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Global Models of Contact Forms

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Abstract. We prove that any contact form α on a closed 3-manifold M^3 admits a certain global expression. Using this expression, we construct a contact immersion of (M^3, α) into the standard contact 5-sphere.

1. Introduction

A Pfaff form α on an oriented odd-dimensional manifold $M = M^{2n+1}$ is called a contact form if $\alpha \wedge (d\alpha)^n$ is a positive volume form on M. Then the pair (M, α) is called a contact manifold. We say that a contact manifold (M, α) is contactomorphic (resp. strictly contactomorphic) to another contact manifold (M', α') if there exists a diffeomorphism $\Phi : M \to M'$ satisfying $\Phi^*\alpha' = f\alpha$ for some positive function f (resp. for $f \equiv 1$) on M. Note that $f\alpha \wedge \{d(f\alpha)\}^n (= f^{n+1}\alpha \wedge (d\alpha)^n)$ is a positive volume form for any positive function f. We assume that all functions and forms are C^∞-smooth. Given a point P on (M, α), we can take a local coordinate system $(z, x_1, y_1, \cdots, x_n, y_n)$ centered at $P = (0, \cdots, 0)$ such that

$$\alpha = dz + \sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$$

by Darboux's theorem. Thus a contact form α on a paracompact manifold admits a global expression in the following form:

$$\alpha = \sum_{i=1}^{N} h_i(f_i dg_i - g_i df_i),$$

where (f_i, g_i) is a pair of real functions and h_i is a non-negative function for each $i = 1, \cdots, N(< \infty)$. Hereafter let $N = N(M, \alpha)$ denote the minimal

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number of the terms $h_i(f_i dg_i - g_i df_i)$ among all such expressions for a fixed contact manifold (M, α). Then it is easy to see that N must be greater than the half of the dimension $d = 2n+1$ of the manifold. On the other hand we can show that N is not greater than $\dfrac{3d-(-1)^n}{2}(> d)$ if M is a closed manifold (see Remark in the next section). We do not know, however, whether there exists a closed contact manifold with $N > d > 3$.

Assume that (M, α) is a three-dimensional closed contact manifold for the present. Then, as is shown in Gonzalo-Varela [3], the number N is equal to two if and only if (M, α) is contactomorphic to the standard contact three-sphere:

$$(\{x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\},\ x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2).$$

We prove that N is equal to 3, the dimension, in the other cases. Namely,

THEOREM. *Let (M^3, α) be a three-dimensional closed contact manifold. Then we can take smooth functions f_i, g_i $(i = 1, 2, 3)$ and h satisfying*

$$\alpha = e^h \sum_{i=1}^{3} (f_i dg_i - g_i df_i) \quad \text{and} \quad \sum_{i=1}^{3} (f_i^2 + g_i^2) = 1.$$

Moreover $\phi = (f_1, g_1, f_2, g_2, f_3, g_3)$ is an immersion of M^3 into the unit hypersphere $S^5 = \left\{ \sum_{i=1}^{3} (x_i^2 + y_i^2) = 1 \right\}$ in $\mathbb{R}^6$ satisfying $\phi^* \sum_{i=1}^{3} (x_i dy_i - y_i dx_i) = e^{-h}\alpha$.

2. Preparatory Model

Let (M^3, α) be a three-dimensional closed contact manifold. We owe it to Ibort-Martinez-Presas [4] that we can take a complex-valued function $\sigma : M \to \mathbb{C}$ and a positive function f with the following properties (1), (2) and (3) (see also Giroux [2]).

(1) The image $\sigma(M^3)$ is the unit disk on $\mathbb{C}$. We put $r = |\sigma|$ and $\theta = \arg \sigma$.

(2) The restriction of f on the domain $\{r \geq \frac{1}{2}\}$ is a constant.

(3) The restriction of $d(f\alpha)$ on the relatively compact surface $P_\theta = \{\theta = \text{const}, r > 0\}$ is a positive symplectic form for each value of $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, i. e., $d\theta \wedge d(f\alpha) > 0$.

