J. Math. Sci. Univ. Tokyo
11 (2004), 425–446.

Invariant Measures for SPDEs with Reflection

By Yoshiki Otobe

Abstract. We investigate the stationary distribution for a time evolution of continuous random fields on $\mathbb{R}$ driven by Langevin equation taking nonnegative values only. The dynamics have a reflecting wall at zero. It is known that a stationary distribution for the dynamics without reflection is expressed by locally transforming a shift-invariant Gaussian measure on $C(\mathbb{R}, \mathbb{R})$ in a proper way. The purpose of this paper is to establish a similar relationship for the dynamics with reflection. It will be shown that a Gibbs measure with hard-wall external potential, which is expressed by using 3-dimensional Bessel bridge, is a reversible (and therefore stationary) measure for such dynamics. When the potential is strictly convex, the stationary distribution and the Gibbs measure are both unique in a class of tempered distributions and therefore coincide with each other.

1. Introduction

Stochastic partial differential equations (SPDEs) appear in several contexts. For instance, Hohenberg and Halperin[13] studied such equations to describe a dynamic phenomena approaching to equilibrium in statistical mechanics. Parisi and Wu[25] discussed them to construct a perturbation theory for continuum gauge theories. We emphasise here that those dynamics take both positive and negative values.

Nualart and Pardoux[22] and Donati-Martin and Pardoux[4] introduced a method to make the solutions of SPDEs nonnegative. Those equations are called SPDEs with reflection. Such equations were derived by studying the equilibrium fluctuation for a Ginzburg-Landau $\nabla\phi$ interface model on a wall, see Funaki and Olla[9]. The hydrodynamic behaviour of this model is investigated by Funaki[8]. The result of [9] shows that the SPDEs with reflection have natural and meaningful bases in the context of statistical mechanical problems related to entropic repulsion phenomena[2, 21].

2000 *Mathematics Subject Classification*. 60H15; 82C21.
Key words: SPDE with reflection, Gibbs measure with hard-wall external potential.

In those settings, our interest stays in studying stationary distributions of the evolutions, especially the relationship with the so-called Gibbs measures. The Gibbs measures are characterised in terms of so-called Dobrushin, Lanford and Ruelle's (DLR) equations [3, 14], namely the probability measures on $C(\mathbb{R}, \mathbb{R})$ of which the finite volume distribution conditioned outside of the volume coincides with the so-called local specifications determined from certain Hamiltonian.

The aim of this paper is to study the stationary distributions for the solutions of the SPDEs with reflection in terms of DLR equation: we will establish the convergence in law of the solution of the SPDEs with reflection on a finite interval $[l, r]$ to that on $\mathbb{R}$ as $[l, r]$ tends to $\mathbb{R}$, see section 4. The existence and uniqueness of the dynamics are summarised in section 2. To show the convergence, however, we shall prepare the concepts of weak solution, which gives the definition of the solution for SPDEs with reflection as a probability law, see section 3. Then we give a definition of Gibbs measures with hard-wall external potential, see Definition 5.1. We shall see that it is reversible under the solution of the SPDEs with reflection on $\mathbb{R}$ and its finite volume distribution conditioned outside of the volume is reversible under the finite volume evolution with the corresponding Dirichlet boundary conditions, see section 5. In addition, if the potential is strictly convex, we shall show the uniqueness of tempered stationary probability measures and accordingly Gibbs measures, respectively.

2. Summary of Known Results for the SPDEs with Reflection

This section summarises the definition and known results for the SPDEs with reflection. We denote by I the spatial interval on which we shall consider the SPDEs with reflection, namely $I = (l, r)$ for $-\infty < l < r < \infty$ or $I = \mathbb{R}$. We consider the following SPDE:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} - f(x, u(x,t)) + \eta + \sigma(x, u(x,t))\dot{W}(x,t) \tag{2.1}$$

for $(x, t) \in I \times \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$. $\dot{W}(x, t)$ denotes the space-time white noise, that is, $\dot{W}(x, t)$ is a (formal) Itô derivative of the white noise process W_t with respect to t. Here we call $\{W_t\}_{t\in\mathbb{R}_+}$ the white noise process if W_t is an $\mathscr{S}'$-valued process such that ${}_{\mathscr{S}'}\langle W_t, \psi\rangle_{\mathscr{S}}$ is a one-dimensional Brownian motion multiplied by $|\psi|_{L^2(I)}$, where $\mathscr{S}$ and $\mathscr{S}'$ denote Schwartz space and its topological dual respectively, see [6, 18, 26] and others. The norm $|\psi|_{L^2(I)}$ is defined by $|\psi|^2_{L^2(I)} = \int_I \psi(x)^2\, dx$ as usual. We further denote by $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}_{t\geq 0})$ the filtered probability space on which $\{\mathscr{F}_t\}$-white noise process is defined, i.e., W_t is $\{\mathscr{F}_t\}$-adapted and, for $t > s \geq 0$, $W_t - W_s$ and $\mathscr{F}_s$ are independent.

Two functions $f, \sigma : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ are called an external force and a diffusion coefficient, respectively. To simplify the notation, we write $f(u_t)$ and $\sigma(u_t)$ instead of $f(x, u(x,t))$ and $\sigma(x, u(x,t))$, respectively.

In the equation (2.1), η denotes (a formal derivative in t of) a random measure on $I \times \mathbb{R}_+$ to keep the condition

$$u(x,t) \geq 0 \text{ for } (x,t) \in I \times \mathbb{R}_+ \tag{2.2}$$

and plays the similar role to the *local time* appearing in the usual Skorokhod–Tanaka equation (see, e.g., [15]), that is, η satisfies

$$\int_0^\infty \int_I u(x,t)\, \eta(dx, dt) = 0. \tag{2.3}$$

Note that, in the case where $I = \mathbb{R}$, the uniqueness fails in $C(\mathbb{R}, \mathbb{R}_+)$ even if $f \equiv \sigma \equiv 0$[19], in which case (2.1) is just a heat equation since η automatically vanishes. This means that $C(\mathbb{R}, \mathbb{R}_+)$ is too large to study our problem and therefore we are lead to introduce a suitable state space.

