|$ for $f \in \mathcal{O}_X^\times|_T$, and ι is an isomorphism $\mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ of extensions of $M_X^{\mathrm{gp}}|_T$ by $\mathbf{Z}(1)$, where $\mathcal{L}$ is obtained from the exact sequence $0 \to \mathbf{Z}(1) \to \mathcal{O}_T \to \mathcal{O}_T/\mathbf{Z}(1) \to 0$ by pull-back by log, and $\mathcal{L}'$ is obtained from the exact sequence $0 \to \mathrm{Cont}_T(-, \mathbf{Z}(1)) \to \mathrm{Cont}_T(-, \mathbf{R}(1)) \to \mathrm{Cont}_T(-, \mathbf{S}^1) \to 0$ by c. Then G is represented by $(X^{\log}, \mathcal{O}_X^{\log})$. The projection $(\log, c, \iota) \mapsto c$ corresponds to forgetting $\mathcal{O}_X^{\log}$.

For $\mathcal{S}$ and X as in (1.1) Step 1, $(\log, c, \iota)$ for $(T, \mathcal{O}_T) = (X^{\log}, \mathcal{O}_X^{\log})$ in (1.1) Step 1 is described as follows; log is induced by $\log : j_*^{\log}(\mathcal{O}_U^\times) \to j_*^{\log}(\mathcal{O}_U)/\mathbf{Z}(1)$, c is already described in (1.2), $\mathcal{L}$ is identified with the subsheaf of $j_*^{\log}(\mathcal{O}_U)$ additively generated by $\tau^{-1}(\mathcal{O}_X)$ and $\log(\mathcal{S}^{\mathrm{gp}}|_{X^{\log}})$, and ι is induced by $\mathcal{O}_U \to \mathrm{Cont}_U(-, \mathbf{R}(1))$; $f \mapsto \frac{1}{2}(f - \bar{f})$.

For another construction of $\mathcal{O}_X^{\log}$, see [KtNk].

We give some basic facts about $\mathcal{O}_X^{\log}$.

The canonical map $\tau^{-1}(\mathcal{O}_X) \to \mathcal{O}_X^{\log}$ is injective.

Denote by $\mathcal{L}_X$ the sheaf $\mathcal{L}$ on $X^{\log}$ in the above categorical definition of $(X^{\log}, \mathcal{O}_X^{\log})$. Then the canonical homomorphism $\mathcal{L}_X \to \mathcal{O}_X^{\log}$ is injective, and we regard $\mathcal{L}_X$ as a subsheaf of $\mathcal{O}_X^{\log}$ via this injection. We have $\tau^{-1}(\mathcal{O}_X) \subset \mathcal{L}_X \subset \mathcal{O}_X^{\log}$. The canonical surjective homomorphism $\mathcal{L}_X \to \tau^{-1}(M_X^{\mathrm{gp}})$, which we denote by exp, extends $\exp : \tau^{-1}(\mathcal{O}_X) \to \tau^{-1}(\mathcal{O}_X^\times)$.

For $x \in X$ and $y \in \tau^{-1}(x)$, if $(l_i)_{1 \le i \le r}$ is a family of elements of $\mathcal{L}_{X,y}$ such that $(\exp(l_i) \bmod \mathcal{O}_{X,x}^{\times})_{1 \le i \le r}$ is a $\mathbf{Z}$-basis of $M_{X,x}^{\mathrm{gp}}/\mathcal{O}_{X,x}^{\times}$, we have an isomorphism of rings

$$\mathcal{O}_{X,x}[T_1, ..., T_r] \xrightarrow{\cong} \mathcal{O}_{X,y}^{\log} \ ; \ T_i \mapsto l_i.$$

(1.4). For an fs log analytic space X, let $L(X)$ be the category of local systems of finite dimensional $\mathbf{C}$-vector spaces on $X^{\log}$ and let $L_{\mathrm{unip}}(X)$ be the full subcategory of $L(X)$ consisting of the objects L satisfying the following condition: for any $y \in X^{\log}$ and any element $g \in \pi_1(\tau^{-1}(\tau(y)))$, the action of g on L_y is unipotent. The category $L_{\mathrm{unip}}(X)$ is denoted by $L_{\mathrm{unip}}(X^{\log})$ in [KtNk].

Definition (1.5). (Cf. [O1], [O2].) An fs log analytic space X is called *ideally log smooth* if X is locally a closed subspace of a log smooth fs log analytic space defined by a log ideal, more precisely, if X satisfies the condition in [KtNk] (0.4), i.e. if there exist an open covering $(U_\lambda)_\lambda$ of X, fs monoids P_λ, and an ideal Σ_λ of P_λ for each λ, such that U_λ is isomorphic to an open analytic subspace of Spec $\big(\mathbf{C}[P_\lambda]/(\Sigma_\lambda)\big)_{\mathrm{an}}$ endowed with the log structure associated to $P_\lambda \longrightarrow \mathcal{O}_{U_\lambda}$.

Let X be an ideally log smooth fs log analytic space. Let $V_{\mathrm{nilp}}(X)$ be the category of vector bundles V on X endowed with an integrable connection ∇ with log poles satisfying the following condition locally on X: There exists a finite family of $\mathcal{O}_X$-subsheaves $(V_i)_{0 \le i \le n}$ of V satisfying $\nabla(V_i) \subset \omega^1_{X/\mathbf{C}} \otimes_{\mathcal{O}_X} V_i$ such that $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, and such that for each $1 \le i \le n$, V_i/V_{i-1} is a vector bundle and the connection induced on V_i/V_{i-1} does not have a pole. Here we say that the connection of V does not have a pole if the image of ∇ is contained in the image of $\Omega^1_{X/\mathbf{C}} \otimes_{\mathcal{O}_X} V \longrightarrow \omega^1_{X/\mathbf{C}} \otimes_{\mathcal{O}_X} V$, with $\Omega^1_{X/\mathbf{C}}$ the usual sheaf of differential forms. Note that $V_{\mathrm{nilp}}(X)$ is denoted by $D_{\mathrm{nilp}}(X)$ in [KtNk].

Now we review the log Riemann-Hilbert correspondences obtained in [KtNk].

Theorem (1.6). *Let X be an ideally log smooth fs log analytic space. Then there is an equivalence of categories*

$$\Phi' \colon L_{\mathrm{unip}}(X) \xrightarrow{\sim} V_{\mathrm{nilp}}(X),$$

given by $\Phi'(L) = \tau_*(\mathcal{O}_X^{\log} \otimes_{\mathbf{C}} L)$*, whose inverse* Φ *is defined by* $\Phi(V) = (\mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(V))^{\nabla=0}$*. Further, for an* $L \in L_{\mathrm{unip}}(X)$*, corresponding to* $V \in V_{\mathrm{nilp}}(X)$*, there is a natural identification*

$$\mathcal{O}_X^{\log} \otimes_{\mathbf{C}} L = \mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} V,$$

where $\mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} V = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{-1}(V)$*.*

The two categories $L_{\mathrm{unip}}(X)$ and $V_{\mathrm{nilp}}(X)$ are abelian categories because $L_{\mathrm{unip}}(X)$ is clearly abelian. The functors Φ and Φ', being equivalences of abelian categories, preserve exact sequences. Further they are compatible with tensor products.

2. Analytic Ket Sites

In this section we introduce the Kummer log étale site on an fs log analytic space. This is nothing but the analytic correspondent to the Kummer log étale site on an fs log scheme introduced in [Fjw], [FK], and [NC1] (see also [I3]). See [KtNk] Section 1 for the definition of fs log analytic spaces etc.

Definition (2.1). (Cf. [KtK1] (4.6) and [NC1] (2.1.2).) Let $h\colon Q \longrightarrow P$ be a homomorphism of fs monoids.

(1) h is said to be *exact* if $Q = (h^{\mathrm{gp}})^{-1}(P)$ in Q^{gp}.

(2) h is said to be *Kummer* if h is injective and for any $a \in P$, there exists an $n \geqq 1$ such that $a^n \in h(Q)$.

A Kummer homomorphism is exact.

Definition (2.2). Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces.

(1) (Cf. [KtK1] (4.6) and [NC1] (1.4), (2.1.2).) f is said to be *strict* (resp. *exact*, resp. *Kummer*) if, for any $x \in X$, the homomorphism of fs monoids $(M_Y/\mathcal{O}_Y^{\times})_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^{\times})_x$ is an isomorphism (resp. exact, resp. Kummer).

(2) (Cf. [KtK1] (3.2).) f is said to be a *strict closed immersion* if f is strict and if the underlying morphism of analytic spaces is a closed immersion.

(3) (Cf. [KtK1] (3.3).) f is said to be *log smooth* (resp. *log étale*) if for any commutative diagram

$$\begin{array}{ccc} T' & \xrightarrow{s} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow[t]{} & Y \end{array}$$

of fs log analytic spaces such that i is a strict closed immersion whose ideal of definition I satisfies $I^2 = (0)$, there exists locally on T (resp. there exists a unique) $g\colon T \longrightarrow X$ such that $gi = s$ and $fg = t$.

(4) f is said to be *Kummer log étale* if f is Kummer and log étale. Sometimes we call such an f *Kummer étale* (or even *ket*) for short.

THEOREM (2.3). *(Cf. [KtK1]* (3.5)*.) Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Then f is log smooth (resp. log étale, resp. Kummer log étale) if and only if locally on X and on Y, there exists a chart $(M_Y \longleftarrow Q \xrightarrow{h} P \longrightarrow M_X)$ of f with Q and P fs satisfying the following conditions* (1) *and* (2) :

(1) h^{gp} *is injective (resp.* h^{gp} *is injective and* $\mathrm{Cok}\,(h^{\mathrm{gp}})$ *is finite, resp. h is Kummer)*;

(2) *The induced morphism* $X \longrightarrow Y \times_{(\mathrm{Spec}\,\mathbf{C}Q)_{\mathrm{an}}} (\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}$ *is a (strict) open immersion.*

PROOF. The first two cases are proved in the same way as [KtK1] (3.5). The last case is easily reduced to the second case. □

(2.4). Let X be an fs log analytic space. The Kummer log étale site (or simply Kummer étale site) X^{ket} of X is defined as follows. The category of X^{ket} is the category of fs log analytic spaces U over X whose structural morphisms are Kummer log étale. The topology is the one associated to the pretopology defined by surjective families $(u_i\colon U_i \longrightarrow U)_{i \in I}$ (surjective means that U is set theoretically the union of the images of the u_i). Since a Kummer morphism of fs log analytic spaces is universally surjective in the category of fs log analytic spaces (the proof of this is the same as in the algebraic context in [NC1] Section 2), X^{ket} is indeed a $\mathcal{U}$-site, where $\mathcal{U}$ is a fixed universe. We denote by the same symbol X^{ket} the induced topos.

The structural sheaf $U \mapsto \mathcal{O}_U(U)$, $U \in \mathrm{Ob}\,X^{\mathrm{ket}}$ of X^{ket} is denoted by $\mathcal{O}_{X^{\mathrm{ket}}} = \mathcal{O}_X^{\mathrm{ket}}$. The sheaf $U \mapsto M_U(U)$ (resp. $M_U^{\mathrm{gp}}(U)$) is denoted by

M_X^{ket} (resp. M_X^{gpket}). (The proof of the fact that the presheaves $\mathcal{O}_{X^{\mathrm{ket}}}$, M_X^{ket}, and M_X^{gpket} are actually sheaves relies on an unpublished result in [KtK2] that the ket topology is coarser than the canonical topology ; in fact, $\mathcal{O}_{X^{\mathrm{ket}}}$ (resp. M_X^{ket}) is represented by the affine line $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathbf{A}^1_{\mathbf{Z}}$ (resp. $X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[\mathbf{N}]$) ; the case of M_X^{gpket} is reduced to that of M_X^{ket}. See also [I3] Section 2.) We have $(M_X^{\mathrm{ket}})^{\mathrm{gp}} \xrightarrow{\cong} M_X^{\mathrm{gpket}}$.

If the log structure of X is trivial (i.e., $M_X = \mathcal{O}_X^\times$), then X^{ket} is identified with the usual topos of open sets of the underlying space of X, and $\mathcal{O}_{X^{\mathrm{ket}}}$ is the same as the usual $\mathcal{O}_X$.

Problem (2.4.1). N. Nakayama ([NN]) introduced the category of ∂-spaces (X, B) as a localization of the category of pairs of a complex analytic space X and its nowhere-dense closed analytic subset B. He also introduced the ∂-étale topology on (X, B). Compare this with the ket topology. For example, let X be an fs log analytic space that is log smooth over $\mathbf{C}$. Let $\underline{X}$ be the ∂-space $(X, X - X_{\mathrm{triv}})$. Then the functor $X^{\mathrm{ket}} \ni U \overset{i}{\mapsto} (U, U - U_{\mathrm{triv}})$ preserves fiber products (cf. Example in p.470 of [NN]) so that it induces a morphism of topoi from X^{ket} to the ∂-étale topos. Is it an equivalence of topoi?

(2.5). Let $f \colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Then f induces a morphism of topoi $f = f^{\mathrm{ket}} \colon X^{\mathrm{ket}} \longrightarrow Y^{\mathrm{ket}}$ (the proof of this fact is the same as in the algebraic context in [NC1] (2.4)). In particular if f is the natural map ε from X to $\overset{\circ}{X}$, where $\overset{\circ}{X}$ denotes the underlying analytic space of X endowed with the trivial log structure, ε induces a natural morphism of topoi $X^{\mathrm{ket}} \longrightarrow \overset{\circ}{X}$, also denoted by ε in the sequel. Note that the morphism $\mathcal{O}_X \longrightarrow \varepsilon_* \mathcal{O}_{X^{\mathrm{ket}}}$ is an isomorphism.

(2.6). A *vector bundle* on X^{ket} is a locally free $\mathcal{O}_{X^{\mathrm{ket}}}$-module V of finite rank; locally here means ket locally, that is, there is a ket covering $(X_i \longrightarrow X)_i$ such that $V|_{X_i}$ is $\mathcal{O}_{X_i^{\mathrm{ket}}}$-free.

An example of a vector bundle of rank one (i.e. a line bundle) on X^{ket} is obtained as follows, from a divisor with $\mathbf{Q}$-coefficients. Let X be a smooth analytic space over $\mathbf{C}$, let D be a divisor on X with normal crossings, and endow X with the log structure associated to D. Let E be a divisor with $\mathbf{Q}$-coefficients on X whose restriction to $X - D$ is with $\mathbf{Z}$-coefficients (i.e., the non-integer coefficients can appear only along D). Then we have a line bundle $\mathcal{O}_{X^{\mathrm{ket}}}(E)$ on X^{ket} defined as follows. For an object $f : U \to X$ of X^{ket}, $\mathcal{O}_{X^{\mathrm{ket}}}(E)(U)$ is the set of all meromorphic functions g on U such that $\mathrm{div}(g) + f^*E \geqq 0$. For instance, if X is the projective line and D is the divisor $t = 0$ (t is the coordinate function), then $\mathcal{O}_{X^{\mathrm{ket}}}(\frac{1}{n}D)$, $n \geq 2$, is the line bundle which is, ket locally, generated by $t^{-\frac{1}{n}}$ around D. This is an n^{th} root of $\mathcal{O}_X(D)$, which doesn't exist for the classical zariski (or étale) topology, since the class of D in $\mathrm{H}^2(X, \mathbf{Z}/n\mathbf{Z}(1))$ is nontrivial.

3. Relation between $X^{\log}$ and X^{ket}

Let X be an fs log analytic space.

(3.1). We define a morphism of topoi $\tau^{\mathrm{ket}} : X^{\log} \to X^{\mathrm{ket}}$.

For a ket morphism $U \to X$ of fs log analytic spaces, the induced map $U^{\log} \to X^{\log}$ is etale, i.e. a local homeomorphism ([KtNk] (2.2)). The functor $U \mapsto U^{\log}$ from the ket site X^{ket} to the category of etale topological spaces over $X^{\log}$ is continuous and preserves fiber products, and induces a morphism of topoi $\tau^{\mathrm{ket}} : X^{\log} \to X^{\mathrm{ket}}$.

We have an (essentially) commutative diagram of topoi

$$\begin{array}{ccc} X^{\log} & \xrightarrow{\tau^{\mathrm{ket}}} & X^{\mathrm{ket}} \\ & \tau \searrow & \downarrow \varepsilon \\ & & X. \end{array}$$

(3.2). We define a sheaf of rings $\mathcal{O}_X^{\mathrm{klog}}$ on $X^{\log}$ by

$$\mathcal{O}_X^{\mathrm{klog}} = \mathcal{O}_X^{\log} \otimes_{\tau^{-1}(\mathcal{O}_X)} \tau^{\mathrm{ket}-1}(\mathcal{O}_X^{\mathrm{ket}}).$$

We have a commutative diagram of ringed topoi

$$\begin{array}{ccc} (X^{\log}, \mathcal{O}_X^{\mathrm{klog}}) & \longrightarrow & (X^{\mathrm{ket}}, \mathcal{O}_X^{\mathrm{ket}}) \\ \downarrow & & \downarrow \\ (X^{\log}, \mathcal{O}_X^{\log}) & \longrightarrow & (X, \mathcal{O}_X). \end{array}$$

Both diagrams are functorial with respect to X.

The sheaf $\mathcal{O}_X^{\mathrm{klog}}$ admits a description similar to that of $\mathcal{O}_X^{\log}$ in (1.1), as follows.

Step 1. First assume $X = \mathrm{Spec}\,(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ with the log structure associated to $\mathcal{S} \to \mathcal{O}_X$, let $U = \mathrm{Spec}\,(\mathbf{C}[\mathcal{S}^{\mathrm{gp}}])_{\mathrm{an}} = \mathrm{Hom}\,(\mathcal{S}^{\mathrm{gp}}, \mathbf{C}^{\times})$ regarded as an open fs log analytic subspace of X, and let $j^{\log} : U \to X^{\log}$ be the map defined in (1.1). Then as a subsheaf of rings of $j^{\log}_*(\mathcal{O}_U)$, $\mathcal{O}_X^{\mathrm{klog}}$ is generated by $\tau^{-1}(\mathcal{O}_X)$, $\log(\mathcal{S}^{\mathrm{gp}}|_{X^{\log}})$, and by n-th roots of local sections of $\mathcal{S}|_{X^{\log}}$ for $n \geq 1$. When $\mathcal{S} \cong \mathbf{N}^r$ for some $r \geq 0$, this is the ring of the Nilsson classes ([D2] III 1).

Step 2. Next assume that X is a locally closed analytic subspace of $Y = \mathrm{Spec}\,(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ endowed with the log structure associated to $\mathcal{S} \to \mathcal{O}_X$. Then

$$\mathcal{O}_X^{\mathrm{klog}} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^{\mathrm{klog}}.$$

Step 3. Finally any fs log analytic space X is covered by open fs log analytic subspaces which are as in Step 2. Again, thanks to the categorical definition explained below, the sheaf $\mathcal{O}_X^{\mathrm{klog}}$ is constructed by glueing those constructed in Step 2.

There are several other equivalent definitions of $\mathcal{O}_X^{\mathrm{klog}}$. In fact each of the following properties of $\mathcal{O}_X^{\mathrm{klog}}$ characterizes it.

1. We have a categorical definition of the ringed space $(X^{\log}, \mathcal{O}_X^{\mathrm{klog}})$ over the ringed topos $(X^{\mathrm{ket}}, \mathcal{O}_{X^{\mathrm{ket}}})$ as in (1.3), by just replacing the ringed space $(X, \mathcal{O}_X)$ in (1.3) by the ringed topos $(X^{\mathrm{ket}}, \mathcal{O}_{X^{\mathrm{ket}}})$, $M_X^{\mathrm{gp}}|_T$ by $M_{X^{\mathrm{ket}}}^{\mathrm{gp}}|_T$, and $\mathcal{O}_X|_T$ by $\mathcal{O}_{X^{\mathrm{ket}}}|_T$.

2. $\mathcal{O}_X^{\mathrm{klog}}$ is defined also as follows (exactly in the same way as the definition of $\mathcal{O}_X^{\log}$ in [KtNk] (3.2)). The proof of the equivalence is left to the reader. Define a sheaf $\mathcal{L}^{\mathrm{klog}}$ of abelian groups on $X^{\log}$ by

$$\mathcal{L}^{\mathrm{klog}} = \varprojlim(\mathrm{Cont}\,(-, \mathbf{R}(1)) \xrightarrow{\exp} \mathrm{Cont}\,(-, \mathbf{S}^1) \xleftarrow{c} (\tau^{\mathrm{ket}})^{-1}(M_X^{\mathrm{ketgp}})),$$

where c is the homomorphism induced by the maps $M_U^{\mathrm{gp}}(U) \longrightarrow \mathrm{Cont}\,(U^{\log}, \mathbf{S}^1)$, $a \mapsto \big((u,h) \mapsto h(a_u)\big)$ for objects U of X^{ket}. Since there is the commutative diagram

$$\begin{array}{ccc} (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_X^{\mathrm{ket}}) & \xrightarrow{\exp} & (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_{X^{\mathrm{ket}}}^{\times}) \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow \\ \mathrm{Cont}\,(-, \mathbf{R}(1)) & \xrightarrow{\exp} & \mathrm{Cont}\,(-, \mathbf{S}^1), \end{array}$$

where a is induced by the maps $\mathcal{O}_U(U) \longrightarrow \mathrm{Cont}\,(U^{\log}, \mathbf{R}(1))$, $f \mapsto \mathrm{Im}(f)$ (Im is the imaginary part), and $b = c \circ \alpha^{-1}$ is by $\mathcal{O}_U(U)^\times \longrightarrow \mathrm{Cont}\,(U^{\log}, \mathbf{S}^1)$, $f \mapsto f/|f|$, we have a homomorphism $h\colon (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_X^{\mathrm{ket}}) \longrightarrow \mathcal{L}^{\mathrm{klog}}$ of sheaves of abelian groups, which fits into the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_X^{\mathrm{ket}}) & \xrightarrow{\exp} & (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_{X^{\mathrm{ket}}}^\times) & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathcal{L}^{\mathrm{klog}} & \xrightarrow{\exp} & (\tau^{\mathrm{ket}})^{-1}(M_X^{\mathrm{ketgp}}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{Z}(1) & \longrightarrow & \mathrm{Cont}\,(-, i\mathbf{R}) & \xrightarrow{\exp} & \mathrm{Cont}\,(-, \mathbf{S}^1) & \longrightarrow & 0. \end{array}$$

Consider commutative $(\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_{X^{\mathrm{ket}}})$-algebras $\mathcal{A}$ on $X^{\log}$ endowed with a homomorphism $\mathcal{L}^{\mathrm{klog}} \longrightarrow \mathcal{A}$ of sheaves of abelian groups which commutes with h. Then $\mathcal{O}_X^{\mathrm{klog}}$ is the universal one among such $\mathcal{A}$.

More explicitly, $\mathcal{O}_X^{\mathrm{klog}}$ is defined by

$$\mathcal{O}_X^{\mathrm{klog}} := ((\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_{X^{\mathrm{ket}}}) \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}^{\mathrm{klog}}))/\mathfrak{a},$$

where $\mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}^{\mathrm{klog}})$ is the symmetric algebra of $\mathcal{L}^{\mathrm{klog}}$ over $\mathbf{Z}$ and $\mathfrak{a}$ is the ideal of $(\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_{X^{\mathrm{ket}}}) \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(\mathcal{L}^{\mathrm{klog}})$ generated locally by local sections of the form

$$f \otimes \underline{1} - 1 \otimes h(f) \text{ for } f \text{ a local section of } (\tau^{\mathrm{ket}})^{-1}(\mathcal{O}_X^{\mathrm{ket}}).$$

Here $\underline{1}$ means the $1 \in \mathbf{Z} = \mathrm{Sym}^0(\mathcal{L}^{\mathrm{klog}})$, whereas $h(f)$ belongs to $\mathcal{L}^{\mathrm{klog}} = \mathrm{Sym}^1(\mathcal{L}^{\mathrm{klog}})$.

3. $\mathcal{O}_X^{\mathrm{klog}}$ is the sheaf on $X^{\log}$ associated to the separated presheaf $U \mapsto \varinjlim_V \Gamma(U, f^{-1}\mathcal{O}_V^{\log})$, where the limit runs over the category of the pairs of an object $V \in X^{\mathrm{ket}}$ and a continuous map $f\colon U \to V^{\log}$ over $X^{\log}$. This is seen as follows. First note that the above f is an open immersion. Then the natural map is induced by the maps $\Gamma(U, f^{-1}\mathcal{O}_V^{\log}) = \Gamma(f(U), \mathcal{O}_V^{\log}) \longrightarrow \Gamma(f(U), \mathcal{O}_V^{\mathrm{klog}}) \xleftarrow{\cong} \Gamma(U, \mathcal{O}_X^{\mathrm{klog}})$. This is an isomorphism at stalks.