Note that (3) together with the fact that $f\alpha$ is a contact form implies that $\alpha \wedge (rdr \wedge d\theta)$ is positive on the domain $\{r < \epsilon\}$ for small $\epsilon > 0$.

Using this picture, we construct a global expression of the contact form α.

Proposition. *We can take real functions $f_1, g_1, \cdots, f_5, g_5 : M^3 \to \mathbb{R}$ and non-negative functions $h_1, \cdots, h_5 : M^3 \to \mathbb{R}_{\geq 0}$ on M^3 satisfying $\alpha = \sum_{i=1}^{5} h_i(f_i dg_i - g_i df_i)$.*

Here the number of the terms $h_i(f_i dg_i - g_i df_i)$ is equal to $\dfrac{3d - (-1)^n}{2} = \dfrac{3 \cdot 3 - (-1)^1}{2}$ and greater than the dimension three. We prepare a lemma to prove the proposition.

Lemma. *A contact form α' satisfying $d\alpha'|_{P_\theta} > 0$ $(\forall \theta \in \mathbb{R}/2\pi\mathbb{Z})$ defines a contact structure isotopic to* $\ker \alpha$.

Proof of Lemma. If a non-negative function δ of r satisfies $\delta \equiv 1$ for $r \geq \epsilon$ and $d\delta/(rdr) > 0$ for $r < \epsilon$ then $\alpha_K = f\alpha + K\delta d\theta$ is a contact form satisfying $d\alpha_K|_{P_\theta} > 0$ $(\forall \theta)$ for any positive constant K. Let α' be another contact form satisfying $d\alpha'|_{P_\theta} > 0$ $(\forall \theta)$. The family $\{\beta_t = (1-t)\alpha_K + t(\alpha' + K\delta d\theta)\}_{t \in [0,1]}$ is a homotopy of contact forms if the positive constant K is large enough. Actually,

$$\begin{aligned}\beta_t \wedge d\beta_t = \; & \{(1-t)f\alpha + t\alpha'\} \wedge \{(1-t)d(f\alpha) + td\alpha'\} \\ & + K[\delta d\theta \wedge \{(1-t)d(f\alpha) + td\alpha'\} \\ & + \{(1-t)f\alpha + t\alpha'\} \wedge \{(d\delta/rdr)(rdr \wedge d\theta)\}]\end{aligned}$$

is a positive volume form for large $K > 0$. Then Gray's stability theorem implies that the plane field $\ker \alpha$ is isotopic to the plane field $\ker \alpha_K$; then to the plane field $\ker(\alpha' + K\delta d\theta)$; finally to the plane field $\ker \alpha'$. This completes the proof. □

The above proposition is proved by constructing a contact form α' satisfying

$$d\alpha'|_{P_\theta} > 0 \quad \text{and} \quad \alpha' = \sum_{i=1}^{5} h_i(f_i dg_i - g_i df_i)$$

for suitable real functions f_i, g_i and non-negative functions h_i $(i = 1, \cdots, 5)$. Moreover we see from the above proof how to construct a contactomorphism $\Phi : (M^3, \alpha') \to (M^3, \alpha)$ and a positive function f satisfying $\Phi^*\alpha = f\alpha'$. The following construction of α' will be improved in the next section in order to prove the theorem.

Construction of Preparatory Model. Let $\sigma = re^{\sqrt{-1}\theta}$ be as above. Put