For functions $p(x)$ from $\mathbb{R}$ to $\mathbb{R}$, define

$$\|p\|_{\lambda,\infty} := \sup_{x\in\mathbb{R}} |p(x)| e^{-\lambda\chi(x)} \text{ for } \lambda > 0,$$

where $\chi(x)$ is a symmetric C^2-class function on $\mathbb{R}$ satisfying that $\chi(x) = |x|$ for $|x| \geq 1$. Let

$$\mathscr{C} := \bigcap_{\lambda>0} \left\{ p \in C(\mathbb{R}, \mathbb{R}); \|p\|_{\lambda,\infty} < \infty \right\}.$$

With the system of norms $\|\cdot\|_{\lambda,\infty}$, $\mathscr{C}$ becomes a Fréchet space and set $\tilde{\mathscr{C}}^+ := \mathscr{C} \cap C(\mathbb{R}, (0, \infty))$. We take the initial condition $u_0(x)$ for the SPDEs with reflection from $\tilde{\mathscr{C}}^+$:

$$u(x,0) = u_0(x) \text{ for } x \in I. \tag{2.4}$$

We further assume that $u_0(x)$ is 1/2-Hölder continuous in x. Note that even if we consider the case of $I = (l, r)$, we take $u_0 \in \tilde{\mathscr{C}}^+$. In the case of $I = (l, r)$ we shall consider the SPDE (2.1) with Dirichlet boundary conditions, that is,

$$u(l, t) = u_0(l), \quad u(r, t) = u_0(r)$$

are satisfied for every $t \geq 0$. Moreover, we extend the solutions of (2.1) by putting

$$u(x, t) = u_0(x) \text{ for } x \in I^c, \ t \geq 0. \tag{2.5}$$

We also extend η to a measure on $\mathbb{R} \times \mathbb{R}_+$ by putting $\eta \equiv 0$ outside of $(l, r) \times \mathbb{R}_+$.

With these settings, we can give the definition of the strong solutions for the SPDEs with reflection. Let $\mathscr{C}^+ := \mathscr{C} \cap C(\mathbb{R}, \mathbb{R}_+)$.

DEFINITION 2.1. We call a pair (u, η) of a function and a measure a strong solution of the SPDEs with reflection (2.1) if:

(1) $u_t := u(\cdot, t)$ is a continuous $\mathscr{C}^+$-valued adapted function.

(2) η is an adapted measure such that, for every T and $\lambda > 0$,

$$\int_0^T \int_I e^{-\lambda\chi(x)} \eta(dx, dt) < \infty \tag{2.6}$$

a.s. Moreover, η satisfies (2.3).

(3) u satisfies the initial condition (2.4). Moreover, in the case of $I = (l, r)$, u satisfies the Dirichlet boundary conditions in the sense of (2.5).

(4) (u, η) satisfies the following stochastic integral equation:

$$\begin{aligned} \langle u_t, \phi \rangle = \langle u_0, \phi \rangle &+ \int_0^t \langle u_s, \phi'' \rangle \, ds - \int_0^t \langle f(u_s), \phi \rangle \, ds \\ &+ \int_0^t \int_I \phi(x) \eta(dx, ds) \\ &+ \int_0^t \int_I \phi(x) \sigma(u_s) W(dx, ds) \text{ a.s.} \end{aligned} \tag{2.7}$$

for every $t > 0$ and $\phi \in C_0^\infty(I)$.

Here, $\langle p, q\rangle \equiv \langle p, q\rangle_I := \int_I p(x)q(x)\,dx$ if the integral is absolutely converging. The last term of (2.7) is considered as a stochastic integral with respect to the white noise process. Moreover, we say u_t and η are adapted if $\langle u_t, \phi\rangle$ and $\eta(\Gamma)$ are $\mathscr{F}_t$-measurable for every $\phi \in C_0^\infty(I)$ and $\Gamma \in \mathscr{B}(\mathbb{R}\times[0,t])$ (recall that η was extended to $\mathbb{R}\times\mathbb{R}_+$), respectively.

Remark 2.1. In the case of $I = (l, r)$, Nualart and Pardoux[22] required an integrability condition: $\eta((l-\delta, r-\delta)\times[0,T]) < \infty$ for every $\delta, T > 0$. In this paper, we assume $u_0(x) > 0$ for $x \in \mathbb{R}$. Hence the support of η is actually contained in $I \times \mathbb{R}$.

The terminology "strong" was used in the usual sense, that is, the above definition are considered for a fixed $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\}, \{W_t\})$. We shall give another weaker definition of solutions for the SPDEs with reflection in the next section.

Now we introduce conditions on the coefficients in (2.1). First of all we will always assume Lipschitz continuity.

(A1) There exists a constant $L > 0$ such that

$$|f(x,z) - f(x,w)| + |\sigma(x,z) - \sigma(x,w)| \le L|z-w|$$

for every $x \in I$ and $z, w \in \mathbb{R}_+$.

When $I = \mathbb{R}$ we further assume the following conditions.

(A2) $f(x,0)$ and $\sigma(x,0)$ are members of $\bigcap_{\lambda>0} L^2(\mathbb{R}, e^{-2\lambda\chi(x)}dx)$.

(A3) In addition, if $\sigma \not\equiv$ Const., there exist constants $0 < q < 1$, $\ell \in \mathbb{R}$ and $M > 0$ such that

$$|f(x,z) - \ell z| + |\sigma(x,z)| \le M(1+z^q) \tag{2.8}$$

for every $x \in \mathbb{R}$ and $z \in \mathbb{R}_+$.

Theorem 2.1 ([4, 22, 24]). *Under the assumptions (A1)–(A3), a solution for the SPDEs with reflection (u, η) exists.*

For the proof of Theorem 2.1, they introduced the following penalised SPDE, which will also play an important role in the present paper: For $\varepsilon > 0$,

$$\frac{\partial u_t^\varepsilon}{\partial t} = \frac{\partial^2 u_t^\varepsilon}{\partial x^2} - f(u_t^\varepsilon) + \frac{1}{\varepsilon}(u_t^\varepsilon)^- + \sigma(u_t^\varepsilon)\dot{W}(x,t) \tag{2.9}$$

with $u_0^\varepsilon = u_0 \in \tilde{\mathscr{C}}^+$ and the same Dirichlet boundary conditions (2.5). In this equation, f and σ are extended to the functions $I \times \mathbb{R} \to \mathbb{R}$ by putting $f(x,z) := f(x,0)$ and $\sigma(x,z) := \sigma(x,0)$ for $z < 0$, respectively, and the solution u_t^ε can take negative values. Here and after z^- denotes the negative part of $z \in \mathbb{R}$, i.e., $z^- := -\min(0,z)$. Then they showed there exists a function u_t such that $\|u_t^\varepsilon - u_t\|_{\lambda,\infty}$ converges monotonically to zero uniformly in $t \in [0,T]$ as $\varepsilon \downarrow 0$ for every $T > 0$ a.s. In the case of $I = (l,r)$, the norm can be taken as $\|\cdot\|_\infty := \|\cdot\|_{0,\infty}$. Moreover they also showed $(u^\varepsilon)^-/\varepsilon$ converges to a positive Schwartz' distribution, which they put η. For the uniqueness, we have only restricted results.