Remark (3.2.1). A. Ogus constructed a ring $\widetilde{\mathcal{O}}_X^{\log}$ in [O2], which is similar to our $\mathcal{O}_X^{\mathrm{klog}}$ and which controls the log Riemann-Hilbert correspondence with arbitrary monodromies, as $\mathcal{O}_X^{\mathrm{klog}}$ controls that with quasi-unipotent monodromies. See (4.4.1).

(3.3). For each point x of an fs log analytic space X, let y be a point of $X^{\log}$ lying over x. Then we have a functor $X^{\mathrm{ket}} \longrightarrow$ (Sets) defined by $F \mapsto F_{x(\log)} := ((\tau^{\mathrm{ket}})^{-1}F)_y$, which does not depend on the choices of y up to non-canonical isomorphisms because this functor can be defined also as in [NC1] (2.5) via log separably closed fields. (For example, in case of $X = (\mathrm{Spec}\,\mathbf{C}, \mathbf{N} \oplus \mathbf{C}^\times)$, taking an element $(y_n)_n \in \varprojlim X_n^{\log}$ with $y_1 = y$, we have an isomorphism $\varinjlim F(X_n) \overset{\cong}{\longrightarrow} F_{x(\log)}$. Here $X_n := (\mathrm{Spec}\,\mathbf{C}, \mathbf{N}^{\frac{1}{n}} \oplus \mathbf{C}^\times), n \geq 1$ and the source does not depend on the choices of y.) These functors $(-)_{x(\log)}$ are points of X^{ket}, and the family $\{(-)_{x(\log)}\ ;\ x \in X\}$ is conservative.

Remark (3.3.1). Let x be a point of X. There always exists a local chart $P \longrightarrow \Gamma(X', M_X)$ around x (X' is an open neighborhood of x) such that the induced map $P \longrightarrow M_{X,x}/\mathcal{O}_{X,x}^\times$ is an isomorphism (the proof is the same as in the algebraic context in [NC1] (1.6)). When we fix such a chart, we have $F_{x(\log)} = \varinjlim_{n\geq 1}(\varepsilon_*(F|_{X_n^{\mathrm{ket}}}))_{x_n}$, where $X_n := X \times_{(\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}} (\mathrm{Spec}\,\mathbf{C}P^{\frac{1}{n}})_{\mathrm{an}}$, and x_n is the unique point of X_n lying over x.

The action of $\pi_1(x^{\log}) = \mathrm{Hom}\,((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x, \mathbf{Z}(1)) \cong \mathrm{Hom}\,(P, \mathbf{Z}(1))$ on $F_{x(\log)}$, where $\mathbf{Z}(1) = \mathbf{Z}\cdot 2\pi i$, extends to a continuous action of $\pi_1(x^{\mathrm{ket}}) := \mathrm{Hom}\,((M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x, \widehat{\mathbf{Z}}(1))$ on $F_{x(\log)}$, where $\widehat{\mathbf{Z}}(1) := \varprojlim_{n\geq 1}\{u \in \mathbf{C}\ ;\ u^n = 1\}$. Here $\pi_1(x^{\log})$ is regarded as a subgroup of $\pi_1(x^{\mathrm{ket}})$ via the injection $\exp\colon \mathbf{Z}(1) \longrightarrow \widehat{\mathbf{Z}}(1)$.

Notation (3.4). Let X be an fs log analytic space and let $X \longrightarrow (\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}$ be a chart with P an fs monoid. Then we write $X \times_{(\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}} (\mathrm{Spec}\,\mathbf{C}P^{\frac{1}{n}})_{\mathrm{an}}$ as X_n for each $n \geqq 1$.

Proposition (3.5). *(Cf. [KtNk]* (3.3)*) Let X be an fs log analytic space, $x \in X$, y a point of $X^{\log}$ with image x in X, and let $P := (M_X/\mathcal{O}_X^\times)_x$.*

(1) *Fix a homomorphism $P \longrightarrow M_{X,x}$ such that $P \longrightarrow M_{X,x} \longrightarrow P$ is the identity. Then we have an $\mathcal{O}_{X,x}$-isomorphism*

$$\mathcal{O}_{X,x(\log)}^{\mathrm{ket}} \cong \mathcal{O}_{X,x} \otimes_{\mathbf{C}[P]} \mathbf{C}[P \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}],$$

where $P \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0} = \bigcup_{n \geq 1} P^{\frac{1}{n}}$.

(2) *Let* $(t_i)_{1 \leq i \leq n}$ *be a family of elements of the stalk* $\mathcal{L}_y$ *whose image under* exp *is a* $\mathbf{Z}$*-basis of* $(M_X^{\mathrm{gp}}/\mathcal{O}_X^{\times})_x$, *where* $\mathcal{L}$ *is the non-ket version of* $\mathcal{L}^{\mathrm{klog}}$, *introduced in [KtNk]. Then, as an* $\mathcal{O}_{X,x(\log)}^{\mathrm{ket}}$*-algebra,* $\mathcal{O}_{X,y}^{\mathrm{klog}}$ *is isomorphic to the polynomial ring* $\mathcal{O}_{X,x(\log)}^{\mathrm{ket}}[T_1, \cdots, T_n]$ *in* n *variables by*

$$\mathcal{O}_{X,x(\log)}^{\mathrm{ket}}[T_1, \cdots, T_n] \longrightarrow \mathcal{O}_{X,y}^{\mathrm{klog}} \ ; \ T_i \mapsto t_i.$$

In particular, $\mathcal{O}_X^{\mathrm{klog}}$ *is* flat *over* $\mathcal{O}_X^{\mathrm{ket}}$.

Proof. (1) We may assume that $P \longrightarrow M_{X,x}$ comes from a chart $P \longrightarrow \Gamma(X, M_X)$. By (3.3.1), it is enough to show $\mathcal{O}_{X_n,x_n} = \mathcal{O}_{X,x} \otimes_{\mathbf{C}[P]} \mathbf{C}[P^{\frac{1}{n}}]$, where x_n is the unique point of X_n lying over x. We may assume that $X = (\operatorname{Spec} \mathbf{C}[P])_{\mathrm{an}}$. Then, by taking a presentation $\mathbf{N}^r \rightrightarrows \mathbf{N}^s \to P$ (exact), we reduce this equality to the case where P is free, which is clear. (That a finitely generated monoid is finitely presented is of course well known, but it's still good to recall an argument. Let $P = \langle x_1, \ldots, x_n \ ; \ f_\lambda(x_1, \ldots, x_n) = g_\lambda(x_1, \ldots, x_n), \lambda \in \Lambda \rangle$ be a presentation of a finitely generated monoid. Then there are finite number of $\lambda_1, \ldots, \lambda_m$ such that $\mathbf{Z}[P] = \mathbf{Z}[x_1, \ldots, x_n]/(f_\lambda - g_\lambda \ ; \ \lambda \in \Lambda) = \mathbf{Z}[x_1, \ldots, x_n]/(f_{\lambda_j} - g_{\lambda_j} \ ; \ 1 \leq j \leq m) = \mathbf{Z}[P']$, where $P' = \langle x_1, \ldots, x_n \ ; \ f_{\lambda_j} = g_{\lambda_j} \ ; \ 1 \leq j \leq m \rangle$. This equality implies $P = P'$.)

(2) is proved as in [KtNk] (3.3). Note that the equality $({M_X}^{\mathrm{gpket}}/\mathcal{O}_{X^{\mathrm{ket}}}^{\times})_{x(\log)} = (M_X^{\mathrm{gp}}/\mathcal{O}_X^{\times})_x \otimes_{\mathbf{Z}} \mathbf{Q}$ implies that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}(({M_X}^{\mathrm{gpket}}/\mathcal{O}_{X^{\mathrm{ket}}}^{\times})_{x(\log)}) = \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Sym}_{\mathbf{Z}}((M_X^{\mathrm{gp}}/\mathcal{O}_X^{\times})_x)$. □

Lemma (3.6). *Let* $f \colon X \longrightarrow Y$ *be a strict morphism of fs log analytic spaces. Then*

(1) $\mathcal{O}_X^{\log} = (f^{\log})^{-1}\mathcal{O}_Y^{\log} \otimes_{(f\tau)^{-1}\mathcal{O}_Y} \tau^{-1}\mathcal{O}_X$.

(2) $\mathcal{O}_X^{\mathrm{ket}} = (f^{\mathrm{ket}})^{-1}\mathcal{O}_Y^{\mathrm{ket}} \otimes_{(f\varepsilon)^{-1}\mathcal{O}_Y} \varepsilon^{-1}\mathcal{O}_X$.

(3) $\mathcal{O}_X^{\mathrm{klog}} = (f^{\log})^{-1}\mathcal{O}_Y^{\mathrm{klog}} \otimes_{(f\tau)^{-1}\mathcal{O}_Y} \tau^{-1}\mathcal{O}_X$
$= (f^{\log})^{-1}\mathcal{O}_Y^{\mathrm{klog}} \otimes_{((f\tau)^{\mathrm{ket}})^{-1}\mathcal{O}_Y^{\mathrm{ket}}} (\tau^{\mathrm{ket}})^{-1}\mathcal{O}_X^{\mathrm{ket}}$.

Proof. These are checked stalkwise by [KtNk] (3.3) and the proposition above. □

Proposition (3.7). *Let X be an fs log analytic space.*

(1) *Let x be a point of X and F a sheaf of $\mathbf{Z}$-modules on X^{ket}. Then $(\mathrm{R}^q\varepsilon_*F)_x \cong \mathrm{H}^q(\pi_1(x^{\mathrm{ket}}), F_{x(\log)})$ for any q, where the right hand side is the cohomology of the pro-finite group $\pi_1(x^{\mathrm{ket}})$ (3.3.1).*

(2) *For a sheaf M of $\mathbf{Q}$-vector spaces on X^{ket}, $\mathrm{R}^q\varepsilon_*M = 0$ for any $q > 0$.*

(3) *For an $\mathcal{O}_X$-module M, the natural homomorphism $M \longrightarrow \mathrm{R}\tau_*(\mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} M)$ is an isomorphism.*

(4) *For an $\mathcal{O}_{X^{\mathrm{ket}}}$-module M, the natural homomorphism $M \longrightarrow \mathrm{R}\tau_*^{\mathrm{ket}}(\mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathcal{O}_{X^{\mathrm{ket}}}} M)$ is an isomorphism.*

(5) *For an $\mathcal{O}_X$-module M, the natural homomorphism $M \longrightarrow \mathrm{R}\varepsilon_*(\mathcal{O}_X^{\mathrm{ket}} \otimes_{\mathcal{O}_X} M)$ is an isomorphism.*

Proof. (1) is proved similarly as [NC1] (4.1). The point is that in the notation in (3.4), $U_n \times_U U_n$ is the disjoint union of n copies of U_n over U as in the proof of [NC1] (4.1).

(2) is deduced from (1), since the cohomology of degree ≥ 1 of a profinite group is a torsion group.

The case of (3) where M is a vector bundle was proved in [M1] 4.6. For reader's convenience, we recall its proof in the case where $\overset{\circ}{X}$ consists of only one point and $M = \mathcal{O}_X$, which will be used below. In this case $X^{\log}$ is homeomorphic to $(\mathbf{S}^1)^r$, where $r = \mathrm{rank}_{\mathbf{Z}}(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)$, and $\mathcal{O}_X^{\log}$ is a locally constant sheaf whose local value is $\mathbf{C}[T_1, \cdots, T_r]$, T_i's are indeterminates, and the action of $\pi_1(X^{\log})$ on the stalk can be described as $g_i(T_j) = T_j + \delta_{ij}2\pi\sqrt{-1}$ $(1 \leq i, j \leq r)$ in taking a suitable $(T_j)_j$ and $(g_i)_i$ such that the set $\{g_1, \cdots, g_r\}$ generates $\pi_1(X^{\log})$. It is enough to show that $\mathrm{H}^q(X^{\log}, \mathcal{O}_X^{\log}) = 0$ for $q > 0$ (resp. $= \mathcal{O}_{X,x}$ for $q = 0$). The case where $r = 1$ is deduced from the exactness of $0 \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}[T_1] \overset{g_1-\mathrm{id}}{\to} \mathcal{O}_{X,x}[T_1] \to 0$. The general case is reduced to this case by the Künneth formula.

Now we prove (3). Let x be a point of X. Since τ is proper (= universally closed) and separated, by proper base change theorem, $(\mathrm{R}\tau_*(\mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} M))_x = \mathrm{R}\tau_*(\mathcal{O}_X^{\log}|_{\tau^{-1}(x)} \otimes_{\mathcal{O}_{X,x}} M_x)$. But $\mathcal{O}_X^{\log}|_{\tau^{-1}(x)} \cong \mathcal{O}_x^{\log} \otimes_{\mathcal{O}_{x,x}} \mathcal{O}_{X,x}$, where x is regarded as an fs log analytic space endowed with the inverse image log structure from X. Since $\mathcal{O}_{x,x} = \mathbf{C}$, we can proceed as $\mathrm{R}\tau_*(\mathcal{O}_X^{\log}|_{\tau^{-1}(x)} \otimes_{\mathcal{O}_{X,x}} M_x) = \mathrm{R}\tau_*(\mathcal{O}_x^{\log} \otimes_{\mathbf{C}} M_x) = \mathrm{R}\tau_*(\mathcal{O}_x^{\log}) \otimes_{\mathbf{C}} M_x = M_x$ by the special case explained above.

(4) Let x be a point of X. Take a local chart $P \longrightarrow M_X$ around x

as in (3.3.1). Then, by proper base change theorem, $(\mathrm{R}^q\tau^{\mathrm{ket}}_*(\mathcal{O}^{\mathrm{klog}}_X\otimes_{\mathcal{O}_{X^{\mathrm{ket}}}} M))_{x(\log)} = \varinjlim_{n\geq 1}\mathrm{H}^q\bigl(x_n^{\log}, \bigl(\mathcal{O}^{\mathrm{klog}}_{X_n}\otimes_{\mathcal{O}_{X_n^{\mathrm{ket}}}} M|_{X_n^{\mathrm{ket}}}\bigr)|_{x_n^{\log}}\bigr)$ in the notation in (3.3.1), where x is regarded as an fs log analytic space as in the proof of (3). On the other hand, the inclusion maps $\tau^{-1}\varepsilon_*(M|_{X_n^{\mathrm{ket}}}) \longrightarrow (\tau^{\mathrm{ket}})^{-1}(M|_{X_n^{\mathrm{ket}}})$ for all $n\geq 1$ induce an isomorphism

$$\varinjlim_{n\geq 1}\mathrm{H}^q\bigl(x_n^{\log}, \bigl(\mathcal{O}^{\log}_{X_n}\otimes_{\mathcal{O}_{X_n}}\varepsilon_*(M|_{X_n^{\mathrm{ket}}})\bigr)|_{x_n^{\log}}\bigr) \xrightarrow{\cong} \varinjlim_{n\geq 1}\mathrm{H}^q\bigl(x_n^{\log}, \bigl(\mathcal{O}^{\log}_{X_n}\otimes_{\mathcal{O}_{X_n}} M|_{X_n^{\mathrm{ket}}}\bigr)|_{x_n^{\log}}\bigr).$$

In the target group, note that $\mathcal{O}^{\log}_{X_n}\otimes_{\mathcal{O}_{X_n}} - = \mathcal{O}^{\mathrm{klog}}_{X_n}\otimes_{\mathcal{O}^{\mathrm{ket}}_{X_n}} -$. By (3), the source group, which is viewed as $\varinjlim_{n\geq 1}\bigl(\mathrm{R}^q\tau_*\bigl(\mathcal{O}^{\log}_{X_n}\otimes_{\mathcal{O}_{X_n}}\varepsilon_*(M|_{X_n^{\mathrm{ket}}})\bigr)\bigr)_{x_n}$, vanishes for $q>0$ and is equal to $\varinjlim_{n\geq 1}(\varepsilon_*(M|_{X_n^{\mathrm{ket}}}))_{x_n}$ for $q=0$, which is nothing but $M_{x(\log)}$.

(5) for a locally free M is a consequence of (2). We use only this case later. See Appendix (A.1) for the proof of general case. □

(3.8). Let $f\colon X\longrightarrow Y$ be a morphism of fs log analytic spaces. Then we have a morphism of ringed topoi with log structures $(X^{\mathrm{ket}}, M_X^{\mathrm{ket}}) \longrightarrow (Y^{\mathrm{ket}}, M_Y^{\mathrm{ket}})$, and we define the complex $\omega^{\cdot,\mathrm{ket}}_{X/Y}$ on X^{ket} of relative analytic forms with log poles in the ket sense exactly in the same way as in [KtNk] (1.7) and (1.9). Further, let $\omega^{p,\mathrm{klog}}_{X/Y} := \mathcal{O}^{\mathrm{klog}}_X\otimes_{\mathcal{O}^{\mathrm{ket}}_X}\omega^{p,\mathrm{ket}}_{X/Y}$ for each $p\geq 0$. Then we have the canonical derivation $d\colon \mathcal{O}^{\mathrm{klog}}_X\longrightarrow\omega^{1,\mathrm{klog}}_{X/Y}$ and the complex $\omega^{\cdot,\mathrm{klog}}_{X/Y}$ of $(f^{\log})^{-1}(\mathcal{O}^{\mathrm{klog}}_Y)$-modules as in [KtNk] (3.5).

4. Quasi-Unipotent Log Riemann-Hilbert Correspondences

(4.1). For an fs log analytic space X, let $L_{\mathrm{qunip}}(X)$ be the full subcategory of $L(X)$ (1.4) consisting of the objects L satisfying the following condition: for any $y\in X^{\log}$ and any element $g\in\pi_1(\tau^{-1}(\tau(y)))$, all the eigenvalues of the action of g on L_y are roots of the unity.

LEMMA (4.2). *For an object L of $L(X)$, the following three conditions* (i)–(iii) *are equivalent.*

(i) *L belongs to* $L_{\mathrm{qunip}}(X)$.

(ii) *The following holds locally on X: There exists a finite family of* $\mathbf{C}$-*subsheaves* $(L_j)_{0\le j\le n}$ *of L such that* $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$, *and such that for each* $1 \le i \le n$, L_i/L_{i-1} *is isomorphic to the inverse image of a local system of finite dimensional* $\mathbf{C}$-*vector spaces on* X^{ket}.

(iii) *The following holds locally on* X^{ket}: *There exists a finite family of* $\mathbf{C}$-*subsheaves* $(L_j)_{0\le j\le n}$ *of L such that* $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$, *and such that for each* $1 \le i \le n$, L_i/L_{i-1} *is isomorphic to the inverse image of a local system of finite dimensional* $\mathbf{C}$-*vector spaces on* X.

Proof. The implications (ii) $\Rightarrow$ (iii) and (iii)$\Rightarrow$ (i) are clear. We prove that (i) implies (ii). Let $x \in X$. We may work locally on X at x, and so we may assume that there is a chart $\mathcal{S} \to M_X$ of M_X such that $\mathcal{S} \to M_{X,x}/\mathcal{O}_{X,x}^{\times}$ is an isomorphism. Let $Y = \mathrm{Spec}\,(\mathbf{C}[\mathcal{S}])_{\mathrm{an}}$ and endow Y with the log structure associated to $\mathcal{S} \to \mathcal{O}_Y$, and let $f\colon X \to Y$ be the induced morphism. Since $\pi_1(x^{\log}) \to \pi_1(Y^{\log})$ is an isomorphism, there exists a local system of finite dimensional $\mathbf{C}$-vector spaces L' on $Y^{\log}$ whose pull back on $x^{\log}$ is isomorphic to the pull back of L. By the proper base change theorem, $\big(\tau_*\mathcal{H}om((f^{\log})^{-1}(L'), L)\big)_x \to \mathrm{Hom}\,((f^{\log})^{-1}(L')|_{x^{\log}}, L|_{x^{\log}})$ is an isomorphism. From this we see that L and $(f^{\log})^{-1}(L')$ are isomorphic on $U^{\log}$ for some open neighbourhood U of x in X. Hence we may assume $L = (f^{\log})^{-1}(L')$. Since the actions of $\pi_1(x^{\log})$ on the stalks of $L|_{x^{\log}}$ are quasi-unipotent, the actions of $\pi_1(Y^{\log})$ on the stalks of L' are quasi-unipotent. Hence we may assume that there exists a finite family of $\mathbf{C}$-subsheaves $(L'_j)_{0\le j\le n}$ of L' such that $0 = L'_0 \subset L'_1 \subset \cdots \subset L'_n = L'$, and such that for each $1 \le i \le n$, L'_i/L'_{i-1} comes from a representation of a finite quotient group of $\pi_1(Y^{\log})$. Then L'_i/L'_{i-1} clearly comes from Y^{ket}. $\square$

(4.2.1). Let $L \in L_{\mathrm{qunip}}(X)$. Then, by the above lemma and (3.7) (4),

$$\Phi'(L) := \tau_*^{\mathrm{ket}}(\mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathbf{C}} L)$$

is a vector bundle on X^{ket} (2.6). (For example, $\Phi'(\mathbf{C}) = \mathcal{O}_{X^{\mathrm{ket}}}$.) Further this vector bundle is endowed with the integrable connection with log poles $\tau_*^{\mathrm{ket}}(d \otimes \mathrm{id})$, where $d\colon \mathcal{O}_X^{\mathrm{klog}} \longrightarrow \omega_{X/\mathbf{C}}^{1,\mathrm{klog}}$ is the canonical derivation explained in (3.8). Thus Φ' defines a functor from $L_{\mathrm{qunip}}(X)$ to the category of vector bundles V on X^{ket} endowed with an integrable connection with log poles $\nabla\colon V \longrightarrow \omega_{X/\mathbf{C}}^{1,\mathrm{ket}} \otimes_{\mathcal{O}_{X^{\mathrm{ket}}}} V$.

(4.3). Let X be an ideally log smooth fs log analytic space (1.5). Let $V_{\rm qnilp}(X)$ be the category of vector bundles V on $X^{\rm ket}$ endowed with an integrable connection ∇ with log poles satisfying the following condition locally on $X^{\rm ket}$: There exists a finite family of $\mathcal{O}_{X^{\rm ket}}$-subsheaves $(V_i)_{0\le i\le n}$ of V satisfying $\nabla(V_i)\subset\omega^{1,{\rm ket}}_{X/\mathbf{C}}\otimes_{\mathcal{O}_{X^{\rm ket}}}V_i$ such that $0=V_0\subset V_1\subset\cdots\subset V_n=V$, and such that for each $1\le i\le n$, V_i/V_{i-1} is a vector bundle and the connection induced on V_i/V_{i-1} does not have a pole. Here we say that the connection of V does not have a pole if the image of ∇ is contained in the image of $\Omega^1_{X/\mathbf{C}}\otimes_{\mathcal{O}_X}V\longrightarrow\omega^{1,{\rm ket}}_{X/\mathbf{C}}\otimes_{\mathcal{O}^{\rm ket}_X}V$, with $\Omega^1_{X/\mathbf{C}}$ the usual sheaf of differential forms.

THEOREM (4.4). *Let X be an ideally log smooth fs log analytic space. Then Φ' induces an equivalence of categories*

$$\Phi'\colon L_{\rm qunip}(X)\xrightarrow{\sim}V_{\rm qnilp}(X),$$

whose inverse Φ is defined as $\Phi(V)=(\mathcal{O}^{\rm klog}_X\otimes_{(\tau^{\rm ket})^{-1}(\mathcal{O}_{X^{\rm ket}})}(\tau^{\rm ket})^{-1}(V))^{\nabla=0}$. Further, for an $L\in L_{\rm qunip}(X)$, corresponding to $V\in V_{\rm qnilp}(X)$, there is a natural identification

$$\mathcal{O}^{\rm klog}_X\otimes_{\mathbf{C}}L=\mathcal{O}^{\rm klog}_X\otimes_{\mathcal{O}_{X^{\rm ket}}}V.$$

The above relates to its non-ket version (1.6) *as follows*: *The fully faithful functor ε^* from the category of vector bundles on X to that for $X^{\rm ket}$ induces an equivalence between $V_{\rm nilp}(X)$ and the full subcategory of $V_{\rm qnilp}(X)$ consisting of the objects (V,∇) such that V, as a vector bundle on $X^{\rm ket}$, is the pull back of a vector bundle on X by ε^*. Further, $\varepsilon^*\colon V_{\rm nilp}(X)\longrightarrow V_{\rm qnilp}(X)$ is compatible with the inclusion $L_{\rm unip}(X)\subset L_{\rm qunip}(X)$ via both correspondences.*

The functors Φ and Φ' are compatible with tensor products and are equivalences of abelian categories, so that they preserve exact sequences.