$$f_1 = r\cos\theta \quad \text{and} \quad g_1 = r\sin\theta.$$

Then we have $f_1 dg_1 - g_1 df_1 = r^2 d\theta$. First we trivialize the surface bundle $P_\theta \mapsto \theta$ except on $[3\pi/4, 5\pi/4]$, that is, regard $\{P_\theta\}_{\theta\in(-3\pi/4,3\pi/4)}$ as $P_0\times(-3\pi/4, 3\pi/4)$. Let $H_2 : \overline{P_0} \to [0,1]$ be a function on the closure $\overline{P_0}$ supported in a collar neighbourhood $C = \partial\overline{P_0} \times [0,1)$ and satisfying $H_2^{-1}(1) = \{(p,q) \in C = \partial\overline{P_0}\times[0,1) \mid q \le 1/2\}$. We consider $(\partial\overline{P_0}, dp)$ as a contact-type border of the symplectic manifold $(P_0, \iota^* d(x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2))$ where ι is a proper embedding (or immersion) of P_0 into the open unit ball in $\mathbb{R}^4$ as a symplectic submanifold. That is, we take four functions F_4, G_4, F_5 and G_5 on $\overline{P_0}$ such that

(1) $d(F_4dG_4 - G_4dF_4 + F_5dG_5 - G_5dF_5)$ is a symplectic form on $\overline{P_0}$ and

(2) the restriction $(F_4dG_4 - G_4dF_4 + F_5dG_5 - G_5dF_5)_C$ coincides with $(1 - q/2)dp$.

Here the restriction $(F_4, G_4, F_5, G_5)|_{P_0}$ is nothing but the embedding (or immersion) $\iota : P_0 \to \mathbb{R}^4$. Put

$$H_3 = \left(1 + \frac{q}{2}\right) H_2 + 1 - H_2 \quad \text{and}$$
$$\beta = H_3(F_4dG_4 - G_4dF_4 + F_5dG_5 - G_5dF_5).$$

Then $d\theta + \mathrm{pr}^*(\beta|_{P_0})$ is a contact form on $P_0 \times (-3\pi/4, 3\pi/4)$ where $\mathrm{pr} : P_0 \times (-3\pi/4, 3\pi/4) \to P_0$ is the projection to the first factor. Next we regard the surface bundle as the mapping torus $P_0 \times [0, 2\pi]/\mu$ for some diffeomorphism $\mu : P_0 \times \{2\pi\} \to P_0 \times \{0\}$, that is, as the quotient space $P_0 \times \mathbb{R}/\widetilde{\mu}$ under the identification map $\widetilde{\mu} : (x, \theta + 2\pi) \mapsto (\mu(x), \theta)$. Suppose that the new projection pr' to the first factor of $P_0 \times [0, 2\pi)$ satisfies

$$\begin{aligned} &\mathrm{pr}'|_{P_0\times[0,3\pi/4)} = \mathrm{pr}\,|_{P_0\times[0,3\pi/4)} \quad \text{and} \\ &\mathrm{pr}'|_{P_0\times(5\pi/4,2\pi)} = \mu^{-1} \circ \mathrm{pr} \circ \widetilde{\mu}|_{P_0\times(5\pi/4,2\pi)}. \end{aligned}$$

We may assume that μ is supported in the complement of a larger collar neighbourhood $C' \supset \overline{C}$ and preserving $d\beta$. Then the restriction of $(\mathrm{pr}')^*(d\beta)$ on each fiber P_θ can also be denoted by $d\beta$. Let $h : M \to [0, 1]$ be a function supported in $(\overline{P_0} \setminus C) \times [\pi/2, 3\pi/2]$ and satisfying $h \equiv 1$ on the domain $D = (\overline{P_0} \setminus C') \times [3\pi/4, 5\pi/4]$. We regard $\mathrm{pr}\,|_{P_0\times(-3\pi/4,3\pi/4)}$ and $\mathrm{pr}'|_{C'\times[3\pi/4,5\pi/4]}$ as the restrictions of the same map pr'' defined on the support of $1-h$. Put $h_2 = h_3 = h\cdot(H_3\circ\mathrm{pr}')$ and $h_4 = h_5 = (1-h)\cdot(H_3\circ\mathrm{pr}'')$. Let $f_2, g_2, \cdots, f_5, g_5$ be functions on M such that

(1) $f_2 = F_4 \circ \mathrm{pr}'$, $g_2 = G_4 \circ \mathrm{pr}'$, $f_3 = F_5 \circ \mathrm{pr}'$ and $g_3 = G_5 \circ \mathrm{pr}'$ hold on the support of the function $h_2(= h_3)$ and

(2) $f_4 = F_4 \circ \mathrm{pr}''$, $g_4 = G_4 \circ \mathrm{pr}''$, $f_5 = F_5 \circ \mathrm{pr}''$ and $g_5 = G_5 \circ \mathrm{pr}''$ hold on the support of the function $h_4(= h_5)$.