THEOREM 2.2 ([22, 24]). *If $\sigma \equiv 1$, the solution (u,η) is unique.*

3. Weak Solutions for SPDEs with Reflection

This section gives another definition of a solution, i.e. a weak solution, of the SPDEs with reflection, which can be described in terms of Stroock–Varadhan's martingale problem[28].

Let $\mathbb{C}^+ := C(\mathbb{R}_+, \mathscr{C}^+)$ and let $\mathbb{M}$ be the space of nonnegative measures on $\mathbb{R}\times\mathbb{R}_+$ satisfying (2.6) for $I = \mathbb{R}$. Recall that $(u^{l,r}, \eta^{l,r})$, a solution of the SPDEs with reflection on (l,r), is extended to $\mathbb{R}\times\mathbb{R}_+$. Hence any strong solution (u,η) (for $I = \mathbb{R}$) or $(u^{l,r}, \eta^{l,r})$ (for $I = (l,r)$) stays in $\mathbb{C}^+ \times \mathbb{M} =: \mathbb{W}$ almost surely.

Let $\theta : \mathbb{W} \to \mathbb{C}^+$ and $\xi : \mathbb{W} \to \mathbb{M}$ be the canonical projections. Moreover let $\theta_t : \mathbb{C}^+ \to \mathscr{C}^+$ also denote the canonical projection (coordinate mapping process). We simply use the symbols θ_t and ξ instead of $\theta_t(\theta\omega)$ and $\xi\omega$ for $\omega \in \mathbb{W}$, respectively, if there is no confusion. Next, we introduce a suitable class of test functions.

DEFINITION 3.1. We say a function $\Psi : \mathscr{C}^+ \to \mathbb{R}$ is in a class $\mathscr{F}C_{b,I}^\infty$ if there exist a function $\psi \equiv \psi(\alpha_1, \dots, \alpha_k) \in C_b^\infty(\mathbb{R}^k)$, $k = 1, 2, \dots$ and

$\{\varphi_i\}_{i=1,\dots,k} \subset C_0^\infty(I)$ satisfying

$$\Psi(u) \equiv \psi\left(\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_k\rangle\right).$$

For $\Psi \in \mathscr{F}C_{b,I}^\infty$, we define a gradient operator ∇ and second derivative ∇^2 by

$$\nabla\Psi(x,u) := \sum_{i=1}^k \frac{\partial\psi}{\partial\alpha_i}\left(\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_k\rangle\right)\varphi_i(x), \quad x \in I \tag{3.1}$$

$$\begin{aligned}&\nabla^2\Psi(x,y,u)\\ &:= \sum_{i,j=1}^k \frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}\left(\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_k\rangle\right)\varphi_i(x)\varphi_j(y), \quad x, y \in I,\end{aligned} \tag{3.2}$$

respectively. Using these operators, we define an operator L_I acting on $\mathscr{F}C_{b,I}^\infty$ by

$$\begin{aligned}(L_I\Psi)(u) :=& \langle \Delta u - f(x,u(x)), \nabla\Psi(u)\rangle\\ &+ \frac{1}{2}\mathrm{Tr}\left(\sigma(x,u(x))\sigma(y,u(y))\nabla^2\Psi(x,y,u)\right),\end{aligned} \tag{3.3}$$

where the coupling $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_I$ and $\Delta u := \partial^2 u/\partial x^2$ are interpreted as Schwartz' distributions. Namely, we have

$$\begin{aligned}(L_I\Psi)(u) = & \sum_{i=1}^k \left\{\langle u, \Delta\varphi_i\rangle - \langle f(\cdot,u(\cdot)), \varphi_i\rangle\right\}\\ &\times \frac{\partial\psi}{\partial\alpha_i}\left(\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_k\rangle\right)\\ &+ \frac{1}{2}\sum_{i,j=1}^k \langle \sigma(\cdot,u(\cdot))\varphi_i, \sigma(\cdot,u(\cdot))\varphi_j\rangle\\ &\times \frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_j}\left(\langle u, \varphi_1\rangle, \dots, \langle u, \varphi_k\rangle\right).\end{aligned} \tag{3.4}$$

Put $\xi_t := 1_{[0,t]}\xi$ and define σ-fields on $\mathbb{W}$ by $\mathscr{G}_t := \bigvee_{0\le s\le t}\sigma\left((\theta_s, \xi_s)\right)$ and $\mathscr{G} := \bigvee_{0\le t}\mathscr{G}_t$. With these settings, now we give a definition of weak solutions.

DEFINITION 3.2. We call a probability measure P on the measurable space $(\mathbb{W}, \mathscr{G})$ a weak solution (more precisely, a law of a weak solution) of the SPDEs on I with reflection, or a solution for L_I-local martingale problem with reflection, with an initial condition $u_0 \in \tilde{\mathscr{C}}^+$ if:

(1) $P(\theta_0 = u_0) = 1$.

(2) For every $\Psi \in \mathscr{F}C_{b,I}^\infty$,

$$\Psi(\theta_t) - \Psi(\theta_0) - \int_0^t (L_I \Psi)(\theta_s)\, ds - \int_0^t \int_I \nabla\Psi(x, \theta_s)\xi(dx, ds)$$

is a local martingale with respect to $(P, \{\mathscr{G}_t\})$.

(3) We have P-almost surely that

$$\int_0^\infty \int_I \theta_t(x)\xi(dx, dt) = 0.$$

Similar problems without reflection terms were studied by several authors, see [6, 18] and others. We shall discuss here only fundamental properties of weak solutions.

PROPOSITION 3.1. *The law on $\mathbb{W}$ of a strong solution gives a weak solution.*

PROOF. We set P the law on $(\mathbb{W}, \mathscr{G})$ determined by a strong solution (u, η). The properties (1) and (3) of Definition 3.2 are automatically satisfied. Let $\Psi \in \mathscr{F}C_{b,I}^\infty$ be given. Then, Itô's formula implies

$$\begin{aligned} d\Psi(u_t) = &\sum_{i=1}^k \frac{\partial \psi}{\partial \alpha_i} \left(\langle u_t, \varphi_1 \rangle, \dots \langle u_t, \varphi_k \rangle \right) d \langle u_t, \varphi_i \rangle \\ &+ \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 \psi}{\partial \alpha_i \partial \alpha_j} \left(\langle u_t, \varphi_1 \rangle, \dots \langle u_t, \varphi_k \rangle \right) \\ &\times d \left[\langle u_\cdot, \varphi_i \rangle, \langle u_\cdot, \varphi_j \rangle \right]_t , \end{aligned} \tag{3.5}$$

where $\left[\langle u_\cdot, \varphi_i \rangle, \langle u_\cdot, \varphi_j \rangle \right]_t$ is a quadratic variational process. Since $\int_0^t \int_I \varphi(x)\eta(dx, ds)$ is of bounded variation in t, $\int_0^t \langle \Delta u_t - f(u_t) + \eta, \varphi \rangle\, ds$ is also of bounded variation. Here, we have used an abbreviation $\int_0^t \langle \eta, \varphi \rangle \, ds \equiv \int_0^t \int_I \varphi(x) \eta(dx, ds)$. Hence, from (2.7), we have

$$d\left[\langle u_\cdot, \varphi_i \rangle, \langle u_\cdot, \varphi_j \rangle\right]_t = \langle \sigma(u_t)\varphi_i, \sigma(u_t)\varphi_j \rangle \, dt. \tag{3.6}$$