PROOF. First, for an $L\in L_{\rm qunip}(X)$, $\Phi'(L)$ belongs to $V_{\rm qnilp}(X)$, the natural functorial homomorphism $\mathcal{O}^{\rm klog}_X\otimes\Phi'(L)\longrightarrow\mathcal{O}^{\rm klog}_X\otimes L$ induced by the adjunction map $(\tau^{\rm ket})^{-1}\Phi'(L)\longrightarrow\mathcal{O}^{\rm klog}_X\otimes L$ is an isomorphism, and this isomorphism induces a functorial isomorphism $\Phi\Phi'(L)\cong L$. This is reduced to the equality $\Phi'(\mathbf{C}) = \mathcal{O}_{X^{\mathrm{ket}}}$ by ket localization with (4.2), and devissage with (3.7) (4).

Next we show that for a $V \in V_{\mathrm{qnilp}}(X)$, $\Phi(V)$ belongs to $L_{\mathrm{qunip}}(X)$. By ket localization, we may assume that there exists a filtration $(V_i)_{0\leq i\leq n}$ in (4.3). Then the natural homomorphism $\mathcal{O}_X^{\mathrm{ket}} \otimes_{\mathcal{O}_X} \varepsilon_* V \longrightarrow V$ is an isomorphism, as seen by reducing to the case $V = \mathcal{O}_X^{\mathrm{ket}}$ by devissage. By (1.6), it is enough to show that the natural homomorphism

$$h\colon (\mathcal{O}_X^{\log} \otimes_{\mathcal{O}_X} \varepsilon_* V)^{\nabla=0} \longrightarrow \Phi(V) = (\mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathcal{O}_X} \varepsilon_* V)^{\nabla=0}$$

is an isomorphism because $\varepsilon_* V$ belongs to $V_{\mathrm{nilp}}(X)$. Since the stalks of the target of h is described as the limit of the groups of the horizontal sections of $\mathcal{O}_{X'}^{\log} \otimes_{\mathcal{O}_{X'}} g^*\varepsilon_* V$, where $g\colon X' \longrightarrow X$ is ket, we reduce this to the fact that the log Riemann-Hilbert correspondences in (1.6) commute with the pull-backs.

We also see that the natural functorial homomorphism $\mathcal{O}_X^{\mathrm{klog}} \otimes \Phi(V) \longrightarrow \mathcal{O}_X^{\mathrm{klog}} \otimes V$ is an isomorphism by reducing to the case that V comes from $V_{\mathrm{nilp}}(X)$ as above, which is seen by tensoring $\mathcal{O}_X^{\mathrm{klog}}$ with the identification in (1.6). Applying τ_*^{ket}, we have a functorial isomorphism $\Phi'\Phi(V) \cong V$ by (3.7) (4).

We already proved the last statement when we saw that h is an isomorphism in the above. Finally, let $(V, \nabla) \in V_{\mathrm{qnilp}}(X)$ and assume that $V = \varepsilon^* V_0$ for some V_0. Then $V_0 = \varepsilon_* V$ and $(V, \nabla) = \varepsilon^*(V_0, \nabla_0)$, where ∇_0 is the induced connection on V_0. Now note that (V_0, ∇_0) belongs to $V_{\mathrm{nilp}}(X)$ if and only if $L := \Phi(V)$ belongs to $L_{\mathrm{unip}}(X)$. The rest is to prove that these equivalent conditions are satisfied. First we may assume that $X = \operatorname{Spec} \mathbf{C}$. Put $P := \Gamma(X, M_X/\mathcal{O}_X^\times)$. We may further assume $P = \mathbf{N}$ because $\operatorname{Hom}(P, \mathbf{N}\cdot 2\pi i)$ generates $\pi_1(X^{\log}) = \operatorname{Hom}(P^{\mathrm{gp}}, \mathbf{Z}(1))$. It is enough to show that any subsystem L' of L of rank one belongs to $L_{\mathrm{unip}}(X)$. Let V' correspond to L'. Let z be the generator of $P = \mathbf{N}$. Then $A := \mathcal{O}_{X,x(\log)}^{\mathrm{ket}}$ is identified with $\big(\bigcup_{n\geq 1} \mathbf{C}[z^{\frac{1}{n}}]\big)/(z)$ (3.5), where $X =: \{x\}$. Let σ be the generator of $\pi_1(X^{\log})$ such that $\sigma(z^p) = e^{2\pi i p} z^p$, $p \in \mathbf{Q}$. Let $e^{2\pi i q}$, $q \in (0,1] \cap \mathbf{Q}$, be the eigenvalue of the action of σ on L'. Then the stalk of V' is isomorphic to Ae, where e is a basis of L' and $\varepsilon_* V' \cong \mathbf{C} z^{1-q} e$. Since $\varepsilon^*\varepsilon_* V' \longrightarrow V'$ is injective, $1 - q = 0$. $\square$

Remark (4.4.1). A. Ogus also generalizes (1.6) in [O2]. His equivalence works for the modules with arbitrary monodromies. (He does not use ket sites.)

Example (4.5). Let $X = \mathbf{C}$ with the coordinate function z, and endow X with the log structure associated to the divisor $D = \{0\}$. Let a be a rational number. Consider the vector bundle of rank 1 on $X_{\mathrm{triv}} = \mathbf{C}^{\times}$ with a basis e endowed with a connection ∇ which sends e to $a \cdot d\log(z) \otimes e$, let $L \in L_{\mathrm{qunip}}(X)$ be the unique extension of the local system $\mathrm{Ker}\,(\nabla)$ on X_{triv}, and let V be the corresponding object of $V_{\mathrm{qnilp}}(X)$. Then

$$V = \mathcal{O}_{X^{\mathrm{ket}}}(aD)e,$$

where $\mathcal{O}_{X^{\mathrm{ket}}}(aD)$ is as in (2.6). Locally on X^{ket}, V has a basis $z^{-a}e$ which belongs to the kernel of the connection, and L is the inverse image of the local system $\mathbf{C}z^{-a}e$ on X^{ket}. Let $V' = \varepsilon_*(V)$ and let ∇' be the induced connection on V'. Then $V' = \mathcal{O}_X(\lfloor a \rfloor D)e$, where $\lfloor a \rfloor$ is the greatest integer which is not strictly bigger than a, and the basis $z^{-\lfloor a \rfloor}e$ of V' satisfies $\nabla'(z^{-\lfloor a \rfloor}e) = (a - \lfloor a \rfloor)d\log(z) \otimes z^{-\lfloor a \rfloor}e$. (Note that for a meromorphic function g on X, $\mathrm{div}\,(g) + aD \geq 0$ if and only if $\mathrm{div}\,(g) + \lfloor a \rfloor D \geq 0$. Hence $\varepsilon_*(\mathcal{O}_{X^{\mathrm{ket}}}(aD)) = \mathcal{O}_X(\lfloor a \rfloor D)$.) This (V', ∇') is the canonical extension of $(\mathcal{O}_{X_{\mathrm{triv}}}e, \nabla)$ on X_{triv} to X, in the sense of [D2] II, 5.5, with respect to the unique section τ of the projection $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ such that $0 \leq \mathrm{Re}\,(\tau) < 1$. More generally we have the following.

PROPOSITION (4.6). *Let*

$$\Delta = \{z \in \mathbf{C}; |z| < 1\}, \quad \Delta^* = \Delta - \{0\}.$$

Let $m, n \geq 0$, $X = \Delta^{n+m}$, $X^* = (\Delta^*)^n \times \Delta^m$, and endow X with the log structure associated to the divisor $X - X^*$. Let $Y_i = \{z \in X; z_i = 0\}$ for $1 \leq i \leq n$ so that $X - X^* = \cup_{1 \leq i \leq n} Y_i$. Let L be an object of $L_{\mathrm{qunip}}(X)$, let V be the corresponding object of $V_{\mathrm{qnilp}}(X)$, and let $V' = \varepsilon_*(V)$. Then as a vector bundle on X with a connection with log poles, V' is the canonical extension of $V|_{X^*}$ in the sense of [D2] II, 5.5 *with respect to the section τ of $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$ such that $0 \leq \mathrm{Re}\,(\tau) < 1$, which means that for any $1 \leq i \leq n$ and for any eigenvalue a of the residue $\mathrm{Res}_i(\nabla')$ of the connection ∇' of V' along Y_i, a is a rational number satisfying $0 \leq a < 1$.*

Proof. By (4.2), we may assume that L is the inverse image of a locally constant sheaf L_1 on X^{ket}. In this case,

$$V = \tau_*^{\text{ket}}(\mathcal{O}_X^{\text{klog}} \otimes_{\mathbf{C}} L) = \mathcal{O}_X^{\text{ket}} \otimes_{\mathbf{C}} L_1.$$

The stalks of $\mathcal{O}_X^{\text{ket}} \otimes_{\mathbf{C}} L_1$ are generated by $\prod_{1 \le j \le n} z_j^{a_j} \otimes l$ over $\mathcal{O}_X$, where a_j are rational numbers such that $0 \le a_j < 1$ and where $l \in L_1$. We have

$$\nabla(\prod_{1 \le j \le n} z_j^{a_j} \otimes l) = \sum_{1 \le k \le n} a_k d\log(z_k) \otimes \prod_{1 \le j \le n} z_j^{a_j} \otimes l,$$

and hence

$$\text{Res}_i(\nabla)(\prod_{1 \le j \le n} z_j^{a_j} \otimes l) = a_i \prod_{1 \le j \le n} z_j^{a_j} \otimes l$$

on Y_i. □

Deligne-Manin's canonical extension $L \mapsto V'$ is not compatible with tensor products, while our $L \mapsto V$ is ; note that there is no contradiction because ε_* is not compatible with tensor products.

5. Log Poincaré Lemma

In this section we prove the following (holomorphic) Poincaré lemma for log smooth morphisms. See [M1], [KtF], [O1] and [O2] for its non-ket analogue.

Theorem (5.1). *Let X and Y be fs log analytic spaces and let $f\colon X \longrightarrow Y$ be a log smooth exact morphism. Then the natural map*

$$(f^{\log})^{-1}(\mathcal{O}_Y^{\text{klog}}) \longrightarrow \omega_{X/Y}^{\cdot,\text{klog}} \tag{5.1.1}$$

is a quasi-isomorphism.

Remark (5.1.1). For a non-exact f, the above is not necessarily valid. For example, let f be a log blow-up along the log structure (6.1.1), then $\omega_{X/Y}^{\cdot,\text{klog}} = \mathcal{O}_X^{\text{klog}}$ and (5.1.1) is not necessarily an isomorphism (cf. [KtF] (1.2)).

We will need the following lemmas.

Lemma (5.2). *Let $\mathcal{S}$ and $\mathcal{S}'$ be fs monoids such that $\mathcal{S}^\times = (\mathcal{S}')^\times = \{1\}$, and let $\mathcal{S}' \longrightarrow \mathcal{S}$ be an exact injective homomorphism. Let A be a finite set, let A' be a subset of A, let*

$$X = \operatorname{Spec}(\mathbf{C}[\mathcal{S}])_{\mathrm{an}} \times \mathbf{C}^A$$

with the log structure associated to $\mathcal{S}$, and let

$$Y = \operatorname{Spec}(\mathbf{C}[\mathcal{S}'])_{\mathrm{an}} \times \mathbf{C}^{A'}$$

with the log structure associated to $\mathcal{S}'$. Consider the natural projection $X \longrightarrow Y$. Let x be a point of $X^{\log}$ lying over the origin of X (= the point of X defined by the map $\mathbf{C}[\mathcal{S}] \longrightarrow \mathbf{C}$ sending $\mathcal{S} - \{1\}$ to 0 and by the origin of $\mathbf{C}^A$), and let $y \in Y^{\log}$ be the image of x. Then the map $(5.1.1)_x \colon \mathcal{O}^{\mathrm{klog}}_{Y,y} \longrightarrow \omega^{\cdot,\mathrm{klog}}_{X/Y,x}$ is a homotopy equivalence of complexes of $\mathcal{O}^{\mathrm{klog}}_{Y,y}$-modules.

Lemma (5.3). *Let $f \colon X' \longrightarrow Y'$ be a log smooth exact morphism of fs log analytic spaces. Let $p' \in X'$, and q' its image in Y'. Then locally on X' and on Y' there is a commutative diagram*

$$\begin{array}{ccc} (X', p') & \longrightarrow & (X, p) \\ \big\downarrow & & \big\downarrow \\ (Y', q') & \xrightarrow{\ i\ } & (Y, q) \end{array}$$

of pointed fs log analytic spaces with i being a strict closed immersion, where $X \longrightarrow Y$ is as in 5.2, and p and q denote the origins of both spaces, such that the induced morphism $X' \longrightarrow X \times_Y Y'$ is a strict open immersion.

First we show that 5.1 follows from 5.2. In fact, let $f \colon X \longrightarrow Y$ be as in 5.1 and let x be a point of $X^{\log}$, and $y \in Y^{\log}$ be the image of x. Then we see by 5.3 that $(X, x) \longrightarrow (Y, y)$ is obtained locally on X and on Y from $(X, x) \longrightarrow (Y, y)$ in 5.2 by a base change with respect to a strict closed immersion. Since the homotopy survives under the base change, 5.2 implies 5.1.

Next we prove 5.3. By (2.3) (see also [NC2] (A.2)), we may assume that there exist a chart by an injection $h \colon Q \oplus \mathbf{Z}^s \longrightarrow P \oplus \mathbf{Z}^r$ for f such that

$Q \longrightarrow (M_{Y'}/\mathcal{O}^{\times}_{Y'})_{q'}$ and $P \longrightarrow (M_{X'}/\mathcal{O}^{\times}_{X'})_{p'}$ are isomorphisms, such that $Y' \longrightarrow (\operatorname{Spec}\mathbf{C}[Q\oplus\mathbf{Z}^s])_{\mathrm{an}}$ is a strict closed immersion, and such that $X' \longrightarrow Y' \times_{(\operatorname{Spec}\mathbf{C}[Q\oplus\mathbf{Z}^s])_{\mathrm{an}}} (\operatorname{Spec}\mathbf{C}[P \oplus \mathbf{Z}^r])_{\mathrm{an}}$ is a strict open immersion. Since $P^{\times} = \{1\}$, the induced map $\mathbf{Z}^s \longrightarrow P$ is zero. We will reduce the problem to the case where $Q \longrightarrow \mathbf{Z}^r$ is also zero. Since $h_0 \colon Q \longrightarrow P$ is injective by the exactness, there exists an $n \geq 1$ such that $Q \longrightarrow \mathbf{Z}^r \longrightarrow \frac{1}{n}\mathbf{Z}^r$ factors through h_0. Since $(\operatorname{Spec}\mathbf{C}[P \oplus (\mathbf{Z}^r \longrightarrow \frac{1}{n}\mathbf{Z}^r)])_{\mathrm{an}}$ is a local isomorphism of log spaces, we may assume that $Q \longrightarrow \mathbf{Z}^r$ factors through h_0. Then by twisting the map $P \oplus \mathbf{Z}^r \longrightarrow \Gamma(X', M_{X'})$ with some $P \longrightarrow \mathbf{Z}^r$ that induces $Q \longrightarrow \mathbf{Z}^r$ via h_0, we may assume that the induced map $Q \longrightarrow \mathbf{Z}^r$ is zero. Thus we can write $h = h_0 \oplus h_1$ for some $h_1 \colon \mathbf{Z}^s \longrightarrow \mathbf{Z}^r$. Factor h_1 into $\mathbf{Z}^s \overset{h'_1}{\hookrightarrow} G \overset{h''_1}{\hookrightarrow} \mathbf{Z}^r$, where $G \cong \mathbf{Z}^s$ and h''_1 is split. Since $(\operatorname{Spec}\mathbf{C}[h'_1])_{\mathrm{an}}$ is a local isomorphism of log spaces, we may assume that h_1 is split. Thus the conclusion follows. □

(5.4). Our proof of 5.2 is a simple modification of the following proof of the fact that for $R = \mathbf{C}[T_1, \cdots, T_n]$ $(n \geq 0)$, $\mathbf{C} \longrightarrow \Omega^{\cdot}_{R/\mathbf{C}}$ is a homotopy equivalence of complexes of $\mathbf{C}$-vector spaces.
For $q \geq 1$, let $s_q \colon \Omega^q_{R/\mathbf{C}} \longrightarrow \Omega^{q-1}_{R/\mathbf{C}}$ be the $\mathbf{C}$-linear map which sends

$$\prod_{i=1}^{n} T_i^{m(i)} \cdot dT_{w(1)} \wedge \cdots \wedge dT_{w(q)} \quad (m(i) \geq 0, w(1) < \cdots < w(q))$$

to 0 if $m(i) \neq 0$ for some $i < w(1)$, and to

$$\Big(\prod_{i \neq w(1)} T_i^{m(i)}\Big) \cdot \frac{1}{m(w(1)) + 1} \cdot T_{w(1)}^{m(w(1))+1} \cdot dT_{w(2)} \wedge \cdots \wedge dT_{w(q)}$$

if $m(i) = 0$ for all $i < w(1)$. Let $s \colon R \longrightarrow \mathbf{C}$ be the $\mathbf{C}$-linear map which sends $\prod_{i=1}^{n} T_i^{m(i)}$ $(m(i) \geq 0)$ to 0 if $m(i) \neq 0$ for some i, and to 1 if $m(i) = 0$ for all i. Then $ds_q + s_{q+1}d$ for $q \geq 1$ and $s + s_1 d$ are the identity maps. Hence $s \colon \Omega^{\cdot}_{R/\mathbf{C}} \longrightarrow \mathbf{C}$ is an inverse of $\mathbf{C} \longrightarrow \Omega^{\cdot}_{R/\mathbf{C}}$ up to homotopy equivalences of complexes of $\mathbf{C}$-vector spaces.

Next we sketch the proof of the case of 5.2 where X is as in 4.5 and $Y = \operatorname{Spec}\mathbf{C}$ with the trivial log structure. Take a point x of $X^{\log}$ lying over $0 \in X$. Take an element $l \in \mathcal{L}^{\mathrm{klog}}_{X,x}$ such that $\exp(l) = z$. Then the $\mathbf{C}$-linear map $s_1 \colon \omega^{1,\mathrm{klog}}_{X,x} \longrightarrow \mathcal{O}^{\mathrm{klog}}_{X,x}$ which sends $l^m dl$ to $\frac{1}{m+1} l^{m+1}$ and $z^{\gamma} l^m dl$

$(\gamma \in \mathbf{Q}_{>0})$ to $\sum_{j=0}^{m}(-1)^j\gamma^{-j-1}\frac{m!}{(m-j)!}z^\gamma l^{m-j}$ is well-defined. Further ds_1 and $s+s_1d$ are the identity maps, where s is the $\mathbf{C}$-linear map $\mathcal{O}^{\mathrm{klog}}_{X,x} \longrightarrow \mathbf{C}$ which sends $z^\gamma l^m$ to zero unless $\gamma = m = 0$. Thus $s\colon \omega^{\cdot,\mathrm{klog}}_{X,x} \longrightarrow \mathbf{C}$ is the desired inverse.

Now we prove 5.2.

Fix a $\mathbf{Q}$-basis B of $\mathcal{S}^{\mathrm{gp}} \otimes \mathbf{Q}$ containing a $\mathbf{Q}$-basis B' of $(\mathcal{S}')^{\mathrm{gp}} \otimes \mathbf{Q}$. For $w \in A$, let l_w be the coordinate function of $\mathbf{C}^A$ corresponding to w. For $w \in B$, fix an element l_w of $\mathcal{L}^{\mathrm{klog}}_{X,x}$ such that $\exp(l_w) = w$. (Note that l_w is not the logarithm of the coordinate function for $w \in A$. This unusual notation is for the simplification of the following calculations.)

Let

$$\widetilde{\mathcal{S}} = \bigcup_{n\geq 1}\mathcal{S}^{1/n} \subset \mathcal{S}^{\mathrm{gp}} \otimes \mathbf{Q}, \qquad \widetilde{\mathcal{S}}' = \bigcup_{n\geq 1}\mathcal{S}'^{1/n} \subset (\mathcal{S}')^{\mathrm{gp}} \otimes \mathbf{Q}.$$

Then, by (3.5),

$$\begin{aligned}
\mathcal{O}_{X,x} &= \mathbf{C}\{\mathcal{S} \times \mathbf{N}^A\} \subset \mathcal{O}^{\mathrm{ket}}_{X,x} = \mathbf{C}\{\widetilde{\mathcal{S}} \times \mathbf{N}^A\},\\
\subset \mathcal{O}^{\mathrm{klog}}_{X,x} &= \mathbf{C}\{\widetilde{\mathcal{S}} \times \mathbf{N}^A\}[\mathbf{N}^B] = \mathbf{C}\{\widetilde{\mathcal{S}} \times \mathbf{N}^A\}[l_w\ ;\ w \in B],\\
\mathcal{O}_{Y,y} &= \mathbf{C}\{\mathcal{S}' \times \mathbf{N}^{A'}\} \subset \mathcal{O}^{\mathrm{ket}}_{Y,y} = \mathbf{C}\{\widetilde{\mathcal{S}}' \times \mathbf{N}^{A'}\},\\
\subset \mathcal{O}^{\mathrm{klog}}_{Y,y} &= \mathbf{C}\{\widetilde{\mathcal{S}}' \times \mathbf{N}^{A'}\}[\mathbf{N}^{B'}] = \mathbf{C}\{\widetilde{\mathcal{S}}' \times \mathbf{N}^{A'}\}[l_w\ ;\ w \in B'].
\end{aligned}$$

(Here $\mathcal{O}_{X,x}$ denotes the stalk of $\mathcal{O}_X$ at the image of x in X, and $\mathcal{O}^{\mathrm{ket}}_{X,x}$ denotes the stalk of $\mathcal{O}^{\mathrm{ket}}_X$ $(= \mathcal{O}_{X^{\mathrm{ket}}})$ at the point of the topoi of sheaves on X^{ket} defined to be the image of x. For $w \in \mathcal{S}$, we identify $w^{1/n} \in \widetilde{\mathcal{S}}$ with $\exp(\frac{1}{n}\cdot l_w) \in \mathcal{O}^{\mathrm{ket}}_{X,x}$. $\{\ \ \}$ means convergent series which will be described below.)

Let $C = (A - A')\coprod(B - B')$ and endow C with a structure of a totally ordered set. For a subset $I = \{w(1), \cdots, w(q)\}$ $(w(1) < \cdots < w(q))$ of C, let

$$\eta_I = dl_{w(1)} \wedge \cdots \wedge dl_{w(q)}.$$

When I ranges over all subsets of C such that $\sharp(I) = q$, η_I forms an $\mathcal{O}^{\mathrm{klog}}_{X,x}$-basis of $\omega^{q,\mathrm{klog}}_{X/Y,x}$.

Let

$$\Lambda = \widetilde{\mathcal{S}} \times \mathbf{N}^{A\sqcup B}.$$

For $\lambda = (\gamma, m)$ $(\gamma \in \widetilde{\mathcal{S}}, m \in \mathbf{N}^{A \sqcup B}) \in \Lambda$, let

$$t_\lambda = \gamma \cdot \prod_{w \in A \sqcup B} l_w^{m(w)} \in \mathcal{O}_{X,x}^{\mathrm{klog}}.$$

Then $\mathcal{O}_{X,x}^{\mathrm{klog}}$ (resp. $\omega_{X/Y,x}^{q,\mathrm{klog}}$) coincides with the set of all formal power series

$$\sum_{\lambda \in \Lambda} a_\lambda \cdot t_\lambda \quad (\text{resp.} \sum_{\substack{\lambda \in \Lambda, I \subset C \\ \sharp(I) = q}} a_{\lambda,I} \cdot t_\lambda \cdot \eta_I)$$

with $a_\lambda \in \mathbf{C}$ (resp. $a_{\lambda,I} \in \mathbf{C}$), satisfying the following (i)–(iii).

(i) There exists an integer $M > 0$ such that if $\lambda = (\gamma, m)$ and a_λ (resp. $a_{\lambda,I}$) $\neq 0$, then $m(w) \leq M$ for any $w \in B$.

(ii) There exists an integer $N > 0$ such that if $\lambda = (\gamma, m)$ and a_λ (resp. $a_{\lambda,I}$) $\neq 0$, then $\gamma \in \mathcal{S}^{1/N}$.

(iii) There exists a homomorphism φ from the multiplicative monoid $\{t_\lambda; \lambda \in \Lambda\}$ to the multiplicative group $\mathbf{R}_{>0}$ such that $\varphi(l_w) = 1$ for any $w \in B$ and such that $\sum_\lambda |a_\lambda| \cdot \varphi(t_\lambda)$ (resp. $\sum_{\lambda,I} |a_{\lambda,I}| \cdot \varphi(t_\lambda)$) converges.