Then putting

$$\beta' = h_2(f_2dg_2 - g_2df_2) + \cdots + h_5(f_5dg_5 - g_5df_5),$$

we have $d\beta'|_{P_\theta} = d\beta > 0$ on each fiber P_θ and $\beta'|_L > 0$ on the fibered link $L = \sigma^{-1}(0)$. Take a small constant $\epsilon > 0$ such that $\beta' \wedge (rdr \wedge d\theta)$ is positive on $\{r < \epsilon\}$. Let $\delta = \delta(r)$ be a non-negative function satisfying $\delta \equiv 1$ for $r \geq \epsilon$ and $d\delta/(rdr) > 0$ for $r < \epsilon$. Then putting

$$h_1 = K\delta/r^2 \quad \text{and} \quad \alpha' = \sum_{i=1}^{5} \{h_i(f_idg_i - g_idf_i)\}$$

for a sufficiently large constant $K > 0$, we have

$$\alpha' \wedge d\alpha' = \beta' \wedge d\beta' + K\{\delta d\theta \wedge d\beta' + (d\delta/rdr)\beta' \wedge (rdr \wedge d\theta)\} > 0.$$

This completes the construction of α' and the proof of the proposition.

Remark. Given a higher dimensional closed contact manifold (M^{2n+1}, α), we can also take a complex-valued function σ with similar properties to the above (see Ibort-Martinez-Presas [4]). Recently Giroux and Mohsen characterized all contact manifolds up to contactomorphism by using a symplectic open-book decomposition by $P_\theta = \sigma^{-1}(\{\arg z = \theta\})$ (see [2]). Then $(P_\theta, d(f\alpha)|_{P_\theta})$ is a Weinstein manifold with contact end $(L, \alpha|L)$, that is, $d(f\alpha)|_{P_\theta}$ is a Kähler form associated with some Stein complex structure on P_θ. Since we can properly immerse a Stein manifold of complex dimension n into $\mathbb{C}^m$ for $m = \dfrac{6n+1-(-1)^n}{4} \left(= \left[\dfrac{3n+1}{2}\right]\right)$, we may assume that the symplectic form on each page is conformally equivalent to the pull-back of the exact 2-form $d\sum(x_i dy_i - y_i dx_i)$ on the open unit ball $B \subset \mathbb{C}^m$ under some proper immersion $P_\theta \to B$(see [1]). Then we see that the number N is not greater than $2m+1 = \dfrac{3d-(-1)^n}{2}(> d)$ as mentioned in the introduction by constructing a similar contact form to the above α'.

3. Model on S^5

The theorem stated in the introduction is proved by improving the above construction. We construct here a contact form α' on M^3 such that

(1) (M, α') is contactomorphic to (M, α) and

(2) (M, α') is contactomorphic to an immersed contact submanifold of the standard contact unit five-sphere $S^5 \subset \mathbb{C}^3$.

Construction of Model on S^5. Let $\sigma = re^{\sqrt{-1}\theta}$, f_1, g_1, H_2, C, (p, q), C', pr, pr$'$, h and pr$''$ be as above. Take a pair of functions (F_3, G_3) on P_0 which induces an immersion of an open neighbourhood of the complement of C into $\mathbb{R}^2$. If we put

$$F_2 = \sqrt{k-q}\cos(2\pi p) \quad \text{and} \quad G_2 = \sqrt{k-q}\sin(2\pi p)$$

for a sufficiently large positive constant k, then we see that the 1-form

$$\beta = H_2(F_2 dG_2 - G_2 dF_2) + F_3 dG_3 - G_3 dF_3$$

satisfies $\beta|_{\partial \overline{P_0}} > 0$ and $d\beta|_{P_0} > 0$. We may assume that μ is supported in the complement of a larger collar neighbourhood $C' \supset \overline{C}$ and preserving $d\beta$.