Combining above computations (3.5), (3.6) with (2.7), we have

$$\begin{aligned}\Psi(u_t) = &\Psi(u_0) + \int_0^t (L_I\Psi)(u_s)\, ds + \int_0^t \int_I \nabla\Psi(x, u_s)\eta(dx, ds) \\ &+ \int_0^t \sum_{i=1}^k \frac{\partial \psi}{\partial \alpha_i} \left(\langle u_s, \varphi_1 \rangle, \dots, \langle u_s, \varphi_k \rangle\right) \langle \sigma(u_s)\varphi_i, dW_s \rangle .\end{aligned}$$

Since the last term is a local martingale, the proof is completed. □

The above proposition combined with Theorem 2.1 implies the existence of a weak solution. The next proposition shows some kind of equivalence between strong solutions and weak ones. The proof is rather standard.

PROPOSITION 3.2. *Let P be a solution for L_I-local martingale problem with reflection. Then there exists a white noise process $\{W_t\}$ on $(P, \mathbb{W}, \mathcal{G}, \{\mathcal{G}_t\})$ such that, replacing (u, η) by (θ, ξ), (2.7) is satisfied.*

PROOF. As the first step, consider a $\mathcal{D}'(I)$-valued process $M_t := \theta_t - \theta_0 - A_t$, where $A_t := \int_0^t (\Delta\theta_s - f(\theta_s) + \xi) \, ds$. Here we used an abbreviation that $\xi \, ds := \xi(\cdot, ds)$. For $\varphi \in C_0^\infty(I)$ and $N > 0$, define a Markov time $\tau := \inf\{t \geq 0; |\langle \theta_t, \varphi \rangle| \geq N\}$. Take $\psi_N \in C_b^\infty(\mathbb{R})$ such that $\psi_N(\alpha) = \alpha$ for $|\alpha| \leq N$ and set $\Psi_N(u) := \psi_N(\langle u, \varphi \rangle) \in \mathcal{F}C_{b,I}^\infty$. Then, since P is a solution for L_I-local martingale problem with reflection, from the property (2) for $\Psi = \Psi_N$, we see that

$$\begin{aligned}\psi_N(\langle \theta_t, \varphi \rangle) - \psi_N(\langle \theta_0, \varphi \rangle) - \int_0^t \frac{\partial \psi_N}{\partial \alpha}(\langle \theta_s, \varphi \rangle) \langle \Delta\theta_s - f(\theta_s) + \xi, \varphi \rangle \, ds \\ - \frac{1}{2} \int_0^t \frac{\partial^2 \psi_N}{\partial \alpha^2}(\langle \theta_s, \varphi \rangle) \langle \sigma(\theta_s)\varphi, \sigma(\theta_s)\varphi \rangle \, ds\end{aligned} \tag{3.7}$$

is a local martingale. Hence $\langle \theta_{t\wedge\tau}, \varphi \rangle - \langle \theta_0, \varphi \rangle - \int_0^{t\wedge\tau} \langle \Delta\theta_s - f(\theta_s) + \xi, \varphi \rangle \, ds$ is a martingale, which proves that $M_t(\varphi) \equiv \langle M_t, \varphi \rangle$ is a local martingale.

Next, re-set $\psi_N \in C_b^\infty(\mathbb{R})$ to satisfy $\psi_N(\alpha) = \alpha^2$ for $|\alpha| \le N$. Then

$$(3.8)\quad H_{t\wedge\tau}(\varphi) := \langle \theta_{t\wedge\tau}, \varphi\rangle^2 - \langle \theta_0, \varphi\rangle^2 - 2\int_0^{t\wedge\tau} \langle \theta_s, \varphi\rangle \langle \Delta\theta_s - f(\theta_s) + \xi, \varphi\rangle \, ds - V_{t\wedge\tau}(\varphi)$$

is a martingale, where $V_t(\varphi) := \int_0^t \langle \sigma(\theta_s)\varphi, \sigma(\theta_s)\varphi\rangle \, ds$. On the other hand,

$$M_t(\varphi)^2 = (M_t(\varphi) + \langle \theta_0, \varphi\rangle)^2 - 2\langle \theta_0, \varphi\rangle (M_t(\varphi) + \langle \theta_0, \varphi\rangle) + \langle \theta_0, \varphi\rangle^2 .$$

Hence we have

$$M_t(\varphi)^2 - V_t(\varphi) \sim (M_t(\varphi) + \langle \theta_0, \varphi\rangle)^2 - \langle \theta_0, \varphi\rangle^2 - V_t(\varphi) - H_t(\varphi),$$

where $X \sim Y$ means that $X - Y$ is a local martingale, and this implies that

$$M_t(\varphi)^2 - V_t(\varphi) \sim A_t(\varphi)^2 - 2\langle \theta_t, \varphi\rangle A_t(\varphi) + 2\int_0^t \langle \theta_s, \varphi\rangle \, dA_s(\varphi).$$

Computing $d\{A_t(\varphi)\langle \theta_t, \varphi\rangle\}$ and recalling that $d\langle \theta_t, \varphi\rangle = dM_t(\varphi) + dA_t(\varphi)$, we finally obtain $M_t(\varphi)^2 \sim V_t(\varphi)$. Now we can apply Lévy's characterisation theorem for the white noise process on some extension of the space (see [18]), which completes the proof. □

Now we move to the uniqueness problem. There are several types of uniqueness theorem for the weak solution. We will, however, give here only Yamada–Watanabe type theorem.

PROPOSITION 3.3. *Assume that the uniqueness for the strong solution of an SPDE with reflection holds. Then the weak solution for the SPDE with reflection is also unique.*

REMARK 3.1. Yamada–Watanabe type theorem for SPDEs had been discussed e.g. by Funaki[6] when σ is somehow a constant. He, however, had pointed out that generic version of Yamada–Watanabe might hold. The proof of this type of theorem for finite dimensional SDEs can be found in several literatures, see e.g. [15]. The difference in our situation mainly comes from the fact that the white noise process $\{W_t\}$ stays in $S := C([0,\infty), \mathscr{S}')$ which is different from $\mathbb{C} := C(\mathbb{R}_+, \mathscr{C})$, the space of solutions u_t. However, since $S \times \mathbb{W}$ is a standard measurable space (see [1, Chapter 9] or [16]), the proof of Proposition 3.3 also goes similarly and therefore it is omitted.