We define $\mathcal{O}_{Y,y}^{\mathrm{klog}}$-linear maps

$$\begin{aligned} s_q \colon \omega_{X/Y,x}^{q,\mathrm{klog}} &\longrightarrow \omega_{X/Y,x}^{q-1,\mathrm{klog}} \quad (q \geq 1) \\ s \colon \mathcal{O}_{X,x}^{\mathrm{klog}} &\longrightarrow \mathcal{O}_{Y,y}^{\mathrm{klog}}. \end{aligned}$$

For $q \geq 1$, define s_q by

$$s_q(\textstyle\sum_{\lambda,I} a_{\lambda,I} \cdot t_\lambda \cdot \eta_I) = \textstyle\sum_{\lambda,I} a_{\lambda,I} s_q(t_\lambda \cdot \eta_I),$$

where $s_q(t_\lambda \cdot \eta_I)$ is as follows.

Write

$$\lambda = (\gamma, m),\ \gamma = \prod_{w \in B} w^{n(w)} \ (\gamma \in \widetilde{\mathcal{S}}, m \in \mathbf{N}^{A \sqcup B}, n(w) \in \mathbf{Q}),$$

and for $w \in A \coprod B$, let

$$t_{\lambda,w} = l_w^{m(w)} \text{ if } w \in A, \qquad t_{\lambda,w} = w^{n(w)} l_w^{m(w)} \text{ if } w \in B.$$

Thus $t_\lambda = \prod_{w\in A\sqcup B} t_{\lambda,w}$. Let v be the smallest element of I. If $t_{\lambda,w} \neq 1$ for some $w \in C$ such that $w < v$, let $s_q(t_\lambda \cdot \eta_I) = 0$. If $t_{\lambda,w} = 1$ for any $w \in C$ such that $w < v$, let

$$s_q(t_\lambda \cdot \eta_I) = (\prod_{\substack{w\in A\sqcup B \\ w\neq v}} t_{\lambda,w}) \cdot (\int t_{\lambda,v} dl_v) \cdot \eta_{I-\{v\}},$$

where

$$\int t_{\lambda,v} dl_v = \frac{1}{m(v)+1} \cdot l_v^{m(v)+1}$$

in the case $v \in A$ or in the case $v \in B$ and $n(v) = 0$, and

$$\int t_{\lambda,v} dl_v = \sum_{j=0}^{m(v)} (-1)^j n(v)^{-j-1} \cdot \frac{m(v)!}{(m(v)-j)!} \cdot v^{n(v)} l_v^{m(v)-j}$$

in the case $v \in B$ and $n(v) \neq 0$. Note

$$d(\int t_{\lambda,v} dl_v) = t_{\lambda,v} dl_v. \tag{5.4.1}$$

We define $s\colon \mathcal{O}^{\mathrm{klog}}_{X,x} \longrightarrow \mathcal{O}^{\mathrm{klog}}_{Y,y}$ by

$$\sum_{\lambda\in\Lambda} a_\lambda \cdot t_\lambda \mapsto \sum_{\lambda\in\Lambda'} a_\lambda \cdot t_\lambda,$$

where $\Lambda' = \widetilde{\mathcal{S}}' \times \mathbf{N}^{A'\sqcup B'} \subset \Lambda$.

Then s_q and s send convergent series to convergent series. We prove this for s_q. The proof for s is similar. For an element $\nu = \sum_{\lambda,I} a_{\lambda,I} \cdot t_\lambda \cdot \eta_I$ of $\omega^{q,\mathrm{klog}}_{X/Y,x}$ $(q \geq 1)$, write $s_q(\nu)$ in the form of a formal power series $\sum_{\lambda,I} b_{\lambda,I} \cdot t_\lambda \cdot \eta_I$. Then the conditions (i)(ii) are satisfied for $s_q(\nu)$ clearly. Take φ with which (iii) is satisfied for ν. We show that (iii) for $s_q(\nu)$ is satisfied with the same φ. Let M and N be as in (i)(ii) for ν. Then by the definition of s_q, we have

$$\sum_{\lambda,I} |b_{\lambda,I}| \cdot \varphi(t_\lambda) \leq c \sum_{\lambda,I} |a_{\lambda,I}| \cdot \varphi(t_\lambda),$$

where

$$c = \max(\varphi(l_w)\ (w \in A),\ M!(M+1)N^{M+1}).$$

Hence (iii) for $s_q(\nu)$ is satisfied.

We prove that $ds_q + s_{q+1}d$ $(q \geq 1)$ and $s + s_1 d$ are the identity maps.

First we prove $ds_q + s_{q+1}d$ is the identity map for $q \geq 1$. Let $\lambda = (\gamma, m) \in \Lambda$, $I \subset C$, $\sharp(I) = q$, and let v be the smallest element of I. If $t_{\lambda,w} \neq 1$ for some $w \in C$ such that $w < v$, then $ds_q(t_\lambda \cdot \eta_I) = 0$. If $t_{\lambda,w} = 1$ for any $w \in C$ such that $w < v$, then by (5.4.1),

$$ds_q(t_\lambda \cdot \eta_I) =$$
$$t_\lambda \cdot \eta_I + \sum_{\substack{w \in A \sqcup B \\ w \neq v}} (\prod_{\substack{u \in A \sqcup B \\ u \neq v,w}} t_{\lambda,u}) \cdot (\int t_{\lambda,v} dl_v) \cdot dt_{\lambda,w} \wedge \eta_{I-\{v\}}.$$

On the other hand,

$$d(t_\lambda \cdot \eta_I) = \sum_{w \in A \sqcup B} x_w, \qquad x_w = (\prod_{\substack{u \in A \sqcup B \\ u \neq w}} t_{\lambda,u}) \cdot dt_{\lambda,w} \wedge \eta_I.$$

Assume first $t_{\lambda,w} \neq 1$ for some $w \in C$ such that $w < v$, and let v' be the smallest element of $\{w \in C \ ; \ t_{\lambda,w} \neq 1\}$. Then for $w \in A \coprod B$,

$$s_{q+1}(x_w) = t_\lambda \cdot \eta_I \quad \text{if} \quad w = v',$$
$$s_{q+1}(x_w) = 0 \quad \text{if} \quad w \neq v',$$

and hence

$$ds_q(t_\lambda \cdot \eta_I) + s_{q+1}d(t_\lambda \cdot \eta_I) = 0 + t_\lambda \cdot \eta_I = t_\lambda \cdot \eta_I$$

in this case. Assume next $t_{\lambda,w} = 1$ for any $w \in C$ such that $w < v$. Then for $w \in A \coprod B$,

$$s_{q+1}(x_w) = -(\prod_{\substack{u \in A \sqcup B \\ u \neq v,w}} t_{\lambda,u}) \cdot (\textstyle\int t_{\lambda,v} dl_v) \cdot dt_{\lambda,w} \wedge \eta_{I-\{v\}} \text{ if } w \neq v,$$
$$s_{q+1}(x_w) = 0 \text{ if } w = v.$$

Hence $ds_q(t_\lambda \cdot \eta_I) + s_{q+1}d(t_\lambda \cdot \eta_I) = t_\lambda \cdot \eta_I$ also in this case.

We next prove $s + s_1 d$ is the identity map. For $\lambda \in \Lambda$, since $\mathcal{S}' \longrightarrow \mathcal{S}$ is exact, $\widetilde{\mathcal{S}} \cap (\mathcal{S}')^{\text{gp}} \otimes \mathbf{Q} = \widetilde{\mathcal{S}}'$, and hence $s_1 dt_\lambda = t_\lambda$ if λ does not belong to Λ' and $s_1 dt_\lambda = 0$ if λ belongs to Λ'.

Hence $s(t_\lambda) + s_1 dt_\lambda = t_\lambda$ for any $\lambda \in \Lambda$.

This proves 5.2.

6. Functoriality of Log Riemann-Hilbert Correspondences

Proposition (6.1). *Let $f\colon X \longrightarrow Y$ be a proper separated log smooth morphism of fs log analytic spaces. Let L be an object of $L_{\mathrm{qunip}}(X)$. Then for any $q \in \mathbf{Z}$, $\mathrm{R}^q f_*^{\log} L$ is an object of $L_{\mathrm{qunip}}(Y)$.*

Definition (6.1.1). Let P be an fs monoid, I its ideal, and $X := (\operatorname{Spec} \mathbf{C}[P])_{\mathrm{an}}$. Then X_I is by definition the saturation of the blow-up with the natural log structure of $(\operatorname{Spec} \mathbf{C}[P])_{\mathrm{an}}$ along the ideal generated by I, i.e. $(\operatorname{Proj}(\bigoplus_n \langle I \rangle^n))^{\mathrm{sat}}$. This is the toric variety with the natural log structure associated to the finite polyhedral cone decomposition of $\operatorname{Hom}(P, \mathbf{Q}_{\geq 0})$ defined by I ([KKMS] I). Note that X_I is covered by $(\operatorname{Spec} \mathbf{C}[P[a^{-1}I]^{\mathrm{sat}}])_{\mathrm{an}}$, $a \in I$, where $P[a^{-1}I]$ is the monoid generated by $\frac{i}{a}$, $i \in I$, in P^{gp}. We say a morphism $f\colon X \longrightarrow Y$ of fs log analytic spaces is a *log blow-up (along the log structure)* if locally on Y, f is the base change with respect to a strict $Y \longrightarrow Y_0 := (\operatorname{Spec} \mathbf{C}[P])_{\mathrm{an}}$ of some $(Y_0)_I \longrightarrow Y_0$, where P is an fs monoid and I is its ideal.

To prove 6.1, we need the following lemma.

Lemma (6.1.2). *Let $a\colon X' \longrightarrow X$ be a log blow-up along the log structure between fs log analytic spaces. Then for any $x \in X$, there exists $x' \in X'$ such that $a(x') = x$ and $(M_X^{\mathrm{gp}}/\mathcal{O}_X^\times)_x \longrightarrow (M_{X'}^{\mathrm{gp}}/\mathcal{O}_{X'}^\times)_{x'}$ is an isomorphism.*

Proof. Taking a chart, we reduce to the case where $X = (\operatorname{Spec} \mathbf{C}P)_{\mathrm{an}}$ and $X' = ((\operatorname{Spec} \mathbf{C}P)_I)_{\mathrm{an}}$. Here P is an fs monoid, $I = \langle a_1, \dots, a_n \rangle$ is its ideal, and we may assume that $P/P^\times \longrightarrow (M_X/\mathcal{O}_X^\times)_x$ is an isomorphism. It is enough to show that there exists a_i such that $(P[a_i^{-1}I]^{\mathrm{sat}})^\times = P^\times$ (because an extension $P[a_i^{-1}I]^{\mathrm{sat}} \longrightarrow \mathbf{C}$ of $P \xrightarrow{x} \mathbf{C}$ gives an x'). To see this, we may assume that $P^\times = \{1\}$. Then the convex hull C of I in $P^{\mathrm{gp}}_{\mathbf{Q}}$ is a rational convex polyhedral set (the convexity here is taken over $\mathbf{Q}$ as in the proof of (A.3.2.2)), and we see that the set of vertices of C is contained in $\{a_1, \dots, a_n\}$ by reducing to the case where $n = 1$ because any vertex of the convex hull of the union of two rational convex polyhedral sets P_1 and P_2 is a vertex of either P_1 or P_2. Take a vertex a_i of C and a supporting hyperplane H of $\{a_i\}$ for C. Then $(P[a_i^{-1}I]^{\mathrm{sat}})^\times = \{1\}$ because $a_i^{-1}I - \{1\}$

is contained in one of the open half spaces determined by the hyperplane $a_i^{-1}H$ so that $P[a_i^{-1}I]^{\mathrm{sat}} \cap a_i^{-1}H = \{1\}$. □

Proof of 6.1. Since $\mathrm{R}^q f_*^{\log} L$ belongs to $L(Y)$ (1.4) by [KjNk] Corollary 0.2, it is enough to treat the quasi-unipotency. First we reduce to the case where f is exact. Since the problem is local on Y, we may assume by (A.4.4) that we have a cartesian diagram

$$\begin{array}{ccc} X & \xleftarrow{\;a\;} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xleftarrow[\;b\;]{} & Y', \end{array}$$

where a, b are blow-ups along log structures such that f' is exact. Further, by (6.1.2), for any $y \in Y$, we can choose $y' \in Y'$ lying over y such that $(M_Y/\mathcal{O}_Y^\times)_y^{\mathrm{gp}} \cong (M_{Y'}/\mathcal{O}_{Y'}^\times)_{y'}^{\mathrm{gp}}$ (so that $\pi_1(y^{\log}) \cong \pi_1(y'^{\log})$). Since $\mathrm{R}f'^{\log}_*(a^{\log})^{-1}L = (b^{\log})^{-1}\mathrm{R}b_*^{\log}Rf'^{\log}_*(a^{\log})^{-1}L = (b^{\log})^{-1}\mathrm{R}f_*^{\log}Ra_*^{\log}(a^{\log})^{-1}L = (b^{\log})^{-1}\mathrm{R}f_*^{\log}L$ (the first and the last equalities are due to [KjNk] Proposition 5.3 (2) and (1) respectively), showing the quasi-unipotency for $\mathrm{R}^q f_*^{\log}L$ at y reduces to that for $\mathrm{R}^q f'^{\log}_*(a^{\log})^{-1}L$ at y'.

We will now prove 6.1 assuming that f is exact. By proper base change, we may assume that $\overset{\circ}{Y} = \operatorname{Spec}\mathbf{C}$. Factor f into $X \xrightarrow{\nu} X' \xrightarrow{f'} Y$ with $\overset{\circ}{\nu}$ being identity and f' strict, and consider the spectral sequence

$$\mathrm{E}_2{}^{p,q} = \mathrm{R}^p f'^{\log}_* \mathrm{R}^q \nu_*^{\log} L \Rightarrow \mathrm{R}^{p+q} f_*^{\log} L.$$

The sheaf $\mathrm{R}^q\nu_*^{\log}L$ is called the sheaf of log nearby cycles, and the commutative group $\pi_1(Y^{\log})$ acts naturally on it because, in virtue of the exactness (or the log injectiveness) of f, $\mathrm{R}^q\nu_*^{\log}L$ on $(X')^{\log} = \overset{\circ}{X} \times Y^{\log}$ is locally on $Y^{\log}$ the pull-back of a sheaf of $\overset{\circ}{X}$. The log nearby cycles will be studied systematically in the last section.

Since the above action of $\pi_1(Y^{\log})$ is clearly compatible with that on $\mathrm{R}^{p+q}f_*^{\log}L$, it is enough to prove that there exist positive integers n, l such that $(T^n - 1)^l = 0$ on $\mathrm{R}^q\nu_*^{\log}L$ for any $T \in \pi_1(Y^{\log})$. Let $x \in X$. Then, by proper base change theorem, the stalk of $\mathrm{R}^q\nu_*^{\log}L$ at a point lying over x is isomorphic to the q-th cohomology group H^q of a fiber of the map

$x^{\log} \longrightarrow x'^{\log}$ with coefficient L, where x (resp. x') is $\{x\}$ endowed with the inverse image log structure from X (resp. Y). Take a chart of X by $P := M_{X,x}/\mathcal{O}^{\times}_{X,x}$, recall that we write as $x_n := x \times_{(\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}} (\mathrm{Spec}\,\mathbf{C}P^{\frac{1}{n}})_{\mathrm{an}}$ (3.4), and take an integer $n > 0$ such that $L|_{x_n^{\log}}$ is unipotent. Let π be the projection $x_n \longrightarrow x$. Then $L|_{x^{\log}} \longrightarrow \pi_*^{\log}(L|_{x_n^{\log}})$ is a split injection and the above H^q is a direct summand of the q-th cohomology group of a fiber of $x_n^{\log} \longrightarrow x'^{\log}$ with coefficient $L|_{x_n^{\log}}$. Putting $l := \dim L|_{x^{\log}}$, we have $(T^n - 1)^l = 0$ on $(\mathrm{R}^q\nu_*^{\log}L)|_{x^{\log}}$. In virtue of (4.2), the compactness of $\overset{\circ}{X}$ implies that the above n can be chosen to be bounded over $\overset{\circ}{X}$. □

THEOREM (6.2). *Let* $f\colon X \longrightarrow Y$ *be a proper separated log smooth morphism in* (fs log analytic space)*. Assume either that* Y *is log smooth over* $\mathbf{C}$ *or that* f *is exact. Let* L *be an object of* $L_{\mathrm{qunip}}(X)$ *and let* $V = \Phi'(L) = \tau_*^{\mathrm{ket}}(\mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathbf{C}} L)$ *(cf.* (6.2.2) (1) *). Then for any* $q \in \mathbf{Z}$*, we have*:

(1) $\mathrm{R}^q f_*^{\log} L$ *is an object of* $L_{\mathrm{qunip}}(Y)$*.*

(2) $\mathrm{R}^q f_*^{\mathrm{ket}} \omega^{\cdot,\mathrm{ket}}_{X/Y}(V)$ *is locally free.*

(3) $\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathbf{C}} \mathrm{R}f_*^{\log} L \cong \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \mathrm{R}f_*^{\mathrm{ket}} \omega^{\cdot,\mathrm{ket}}_{X/Y}(V)$*.*

(4) *In case that* Y *is ideally log smooth,* $\mathrm{R}^q f_*^{\mathrm{ket}} \omega^{\cdot,\mathrm{ket}}_{X/Y}(V)$*, endowed with the Gauss-Manin connection, is an object of* $V_{\mathrm{qnilp}}(Y)$*, which corresponds to* $\mathrm{R}^q f_*^{\log} L$ *with respect to the log Riemann-Hilbert correspondence in* (4.4)*.*

PROOF. (1) is by (6.1).

Consider the "log Poincaré map"

(i) $\quad \mathcal{O}_Y^{\mathrm{klog}} \otimes L \longrightarrow \omega^{\cdot,\mathrm{klog}}_{X/Y}(V),$

which is induced by the equality $\mathcal{O}_X^{\mathrm{klog}} \otimes L = \mathcal{O}_X^{\mathrm{klog}} \otimes V$.

If f is exact, (i) is an isomorphism. This is reduced to the log Poincaré lemma (5.1) by ket localization and dévissage.

Let $W := \mathrm{R}\tau_*^{\mathrm{ket}}\mathrm{R}f_*^{\log}(\mathcal{O}_Y^{\mathrm{klog}} \otimes L)$. By (1), (3.7) (4), and the projection formula, we have that $W = \tau_*^{\mathrm{ket}}(\mathcal{O}_Y^{\mathrm{klog}} \otimes \mathrm{R}f_*^{\log} L)$ and that the m-th cohomology sheaf of it is isomorphic to $\Phi'(\mathrm{R}^m f_*^{\log}(L))$, that is, the vector bundle on Y^{ket} corresponding to $\mathrm{R}^m f_*^{\log}(L)$, and hence is locally free. By applying $\mathrm{R}\tau_*^{\mathrm{ket}}\mathrm{R}f_*^{\log} = \mathrm{R}f_*^{\mathrm{ket}}\mathrm{R}\tau_*^{\mathrm{ket}}$ to (i) (we apply l.h.s. to l.h.s. and r.h.s. to r.h.s. and use (3.7) (4) in r.h.s.), we obtain a canonical map

(ii) $\quad W \longrightarrow \mathrm{R}f_*^{\mathrm{ket}}(\omega^{\cdot,\mathrm{ket}}_{X/Y}(V)),$

which is an isomorphism if f is exact. Further (ii) is an isomorphism in the case where Y is log smooth also. This is reduced to the case where f is exact by log blow-up as follows: We may assume that there is a commutative diagram as in the proof of (6.1). We already know that $\mathrm{R}^m f_*^{\log} L$ corresponds to $\mathcal{H}^m(W)$ via the log Riemann-Hilbert correspondence. Hence $(b^{\log})^{-1}\mathrm{R}^m f_*^{\log} L = \mathrm{R}^m f_*'^{\log}(a^{\log})^{-1}L$ corresponds to $b^{\mathrm{ket}*}\mathcal{H}^m(W)$. Hence $b^{\mathrm{ket}*}\mathcal{H}^m(W) = \mathrm{R}^m f_*'^{\mathrm{ket}}(\omega^{\cdot,\mathrm{ket}}_{X'/Y'}(a^{\mathrm{ket}*}V))$ by the exact case. By applying b_*^{ket} to this and using (6.2.3) below, we obtain $\mathcal{H}^m(W) = \mathrm{R}^m f_*^{\mathrm{ket}}(\omega^{\cdot,\mathrm{ket}}_{X/Y}(V))$. Hence (ii) is an isomorphism. Thus we have (2) and (4).

Next let (iii) be the isomorphism obtained from (ii) by applying $\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} -$. Consider also the canonical map

(iv) $\quad \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} W \longrightarrow \mathcal{O}_Y^{\mathrm{klog}} \otimes \mathrm{R} f_*^{\log}(L).$

This is also an isomorphism by $\mathcal{H}^m(W) = \Phi'(\mathrm{R}^m f_*^{\log}(L))$ for any m.

By composing (iii) and (iv), we obtain (3). □

Problem (6.2.1). We assumed in (6.2) either that Y is log smooth over $\mathbf{C}$ or that f is exact. Find a unified condition under which the conclusions of (6.1) (2)–(4) hold.

Remarks (6.2.2). (1) In (6.2), note that $L_{\mathrm{qunip}}(X)$ was defined for any fs log analytic space X in (4.1). (When $L = \mathbf{C}$, $V = \mathcal{O}_X^{\mathrm{ket}}$ on any X.) When X is ideally log smooth, V is the corresponding object in $V_{\mathrm{qnilp}}(X)$.

(2) We can prove the case of (6.2) where f is exact without the use of (6.1) as follows: First we prove (3). For simplicity we explain here the case of (3) where $L = \mathbf{C}$. By log Poincaré lemma (5.1) and the projection formula, we have $\mathcal{O}_Y^{\mathrm{klog}} \otimes \mathrm{R} f_*^{\log}\mathbf{C} = \mathrm{R} f_*^{\log}\omega^{\cdot,\mathrm{klog}}_{X/Y}$. Since f is saturated after ket localization by (A.4.3), we deduce (3) from the above equality and (8.6.6). For general L, see (8.6.2). Once we have (3), the rest follows: Since $\mathrm{R}^q f_*^{\log} L \in L(Y)$ for any q by [KjNk] Corollary 0.2, (3) implies (1). Further, by (1), (3), and (3.7) (4), $\Phi'(\mathrm{R}^q f_*^{\log} L) = \mathrm{R}^q f_*^{\mathrm{ket}}\omega^{\cdot,\mathrm{ket}}_{X/Y}(V)$. Thus we have (2) and (4). ((2) can be deduced from (3) also by (A.2) without the use of quasi-unipotency.)

Lemma (6.2.3). *Let $a\colon X' \longrightarrow X$ be a log blow-up along log structure between log smooth fs log analytic spaces. Then $\mathcal{O}_X^{\mathrm{ket}} \longrightarrow \mathrm{R}a_*^{\mathrm{ket}}\mathcal{O}_{X'}^{\mathrm{ket}}$ is an isomorphism.*

Proof. By [KKMS] I, Corollary 1 c) to Theorem 12, GAGA ([SGA 1] XII 4.2), and (3.7) (2). □

Theorem (6.3). *In* (6.2), *assume that f satisfies the condition that for any $x \in X$, the cokernel of $M^{\mathrm{gp}}_{Y,f(x)}/\mathcal{O}^{\times}_{Y,f(x)} \longrightarrow M^{\mathrm{gp}}_{X,x}/\mathcal{O}^{\times}_{X,x}$ is torsion free, and that L is in $L_{\mathrm{unip}}(X)$. Let $V = \tau_*(\mathcal{O}^{\log}_X \otimes_{\mathbf{C}} L)$. Then for any $q \in \mathbf{Z}$, we have*:

(1) $\mathrm{R}^q f^{\log}_* L$ *is in* $L_{\mathrm{unip}}(Y)$.

(2) $\mathrm{R}^q f_* \omega^{\cdot}_{X/Y}(V)$ *is locally free.*

(3) $\mathcal{O}^{\log}_Y \otimes_{\mathbf{C}} \mathrm{R}f^{\log}_* L \cong \mathcal{O}^{\log}_Y \otimes_{\mathcal{O}_Y} \mathrm{R}f_* \omega^{\cdot}_{X/Y}(V)$.

(4) *In case that Y is ideally log smooth,* $\mathrm{R}^q f_* \omega^{\cdot}_{X/Y}(V)$, *endowed with the Gauss-Manin connection, is an object of* $V_{\mathrm{nilp}}(Y)$, *which corresponds to* $\mathrm{R}^q f^{\log}_* L$ *with respect to the log Riemann-Hilbert correspondence in* (1.6).

Proof. First (1) is proved similarly to (6.1): By the torsion freeness assumption, the n in the proof of 6.1 can be taken to 1 ; note that the torsion freeness of the cokernel of $f^{-1}(M^{\mathrm{gp}}_Y/\mathcal{O}^{\times}_Y) \longrightarrow M^{\mathrm{gp}}_X/\mathcal{O}^{\times}_X$ is stable under base change with respect to log blow-ups. In fact the assumption that Y is log smooth or that f is exact is not necessary for (1).