Then the restriction of $(p')^*(d\beta)$ on each fiber P_θ can also be denoted by $d\beta$. Put $h_3 = 1 - h$ and $h_2 = H_2 \circ \mathrm{pr}' + h$. Let f_2, g_2, f_3 and g_3 be functions on M such that

(1) $f_2 = F_2 \circ \mathrm{pr}'$ and $g_2 = G_2 \circ \mathrm{pr}'$ hold on the support of $H_2 \circ \mathrm{pr}'$,

(2) $f_2 = F_3 \circ \mathrm{pr}'$ and $g_2 = G_3 \circ \mathrm{pr}'$ hold on the support of h and

(3) $f_3 = F_3 \circ \mathrm{pr}''$ and $g_3 = G_3 \circ \mathrm{pr}''$ hold on the support of h_3.

Then putting

$$\beta' = h_2(f_2 dg_2 - g_2 df_2) + h_3(f_3 dg_3 - g_3 df_3),$$

we have $d\beta'|_{P_\theta} = d\beta > 0$ on each fiber P_θ and $\beta'|_L > 0$ on the fibered link $L = \sigma^{-1}(0)$. Take a small constant $\epsilon > 0$ such that $\beta' \wedge (rdr \wedge d\theta)$ is positive on $\{r < \epsilon\}$. Let $\delta = \delta(r)$ be a non-negative function satisfying $\delta \equiv 1$ for $r \geq \epsilon$ and $d\delta/(rdr) > 0$ for $r < \epsilon$. Put

$$h_1 = K\delta/r^2 \quad \text{and} \quad \alpha_0 = \sum_{i=1}^{3}\{h_i(f_i dg_i - g_i df_i)\}.$$

Then we see that α_0 is a contact form.

By deforming the contact form α_0, we construct the desired functions in the theorem.

Put

$$\alpha_t = \sum_{i=1}^{3}\{(h_i + t)(f_i dg_i - g_i df_i)\} \quad (t > 0)$$

Then $\{\alpha_t\}_{t\in(0,\epsilon')}$ is a family of contact forms for sufficiently small $\epsilon' > 0$. Since the limit contact form α_0 satisfies $d\alpha_0|_{P_\theta} = d\beta > 0$, we see that (M^3, α_t) is contactomorphic to (M^3, α) by using Lemma in the previous section and Gray's stability theorem. Then putting

$$h' = \ln \sum_{i=1}^{3}\{(h_i + t)({f_i}^2 + {g_i}^2)\}, \quad f'_i = e^{-h'/2}\sqrt{h_i + t}\, f_i \quad \text{and}$$
$$g'_i = e^{-h'/2}\sqrt{h_i + t}\, g_i,$$

we have

$$\sum_{i=1}^{3}(f_i'^2 + g_i'^2) = 1 \quad \text{and} \quad \alpha_t = e^{h'}\sum_{i=1}^{3}(f_i' dg_i' - g_i' df_i').$$

Moreover $(f_1', g_1', \cdots, f_3', g_3')$ defines an immersion of M^3 into the unit hypersphere in $\mathbb{R}^6$ since the volume form $\alpha_t \wedge d\alpha_t$ is the pull-back of a 3-form. This completes the construction.

Proof of the Theorem. Since we know how to construct a diffeomorphism $\Phi : M^3 \to M^3$ and a positive function f with $\Phi^*\alpha = f\alpha'$, we can take smooth functions f_i, g_i $(i = 1, 2, 3)$ satisfying the condition of the theorem. This completes the proof. □

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