COROLLARY 3.4. *If* $\sigma \equiv \mathrm{Const.}$*, then the law of* (u, η) *on* $\mathbb{W}$ *is uniquely determined.*

REMARK 3.2. For the generic diffusive case ($\sigma \not\equiv \mathrm{Const.}$) there is no uniqueness result for the SPDEs with reflection.

REMARK 3.3. As usual, if we have the uniqueness result for the solution of L_I-local martingale problem with reflection, it determines a diffusion process in $\mathscr{C}^+$. A direct proof of this assertion can be found in [29] for $L^2([l, r])$-valued process, of which method can be applied to the case where $\sigma \not\equiv \mathrm{Const.}$, that is, the solution obtained by taking limit for the solution of penalised equation (2.9) is a diffusion process.

4. Convergence of Dynamics

From now on we assume that $\sigma \equiv 1$, see Remark 4.1 for the case where $\sigma \not\equiv \mathrm{Const.}$ We denote by $(u^{l,r}, \eta^{l,r})$ the (unique) solution of the SPDE with reflection on $I = (l, r)$. Note that $(u^{l,r}, \eta^{l,r})$ is extended to $\mathbb{R} \times \mathbb{R}_+$ as (2.5). We simply write the solution (u, η) in the case of $I = \mathbb{R}$.

Let $P^{l,r}$ and P be probability laws on $\mathbb{W}$ induced by $(u^{l,r}, \eta^{l,r})$ and (u, η), respectively. In this section we shall discuss a converging result of $(u^{l,r}, \eta^{l,r})$ to (u, η) as (l, r) tends to $\mathbb{R}$ in the following sense.

THEOREM 4.1. $P^{l,r}$ *converges to* P *as* (l, r) *tends to* $\mathbb{R}$ *in the sense of probability law.*

To prove the theorem, since we have uniqueness (Corollary 3.4), it is sufficient to prove the following proposition.

PROPOSITION 4.2. *The family* $\{P^{l,r}\}_{(l,r)\subset\mathbb{R}}$ *is relatively compact.*

PROOF. We prepare the following SPDE:

$$
\begin{cases}
\dfrac{\partial v_t^{l,r}}{\partial t} = \dfrac{\partial^2 v_t^{l,r}}{\partial x^2} - f(u_t^{l,r}) + \dot{W}(x,t) \quad x \in (l, r); \\
v_0^{l,r}(x) = u_0(x) \quad x \in (l, r); \\
v_t^{l,r}(x) = u_0(x) \quad x \notin (l, r) \ t \geq 0.
\end{cases}
\tag{4.1}
$$

Then it is standard to show that the law of $v^{l,r}$ on $\mathbb{C}$ is tight, see Lemma 5.4 of [18]. Now we use Skorokhod's representation theorem, that is, for every limit point of the law of $v^{l,r}$, there exists a random variable $\bar{v}$ on some probability space that $\bar{v}^{l,r}$ converges to $\bar{v}$ in $\|\cdot\|_{\lambda,\infty}^{T}$-norm for every $\lambda > 0$ and $T > 0$ almost surely, where $\bar{v}^{l,r}$'s have same laws with $v^{l,r}$'s for every $-\infty < l < r < \infty$. After here, we always consider the subsequence that $\bar{v}^{l,r} \to \bar{v}$. Moreover, without confusion, we write $v^{l,r} \to \bar{v}$ almost surely with $(l,r) \to \mathbb{R}$ to simplify the notation.

Now let us consider the following deterministic PDEs for $(w^{l,r}, \xi^{l,r})$ and $(\bar{w}, \bar{\xi})$.

$$
\begin{cases}
\dfrac{\partial w_t^{l,r}}{\partial t} = \dfrac{\partial^2 w_t^{l,r}}{\partial x^2} + \xi^{l,r} \quad x \in (l,r), \\
w_0^{l,r}(x) = 0 \quad x \in (l,r), \\
w_t^{l,r}(x) = 0 \quad x \notin (l,r),\ t \geq 0, \\
w_t^{l,r}(x) \geq -v_t^{l,r}(x) \quad x \in (l,r), \\
\displaystyle\int_0^T \int_l^r \xi^{l,r}(dx,dt) < \infty \quad T > 0, \\
\displaystyle\int_0^\infty \int_l^r (w_t^{l,r}(x) + v_t^{l,r}(x))\xi^{l,r}(dx,dt) = 0.
\end{cases} \tag{4.2}
$$

$$
\begin{cases}
\dfrac{\partial \bar{w}_t}{\partial t} = \dfrac{\partial^2 \bar{w}_t}{\partial x^2} + \bar{\xi} \quad x \in \mathbb{R}, \\
\bar{w}_0(x) = 0, \\
\bar{w}_t(x) \geq -\bar{v}_t(x), \\
\displaystyle\int_0^T \int_{\mathbb{R}} e^{-\lambda\chi(x)} \bar{\xi}(dx,dt) < \infty \quad \lambda, T > 0, \\
\displaystyle\int_0^\infty \int_{\mathbb{R}} (\bar{w}_t(x) + \bar{v}_t(x))\bar{\xi}(dx,dt) = 0.
\end{cases} \tag{4.3}
$$

These PDEs are considered by [22, 24], in which it is shown that $\xi^{l,r}$ and $\bar{\xi}$ are obtained by

$$
\xi^{l,r}(dx,dt) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left((w^{l,r})_t^{\varepsilon}(x) + v_t^{l,r}(x) \right)^- \, dx\,dt,
$$
$$
\bar{\xi}(dx,dt) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\bar{w}_t^{\varepsilon}(x) + \bar{v}_t(x) \right)^- \, dx\,dt,
$$

where $(w^{l,r})^\varepsilon$ and $\bar{w}^\varepsilon$ are solutions of the following penalised PDEs, respectively:

$$
(4.4)\qquad \begin{cases} \dfrac{\partial (w^{l,r})^\varepsilon}{\partial t} = \dfrac{\partial^2 (w^{l,r})^\varepsilon}{\partial x^2} + \dfrac{1}{\varepsilon}\left((w^{l,r})^\varepsilon + v^{l,r}\right)^- \quad x \in (l,r), \\ (w^{l,r})^\varepsilon_0(x) = 0 \quad x \in (l,r), \\ (w^{l,r})^\varepsilon_t(x) = 0 \quad x \notin (l,r),\ t \geq 0. \end{cases}
$$

$$
(4.5)\qquad \begin{cases} \dfrac{\partial \bar{w}^\varepsilon}{\partial t} = \dfrac{\partial^2 \bar{w}^\varepsilon}{\partial x^2} + \dfrac{1}{\varepsilon}\left(\bar{w}^\varepsilon + \bar{v}\right)^- \quad x \in \mathbb{R}; \\ \bar{w}^\varepsilon_0(x) = 0. \end{cases}
$$

It is shown in [22, 24] that $(w^{l,r}, \xi^{l,r})$ and $(\bar{w}, \bar{\xi})$ exist uniquely and $w^{l,r}, \bar{w} \in C(\mathbb{R}_+, \mathscr{C})$. In addition, $\|(w^{l,r})^\varepsilon - w^{l,r}\|^T_{0,\infty} \to 0$ and $\|\bar{w}^\varepsilon - \bar{w}\|^T_{\lambda,\infty} \to 0$ as $\varepsilon \downarrow 0$ for each l, r and the convergences are monotone.