Next, writing $L^q := \mathrm{R}^q f^{\log}_* L$, we see that $\tau_*(\mathcal{O}^{\log}_Y \otimes L^q) \longrightarrow \varepsilon_* \tau^{\mathrm{ket}}_*(\mathcal{O}^{\mathrm{klog}}_Y \otimes L^q)$ is an isomorphism by (1), (3.7) (3), (4), and (5). On the other hand, by (6.2) (3), $\varepsilon_* \tau^{\mathrm{ket}}_*(\mathcal{O}^{\mathrm{klog}}_Y \otimes L^q) \cong \varepsilon_* \mathrm{R}^q f^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y}(\varepsilon^* V) \overset{3.7\ (2)}{=} \mathrm{R}^q f_* \varepsilon_* \omega^{\cdot,\mathrm{ket}}_{X/Y}(\varepsilon^* V) = \mathrm{R}^q f_* \omega^{\cdot}_{X/Y}(V) =: V^q$. Thus we have (2), (4), and a natural isomorphism $\mathcal{O}^{\log}_Y \otimes L^q = \mathcal{O}^{\log}_Y \otimes V^q$ for any q. The rest is to construct a natural map between both sides of (3). As in (6.2), let W be $\tau_*(\mathcal{O}^{\log}_Y \otimes \mathrm{R}f^{\log}_* L)$. Then we have a map $W \longrightarrow \mathrm{R}f_*(\omega^{\cdot}_{X/Y}(V))$. By composing τ^* of it with the isomorphism $\mathcal{O}^{\log}_Y \otimes_{\mathcal{O}_Y} W \xrightarrow{\cong} \mathcal{O}^{\log}_Y \otimes \mathrm{R}f^{\log}_* L$, we obtain the required map. □

Remark (6.3.1). The torsion freeness condition in (6.3) was introduced in [KtF]. As [KtF], (6.3) may be directly proved by the "non-ket version of the log Poincaré lemma" not via (6.2). We do not pursue it in this paper.

Remark (6.3.2). This (6.3) is a generalization of results of T. Matsubara [M1], [M2], [M3], F. Kato [KtF], and S. Usui [U1], [U2].

REMARK (6.3.3). Here let $f\colon X \longrightarrow Y$ be a log smooth morphism of fs log analytic spaces. When f is integral, the torsion freeness condition in (6.3), that is,

(a) for any $x \in X$, the cokernel of $M^{\mathrm{gp}}_{Y,f(x)}/\mathcal{O}^{\times}_{Y,f(x)} \longrightarrow M^{\mathrm{gp}}_{X,x}/\mathcal{O}^{\times}_{X,x}$ is torsion free

is equivalent to

(b) each fiber of $\overset{\circ}{f}$ is reduced;

and is equivalent also to

(c) f is saturated ((A.4)).

For a general (not necessarily integral) log smooth f, (c) implies (a) and (b). These are corollaries of Tsuji's results in [T1]. See Appendix (A.5) for the proofs. We remark that for general log smooth f, neither (a) nor (b) does not imply the other.

THEOREM (6.4). *In* (6.2)*, assume that each stalk of $M_Y/\mathcal{O}^{\times}_Y$ is a free monoid. Then $\mathrm{R}^q f_* \omega^{\cdot}_{X/Y}(V)$ is locally free for any q.*

PROOF. As in the proof of (6.3), $\varepsilon_* \mathrm{R}^q f^{\mathrm{ket}}_* \omega^{\cdot,\mathrm{ket}}_{X/Y}(\varepsilon^* V) = \mathrm{R}^q f_* \omega^{\cdot}_{X/Y}(V)$. Thus 6.2 (2) together with the following lemma implies the theorem. □

LEMMA (6.4.1). *Let X be an fs log analytic space such that each stalk of $M_X/\mathcal{O}^{\times}_X$ is a free monoid. Then for a locally free $\mathcal{O}_{X^{\mathrm{ket}}}$-module F of finite rank on X^{ket}, $\varepsilon_*(F)$ is a locally free $\mathcal{O}_X$-module of finite rank.*

PROOF. It is sufficient to prove that the stalk $\varepsilon_*(F)_x$ is flat over $\mathcal{O}_{X,x}$ for any $x \in X$. We may assume that there are a chart $P \to M_X$ and an integer $n \geq 1$ such that $P \cong \mathbf{N}^r$ for some $r \geq 0$ and such that the pull back of F on the ket site of $X_n = X \times_{(\mathrm{Spec}\,\mathbf{C}P)_{\mathrm{an}}} (\mathrm{Spec}\,\mathbf{C}P^{\frac{1}{n}})_{\mathrm{an}}$ comes from a locally free $\mathcal{O}_{X_n}$-module F' of finite rank on X_n. Let $G = \mathrm{Aut}\,(X_n/X)$. Then $\varepsilon_*(F)_x$ is identified with the G-invariant part of $\oplus_y F'_y$ where y ranges over all points of X_n lying over x. Since F'_y are flat over $\mathcal{O}_{X_n,y}$ for all y and since the underlying morphism of analytic spaces $X_n \to X$ is flat, $\oplus_y F'_y$ is flat over $\mathcal{O}_{X,x}$. The G-invariant part is a direct summand over $\mathcal{O}_{X,x}$, and hence it is also flat over $\mathcal{O}_{X,x}$. □

7. Log Hodge to de Rham Degeneration

Recall the following classical result, due to Deligne ([D1], 5.5). Let Y be a scheme of characteristic zero and let $f : X \to Y$ be a proper and smooth morphism. Then the relative Hodge to de Rham spectral sequence

$$E_1^{pq} = R^q f_* \Omega^p_{X/Y} \Rightarrow R^{p+q} f_* \Omega^\cdot_{X/Y}$$

degenerates at E_1 and the sheaves $R^q f_* \Omega^p_{X/Y}$ are locally free of finite type (so that their formation commutes with any base change). Variants of this result for Hodge and de Rham cohomology with log poles were given first by Steenbrink ([Ste1], [Ste2]), then much later by Illusie [I1] and Cailotto [C] using the method of [DI], and by Fujisawa [Fjs1], Kato-Matsubara-Nakayama [KMN] and Kawamata [Kw] using different approaches. We give here a generalization of these, based on the method of [DI], combined with the results of Section 6. Before stating it, recall that to any fs log scheme X is associated the *Kummer étale site* of X, denoted X_{ket}, whose objects are fs log schemes which are Kummer étale over X ([NC1], [I3]). There is a natural morphism of ringed sites

$$\varepsilon : X_{\mathrm{ket}} \to X_{\mathrm{et}},$$

where X_{et} denotes the classical étale site of the underlying scheme. For a map $f : X \to Y$ of fs log schemes, we denote by $\omega^p_{X/Y}$ the sheaf of relative p-differential forms, and we put

$$\omega^p_{X/Y,\mathrm{ket}} := \varepsilon^* \omega^p_{X/Y}.$$

If U is Kummer étale over X, then $\omega^p_{X/Y,\mathrm{ket}}|U = \omega^p_{U/Y}$. We denote by $\omega^\cdot_{X/Y,\mathrm{ket}}$ the corresponding de Rham complex. If $f : X \to Y$ is a map of fs log schemes, we still denote by f the induced map on Kummer étale sites.

Theorem (7.1). *Let Y be an fs log scheme of characteristic zero, and let $f : X \to Y$ be a proper, log smooth and exact morphism.*

Then :

(1) The sheaves $R^q f_* \omega^p_{X/Y,\mathrm{ket}}$ are locally free of finite type over Y_{ket} and commute with base change by any morphism $g : Y' \to Y$ of fs log schemes (i. e. if $f' : X' \to Y'$ is the fs pull-back of f by g, the canonical base change map $g^* R^q f_* \omega^p_{X/Y,\mathrm{ket}} \to R^q f'_* \omega^p_{X'/Y',\mathrm{ket}}$ is an isomorphism).

(2) The relative Hodge to de Rham spectral sequence

$$E_1^{pq} = R^q f_* \omega^p_{X/Y,\mathrm{ket}} \Rightarrow R^{p+q} f_* \omega^{\cdot}_{X/Y,\mathrm{ket}}$$

degenerates at E_1.

We prove this theorem.

(7.1.1). First observe that by the argument in 3.7 (1) (or Kummer étale descent, using ([I3], 3.6)), the adjunction map

$$\mathcal{O}_X \to R\varepsilon_* \mathcal{O}_{X,\mathrm{ket}}$$

is an isomorphism, hence, by the projection formula, so is the adjunction map

$$\omega^p_{X/Y} \to R\varepsilon_* \omega^p_{X/Y,\mathrm{ket}} \ \ (\text{resp. } \omega^{\cdot}_{X/Y} \to R\varepsilon_* \omega^{\cdot}_{X/Y,\mathrm{ket}}).$$

It follows that the sheaf $R^q f_* \omega^p_{X/Y,\mathrm{ket}}$ (resp. $R^n f_* \omega^{\cdot}_{X/Y,\mathrm{ket}}$) on Y_{ket} is the sheaf associated to the presheaf $U \mapsto H^q(X_U, \omega^p_{X_U/Y_U})$ (resp. $H^n(X_U, \omega^{\cdot}_{X_U/Y_U})$). In particular, if $\tilde{y}$ is a *log geometric* point of Y, and $Y_{(\tilde{y})}$ denotes the strict log localization of Y at $\tilde{y}$ and $X_{(\tilde{y})} = X \times_Y Y_{(\tilde{y})}$, then we have

$$(R^q f_* \omega^p_{X/Y,\mathrm{ket}})_{\tilde{y}} = H^q(X_{(\tilde{y})}, \omega^p_{X/Y} | X_{(\tilde{y})})$$

and a similar formula for the stalk of de Rham cohomology sheaves.

(7.1.2). We first treat the case where the underlying scheme of Y is $\operatorname{Spec} \mathbb{C}$. Since f is of finite type, it follows from A.4.3 that after a Kummer étale extension of Y, f becomes saturated. So we may assume that f is saturated. Choose a chart of Y modeled on a sharp fs monoid P. Since the ket site of Y is generated by the $Y_n = (\operatorname{Spec} \mathbb{C}, \frac{1}{n}P)$, in view of 7.1.1 it then suffices to show :

(2') The Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(X, \omega^p_{X/Y}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X/Y)$$

degenerates at E_1.

We prove (2') by the method of ([DI] 2.7). We may assume X connected, of dimension d. Fix a chart $P \to \mathbb{C}$ of Y with $P^\times = 1$. By the standard technique of spreading out (cf. [T2] 4.11.1), one can find an fs log scheme $\mathcal{Y}$

with a chart $P \to \mathcal{O}_{\mathcal{Y}}$, $a \mapsto 0$ if $a \neq 1$, whose underlying scheme is of finite type over $\mathbb{Z}$, a proper, log smooth and saturated map $\mathbf{f} : \mathcal{X} \to \mathcal{Y}$ of relative dimension d, a strict map $g : Y \to \mathcal{Y}$, and a closed point y of $\mathcal{Y}$ having the following properties :

(i) X/Y is the pull-back of $\mathcal{X}/\mathcal{Y}$ by g ;

(ii) the characteristic p of y is $> d$;

(iii) the map $y = \operatorname{Spec} k(y) \to \mathcal{Y}$ extends to $W_2(y)$;

(iv) the sheaves $R^j\mathbf{f}_*\omega^i_{\mathcal{X}/\mathcal{Y}}$ (resp. $R^n\mathbf{f}_*\omega^{\cdot}_{\mathcal{X}/\mathcal{Y}}$) are locally free of finite type (hence commute with any strict base change), and are of constant rank.

Since $\mathbf{f}$ is saturated, the fiber $\mathbf{f}_y : \mathcal{X}_y \to y$ is of Cartier type by ([T1], II 2.14). Thanks to (ii) and (iii), the Hodge to de Rham spectral sequence of $\mathcal{X}_y/y$ degenerates at E_1 by ([KtK1], (4.12)(3)) (in (iii) of (loc. cit.), as a lifting of $(\mathcal{X}_y)'$ over $W_2(y)$ (with log structure induced by that of $\mathcal{Y}$, i. e. the Teichmüller one) one can take the pull-back of $\mathcal{X}_{W_2(y)}$ by the Frobenius endomorphism of $W_2(y)$). The degeneration (2') then follows from (i) and (iv).

Let us now prove (7.1) in the general case. We follow the method of ([D1], 5.5). By the Lefschetz principle and standard limit arguments, we may assume that the underlying scheme of Y is the spectrum of a $\mathbb{C}$-algebra A of finite type. Since f is proper, it follows from 7.1.1 that the sheaves $R^qf_*\omega^p_{X/Y,\text{ket}}$ (resp. $R^nf_*\omega^{\cdot}_{X/Y,\text{ket}}$) on Y_{ket} are of finite presentation (as $\mathcal{O}_{Y,\text{ket}}$-modules) and (ket locally) of the form $\varepsilon^*R^qf_*\omega^p_{X/Y}$ (resp. $\varepsilon^*R^nf_*\omega^{\cdot}_{X/Y}$). Moreover, by (6.2) the sheaves $R^nf_*\omega^{\cdot}_{X/Y,\text{ket}}$ are locally free of finite type, hence commute with any strict base change. Therefore, we may further reduce to the case where A is an artinian $\mathbb{C}$-algebra, and it suffices to show that, for any n, we have

$$(*) \qquad \lg_A(R^nf_*\omega^{\cdot}_{X/Y,\text{ket}}) = \sum_{p+q=n} \lg_A R^qf_*\omega^p_{X/Y,\text{ket}}.$$

Here $\lg_A$ denotes the length of an A-module. Since the sheaves $R^nf_*\omega^{\cdot}_{X/Y,\text{ket}}$ are locally free of finite type, we have

$$(**) \qquad \lg_A(R^nf_*\omega^{\cdot}_{X/Y,\text{ket}}) = (\lg A)(\text{rk}_{\mathbb{C}}R^n(f_y)_*\omega^{\cdot}_{X_y/y,\text{ket}}),$$

where y is the spectrum of the residue field of A with the induced log structure. On the other hand, by the remark above on the structure of the sheaves $R^q f_* \omega^p_{X/Y,\mathrm{ket}}$ and (EGA III, (6.10.5)), we have

$$(***) \qquad \lg_A R^q f_* \omega^p_{X/Y,\mathrm{ket}} \leq \lg(A) \mathrm{rk}_{\mathbb{C}} R^q (f_y)_* \omega^p_{X_y/y,\mathrm{ket}}.$$

Now, by the particular case already treated, we have

$$(****) \qquad \mathrm{rk}_{\mathbb{C}}(R^n (f_y)_* \omega^{\cdot}_{X_y/y,\mathrm{ket}}) = \sum_{p+q=n} \mathrm{rk}_{\mathbb{C}} R^q f_* \omega^p_{X_y/y,\mathrm{ket}}.$$

Putting (**), (***), (****) together we get the desired formula (*).

COROLLARY (7.2). *Under the assumptions of (7.1), the Hodge to de Rham spectral sequence*

$$E_1^{pq} = R^q f_* \omega^p_{X/Y} \Rightarrow R^{p+q} f_* \omega^{\cdot}_{X/Y}$$

degenerates at E_1. If we assume moreover that each stalk of $M_Y/\mathcal{O}_Y^\times$ is a free monoid, then E_1^{pq} is locally free of finite type for any p and q.

PROOF. By (7.1.1) we have

$$R^n f_* \omega^{[a,b]}_{X/Y} = R\varepsilon_* R^n f_* \omega^{[a,b]}_{X/Y,\mathrm{ket}} = \varepsilon_* R^n f_* \omega^{[a,b]}_{X/Y,\mathrm{ket}}$$

for any interval $[a, b]$. Therefore the degeneration of the Hodge to de Rham spectral sequence follows from the (ket) one. The last assertion is a consequence of a variant of (6.4.1). □

REMARKS (7.3). (a) In the case of generalized semistable reduction over a disc, statements essentially equivalent to (7.1) were proven by Steenbrink ([Ste2], (2.9), (2.10)). The proof of (7.2) as a corollary of (7.1) is similar to the deduction of ([Ste2] (2.11)) from those results. The case of semistable reduction along a divisor with normal crossings was considered in [Fjs1] and [I1]. The case of weakly semistable morphisms was dealt with in [Kw]. Let us also mention that versions of 7.2 in the semistable reduction case over log points were proven (independently) by Steenbrink ([Ste3], 4.13) and Kawamata-Namikawa ([KwNm], 4.1).

(b) See [KMN] for analytic versions of (7.1) and (7.2) with coefficients.

(c) We don't know whether (1) and (2) of 7.1 hold when Y is log smooth and f is proper and log smooth, but not necessarily exact. (They hold when Y is log smooth and f is projective, vertical and log smooth, but not necessarily exact ([KMN]). See also [Fjs2], which treats some nonexact cases.)

8. The Nearby Cycles Functor and the Log de Rham Complex

We recall the definition of the classical nearby cycles functor ([SGA 7] XIV). Let X be an analytic space over the open unit disc Δ and let $F \in \mathrm{D}^+(X^*, \mathbf{Z})$, where $X^* = X \times_\Delta \Delta^* = X - X_0$ (X_0 denotes the fiber of X on $0 \in \Delta$). Let $\tilde{\Delta}^*$ be the universal cover of Δ^*, $\overline{X^*} = X \times_\Delta \tilde{\Delta}^*$. We denote by i and $\overline{j}$ the natural maps $X_0 \longrightarrow X$ and $\overline{X^*} \longrightarrow X$ respectively. Let $I = \pi_1(\Delta^*)$. Then the nearby cycles complex $R\Psi F \in \mathrm{D}^+(X_0, \mathbf{Z}[I])$ is defined by $i^* R\overline{j}_* \overline{j}^* F$ endowed with the natural action of I. Here the last $\overline{j}$ means the natural map $\overline{X^*} \longrightarrow X^*$ by abuse of notation.

In this section we define the nearby cycles functor for any morphism of fs log analytic spaces.

(8.1). Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Let $X' := Y \times_{\overset{\circ}{Y}} \overset{\circ}{X} = (\overset{\circ}{X}, \overset{\circ}{f}{}^* M_Y)$. Consider the commutative diagram of fs log analytic spaces, with cartesian square

$$\begin{array}{ccccc} X & \overset{\nu=\nu_{X/Y}}{\longrightarrow} & X' & \longrightarrow & \overset{\circ}{X}. \\ & {\scriptstyle f}\searrow & \downarrow & & \downarrow \overset{\circ}{f} \\ & & Y & \longrightarrow & \overset{\circ}{Y} \end{array}$$

By functoriality of $(-)^{\log}$, we get a commutative diagram of topological spaces with cartesian square

$$\begin{array}{ccccc} X^{\log} & \overset{\nu^{\log}}{\longrightarrow} & X'^{\log} & \overset{\tau_{X'}}{\longrightarrow} & \overset{\circ}{X}, \\ & {\scriptstyle f^{\log}}\searrow & \downarrow & & \downarrow \overset{\circ}{f} \\ & & Y^{\log} & \underset{\tau_Y}{\longrightarrow} & \overset{\circ}{Y} \end{array}$$

where the composite of the upper row is τ_X. In particular, $\nu^{\log}$ is a *proper* map. For $F \in \mathrm{D}^+(X^{\log}, \mathbf{Z})$, the complex

$$R\Psi^{\log}_{X/Y}(F) := R\nu^{\log}_* F \in \mathrm{D}^+(X'^{\log}, \mathbf{Z}) \tag{8.1.1}$$

is called the *complex of log nearby cycles* of F with respect to f. We call $R\Psi^{\log}_{X/Y}\colon \mathrm{D}^+(X^{\log}, \mathbf{Z}) \longrightarrow \mathrm{D}^+(X'^{\log}, \mathbf{Z})$ *the log nearby cycles functor.*

(8.2). Take for Y the unit disc Δ with the log structure defined by the origin and let $f : X \longrightarrow \Delta$ be a vertical morphism, where X is log smooth over $\mathbb{C}$. In this case, $\Delta_{\text{triv}} = \Delta^* = \Delta - \{0\}$ and the verticality of f means that $X_{\text{triv}} = X - X_0 = X^*$. As X is log smooth over $\mathbb{C}$, this implies that the log structure of X is the direct image of the trivial one on X^* (and in particular is determined by the map of analytic spaces underlying f). Since $f' : X' \longrightarrow \Delta$ is a strict map, the topological space $X'^{\log}$ (resp. $X_0'^{\log}$) is deduced from X (resp. X_0) by pull-back by $\Delta^{\log} \longrightarrow \Delta$ (resp. $\{0\}^{\log} = \mathbf{S}^1 := \{z \in \mathbf{C}^\times \ ; \ |z| = 1\} \longrightarrow \{0\}$):

$$X'^{\log} = \Delta^{\log} \times_\Delta X, \quad X_0'^{\log} = \mathbf{S}^1 \times X_0.$$

We also have (trivially) $X'^{\log} - X_0'^{\log} = X^*$. Let $F \in \mathrm{D}^+(X^{\log}, \mathbf{Z})$. By proper base change applied to the cartesian square

$$\begin{array}{ccc} X_0^{\log} & \longrightarrow & X^{\log}, \\ \downarrow & & \downarrow \\ X_0'^{\log} & \longrightarrow & X'^{\log} \end{array}$$

where the vertical maps are given by $\nu^{\log}$, we have a canonical isomorphism

$$R\Psi^{\log}_{X/\Delta}(F)|X_0'^{\log} \xrightarrow{\sim} R\Psi^{\log}_{X_0/\{0\}}(F|X_0^{\log}).$$

We shall compare $R\Psi^{\log}_{X_0/\{0\}}(F|X_0^{\log})$ with the classical nearby cycles complex $R\Psi(F|X^*)$ in the case F is a *locally constant* abelian sheaf. Recall that if $j : X^* \longrightarrow X$ denotes the inclusion, $(j^{\log})^*$ induces an equivalence from the category of locally constant sheaves on $X^{\log}$ to that of locally constant sheaves on X^* (1.2.1), with inverse given by $j_*^{\log}$. In order to state the comparison result we need to relate $\mathrm{D}(X_0, \mathbf{Z}[I])$ and $\mathrm{D}(X_0'^{\log}, \mathbf{Z})$, to which $R\Psi(F|X^*)$ and $R\Psi^{\log}_{X_0/\{0\}}(F|X_0^{\log})$ respectively belong. View I as the automorphism group of the universal cover of $\mathbf{S}^1$:

$$e : \mathbf{R} \longrightarrow \mathbf{S}^1, \quad t \mapsto \exp(2\pi i t).$$

The pull-back by $Id \times e$ sends sheaves on $X_0'^{\log} = X_0 \times \mathbf{S}^1$ to sheaves on $X_0 \times \mathbf{R}$ endowed with an action of I compatible with that of I on $X_0 \times \mathbf{R}$ through the second factor. The push-out by $Id \times p$, where $p : \mathbf{R} \longrightarrow \{0\}$ is the projection, sends I-sheaves on $X_0 \times \mathbf{R}$ to I-sheaves on X_0. Denote by σ the composite

$$\sigma = R(Id \times p)_* \circ (Id \times e)^* : \mathrm{D}^+(X_0'^{\log}, \mathbf{Z}) \longrightarrow \mathrm{D}^+(X_0, \mathbf{Z}[I]).$$

Theorem (8.3). *Let F be a locally constant abelian sheaf on $X^{\log}$. There is a natural isomorphism in* $\mathrm{D}(X_0, \mathbf{Z}[I])$

$$\sigma R\Psi^{\log}_{X_0/\{0\}}(F|X_0^{\log}) \simeq R\Psi(F|X^*).$$

In particular, $\sigma R\Psi^{\log}_{X_0/\{0\}}(\mathbf{Z}_{X_0^{\log}}) \simeq R\Psi(\mathbf{Z})$.