Note that $(w^{l,r} + v^{l,r}, \xi^{l,r})$ has the same law with $(u^{l,r}, \eta^{l,r})$. Now we further prepare the following PDE:

$$
(4.6)\qquad \begin{cases} \dfrac{\partial (\bar{w}^{l,r})^\varepsilon}{\partial t} = \dfrac{\partial^2 (\bar{w}^{l,r})^\varepsilon}{\partial x^2} + \dfrac{1}{\varepsilon}\left((\bar{w}^{l,r})^\varepsilon + \bar{v}\right)^- \quad x \in (l,r), \\ (\bar{w}^{l,r})^\varepsilon_0(x) = 0 \quad x \in (l,r), \\ (\bar{w}^{l,r})^\varepsilon_t(x) = 0 \quad x \notin (l,r)\ t \geq 0. \end{cases}
$$

Then we can show:

LEMMA 4.3. *For every $\lambda, T > 0$, we have the following estimate independent of ε:*

$$
\|(\bar{w}^{l,r})^\varepsilon - (w^{l,r})^\varepsilon\|^T_{\lambda,\infty} \leq e^{K_\lambda T} \|\bar{v} - v^{l,r}\|^T_{\lambda,\infty},
$$

where $K_\lambda := \sup\{\lambda\chi''(x) + (\lambda\chi'(x))^2\}$.

The proof of this lemma is essentially given in [22, 24]; consider a PDE satisfied by $\left((\bar{w}^{l,r})^\varepsilon_t(x) - (w^{l,r})^\varepsilon_t(x)\right) e^{-K_\lambda t} e^{-\lambda\chi(x)} - \|e^{-K_\lambda t}(\bar{v} - v^{l,r})\|^T_{\lambda,\infty}$, we can easily show that it is non-positive and we obtain the lemma by symmetry. We omit the detailed proof.

Now we have the following estimate:

$$
(4.7)\qquad \begin{aligned} \|w^{l,r} - \bar{w}\|^T_{\lambda,\infty} &\leq \|w^{l,r} - (w^{l,r})^\varepsilon\|^T_{\lambda,\infty} + \|\bar{w} - \bar{w}^\varepsilon\|^T_{\lambda,\infty} \\ &\quad + \|(w^{l,r})^\varepsilon - (\bar{w}^{l,r})^\varepsilon\|^T_{\lambda,\infty} + \|(\bar{w}^{l,r})^\varepsilon - \bar{w}^\varepsilon\|^T_{\lambda,\infty} \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned}
$$

Two terms I and II converge to zero as ε tends to zero, while III is estimated by $\|v^{l,r} - \bar{v}\|$ from Lemma 4.3. It only remains to estimate IV. To this end, we prepare the following function:

$$(\tilde{w}^{l,r})^{\varepsilon}_t(x) := \int_0^t \left\{ \frac{\partial}{\partial y} G^{l,r}_{t-s}(x,l)\bar{w}^{\varepsilon}_s(l) - \frac{\partial}{\partial y} G^{l,r}_{t-s}(x,r)\bar{w}^{\varepsilon}_s(r) \right\} ds, \tag{4.8}$$

where $G^{l,r}_t(x,y)$ denotes the heat kernel on (l,r). Then from (4.5) and (4.6),

$$\bar{w}^{\varepsilon}_t(x) = (\tilde{w}^{l,r})^{\varepsilon}_t(x) + \frac{1}{\varepsilon} \int_0^t \int_l^r G^{l,r}_{t-s}(x,y) \left(\bar{w}^{\varepsilon}_s(y) + \bar{v}_s(y)\right)^- \, dy\, ds$$

and

$$(\bar{w}^{l,r})^{\varepsilon}_t(x) = \frac{1}{\varepsilon} \int_0^t \int_l^r G^{l,r}_{t-s}(x,y) \left((\bar{w}^{l,r})^{\varepsilon}_s(y) + \bar{v}_s(y) \right)^- \, dy\, ds$$

on $x \in (l,r)$, respectively. We have, however,

$$\begin{aligned}
\left(\bar{w}^{\varepsilon}_s(y) + \bar{v}_s(y)\right)^- &\le \left(\bar{w}^{\varepsilon}_s(y) - (\bar{w}^{l,r})^{\varepsilon}_s(y) \right)^- + \left((\bar{w}^{l,r})^{\varepsilon}_s(y) + \bar{v}_s(y) \right)^- \\
&\le \left((\bar{w}^{l,r})^{\varepsilon}_s(y) + \bar{v}_s(y) \right)^- .
\end{aligned}$$

Hence we have, since $G^{l,r}_t(x,y) \ge 0$, $0 \le \bar{w}^{\varepsilon}_t(x) - (\bar{w}^{l,r})^{\varepsilon}_t(x) \le (\tilde{w}^{l,r})^{\varepsilon}_t(x)$. Simple computations (see also [18]) lead us to

$$\begin{aligned}
&\int_0^t \frac{\partial}{\partial y} G^{l,r}_{t-s}(x,l)\bar{w}^{\varepsilon}_s(l)\, ds \\
&\qquad \le \sup_{0 \le s \le t} \left| \bar{w}_s(l) e^{-\lambda l} \right| \cdot 2e^{\frac{\lambda^2 t}{2}} \left(1 - e^{-2\lambda(r-l)} \right)^{-1} e^{2\lambda l} e^{-\lambda x}.
\end{aligned}$$

$\int_0^t \frac{\partial}{\partial y} G^{l,r}_{t-s}(x,r)\bar{w}^{\varepsilon}_s(r)\, ds$ can be estimated similarly. Hence IV $\to 0$ as $(l,r) \to \mathbb{R}$ uniformly in ε. Then, letting $\varepsilon \downarrow 0$ in (4.7), we obtain $\|w^{l,r} - \bar{w}\|^T_{\lambda,\infty} \to 0$ as $(l,r) \to \mathbb{R}$, and in particular

$$\sup_{(l,r) \subset \mathbb{R}} \|w^{l,r}\|^T_{\lambda,\infty} < \infty. \tag{4.9}$$

It still remains that $\xi^{l,r}$ converges to some measure $\tilde{\xi}$. Returning to the equation (4.2), we have

$$\int_0^t \int_{\mathbb{R}} \phi(x) \xi^{l,r}(dx,ds) = \left\langle w^{l,r}_s, \phi \right\rangle - \int_0^t \left\langle w^{l,r}_s, \phi'' \right\rangle ds, \tag{4.10}$$

where $\phi \in C_0^2(\mathbb{R})$ and we took (l, r) sufficiently large such that $\mathrm{supp}\phi \subset (l, r)$. Letting $(l, r) \to \mathbb{R}$, we obtain that $\xi^{l,r}$ converges to some Schwartz' distribution $\tilde{\xi}$. Since $\xi^{l,r}$'s are positive, so is $\tilde{\xi}$. After that, letting $\phi(x) \to e^{-\lambda'\chi(x)}$ $(\lambda' > \lambda)$ shows

$$\int_0^T \int_{\mathbb{R}} e^{-\lambda\chi(x)}\tilde{\xi}(dx, dt) < \infty$$

from (4.9). It is easy to show $\tilde{\xi} = \bar{\xi}$. The tightness of the law of $v^{l,r}$ combined with (4.9) and the convergence of $\xi^{l,r}$ proves the conclusion. □

Remark 4.1. Proposition 4.2 can be shown in the case where $\sigma \not\equiv$ Const., if the solution $(u^{l,r}, \eta^{l,r})$ on (l, r) is unique for each l and r. The proof is modified by considering the following equation instead of (4.1):

$$\frac{\partial v_t^{l,r}}{\partial t} = \frac{\partial^2 v_t^{l,r}}{\partial x^2} - f(u_t^{l,r}) + \sigma(u_t^{l,r})\dot{W}(x, t).$$

For the tightness of $v^{l,r}$, it is sufficient to prove

$$\sup_{(l,r)\subset\mathbb{R}} \sup_{\varepsilon>0} E\left[\sup_{x\in\mathbb{R}, 0\le t\le T} \left|(u^{l,r})_t^{\varepsilon}(x)e^{-\lambda\chi(x)}\right|^p\right] < \infty,$$

which is essentially given in [24]. The uniqueness of $(u^{l,r}, \eta^{l,r})$ is required for the uniqueness of $v^{l,r}$, and accordingly $\xi^{l,r}$.

5. Gibbs Measures with Hard-wall External Potential

This section is devoted to discussing Gibbs measures associated with the SPDEs with reflection with $\sigma \equiv 1$. We call a function $U = U(x, z)$, $(x, z) \in \mathbb{R} \times \mathbb{R}_+$ the potential if it satisfies $\nabla U(x, z) = f(x, z)$, where ∇ denotes the partial derivative with respect to the second variable z. The relationship between Gibbs measures and reversible (probability) measures associated with the SPDEs were discussed by several authors, see e.g., [7, 11, 17]. Here we say a probability measure μ is reversible if $E^\mu[F(u_0)G(u_t)] = E^\mu[F(u_t)G(u_0)]$, where $F, G \in \mathscr{F}C_b^\infty \equiv \mathscr{F}C_{b,\mathbb{R}}^\infty$ and μ is the initial distribution of u_0. Note that $u^{l,r}$ and u determine diffusion processes on $\mathscr{C}^+$.

In the first place, we shall discuss reversible measures associated with $u^{l,r}$. In the following theorem, $u_0(l)$ and $u_0(r)$ may be zero (the case where Nualart and Pardoux[22] studied).

We denote by $\beta_\psi^{l,r}$ the probability measure on $C([l,r],\mathbb{R})$ induced by the pinned Brownian motion $\{B_x\}_{l\le x\le r}$ satisfying $B_l=\psi(l)$ and $B_r=\psi(r)$ for $\psi\in C(\mathbb{R},\mathbb{R}_+)$.

THEOREM 5.1. *Let $\tilde f$ be the extension of f to $I\times\mathbb{R}\to\mathbb{R}$ by putting $\tilde f(x,z)=f(x,0)$ for $z<0$ and $\tilde U$ be its corresponding potential. Suppose that $\tilde U$ is normalisable, i.e.,*

$$\int_{C([l,r],\mathbb{R})}\exp\left\{-2\int_l^r \tilde U(x,\phi(x))\,dx\right\}\beta_{u_0}^{l,r}(d\phi)<\infty.$$

Then, a probability measure

$$\mu_{u_0}^{l,r}(d\phi)=\frac{1}{Z_{u_0}^{l,r}}\exp\left\{-2\int_l^r U(x,\phi(x))\,dx\right\}\nu_{u_0}^{l,r}(d\phi) \tag{5.1}$$

is reversible under the solution $u^{l,r}$ of the SPDEs (2.1) *with reflection for $I=(l,r)$ with Dirichlet boundary condition* (2.5), *where $\nu_{u_0}^{l,r}$ is the probability law on $C([l,r],\mathbb{R}_+)$ induced by 3-dimensional Bessel bridge $\{B_x\}_{l\le x\le r}$ with $B_l=u_0(l)$ and $B_r=u_0(r)$. $Z_{u_0}^{l,r}$ is the normalising constant.*

PROOF. First, let us assume $u_0(l),u_0(r)>0$. Note that $\frac{1}{\varepsilon}(z)^-=-\left(\frac{1}{2\varepsilon}(z\wedge 0)^2\right)'$ for $z\neq 0$. It is known that a probability measure

$$\begin{aligned}\left(\mu_{u_0}^{l,r}\right)^\varepsilon(d\phi):=\frac{1}{A_{u_0}^{l,r}}&\exp\left\{-2\int_l^r\tilde U(x,\phi(x))\,dx\right\}\\&\times\exp\left\{-\frac{1}{\varepsilon}\int_l^r(\phi(x)\wedge 0)^2\,dx\right\}\beta_{u_0}^{l,r}(d\phi)\end{aligned}$$

is reversible under the penalised solution $(u^{l,r})^\varepsilon$, see [5, 7]. However, it is easily seen that $\exp\left\{-\frac{1}{\varepsilon}\int_l^r(\phi(x)\wedge 0)^2\,dx\right\}\to 1_{\{\phi(x)\ge 0,\forall x\in[l,r]\}}(\phi)$ as ε tends to zero. From Williams' theorem (see [20]), it is shown that $\left(\mu_{u_0}^{l,r}\right)^\varepsilon\to\mu_{u_0}^{l,r}$ weakly as $\varepsilon\downarrow 0$. Note that the penalised solution $(u^{l,r})^\varepsilon$ converges to $u^{l,r}$ uniformly as $\varepsilon\downarrow 0$. Hence we have

$$E^{\left(\mu_{u_0}^{l,r}\right)^\varepsilon}\left[F((u^{l,r})_0^\varepsilon)G((u^{l,r})_t^\varepsilon)\right]\to E^{\mu_{u_0}^{l,r}}\left[F(u_0^{l,r})G(u_t^{l,r})\right]$$

for every $F, G \in \mathscr{F}C_b^\infty$. The convergence when $u_0(l), u_0(r) \to 0$ can be easily obtained by a comparison theorem with respect to Dirichlet boundary conditions[9]. $\square$

Remark 5.1. The above proof is taken from [23]. Zambotti[29] gave another proof of Theorem 5.1.