Proof. Let $\tilde{\Delta}^{\log}$ be the universal cover $\mathbf{R}_{\geq 0} \times \mathbf{R}$ of $\Delta^{\log} = \mathbf{R}_{\geq 0} \times \mathbf{S}^1$, and let $\tilde{\Delta}^*$ be the inverse image of $\Delta^* = \Delta - \{0\}$ in $\tilde{\Delta}^{\log}$. Consider the diagram (we call this diagram the big diagram)

$$\begin{array}{ccccc}
\mathrm{D}^+(X^*) & \underset{\text{pull}}{\longrightarrow} & \mathrm{D}^+(X^* \times_{\Delta^*} \tilde{\Delta}^*) & \underset{\text{push}}{\longrightarrow} & \mathrm{D}^+(X^*) \\
\downarrow^{\text{push}} & (1) & \downarrow^{\text{push}} & (2) & \downarrow^{\text{push}} \\
\mathrm{D}^+(X^{\log}) & \underset{\text{pull}}{\longrightarrow} & \mathrm{D}^+(X^{\log} \times_{\Delta^{\log}} \tilde{\Delta}^{\log}) & \underset{\text{push}}{\longrightarrow} & \mathrm{D}^+(X^{\log}) \\
\downarrow^{\text{pull}} & (3) & \downarrow^{\text{pull}} & (4) & \downarrow^{\text{pull}} \\
\mathrm{D}^+(X_0^{\log}) & \underset{\text{pull}}{\longrightarrow} & \mathrm{D}^+(X_0^{\log} \times_{\mathbf{S}^1} \mathbf{R}) & \underset{\text{push}}{\longrightarrow} & \mathrm{D}^+(X_0^{\log}) \\
\downarrow^{\text{push}} & (5) & \downarrow^{\text{push}} & (6) & \downarrow^{\text{push}} \\
\mathrm{D}^+(X_0 \times \mathbf{S}^1) & \underset{\text{pull}}{\longrightarrow} & \mathrm{D}^+(X_0 \times \mathbf{R}) & \underset{\text{push}}{\longrightarrow} & \mathrm{D}^+(X_0 \times \mathbf{S}^1) \\
 & & & & \downarrow^{\text{push}} \\
 & & & & \mathrm{D}^+(X_0)
\end{array}$$

The squares (1), (2), (3), (5), (6) are commutative. ((1) is commutative since the pull backs in (1) are pull backs by etale morphisms. (5) is commutative by the proper base change theorem applied to the proper map $X_0^{\log} \longrightarrow X_0 \times \mathbf{S}^1$. The commutativities of the squares (2)(3)(6) are clear. The square (4) is not commutative, and this point will be considered in 8.3.1.)

By definition, the functor $\mathrm{R}\Psi$ is the composite

$$\mathrm{D}^+(X^*) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X^* \times_{\Delta^*} \tilde{\Delta}^*) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(X) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X_0).$$

By the proper base change theorem applied to the proper map $X^{\log} \longrightarrow X$, this composite coincides with

(composite of right vertical arrows) $\circ$ (composite of upper rows)

in the big diagram.

On the other hand, since X is log smooth over $\mathbf{C}$ and F is locally constant, the image of $F|_{X^*}$ in $\mathrm{D}^+(X^{\log})$ is F ((1.2.1)).

By the following (8.3.1), we have that the image of $F|_{X^*}$ under

(composite of right vertical arrows) $\circ$ (composite of upper rows)

coincides with the image of $F|_{X^*}$ under

(push to $\mathrm{D}^+(X_0)$) $\circ$ (composite of lower rows) $\circ$ (composite of left vertical arrows) in the big diagram.

Hence $\mathrm{R}\Psi(F|_{X^*})$ is canonically isomorphic to the l.h.s. of the isomorphism in (8.3). It is clear that this isomorphism preserves the action of the group $\mathbf{Z} = \pi_1(\Delta^*) = \pi_1(\mathbf{S}^1)$. □

LEMMA (8.3.1). *For locally constant sheaves on $X^{\log}$, the two composites*

$\mathrm{D}^+(X^{\log}) \longrightarrow \mathrm{D}^+(X_0^{\log})$;

(right of (4)) $\circ$ (upper of (4)) $\circ$ (upper of (3)) *and*

(lower of (4)) $\circ$ (lower of (3)) $\circ$ (left of (3))

in the big diagram coincide.

PROOF OF (8.3.1). We have a canonical morphism of functors

(right of (4)) $\circ$ (upper of (4)) $\circ$ (upper of (3))

$\longrightarrow$ (lower of (4)) $\circ$ (lower of (3)) $\circ$ (left of (3))

and we are showing that this morphism gives an isomorphism when it is applied to a locally constant sheaf on $X^{\log}$. This can be checked locally on $X^{\log}$. Since $X^{\log} \times_{\Delta^{\log}} \tilde{\Delta}^{\log}$ is an etale Galois covering of $X^{\log}$ with Galois group $\mathbf{Z}$, locally on $X^{\log}$, the composites

(upper of (4)) $\circ$ (upper of (3)) and

(lower of (4)) $\circ$ (lower of (3))

send a sheaf $\mathcal{F}$ to $R\mathcal{M}ap(\mathbf{Z}, \mathcal{F})$.

Further $X^{\log}$ is a topological manifold with the boundary $X_0^{\log}$ by (1.2.2). So both $X^{\log}$ and $X_0^{\log}$ are locally contractible. Hence, if $\mathcal{F}$ is locally constant, the stalk of $R\mathcal{M}ap(\mathbf{Z}, \mathcal{F}) = \mathcal{M}ap(\mathbf{Z}, \mathcal{F})$ at a point is Map($\mathbf{Z}$, the stalk of $\mathcal{F}$). This proves the lemma. □

The above (8.3) is equivalent to the next.

THEOREM (8.4). *Let $X \longrightarrow \Delta$ be as in* (8.2). *Let F be a locally constant sheaf on X^*. Then the image of F under*

$$\mathrm{D}^+(X^*) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(X^{\log}) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X_0^{\log}) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(X_0 \times \mathbf{S}^1) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X_0 \times \mathbf{R})$$

is canonically isomorphic, as an object with an action of $\mathrm{Aut}\,(\mathbf{R}/\mathbf{S}^1) = \pi_1(\mathbf{S}^1) = \mathbf{Z}$, *to the pull back of* $\mathrm{R}\Psi(F)$ *under* $X_0 \times \mathbf{R} \longrightarrow X_0$.

(Here, as before, the map $X_0^{\log} \longrightarrow X_0 \times \mathbf{S}^1$ is $\nu^{\log}$ for $X_0 \longrightarrow 0$, and $\mathbf{R} \longrightarrow \mathbf{S}^1$ is defined by $t \mapsto \exp(2\pi i t)$.)

(8.5). We prove the equivalence of (8.3) and (8.4). First, by the following lemma, (8.4) implies (8.3).

LEMMA (8.5.1). *Let X be a topological space, and F an abelian sheaf on X. Let p be the first projection $X \times \mathbf{R} \longrightarrow X$. Then $F \xrightarrow{\cong} \mathrm{R}p_*p^*F$.*

PROOF. It follows from [KS] 2.7.8. □

We prove that conversely (8.4) is deduced from (8.3). Let G be the image of F in $\mathrm{D}^+(X_0 \times \mathbf{S}^1)$ under the composite in (8.4), let $\tilde{G}$ be the pull back of G in $\mathrm{D}^+(X_0 \times \mathbf{R})$, and let $p\colon X_0 \times \mathbf{R} \longrightarrow X_0$ be the projection. The isomorphism $\mathrm{R}\Psi(F) \cong \mathrm{R}p_*(\tilde{G})$ of (8.3) induces $p^*\mathrm{R}\Psi(F) \longrightarrow \tilde{G}$. Our task is to prove that the last morphism is an isomorphism. By (8.5.1) and (8.5.2) below, each cohomology sheaf $\mathcal{H}^q(\tilde{G})$ of $\tilde{G}$ satisfies $\mathrm{R}^s p_*\mathcal{H}^q(\tilde{G}) = 0$ for all $s \neq 0$. Hence $\mathcal{H}^q(\mathrm{R}p_*(\tilde{G})) \cong p_*\mathcal{H}^q(\tilde{G})$. Again by (8.5.1) and (8.5.2), $p^*p_*\mathcal{H}^q(\tilde{G}) \xrightarrow{\cong} \mathcal{H}^q(\tilde{G})$. Hence $p^*\mathrm{R}^q\Psi(F) \xrightarrow{\cong} p^*\mathcal{H}^q(\mathrm{R}p_*(\tilde{G})) \cong p^*p_*\mathcal{H}^q(\tilde{G}) \cong \mathcal{H}^q(\tilde{G})$. This proves that $p^*\mathrm{R}\Psi(F) \longrightarrow \tilde{G}$ is an isomorphism.

LEMMA (8.5.2). *For each $q \in \mathbf{Z}$, $\mathcal{H}^q(\tilde{G})$ is isomorphic to the pull back of a sheaf on X_0 by p.*

PROOF OF (8.5.2). By [KS] 2.7.8, it is sufficient to prove that for each $x \in X_0$, the pull back of $\mathcal{H}^q(\tilde{G})$ to $p^{-1}(x) = \mathbf{R}$ is a constant sheaf. Hence it is sufficient to prove that for each $x \in X_0$, the pull back of $\mathcal{H}^q(G)$ to $\tau^{-1}(x) = \mathbf{S}^1$ is a locally constant sheaf. By proper base change theorem applied to the proper map $X_0^{\log} \longrightarrow X_0 \times \mathbf{S}^1$, the pull back of $\mathcal{H}^q(G)$ to $\mathbf{S}^1$ is isomorphic to the q-th cohomology sheaf of the image of F under

$$\text{(8.5.2.1)} \qquad \mathrm{D}^+(X^*) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(X^{\log}) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X_0^{\log}) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(\{x\}^{\log}) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(\mathbf{S}^1).$$

Let $P = M_{X,x}/\mathcal{O}_{X,x}^{\times}$. Then $\{x\}^{\log} = \mathrm{Hom}\,(P^{\mathrm{gp}}, \mathbf{S}^1)$, and the map $\{x\}^{\log} \longrightarrow \mathbf{S}^1$ is the map $\mathrm{Hom}\,(P^{\mathrm{gp}}, \mathbf{S}^1) \longrightarrow \mathbf{S}^1$ induced by the canonical map $\mathbf{Z} = M_{\Delta,0}/\mathcal{O}_{\Delta,0}^{\times} \longrightarrow M_{X,x}/\mathcal{O}_{X,x}^{\times}$, hence is a locally trivial fibration.

This shows that the higher direct images of a locally constant sheaf on $\{x\}^{\log}$ under $\{x\}^{\log} \longrightarrow \mathbf{S}^1$ are locally constant sheaves. Since the image of F in $\mathrm{D}^+(\{x\}^{\log})$ is a locally constant sheaf, this shows that the cohomology sheaves of the image of F in $\mathrm{D}^+(\mathbf{S}^1)$ under (8.5.2.1) are locally constant sheaves. □

THEOREM (8.6). *Let the notation be as in* 8.1. *Assume that f is log smooth and exact. Let L be an object of $L_{\mathrm{qunip}}(X)$ (*4.1*) and let $V = \tau_*^{\mathrm{ket}}(\mathcal{O}_X^{\mathrm{klog}} \otimes_{\mathbf{C}} L)$ (cf.* (6.2.2) (1)*). Then we have*

$$\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathbf{C}} \mathrm{R}\nu_*^{\log} L \cong \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \omega_{X/Y}^{\cdot,\mathrm{ket}/Y}(V) \ \ \textit{on} \ \ X'^{\log},$$

where $\omega_{X/Y}^{q,\mathrm{ket}/Y}(V)$ is defined as $(X'^{\mathrm{ket}} \ni U \mapsto \Gamma(U \times_{X'} X, \omega_{X/Y}^{q,\mathrm{ket}}(V)))$, that is, as $\nu_*^{\mathrm{ket}}\omega_{X/Y}^{q,\mathrm{ket}}(V)$.

PROOF. This is obtained by the isomorphisms

$$\begin{aligned} \text{l.h.s.} &= \mathrm{R}\nu_*^{\log}(\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathbf{C}} L) \\ &= \mathrm{R}\nu_*^{\log}(\omega_{X/Y}^{\cdot,\mathrm{klog}}(V)) \\ &= \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \nu_*^{\mathrm{ket}}\omega_{X/Y}^{\cdot,\mathrm{ket}}(V). \end{aligned}$$

Here the first isomorphism is by the projection formula, the second is induced by the isomorphism (i) in the proof of (6.2), and the last one is by the equality $\mathrm{R}\nu_*^{\log}\mathcal{O}_X^{\mathrm{klog}} = \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \nu_*^{\mathrm{ket}}\mathcal{O}_X^{\mathrm{ket}}$. This equality is by (8.6.6) and (8.6.3) (1). The fact that ν is saturated after ket localization is by (A.3.3), (A.3.4) and (A.4.2) of T. Tsuji as follows: By (A.3.3), there is locally a chart of f by a $\mathbf{Q}$-integral homomorphism h. This h gives also a chart of ν. By (A.3.4) and (A.4.2), after ket localization, h becomes saturated so that ν becomes saturated. $\square$

Remark (8.6.1). In the case of a semistable family of (8.4) (resp. in the situation of (8.8) (1)), by restricting the isomorphism in (8.6) to a fiber of $\overset{\circ}{X}_0 \times \mathbf{S}^1 \longrightarrow \mathbf{S}^1$ (resp. $\overset{\circ}{X} \times \mathbf{S}^1 \longrightarrow \mathbf{S}^1$), tensoring it with $\mathbf{C}$ by the $\mathbf{C}$-homomorphism $\mathcal{O}_0^{\mathrm{klog}} \longrightarrow \mathbf{C}$, and using (3.6) (2) and (8.6.5) (1), we revisit the (non-canonical) isomorphism (in $\mathrm{D}^+(\overset{\circ}{X}_0, \mathbf{C})$) of Steenbrink in [Ste1] (4.14) (defined by φ and ψ in loc. cit. 2.6): $\mathrm{R}\Psi(\mathbf{C}) \cong \omega^{\cdot}_{X_0/0}$ (resp. the isomorphism of Kawamata-Namikawa in the proof of [KwNm] 4.1: $\mathrm{R}\tilde{\rho}_*\mathbf{C}_{\tilde{X}} \cong \omega^{\cdot}_{X/0}$).

Remark (8.6.2). We can deduce the case of (6.2) (3) where f is exact from (8.6) (cf. (6.2.2) (2)) by applying $\mathrm{R}f'^{\log}_*$ to the isomorphism in (8.6), where f' is the induced morphism $X' \longrightarrow Y$. In fact $\mathrm{R}f'^{\log}_*$ of l.h.s. of (8.6) is that of (3) by the projection formula. On the other hand, $\mathrm{R}f'^{\log}_*$ of r.h.s. of (8.6) is $\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \mathrm{R}f'^{\log}_*(\tau^{\mathrm{ket}}_{X'})^{-1}\nu^{\mathrm{ket}}_*\omega^{\cdot,\mathrm{ket}}_{X/Y}(V)$ by the projection formula, which coincides with r.h.s. of (3) by (8.6.4) and (8.6.3) (1).

Here we prove facts referred in the above proof.

Proposition (8.6.3). *Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Assume that the underlying morphism of f is finite. Then the following hold.*

(1) *For a sheaf M of $\mathbf{Q}$-vector spaces on X^{ket}, $\mathrm{R}^q f^{\mathrm{ket}}_* M = 0$ for any $q > 0$.*

(2) $\mathrm{R}^q f^{\log}_* \mathcal{O}_X^{\mathrm{klog}} = 0$ *for any $q > 0$.*

Proof. (1) This is a generalization of (3.7) (2), and easily reduced to it by ket localization on Y. Note that the underlying morphism of any base change morphism of f in the category of fs log analytic spaces is also finite.

(2) We may assume that the underlying morphism of f is an isomorphism. We prove the vanishing in the stalk at $x = y \in X$. Let $P := (M_Y/\mathcal{O}_Y^\times)_y$, $Q := (M_X/\mathcal{O}_X^\times)_x$, and $P_1 := \mathrm{Image}\,(P^{\mathrm{gp}}_{\mathbf{Q}} \longrightarrow Q^{\mathrm{gp}}_{\mathbf{Q}}) \cap Q$ $(\subset Q^{\mathrm{gp}}_{\mathbf{Q}})$.

Then $h\colon P \longrightarrow Q$ factors into $P \xrightarrow{h_1} P_1 \xrightarrow{h_2} Q$ so that $\mathrm{Hom}\,(h, \mathbf{S}^1)$ is the composite of $\mathrm{Hom}\,(h_i, \mathbf{S}^1)$, $i = 1, 2$. Since $\mathrm{Hom}\,(h_1, \mathbf{S}^1)$ is a finite map, the rest is to prove $\mathcal{O}_X^{\mathrm{klog}}$ is acyclic for $\mathrm{Hom}\,(h_2, \mathbf{S}^1)$. Since the cokernel of h_2^{gp} is torsion free, the stalk of the direct image sheaf at a point over x is $\mathrm{H}^q(\mathrm{Ker}\,(\mathrm{Hom}\,(h_2, \mathbf{Z}(1))), \mathcal{O}_{X,x} \otimes_{\mathbf{C}[Q]} \mathbf{C}[Q \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}][l_1, \dots, l_r])$, where r is the rank of Q^{gp}. This vanishes by the next lemma. □

Lemma (8.6.3.1). *Let Q be an fs monoid with $Q^{\times} = \{1\}$. Let $l_1, \dots, l_r$ be a basis of Q^{gp}. Let A be a $\mathbf{C}[Q]$-module, $n \geq 1$, $B := A \otimes_{\mathbf{C}[Q]} \mathbf{C}[Q^{\frac{1}{n}}]$, and $M := B \otimes_{\mathbf{C}} \mathbf{C}[l_1, \dots, l_r]$, where $l_1, \dots, l_r$ are considered as indeterminates, endowed with a $\mathbf{C}$-linear action of $\check{Q} := \mathrm{Hom}\,(Q, \mathbf{Z}(1))$ defined by $g \cdot (a \otimes t \otimes \prod l_j^{m_j}) = a \otimes e^{g(t)} t \otimes \prod (l_j + g(l_j))^{m_j}$, $g \in \check{Q}$, $a \in A$, $t \in Q^{\frac{1}{n}}$, $m_j \geq 0$, $1 \leq j \leq r$. Let $g_1, \dots, g_r \in \check{Q}$ be the dual basis of $l_1, \dots, l_r$, and $K := \langle g_1, \dots, g_s \rangle$, $0 \leq s \leq r$. Then*

(1) $\mathrm{H}^0(K, M) = B^K \otimes_{\mathbf{C}} \mathbf{C}[l_{s+1}, \dots, l_r]$.

(2) $\mathrm{H}^q(K, M) = 0$ *for any* $q > 0$.

Proof. (1) This is proved by induction on s. The case $s = 0$ is clear. To proceed from s to $s+1$, it is enough to show that the equality $\sum_{d \geq 0} b_d l^d = \sum_{d \geq 0} (g \cdot b_d)(l + 2\pi i)^d$, $b_d \in B^K$, implies $b_d = 0$, $d > 0$. Here we write $l = l_{s+1}$, $g = g_{s+1}$ for simplicity. Assume that $b_{d'} = 0$ for $d' > d \geq 1$. Then b_d is g-fixed and $b_{d-1} = 2\pi i d \cdot b_d + g \cdot b_{d-1}$ in B. This shows $b_d = 0$ with the fact that any $b \in B^K$ is uniquely written as the sum of $b'_0, \dots, b'_{n-1} \in B^K$ such that $g \cdot b'_k = e^{\frac{2\pi i k}{n}} b'_k$, $0 \leq k \leq n-1$.

(2) It is enough to show that $g_{j+1} - \mathrm{id}$ is a surjective endomorphism on $M^{\langle g_1, \dots, g_j \rangle}$ for $j = 0, \dots, s-1$. Let $G := \langle g_1, \dots, g_j \rangle$. By (1), it is enough to show that bl^d, $b \in B^G$, $d \geq 0$, belongs to $(g - \mathrm{id})(M^G)$, where we write $l = l_{j+1}$, $g = g_{j+1}$ for simplicity. We proceed by the induction on d. Writing $b = b_0 + \cdots + b_{n-1}$ as in the proof of (1), we may assume that $b = b_k$ for some k. In case $k = 0$, observe $(g - \mathrm{id})(bl^{d+1}) = b((l + 2\pi i)^{d+1} - l^{d+1})$. Then any bl^d belongs to the image. The case $k > 0$ is also done with $(g - \mathrm{id})(bl^d) = b(e^{\frac{2\pi i k}{n}}(l + 2\pi i)^d - l^d)$. □

Proposition (8.6.4). *Let $f\colon X \longrightarrow Y$ be a strict proper separated morphism in* (fs log analytic space). *Then the natural morphism*

$(\tau^{\mathrm{ket}})^{-1}\mathrm{R}f_*^{\mathrm{ket}} \longrightarrow \mathrm{R}f_*^{\log}(\tau^{\mathrm{ket}})^{-1}$ *of functors from* $\mathrm{D}^+(X^{\mathrm{ket}}, \mathbf{Q})$ *to* $\mathrm{D}^+(Y^{\log}, \mathbf{Q})$ *is an isomorphism.*

PROOF. Let $q \geq 0$ and F a sheaf of $\mathbf{Q}$-modules on X^{ket}. It is enough to show that the stalk of $(\tau^{\mathrm{ket}})^{-1}\mathrm{R}^q f_*^{\mathrm{ket}} F \longrightarrow \mathrm{R}^q f_*^{\log}(\tau^{\mathrm{ket}})^{-1}F$ at a point y' of $Y^{\log}$ is an isomorphism. Let y be the image of y' in Y. We may assume that there is a chart $P \longrightarrow \Gamma(Y, M_Y)$ such that the induced map $P \longrightarrow M_{Y,y}/\mathcal{O}_{Y,y}^\times$ is an isomorphism. Let y_n be the unique point of Y_n lying over y and let $(y_n' \in Y_n^{\log})_n$ be their compatible liftings such that $y_1' = y'$. Then the stalk of the source is isomorphic to $(\mathrm{R}^q f_*^{\mathrm{ket}} F)_{y(\log)} = \varinjlim_n (\mathrm{R}^q f_{n*}\varepsilon_*(F|_{X_n}))_{y_n} = \varinjlim_n \mathrm{H}^q(Z_n, (\varepsilon_*(F|_{X_n}))|_{Z_n})$, where $f_n \colon X_n \longrightarrow Y_n$, and $Z_n := (f_n^{\log})^{-1}(y_n') \cong f_n^{-1}(y_n)$. Here the first equality comes from (3.7) (2), and the second one is by proper base change theorem. On the other hand, the stalk of the target is isomorphic to $\mathrm{H}^q(Z_1, ((\tau^{\mathrm{ket}})^{-1}F)|_{Z_1})$. Thus, identifying compact Hausdorff spaces Z_n, $n \geq 1$, by the natural homeomorphisms $Z_n \longrightarrow Z_1 =: Z$, we can reduce the problem to the equality $\varinjlim_n (\varepsilon_*(F|_{X_n}))|_Z = ((\tau^{\mathrm{ket}})^{-1}F)|_Z$. This is checked stalkwise as $\varinjlim_n (\varepsilon_*(F|_{X_n}))_{x_n} = F_{x(\log)}, x \in f^{-1}(y)$. □

In the propositions below, we use the theory of saturated morphisms. See (A.4) for their definition and basic properties.

PROPOSITION (8.6.5). *Let* $f \colon X \longrightarrow Y$ *be a saturated morphism of fs log analytic spaces. Assume that the underlying morphism of* f *is an isomorphism. Then*

(1) $\mathcal{O}_Y^{\mathrm{ket}} \xrightarrow{\cong} \mathrm{R}f_*^{\mathrm{ket}}\mathcal{O}_X^{\mathrm{ket}}$.

(2) $\mathcal{O}_Y^{\mathrm{klog}} \xrightarrow{\cong} \mathrm{R}f_*^{\log}\mathcal{O}_X^{\mathrm{klog}}$.

PROOF. By (8.6.3), the cohomologies of higher degrees of r.h.s. of (1) and (2) vanish.

(1) We show $\mathcal{O}_Y^{\mathrm{ket}} \cong f_*^{\mathrm{ket}}\mathcal{O}_X^{\mathrm{ket}}$. Let $Y' \longrightarrow Y$ be a ket morphism. Then the fiber product $X' := X \times_Y Y'$ in the category of log analytic spaces is already saturated by (A.4) (2). Thus $\Gamma(Y', \mathcal{O}_{Y'}) \xrightarrow{\cong} \Gamma(X', \mathcal{O}_{X'})$.

(2) We show $\mathcal{O}_Y^{\mathrm{klog}} \cong f_*^{\log}\mathcal{O}_X^{\mathrm{klog}}$. We adopt the same notation in the proof of (8.6.3). Since the cokernel of h^{gp} is torsion free by (A.4.1), the stalk of

$f_*^{\log}\mathcal{O}_X^{\text{klog}}$ at a point over x is $\mathrm{H}^0(\mathrm{Ker}\,(\mathrm{Hom}\,(h,\mathbf{Z}(1))),\mathcal{O}_{X,x}\otimes_{\mathbf{C}[Q]}\mathbf{C}[Q\otimes_{\mathbf{N}}\mathbf{Q}_{\geq 0}][l_1,\dots,l_r])$. The rest calculation is reduced to (8.6.3.1) (1) and the next lemma. □

Lemma (8.6.5.1). *In the situation of* (8.6.3.1), *let* P *be an fs submonoid of* Q. *Assume that* $P\longrightarrow Q$ *is saturated and* $P^{\mathrm{gp}}=\langle l_{s+1},\dots,l_r\rangle$. *Then* $A\otimes_{\mathbf{C}P}\mathbf{C}P^{\frac{1}{n}}\longrightarrow B^K$ *is an isomorphism.*

Proof. First we show $B^K=\{\sum_q a_q\otimes_{\mathbf{C}Q}q\ ;\ q\in Q^{\frac{1}{n}},\ a_q\in A,\ g(q)\in\mathbf{Z}(1)$ in $\frac{1}{n}\mathbf{Z}(1)$ for any $g\in K\}$. For any element $b=\sum_{q\in Q^{\frac{1}{n}}}a_q\otimes q$ of B, we can associate a decomposition $b=\sum_{k\in(\mathbf{Z}/n\mathbf{Z})^s}b_k$ to the decomposition as sets $Q^{\frac{1}{n}}=\coprod_{k=(k_1,\dots,k_s)}Q_k^{\frac{1}{n}}$, where $Q_k^{\frac{1}{n}}=\{q\in Q^{\frac{1}{n}}\ ;\ g_j(q)\equiv\dfrac{k_j}{n}2\pi\sqrt{-1}\bmod\mathbf{Z}(1)$ for $1\leq j\leq s\}$. Then b is K-fixed if and only if $b=b_0$ because for $q\in Q_k^{\frac{1}{n}}$, $g_j\cdot q=e^{\frac{2\pi\sqrt{-1}k_j}{n}}q$ $(1\leq j\leq s)$.

Next we show $P^{\frac{1}{n}}\oplus Q\longrightarrow Q_0^{\frac{1}{n}}$ is surjective. Let $a\in Q_0^{\frac{1}{n}}$. Then a is considered as an element of $\mathrm{Hom}\,(Q^*,\frac{1}{n}\mathbf{Z}(1))$, inducing a homomorphism $K\xrightarrow{a'}\mathbf{Z}(1)$. Take an extension $Q^*\xrightarrow{p}\mathbf{Z}(1)$ of a'. Then $pa^{-1}\in(Q^{\frac{1}{n}})^{\mathrm{gp}}$ kills K, so that $(pa^{-1})^n\in Q^{\mathrm{gp}}$ belongs to P^{gp}. Thus $pa^{-1}\in(P^{\frac{1}{n}})^{\mathrm{gp}}$, and a is in the image of $(P^{\frac{1}{n}})^{\mathrm{gp}}\oplus Q^{\mathrm{gp}}$. By [T1] I Proposition 3.8, $P^{\frac{1}{n}}\overset{\mathrm{int}}{\oplus}_P Q\longrightarrow Q^{\frac{1}{n}}$ is exact, where int means that the push out in the category of integral monoids. Thus a is in the image of $P^{\frac{1}{n}}\oplus Q$.

The rest is to show $A\otimes_{\mathbf{C}P}\mathbf{C}P^{\frac{1}{n}}\longrightarrow A\otimes_{\mathbf{C}Q}\mathbf{C}Q^{\frac{1}{n}}$ is injective. Since $Q\oplus_P P^{\frac{1}{n}}\longrightarrow Q^{\frac{1}{n}}$ is injective and exact by saturatedness and (A.4.1), we reduce the problem to

Claim. Let $h\colon P\longrightarrow Q$ be an exact injection of integral monoids. Then $\mathbf{C}[h]$ is universally injective in the category of $\mathbf{C}P$-modules.

Proof of Claim. Let M be a $\mathbf{C}P$-module. Let $s\colon M\otimes_{\mathbf{C}P}\mathbf{C}Q\longrightarrow M$ be the homomorphism which sends $m\otimes q$ $(q\in Q)$ to qm if $q\in P$, and to 0 if $q\notin P$. This is well-defined by the exactness of h and $s\circ(\mathrm{id}_M\otimes\mathbf{C}[h])$ is the identity. □

PROPOSITION (8.6.6). *Let $f\colon X \longrightarrow Y$ be a proper separated morphism of fs log analytic spaces. Assume that, ket locally on Y, f is saturated, that is, there is a ket covering $Y' \longrightarrow Y$ such that the base change morphism $X \times_Y Y' \longrightarrow Y'$ is saturated. Then*

$$\mathrm{R}f_*^{\log}\mathcal{O}_X^{\mathrm{klog}} \overset{\cong}{\longleftarrow} \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \mathrm{R}f_*^{\mathrm{ket}}\mathcal{O}_X^{\mathrm{ket}}.$$

PROOF. Since the problem is ket local on Y, we may assume that f is saturated. Further $f = f' \circ \nu$ as in the proof of (6.1). Applying (8.6.5) to ν, we may assume that $f = f'$, that is, f is strict. Then l.h.s. $\overset{(3.6)\ (3)}{=}$ $\mathrm{R}f_*^{\log}(\mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \mathcal{O}_X^{\mathrm{ket}}) = \mathcal{O}_Y^{\mathrm{klog}} \otimes_{\mathcal{O}_Y^{\mathrm{ket}}} \mathrm{R}f_*^{\log}(\tau^{\mathrm{ket}})^{-1}\mathcal{O}_X^{\mathrm{ket}} =$ r.h.s. by (8.6.4). □

(8.7). In this subsection we collect some variants in the preceeding results (8.3) and (8.4). The first is suggested by A. Ogus to the authors: Let Y be the polydisk Δ^n endowed with the log structure given by $\Delta^n - (\Delta^*)^n$. Let $\mathfrak{h}^n \longrightarrow (\Delta^*)^n$ be a universal cover of $(\Delta^*)^n$, where $\mathfrak{h}$ denotes the upper half plane. Let $f\colon X \longrightarrow Y$ be a vertical morphism of fs log analytic spaces. Let F be an abelian sheaf on $X^* = f^{-1}((\Delta^*)^n)$ and let $\mathrm{R}\Psi(F) := i^{-1}\mathrm{R}j_*\mathrm{R}p_*p^{-1}F$, where $i\colon X_0 \hookrightarrow X$ and $j\colon X^* \hookrightarrow X$ are strict immersions ($X_0 \longrightarrow 0$ is the special fiber of f) and p is the base change map $X^* \times_{(\Delta^*)^n} \mathfrak{h}^n \longrightarrow X^*$.

PROPOSITION (8.7.1). *In the above, assume that X is log smooth over $\mathbf{C}$, that for any $x \in X$, $(M_Y/\mathcal{O}_Y^\times)_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^\times)_x$ is injective and that F is locally constant. Then*

(1) *The complex of log nearby cycles $\mathrm{R}\nu^{\log}_{X_0/\{0\}*}((i^{\log})^{-1}j_*^{\log}F)$ is endowed with the natural action of $\pi_1(Y^{\log}) = \pi_1(0^{\log})$, and coincides with $\mathrm{R}\Psi(F)$ paired with the action of $\pi_1(Y^{\log})$, that is, $\mathrm{R}\tau'_*\mathrm{R}p_*p^{-1}\mathrm{R}\nu_*^{\log}((i^{\log})^{-1}j_*^{\log}F) \overset{\cong}{\longrightarrow} \mathrm{R}\Psi(F)$ as objects with actions of $\pi_1(0^{\log})$, where τ' is the natural map $(X_0)'^{\log} \longrightarrow (\mathring{X}_0)' = \mathring{X}_0$ ($(X_0)'$ is as in (8.1)) and p is the projection $X_0 \times \mathbf{R}^n \longrightarrow X_0 \times 0^{\log}$ ($\mathbf{R}^n$ is regarded as a universal cover of $0^{\log}$).*

(2) *$p^{-1}\mathrm{R}\nu_*^{\log}((i^{\log})^{-1}j_*^{\log}F) \cong p^{-1}\tau'^{-1}\mathrm{R}\Psi(F)$ as objects with actions of $\pi_1(Y^{\log})$.*

Proof. (1) and (2) are variants of (8.3) and (8.4) respectively. The proofs are similar. We use the injectivity assumption when we show that the higher direct image of $(i^{\log})^{-1}j_*^{\log}F|_{x^{\log}}$ by $x^{\log} \longrightarrow x'^{\log}$ $(x \in X, x' = \nu(x))$ is locally constant in the proof of the statement generalizing (8.5.2): By the assumption, $x^{\log} \longrightarrow x'^{\log}$ is homeomorphic to $\mathrm{Hom}\,(h, \mathbf{S}^1)$ for some injection h of abelian groups, hence a topological fibration. Hence the associated higher direct image functors preserve local constantness. □

The proof for the next is essentially a part of that for the above.

Proposition (8.7.2). *Let Y be an fs log analytic space whose underlying analytic space is* $\mathrm{Spec}\,\mathbf{C}$. *Let $\mathbf{R}^n \longrightarrow Y^{\log}$ be a universal cover. Let $f\colon X \longrightarrow Y$ be a morphism of fs log analytic spaces. Assume that for any $x \in X$, $(M_Y/\mathcal{O}_Y^{\times})_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^{\times})_x$ is injective. Let F be a locally constant sheaf F on $X^{\log}$ and let* $\mathrm{R}\Psi(F) := \mathrm{R}\tau_*\mathrm{R}p_*p^{-1}F$, *where p is the base change map $X^{\log} \times_{Y^{\log}} \mathbf{R}^n \longrightarrow X^{\log}$. Then*

(1) *The complex of log nearby cycles* $\mathrm{R}\nu^{\log}_{X/Y*}F$ *is endowed with the natural action of* $\pi_1(Y^{\log})$, *and coincides with* $\mathrm{R}\Psi(F)$ *paired with the action of* $\pi_1(Y^{\log})$, *that is,* $\mathrm{R}\tau'_*\mathrm{R}p_*p^{-1}\mathrm{R}\nu_*^{\log}F \xrightarrow{\cong} \mathrm{R}\Psi(F)$ *as objects with actions of* $\pi_1(Y^{\log})$, *where τ' is the natural map $X'^{\log} \longrightarrow \overset{\circ}{X}{}' = \overset{\circ}{X}$ (X' is as in* (8.1)*).*

(2) $p^{-1}\mathrm{R}\nu_*^{\log}F \cong p^{-1}\tau'^{-1}\mathrm{R}\Psi(F)$ *as objects with actions of* $\pi_1(Y^{\log})$.

Remark (8.7.2.1). In case of normal crossing varieties over the standard log point ((8.8) (1)), the above $\mathrm{R}\Psi$ was introduced in [FN] as a nearby cycles complex.

Remarks (8.8). (1) Y. Kawamata and Y. Namikawa made a similar construction to our (8.1) in the proof of 4.1 in [KwNm]: Let 0 be the standard log point. Let $f\colon X \longrightarrow 0$ be a morphism of fs log analytic spaces which is, locally on X, isomorphic over 0 to the special fiber of a semistable family endowed with the natural log structure (such an f is called a normal crossing variety in [KwNm]). For such an f, they introduced the object $\mathrm{R}\tilde{\rho}_*\mathbf{Z}_{\tilde{X}}$ on $\overset{\circ}{X}$ as a complex of nearby cycles, where $\tilde{\rho}\colon \widetilde{X} \longrightarrow \overset{\circ}{X}$ is the base change of $\nu^{\log}$ by the inclusion $\{1\} \longrightarrow \mathbf{S}^1$ so that $\mathrm{R}\tilde{\rho}_*\mathbf{Z}_{\tilde{X}}$ is the restriction of $\mathrm{R}\nu_*^{\log}\mathbf{Z}$. (Note that in this situation, $\mathrm{R}\nu_*^{\log}\mathbf{Z}$ is an object on $X'^{\log} = \overset{\circ}{X} \times \mathbf{S}^1$.)

(2) Clearly the construction in (8.1) can be done in an l-adic setting: Let $f\colon X \longrightarrow Y$ be a morphism of fs log schemes, n an integer invertible

on Y, and F in $\mathrm{D}^+(X_{\mathrm{ket}}, \mathbf{Z}/n\mathbf{Z})$. Then we can define as $R\Psi^{\log}_{X/Y}(F) := R(\nu^{\mathrm{ket}}_{X/Y})_*F \in \mathrm{D}^+((X')_{\mathrm{ket}}, \mathbf{Z}/n\mathbf{Z})$, where X' and $\nu_{X/Y}$ are defined similarly as in (8.1). When Y is locally of finite type over $\mathbf{C}$ and f is log injective in the sense of [NC1] (5.5.1), the comparison theorem [KjNk] Theorem C.1 implies

$$\varepsilon^{-1}R\Psi^{\log}_{X/Y}(F) = R\Psi^{\log}_{X_{\mathrm{an}}/Y_{\mathrm{an}}}(\varepsilon^{-1}F),$$

where ε denotes the morphism of topoi $(X^{\log}_{\mathrm{an}})^{\sim} \longrightarrow (X_{\mathrm{ket}})^{\sim}$ or $(Y^{\log}_{\mathrm{an}})^{\sim} \longrightarrow (Y_{\mathrm{ket}})^{\sim}$. A problem to be investigated is to compare the algebraic log nearby cycles here with the one studied in [NC2], [Vi1] and [Vi2] (see also [I3]). A partial answer can be found in [NC3] A.2, which is an l-adic analogue of (8.3) and in which $R\nu^{\mathrm{ket}}_*(\mathbf{Z}/n\mathbf{Z})$ for a log smooth family over a trait first appeared.

(3) As explained in 8.2, by proper base change theorem, the composite $\mathrm{D}^+(X^{\log}) \longrightarrow \mathrm{D}^+(X_0 \times \mathbf{S}^1)$ in (8.4) coincides with the composite

$$\mathrm{D}^+(X^{\log}) \underset{\text{push}}{\longrightarrow} \mathrm{D}^+(X \times_\Delta \Delta^{\log}) \underset{\text{pull}}{\longrightarrow} \mathrm{D}^+(X_0 \times \mathbf{S}^1).$$

(4) It is probable that there is a non-ket analogue of (8.6) under some additional assumptions (cf. (6.3)). We do not pursue it here (cf. (6.3.1)).

Problems (8.9). (1) In (8.1), the dimension of the base Y is arbitrary. In [L], G. Laumon also defined a vanishing cycles functor over any base (scheme). What is the relationship of these two? Further, exact morphisms could be a log geometric analogue of morphisms without blow-ups in C. Sabbah's [Sab]. Compare our $\mathrm{R}\Psi^{\log}$ for exact morphisms with the one in [Sab] for morphisms without blow-ups.

(2) Does $\mathrm{R}\Psi^{\log}$ preserve the constructibility or perversity in a suitable sense (cf. [GM])? More specifically, can 8.4 be generalized to the case that F is an analytically constructible sheaf in the sense of [Ve2] 2.1.1 or that F is a weakly $\mathbf{R}$-constructible object of $\mathrm{D}^b(X^*, \mathbf{Z})$ in the sense of [KS] 8.4? How about the commutativity of $\mathrm{R}\Psi^{\log}$ with the Verdier's dualizing functor ([Ve1])?

(3) Compare the isomorphism in (8.6) with results of M. Kashiwara in [Ks].

Appendix

(A.1). Here we prove (3.7) (5). By (3.7) (2), it is enough to show that $M \longrightarrow \varepsilon_*\varepsilon^*M$ is an isomorphism. Let $x \in X$, $n \geq 1$, and by (3.7) (1) and (3.5) (1), it is enough to show that $M_x \xrightarrow{\cong} \mathrm{H}^0(\pi_1^{\mathrm{ket}}(x), M_x \otimes_{\mathbf{C}P} \mathbf{C}P^{\frac{1}{n}})$ is an isomorphism, where P is as in (3.5). This is by (8.6.5.1) in taking P, $\{1\}$, and M_x as Q, P, and A there.

The next is mentioned in (6.2.2) (2).

Proposition (A.2). *Let X be an fs log analytic space and V an $\mathcal{O}_X$-module (resp. an $\mathcal{O}_{X^{\mathrm{ket}}}$-module). If the $\mathcal{O}_X^{\log}$-module τ^*V (resp. the $\mathcal{O}_X^{\mathrm{klog}}$-module $\tau^{\mathrm{ket}*}V$) is locally free of finite rank, then so is V.*

Proof. We will prove the case where V is an $\mathcal{O}_X^{\mathrm{ket}}$-module. The other case is similar and simpler. Let $y \in X^{\log}$, and $x = \tau(y)$. Since $\mathcal{O}_{X,x(\log)}^{\mathrm{ket}} \longrightarrow \mathcal{O}_{X,y}^{\mathrm{klog}}$ has a ring-theoretic section by (3.5) (2), we have an isomorphism $V_{x(\log)} \cong (\mathcal{O}_{X,x(\log)})^{\oplus n}$ $(n \geq 0)$ as $\mathcal{O}_{X,x(\log)}$-modules. Extend it on a ket neighborhood U of x: $(\mathcal{O}_X^{\mathrm{ket}}|_U)^{\oplus n} \xrightarrow{f} V|_U$, and consider the homomorphism $\tau^{\mathrm{ket}*}f$ of locally free $\mathcal{O}_U^{\mathrm{klog}}$-modules. To prove $V|_U$ is locally free of finite rank, we may assume that $X = U$. Since $\tau^{\mathrm{ket}*}f$ is an isomorphism at each point of $\tau^{-1}(x)$, so is on a neighborhood W of $\tau^{-1}(x)$. Hence f is an isomorphism on $\tau(W)$ (take a section again), which is a neighborhood of x because τ is a closed map. □

(A.3). Here we collect basic facts on $\mathbf{Q}$-integral homomorphisms. See also the book [O3].

Definition (A.3.1). Let $h \colon P \longrightarrow Q$ be a homomorphism of integral monoids.

(1) ([T1] I 2.2) h is said to be *integral* if for any homomorphism $P \longrightarrow P'$ of integral monoids, the push out of $P' \longleftarrow P \longrightarrow Q$ in the category of monoids is integral.

(2) h is said to be $\mathbf{Q}$-*integral* if $h \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ is integral.

(3) h is said to be *GD* (it means "going-down") if for any $\mathfrak{q} \in \operatorname{Spec} Q$ and $\mathfrak{p} \in \operatorname{Spec} P$ such that $\mathfrak{p} \subset h^{-1}(\mathfrak{q})$, there exists $\mathfrak{q}' \in \operatorname{Spec} Q$ lying over $\mathfrak{p}$ such that $\mathfrak{q}' \subset \mathfrak{q}$.

(4) h is said to be *weak GD* if for any $\mathfrak{q} \in (\operatorname{Spec} h)^{-1}(P - P^{\times})$, $\operatorname{Spec}(P \longrightarrow Q_{\mathfrak{q}})$ is surjective.

An integral homomorphism is $\mathbf{Q}$-integral by the next criterion (A.3.1.1). A local $\mathbf{Q}$-integral homomorphism of saturated monoids are exact (local means that the inverse image of the maximal ideal is the maximal ideal). Further, a $\mathbf{Q}$-integral homomorphism $h\colon P \longrightarrow Q$ and for submonoids S and T of P and Q respectively such that $h(S) \subset T$, $P_S \longrightarrow Q_T$ is also $\mathbf{Q}$-integral; the composite of $\mathbf{Q}$-integral homomorphisms is $\mathbf{Q}$-integral (cf. [T1] I Propositions 2.8, 2.7 and 2.3).

Proposition (A.3.1.1). *A homomorphism $h\colon P \longrightarrow Q$ of integral monoids is integral (resp. $\mathbf{Q}$-integral) if and only if for any a_1, $a_2 \in P$, b_1, $b_2 \in Q$ such that $h(a_1)b_1 = h(a_2)b_2$, there exist (resp. there exist an integer $n \geq 1$,) a_3, $a_4 \in P$ and $b \in Q$ such that $b_1 = h(a_3)b$ and $a_1a_3 = a_2a_4$ (resp. $b_1^n = h(a_3)b$ and $a_1^n a_3 = a_2^n a_4$).*

Proof. By [KtK1] (4.1). □

Proposition (A.3.2). *Let $h\colon P \longrightarrow Q$ be a homomorphism of integral monoids. Let $P' := P_{h^{-1}(Q^{\times})}$. Consider the following conditions.*

(i) *(cf. [T1]* I 2.11*)* $(P'/P'^{\times} \longrightarrow Q/Q^{\times}) \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ *is injective and for any $b \in Q_{\mathbf{Q}_{\geq 0}}$, there exists $b' \in Q_{\mathbf{Q}_{\geq 0}}$ such that $h^{\mathrm{gp}}_{\mathbf{Q}}(P^{\mathrm{gp}}_{\mathbf{Q}})b \cap Q_{\mathbf{Q}_{\geq 0}} = h^{\mathrm{gp}}_{\mathbf{Q}}(P'_{\mathbf{Q}_{\geq 0}})b'$, where $(-)_{\mathbf{Q}_{\geq 0}}$ and $(-)_{\mathbf{Q}}$ denote $(-) \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ and $(-) \otimes_{\mathbf{Z}} \mathbf{Q}$ respectively.*

(ii) *h is $\mathbf{Q}$-integral.*

(iii) *For any $\mathfrak{q} \in \operatorname{Spec} Q$, $(P_{h^{-1}(\mathfrak{q})} \longrightarrow Q_{\mathfrak{q}}) \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ is exact.*

(iv) *h is GD.*

(v) *$P' \longrightarrow Q$ is weak GD.*

Then

(1) *we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v);

(2) *when P and Q are fine, all the conditions are equivalent.*

Remarks. (a) All the implications in (1) are strict.

(b) (iii) is satisfied if

(iii)$'$ $\operatorname{Spec} \mathbf{C}[h]$ is exact.

When P is saturated, (iii) is equivalent to (iii)$'$. (These are proved easily. Cf. (A.3.2.1) below.)

Proof. (1) (i)$\Rightarrow$(ii). Since $P \longrightarrow P'$ is integral, we may assume that $P = P'$. By replacing h with $h_{\mathbf{Q}_{\geq 0}}$, we may assume that P and Q are $\mathbf{Q}_{\geq 0}$-monoids (= monoids on which $\mathbf{Q}_{\geq 0}$ acts). (Here we use $(P/P^{\times}) \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0} = P_{\mathbf{Q}_{\geq 0}}/(P_{\mathbf{Q}_{\geq 0}})^{\times}$ for any monoid P.) Further, by replacing P and Q with $P/P^{\times}$ and $Q/Q^{\times}$, we may assume that $P^{\times} \cong Q^{\times} = \{1\}$ ([T1] I Proposition 2.5) so that h is injective. Now let $a_1,\ a_2 \in P$, $b_1,\ b_2 \in Q$ such that $h(a_1)b_1 = h(a_2)b_2$. Then there exist $a_3,\ a_4 \in P$ and $b \in Q$ such that $b_1 = h(a_3)b$ and $b_2 = h(a_4)b$. Hence $h(a_1a_3) = h(a_2a_4)$, which implies $a_1a_3 = a_2a_4$ because h is injective. Thus h is integral by (A.3.1.1).

(ii)$\Rightarrow$(iii). For any $\mathfrak{q} \in \operatorname{Spec} Q$, $P_{h^{-1}(\mathfrak{q})} \longrightarrow Q_{\mathfrak{q}}$ is local and $\mathbf{Q}$-integral so that exact after tensoring with $\mathbf{Q}_{\geq 0}$ ([T1] I Proposition 2.8).

(iii)$\Rightarrow$(iv) is seen by applying the next lemma (1) to $P_{h^{-1}(\mathfrak{q})} \longrightarrow Q_{\mathfrak{q}}$ for each $\mathfrak{q} \in \operatorname{Spec} Q$.

(iv)$\Rightarrow$(v). Since $\operatorname{Spec}(P \longrightarrow P')$ is injective, $P' \longrightarrow Q$ is GD.

(2) (v)$\Rightarrow$(i). We may assume that $P = P'$ and that $P^{\times} \cong Q^{\times} = \{1\}$. (Note that $\operatorname{Spec} P/P^{\times} \xrightarrow{\cong} \operatorname{Spec} P$ (P is a monoid) is a natural equivalence of functors from the category of monoids to that of lattices.) Then $\operatorname{Spec}(h)$ is surjective by the assumption. Hence h^{sat} is exact by the next lemma (2). Thus h is injective and $h \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ is also. Since $\iota\colon \operatorname{Spec}(P_{\mathbf{Q}_{\geq 0}}) \xrightarrow{\cong} \operatorname{Spec} P$ (P is a monoid) is another natural equivalence of functors to the category of lattices, we see that the problem is reduced to the following (A.3.2.2) (1) by replacing h with $h \otimes \mathbf{Q}_{\geq 0}$. □

Lemma (A.3.2.1). *Let h be as in* (A.3.2). *Consider the following conditions.*

(i) $h \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ *is exact.*

(i)$'$ h^{sat} *is exact.*

(ii) $\operatorname{Spec} h$ *is surjective.*

Then

(1) *we have* (i) $\Leftrightarrow$ (i)$'$ $\Rightarrow$ (ii);

(2) *when P and Q are fine,* (ii) $\Rightarrow$ (i).

Remarks. (a) The implication (i)$'$ $\Rightarrow$ (ii) in (1) is strict.

(b) When P and Q are fine, the above conditions are equivalent also to

(iii) $\operatorname{Hom}(h, \mathbf{Q}_{\geq 0})$ is surjective.

Here $\mathbf{Q}_{\geq 0}$ is regarded as a monoid by addition. (This equivalence is not used in our text.)

Proof. (1) It is easy to see (i)$\Leftrightarrow$(i)$'$. To prove (i)$' \Rightarrow$ (ii), we may assume that P and Q are saturated by replacing h with h^{sat}. (Note that $\operatorname{Spec} P = \operatorname{Spec} P^{\mathrm{sat}}$ for any integral monoid P.) Let $\mathfrak{p} \in \operatorname{Spec} P$ and we have to show that $\mathfrak{p}$ is in the image of $\operatorname{Spec} h$. Replacing h with $P_{\mathfrak{p}} \longrightarrow Q_{h(P-\mathfrak{p})}$, we may assume that $\mathfrak{p}$ is the maximal ideal $P - P^{\times}$. It is clear that an exact homomorphism is local.

(2) We may assume that P and Q are fs and that $P^{\times}$ is trivial. By replacing Q by $(h^{\mathrm{gp}})^{-1}(Q)$, we may further assume that $Q \subset P^{\mathrm{gp}}$. By the assumption, $\dim P \leq \dim Q = \operatorname{rank}_{\mathbf{Z}}(Q^{\mathrm{gp}}/Q^{\times})$ so that $Q^{\times} = \{1\}$. If $P \neq Q$, there exists a proper face of P that contains an interior point of Q. Thus $\operatorname{Spec} h$ is not surjective. $\square$

Proposition (A.3.2.2). *Let $h\colon P \longrightarrow Q$ be a local homomorphism of integral $\mathbf{Q}_{\geq 0}$-monoids that are finitely generated as $\mathbf{Q}_{\geq 0}$-monoids. Assume that $P^{\times} \cong Q^{\times} = \{1\}$ and that h is weak GD. Then the following hold.*

(1) *(cf. [T1]* I 2.11*) For any $b \in Q$, there exists (a unique) $b' =: \pi(b) \in Q$ such that $h^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = h^{\mathrm{gp}}(P)b'$.*

(2) *The map $\pi\colon Q \longrightarrow Q$ defined in* (1) *satisfies the following.*

(2a) *The image of π is the union of the faces F of Q such that $F \cap P = \{1\}$ (or equivalently $F^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$). Here and hereafter we identify P with $h(P)$.*

(2b) *Q is the union of its submonoids FP for such F's.*

(2c) *For such an F, $F \times P \longrightarrow FP$ is an isomorphism of monoids and $\pi|_{FP}$ corresponds to the first projection. In particular, $\pi|_{FP}$ is a homomorphism of monoids.*

Proof. We identify P with its image $h(P)$.

(1) Let $C := bP^{\mathrm{gp}} \cap Q$. This is a (rational) convex polyhedral set (= the intersection of finite number of affine half spaces) in bP^{gp}. (We omit the adjective "rational" in this paper. All the objects in the geometry of convex bodies are being considered rationally.) Let $v_1, \ldots, v_n$ be all the vertices of C.

Claim. For any facet (= a face of codimension one) F of P, there exists a hyperplane H in bP^{gp} which is parallel to F^{gp} such that C is contained in an affine half space in bP^{gp} determined by H.

Proof of Claim. Let F' be the minimal face of Q that contains F. By the assumption, $\operatorname{Spec} h$ is surjective, so that $F' \cap P = F$. In particular, $F' \neq Q$. Let H' be a supporting hyperplane of F' for Q. Then $H' \cap P = F' \cap P = F$. Hence $H' \cap P^{\mathrm{gp}} = F^{\mathrm{gp}}$, and $H := H' \cap bP^{\mathrm{gp}}$ has the required property. □

By the claim, we have a unique facet C_F of C which is parallel to F^{gp}. Assume that C is not simply generated as a P-set. Then there is a facet F of P such that $v := v_1 \notin C_F$. (Otherwise, $v \in \bigcap_F C_F$ implies $C \subset vP$). Let F_0 be the minimal face of Q that owes v. Then we prove $F_0 \cap P = \{1\}$. To see it, it is enough to show that $p \mid v$ in Q implies $p = 1$. Since v is a vertex of C, $vP^{-1} \cap C = \{v\}$. This implies the above. Now by the assumption of being weak GD, there exists a face F_1 of Q such that $F_1 \supset F_0$ and $F_1 \cap P = F$. Since $F_0 \ni v$, we have $F_1 \supset vF$. Then $F_1 \supset C$ because $v \notin C_F$. Hence $F_1 \supset aP$, where a is any element of C, and $F_1 \supset P$, that is a contradiction.

(2) In the notation of (1), $\pi(b) = v \in F_0$ and $F_0 \cap P = \{1\}$. We will show that for any face F of Q such that $F \cap P = \{1\}$ and for any $b \in F$, $b = \pi(b)$. Let $f \in bP^{\mathrm{gp}} \cap F \subset bP^{\mathrm{gp}} \cap Q = \pi(b)P$. Then there exists a $p \in P$ that satisfies $f = \pi(b)p$. Since F is a face, $F \ni p$ and $p \in F \cap P = \{1\}$. Hence $f = \pi(b)$ and $bP^{\mathrm{gp}} \cap F = \{\pi(b)\}$. In particular $b = \pi(b)$. Further $F^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$ is shown as follows: it is enough to show that $f_1p_1 = f_2p_2$ implies $f_1 = f_2$ ($f_i \in F$, $p_i \in P$, $i = 1, 2$). This is because $\pi(f_ip_i) = f_i$, $i = 1, 2$. This argument also shows $F \times P = FP$. Thus we have (2a) and (2c). Finally (2b) is by (2a). □

Proposition (A.3.3). *Let $f\colon X \longrightarrow Y$ be an exact log smooth morphism of fs log analytic spaces. Then there exists locally on X and on Y a chart $\Gamma(Y, \mathcal{O}_Y) \longleftarrow P \xrightarrow{h} Q \longrightarrow \Gamma(X, \mathcal{O}_X)$ of f with P and Q fs such that h is $\mathbf{Q}$-integral and such that the induced morphism $X \xrightarrow{i} Y \times_{(\operatorname{Spec} \mathbf{C}P)_{\mathrm{an}}} (\operatorname{Spec} \mathbf{C}Q)_{\mathrm{an}}$ is a strict open immersion.*

Proof. Let $x \in X$, $y = f(x)$, and let $P \longrightarrow \Gamma(Y, \mathcal{O}_Y)$ be a chart of Y with P fs such that $P/P^{\times} \xrightarrow{\cong} (M_Y/\mathcal{O}_Y^{\times})_y$. Take a chart $\Gamma(Y, \mathcal{O}_Y) \longleftarrow P \xrightarrow{h} Q \longrightarrow \Gamma(X, \mathcal{O}_X)$ of f with P and Q fs such that the induced morphism $X \xrightarrow{i} Y \times_{(\operatorname{Spec} \mathbf{C}P)_{\mathrm{an}}} (\operatorname{Spec} \mathbf{C}Q)_{\mathrm{an}}$ is a strict open immersion (cf. [KtK1] (3.5)). Localizing Q if necessary, we may assume that $Q/Q^{\times} \xrightarrow{\cong}$

$(M_X/\mathcal{O}_X^\times)_x$. Then, by the assumption that f is exact and the fact that the image of the induced morphism i is an open set, the local homomorphism $P \longrightarrow Q$ satisfies the condition (v) in (A.3.2). Hence it is $\mathbf{Q}$-integral by (2) there. □

Remark. A morphism $f\colon X \longrightarrow Y$ of fs log analytic spaces is said to be $\mathbf{Q}$-integral if for any $x \in X$, the homomorphism $M_{Y,f(x)}/\mathcal{O}_{Y,f(x)}^\times \longrightarrow M_{X,x}/\mathcal{O}_{X,x}^\times$ is $\mathbf{Q}$-integral. In general, there are the implications

$$\text{integral} \Rightarrow \mathbf{Q}\text{-integral} \Rightarrow \text{exact} \Rightarrow \text{log injective}.$$

Further, we have log smooth + log injective ⇒ $\mathbf{Q}$-integral. A. Ogus communicated to the authors that A. Gray showed that when each stalk of $M_Y/\mathcal{O}_Y^\times$ is a free monoid, then the $\mathbf{Q}$-integrality implies the integrality.

Proposition (A.3.4). *Let $h\colon P \longrightarrow Q$ be a $\mathbf{Q}$-integral homomorphism of fs monoids. Then there exists an integer $n \geq 1$ such that the cobase change map $P_n \longrightarrow Q'$; $p \mapsto [p,1]$ of h with respect to $n\colon P \longrightarrow P =: P_n$ in the category of fs monoids is integral, where $Q' := \varinjlim(P \overset{n}{\longleftarrow} P \longrightarrow Q)$ and $[p,q]$ denotes the element represented by $p \in P_n$ and $q \in Q$.*

Proof. Replacing P with $P_{h^{-1}(Q^\times)}$, we may assume that h is local. Further we may assume that $P^\times \cong Q^\times = \{1\}$ since $\varinjlim(P/P^\times \overset{n}{\longleftarrow} P/P^\times \longrightarrow Q/Q^\times) = Q'/A$, where $A = [P_n^\times, Q^\times]$ is the group generated by the images of $P_n^\times$ and $Q^\times$ ([T1] I Proposition 2.5). In the following, we denote $\widetilde{P} = P \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$, $\widetilde{Q} = Q \otimes_{\mathbf{N}} \mathbf{Q}_{\geq 0}$ for short. By (A.3.2.2), we have the map $\pi\colon \widetilde{Q} \longrightarrow \widetilde{\widetilde{Q}}$. The image of Q by this π is the union of the images of $Q \cap \widetilde{P}F$ by the homomorphisms $\pi|_{\widetilde{P}F}$ for faces F of $\widetilde{Q}$ such that $F \cap \widetilde{P} = \{1\}$, which are finitely generated submonoids of $\widetilde{Q}$. Thus there exist finite number of elements $a_1, \dots, a_m \in \pi(Q)$ such that the submonoid of $\widetilde{Q}$ generated $a_1, \dots, a_m$ over Q contains $\pi(Q)$.

Claim 1. There exists an integer $n \geq 1$ such that
(A) For each j, $a_j^{-1}Q \cap \widetilde{P} \subset P^{\frac{1}{n}}$.
(B) $Q^{\mathrm{gp}} \cap \widetilde{P}^{\mathrm{gp}} \subset (P^{\frac{1}{n}})^{\mathrm{gp}}$. (Here and hereafter we denote P_n by $P^{\frac{1}{n}}$.)

Proof of Claim 1. For each $a = a_j$ or 1, there exists an $n \geq 1$ such that $a^{-1}Q^{\mathrm{gp}} \cap \widetilde{P}^{\mathrm{gp}} \subset (P^{\frac{1}{n}})^{\mathrm{gp}}$. Thus (B) follows. Since $\widetilde{P} \cap (P^{\frac{1}{n}})^{\mathrm{gp}} = P^{\frac{1}{n}}$, (A) follows. □

Claim 2. $Q_1 := Q'/\mathrm{tors} = \widetilde{Q} \cap ((P^{\frac{1}{n}})^{\mathrm{gp}} \cdot Q^{\mathrm{gp}})$ in $\widetilde{Q}^{\mathrm{gp}}$.

Proof of Claim 2. In $Q_1^{\mathrm{gp}} = ((P^{\frac{1}{n}})^{\mathrm{gp}} \oplus_{P^{\mathrm{gp}}} Q^{\mathrm{gp}})/\mathrm{tors} = (P^{\frac{1}{n}})^{\mathrm{gp}} \cdot Q^{\mathrm{gp}}$, Q_1 is the saturation of the submonoid generated by $P^{\frac{1}{n}}$ and Q. Then $(\widetilde{Q} \cap Q_1^{\mathrm{gp}})^n \subset Q$, and $Q_1 = \widetilde{Q} \cap Q_1^{\mathrm{gp}} =$ r.h.s. □

Claim 3. $\pi(Q) \subset Q_1$.

Proof of Claim 3. It is enough to show that each a_j belongs to Q_1. Since $a_j \in \pi(Q)$, there exists $p \in \widetilde{P}$, $q \in Q$ such that $a_j p = q$. By (A), $p \in P^{\frac{1}{n}}$. Hence $a_j \in (P^{\frac{1}{n}})^{\mathrm{gp}} \cdot Q^{\mathrm{gp}}$. By Claim 2, $a_j \in Q_1$. □

Now we prove that $P^{\frac{1}{n}} \xrightarrow{h'} Q_1$ is integral. We will use the next proposition. It is easy to see $Q_1^{\times} = \{1\}$ and h' is injective. Let $b \in Q_1$. Then, by the assumption of **Q**-integrality and (A.3.2) (2), $\widetilde{P}^{\mathrm{gp}} b \cap \widetilde{Q} = \widetilde{P} b'$, where $b' = \pi(b)$ belongs to Q_1 because $\pi(Q_1) = \pi(Q) \subset Q_1$ by Claim 3. We will show $(P^{\frac{1}{n}})^{\mathrm{gp}} b \cap Q_1 = P^{\frac{1}{n}} b'$. Let $a \in (P^{\frac{1}{n}})^{\mathrm{gp}}$ such that $ab \in Q_1$. Then there exists $a' \in \widetilde{P}$ such that $ab = a'b'$. Since $bb'^{-1} \in Q_1^{\mathrm{gp}} = (P^{\frac{1}{n}})^{\mathrm{gp}} \cdot Q^{\mathrm{gp}}$, there exist $c \in (P^{\frac{1}{n}})^{\mathrm{gp}}$ and $d \in Q^{\mathrm{gp}}$ such that $a'a^{-1} = cd$. Since $a'a^{-1} \in \widetilde{P}^{\mathrm{gp}}$, $d \in Q^{\mathrm{gp}} \cap \widetilde{P}^{\mathrm{gp}} \subset (P^{\frac{1}{n}})^{\mathrm{gp}}$ by (B). Hence $a' = acd \in (P^{\frac{1}{n}})^{\mathrm{gp}} \cap \widetilde{P} = P^{\frac{1}{n}}$. Thus we conclude that h' is integral by the next proposition (2). □

Proposition (A.3.4.1). *(Cf. [T1]* I 2.11*) Let h be as in* (A.3.2). *Let $P' := P_{h^{-1}(Q^{\times})}$. Consider the following conditions.*

(i) $P'/P'^{\times} \longrightarrow Q/Q^{\times}$ *is injective and for any $b \in Q$, there exists $b' \in Q$ such that $h^{\mathrm{gp}}(P^{\mathrm{gp}})b \cap Q = h^{\mathrm{gp}}(P')b'$.*

(ii) *h is integral.*

Then

(1) *we have* (i) $\Rightarrow$ (ii);

(2) *when P and Q are fine,* (ii) $\Rightarrow$ (i).

Remark. The implication in (1) is strict.

Proof. This is a slight modification of a result of T. Tsuji ([T1] I Proposition 2.11). The proof of (1) is similar to (1) (i) $\Rightarrow$ (ii) in (A.3.2). (2) is easily reduced to the case where h is local and to the result of T. Tsuji mentioned above. □

Next we collect basic facts on saturated morphisms.

Proposition and Definition (A.4). (1) ([T1] I Definition 3.12) A homomorphism $h\colon P \longrightarrow Q$ of integral monoids is said to be *saturated* if h is integral and quasi-saturated.

(2) When P and Q are saturated, h is saturated if and only if h is integral and for any homomorphism $P \longrightarrow P'$ of saturated monoids, the push out of $P' \longleftarrow P \longrightarrow Q$ in the category of monoids is saturated.

(3) (Cf. [T1] II Definition 2.10.) A morphism of fs log analytic spaces $f\colon X \longrightarrow Y$ is said to be *saturated* if, for any $x \in X$, the homomorphism of fs monoids $(M_Y/\mathcal{O}_Y^{\times})_{f(x)} \longrightarrow (M_X/\mathcal{O}_X^{\times})_x$ is saturated, or equivalently the homomorphism of saturated monoids $M_{Y,f(x)} \longrightarrow M_{X,x}$ is saturated.

Proof. (2) [T1] I Proposition 3.14. Use [T1] I Proposition 3.16 for the equivalence in (3). □

A saturated morphism of fs log analytic spaces admits local charts by saturated homomorphisms. Conversely, a morphism of fs log analytic spaces admitting a chart by a saturated homomorphism is saturated ([T1] I Propositions 3.16 and 3.18).

Lemma (A.4.1). *Let $h\colon P \longrightarrow Q$ be a saturated homomorphism of fs monoids with $Q^{\times} = \{1\}$. Then the cokernel of h^{gp} is torsion free.*

Proof. We may assume that h is local, and that $P^{\times} = \{1\}$. Then $Q' := P^{\frac{1}{p}} \oplus_P Q$ satisfies $Q'^{\times} = \{1\}$ for any prime number p. Let $b \in Q^{\mathrm{gp}}$, and $a \in P^{\mathrm{gp}}$ such that $b^p = h^{\mathrm{gp}}(a)$. It is enough to show that $b \in h^{\mathrm{gp}}(P^{\mathrm{gp}})$. Consider the element $[a^{\frac{1}{p}}, b^{-1}]$ of Q'^{gp}. Since it maps to $1 \in (Q^{\frac{1}{p}})^{\mathrm{gp}}$ by $[h^{\frac{1}{p}},$ inclusion$]\colon Q' \longrightarrow Q$, the definition of quasi-saturatedness implies that $[a^{\frac{1}{p}}, b^{-1}] \in Q'^{\times} = \{1\}$. Thus $a^{\frac{1}{p}} = b$ in Q'^{gp} and there is an element of P^{gp} whose image by h^{gp} coincides with b. □

Theorem (A.4.2) (T. Tsuji). *Let $h\colon P \longrightarrow Q$ be an integral homomorphism of fs monoids. Then there exists an integer $n \geq 1$ such that the cobase change map $P_n \longrightarrow Q'$; $p \mapsto [p, 1]$ of h with respect to $n\colon P \longrightarrow P =: P_n$ in the category of fs monoids is saturated, where $Q' := \varinjlim(P \xleftarrow{n} P \longrightarrow Q)$ and $[p, q]$ denotes the element represented by $p \in P_n$ and $q \in Q$.*

PROOF. [T1] I Corollary 5.4. □

Together with (A.3.3), (A.3.4), we have the following.

PROPOSITION (A.4.3). *Let $f\colon X \longrightarrow Y$ be an exact log smooth morphism of fs log analytic spaces. Then locally on X and on Y, after suitable ket base change $Y' \longrightarrow Y$, f becomes saturated.*

To make a general map exact, we use the following lemma.

PROPOSITION (A.4.4). *Let $h\colon Q \longrightarrow P$ be a morphism of fs monoids. Let $f\colon X \longrightarrow Y$ be the induced morphism* $(\mathrm{Spec}\,\mathbf{C}[h])_{\mathrm{an}}$ *of fs log analytic spaces. Then there is a non-empty, finitely generated ideal I of Q such that the base change $X \times_Y Y_I \longrightarrow Y_I$ in the category of fs log analytic spaces is exact, where Y_I is defined in 6.1.1.*

PROOF. Fix a prime $\mathfrak{p} \in \mathrm{Spec}\,P$. Let Q' be the inverse image of $P_{\mathfrak{p}}$ in Q^{gp} by h^{gp}. Take a set of generators $\frac{q_1}{s}, \dots, \frac{q_n}{s}$ of Q' ($q_1, \dots, q_n,\ s \in Q$) and let $J = J_{\mathfrak{p}}$ be the ideal of Q generated by $q_1, \dots, q_n, s$. If we show that $X' := X \times_Y Y_J \longrightarrow Y_J$ is exact at each point $x' \in X'$ lying over $\mathfrak{p} \in \mathrm{Spec}\,P$, then $I := \prod_{\mathfrak{p} \in \mathrm{Spec}\,P} J_{\mathfrak{p}}$ is the desired ideal. Let $x \in X$, $y \in Y$ and $y' \in Y_J$ the images of x', and let $P_x := (M_X/\mathcal{O}_X^{\times})_x \cong P_{\mathfrak{p}}/P_{\mathfrak{p}}^{\times}$, $Q_y := (M_Y/\mathcal{O}_Y^{\times})_y$, and $Q_{y'} := (M_{Y_J}/\mathcal{O}_{Y_J}^{\times})_{y'}$. We claim that s divides all the q_i in $Q_{y'}$. To prove this, we may assume that some q_i divides the other q_j and s in $Q_{y'}$. Since s divides q_i in P_x, $q_i = s$ in $P_{x'} := (M_{X'}/\mathcal{O}_{X'}^{\times})_{x'}$. This implies $q_i = s$ in $Q_{y'}$ so that the claim follows. (Alternatively we can use some general facts to prove this claim. See [NC1] (2.2.5.1) and (2.2.6) (i).)

Now we show that $Q_{y'} \longrightarrow P_{x'}$ is exact. Let $a \in Q_{y'}^{\mathrm{gp}}$ and suppose that the image of a belongs to $P_{x'}$ in $P_{x'}^{\mathrm{gp}}$. Then we have $\overline{(1, a^m)} = \overline{(p, q')}$ in $P_x^{\mathrm{gp}} \oplus_{Q_y^{\mathrm{gp}}} Q_{y'}^{\mathrm{gp}}$ for some $m \geqq 1$, $p \in P_x$ and $q' \in Q_{y'}$. Hence there is an element $q \in Q_y^{\mathrm{gp}}$ that maps to p and $a^m q'^{-1}$. By the definition of Q', q is in the image of Q'. Thus the above claim implies $a^m q'^{-1} \in Q_{y'}$ and $a \in Q_{y'}$. □

(A.5). Here we prove facts announced in (6.3.3). Let the notation be as in there. First assume that f is saturated. Let $x \in X$, $y = f(x)$ and $h\colon P \longrightarrow Q$ a (local) chart of f around x and y with P and Q fs such that $P \xrightarrow{\cong} (M_Y/\mathcal{O}_Y^{\times})_y$ and $Q/Q^{\times} \xrightarrow{\cong} (M_X/\mathcal{O}_X^{\times})_x$. Then h is saturated and

$A := \mathbf{C}[Q]/\langle P - \{1\}\rangle$ is reduced by [T1] I Theorem 6.3 (1)⇒(8). Since a neighborhood of x in the fiber $f^{-1}(y)$ is an open subspace of $(\operatorname{Spec} A)_{\mathrm{an}}$, it is reduced. On the other hand, since $h_F\colon P/F \cap P \longrightarrow Q/F$ is saturated for any face F of Q, the cokernel of h_F^{gp} is torsion free by (A.4.1). Thus we proved (c)⇒(a) and (b).

Next assuming that f is integral, we will prove (a)⇒(c) and (b)⇒(c). Take x, y, and $h\colon P \longrightarrow Q$ as above. Then h is integral. [T1] I Theorem 5.1 and I Theorem 6.3 (9)⇒(1) tell us that to prove that h is saturated, it is enough to check that for any facet F of Q such that $P \not\subset F$, $Q^{\mathrm{gp}}/P^{\mathrm{gp}}F^{\mathrm{gp}}$ is trivial or that $\operatorname{Spec}(\mathbf{C}Q/\langle P-1\rangle)$ satisfies (R_0). When (b) is satisfied, $\operatorname{Spec}(\mathbf{C}Q/\langle P-1\rangle)_{\mathrm{an}}$ is reduced at least at the origin. Then the latter condition for saturatedness follows. When (a) is satisfied, we have that $Q^{\mathrm{gp}}/P^{\mathrm{gp}}F'^{\mathrm{gp}}$ is torsion free for any face F' of Q such that $F' \cap P = \{1\}$. We will prove the above condition on the facets. We may assume that $Q^{\times} = \{1\}$. Then we reduce the problem to the next claim.

Claim. Let F be a facet of Q such that $P \not\subset F$. Then there exists a subface F' of F such that $F' \cap P = \{1\}$ and $Q^{\mathrm{gp}}/P^{\mathrm{gp}}F'^{\mathrm{gp}}$ is torsion.

Proof of Claim. We reduce this to the similar claim of $\mathbf{Q}_{\geq 0}$-monoids by tensoring with $\mathbf{Q}_{\geq 0}$. For simplicity, we use the same symbols P, Q, and so on for the $\mathbf{Q}_{\geq 0}$-monoids. By (A.3.2) (1), we can apply (A.3.2.2) to h. Let F' be a face of the maximal dimension among faces contained in $\pi(F)$. Then $F' \subset \pi(F) \subset F$ and $F'^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$. The rest is to show $\dim F' + \dim P = \dim Q$. But we easily see that we can apply (A.3.2.2) also to $h'\colon P \cap F \longrightarrow F$ and that π for h' is compatible with π for h, and (A.3.2.2) (2a) for h' implies $\dim F' = \dim F - \dim(P \cap F)$, which equals to $\dim Q - \dim P$. $\square$

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