Remark 5.2. We supposed that $\tilde{U}$ is normalisable. It is satisfied if, e.g., $\tilde{U}$ is bounded from below. Several conditions for $\tilde{U}$ are known (see [27]). In [29], $-\rho|z| \le f(x, z)$ with $0 \le \rho < \pi^2/(r-l)$ is given.

Remark 5.3. Recall that 3-dimensional Bessel bridge does not hit zero almost surely. Hence the condition that the initial value $u_0 \in \tilde{\mathscr{C}}^+$ is natural from a view point of reversible dynamics.

We now define the Gibbs measure with hard-wall external potential.

Definition 5.1. We call a probability measure μ on $\mathscr{C}^+$ Gibbs measure with hard-wall external potential if μ satisfies the following DLR-equation:

$$\mu(\,\cdot\;|\,\mathscr{F}_{l,r})(\psi) = \mu_\psi^{l,r}(\,\cdot\,), \quad \mu\text{-a.e. } \psi, \tag{5.2}$$

where the left hand side denotes the regular conditional probability measure of μ with respect to the σ-field $\mathscr{F}_{l,r}$ generated by $\mathscr{C}^+|_{[l,r]^c}$. In the right hand side, $\mu_\psi^{l,r}$ is a probability measure defined by (5.1) replacing u_0 by ψ.

Under our assumptions (A1) and (A2), there exist Gibbs measures. General conditions are given by Hariya[12].

Theorem 5.2. *Gibbs measure with hard-wall external potential is reversible under the solution u of the SPDEs* (2.1) *with reflection for $I = \mathbb{R}$.*

Proof. We trace a method used, for instance, by Funaki and Spohn [10]. From Theorem 5.1, we have

$$E^{\mu_\psi^{l,r}}[F(u_0^{l,r})G(u_t^{l,r})] = E^{\mu_\psi^{l,r}}[G(u_0^{l,r})F(u_t^{l,r})] \tag{5.3}$$

for $F, G \in \mathscr{F}C_b^\infty$ and (l, r) enough large to contain the support of F, G. Define

$$W_\psi^{l,r} := \{g \in \mathscr{C}^+; g(x) = \psi(x) \text{ for } x \in [l, r]^c\}.$$

We can rewrite the reversibility (5.3) using $W_\psi^{l,r}$ (we extend $\mu_\psi^{l,r}$ to a measure on $W_\psi^{l,r}$ naturally):

$$\int_{W_\psi^{l,r}} F(\phi) E_{\phi,\psi}^{l,r}[G(u_t^{l,r})] \mu_\psi^{l,r}(d\phi) = \int_{W_\psi^{l,r}} G(\phi) E_{\phi,\psi}^{l,r}[F(u_t^{l,r})] \mu_\psi^{l,r}(d\phi), \tag{5.4}$$

where $P_{\phi,\psi}^{l,r}$ is the law of $u^{l,r}$ such that $u(x, 0) = \phi(x)$ for $x \in [l, r]$ and $u(x, 0) = \psi(x)$ for $x \in [l, r]^c$.

Integrating both sides of (5.4) over $\mathscr{C}^+$ with respect to the Gibbs measure μ with hard-wall external potential and using DLR property, we have

$$\begin{aligned} &\int_{\mathscr{C}} \int_{W_\psi^{l,r}} F(\phi) E_{\phi,\psi}^{l,r}[G(u_t^{l,r})] \mu_\psi^{l,r}(d\phi) \mu(d\psi) \\ &\qquad = \int_{\mathscr{C}} F(\psi) E_\psi^{l,r}[G(u_t^{l,r})] \, \mu(d\psi), \end{aligned} \tag{5.5}$$

where $P_\psi^{l,r}$ denotes the law of u with $u(x, 0) = \psi(x)$ for $x \in \mathbb{R}$ and $u(x, t) = \psi(x)$ for $x \in [l, r]^c$, $t \geq 0$. Using Theorem 4.1, the proof is completed. $\square$

Next, we show an energy inequality.

PROPOSITION 5.3. *Assume that $U(z) \in C^2(\mathbb{R}_+)$ is uniformly convex in the following sense that there exists a constant $c := \inf_{z \in \mathbb{R}_+} U''(z) > 0$ such that*

$$(z - \bar{z})(f(z) - f(\bar{z})) \geq c(z - \bar{z})^2.$$

Let (u, η) and $(\bar{u}, \bar{\eta})$ be two (unique) solutions with initial values u_0 and $\bar{u}_0$, respectively. Then, there exists a constant $K > 0$ such that

$$|u_t - \bar{u}_t|_\lambda^2 \leq e^{-Kt} |u_0 - \bar{u}_0|_\lambda^2. \tag{5.6}$$

PROOF. Note that both u_t and $\bar{u}_t$ satisfy the SPDE with reflection (2.1) with $\sigma \equiv 1$ for $I = \mathbb{R}$. Hence, at least formally speaking, taking the difference of both hand sides of the equation leads us to a deterministic PDE satisfied by $w_t := (u_t - \bar{u}_t)$ with initial value $w_0 := (u_0 - \bar{u}_0)$, that is,

$$\frac{\partial w_t}{\partial t} = \frac{\partial^2 w_t}{\partial x^2} - (f(u_t) - f(\bar{u}_t)) + \eta - \bar{\eta}.$$

Formally speaking (the computations actually can be justified by using mollifier technique[22]), multiplying both hand sides by $e^{Kt} w_t(x)\phi(x)^2$ with $\phi \in C_0^\infty(\mathbb{R})$ and integrating, we obtain

$$\begin{aligned}\int_0^t \left(\frac{\partial w_s}{\partial s}, e^{Ks} w_s \phi^2 \right) ds = &\int_0^t \left(\frac{\partial^2 w_s}{\partial x^2}, e^{Ks} w_s \phi^2 \right) ds \\ &- \int_0^t \left(f(u_s) - f(\bar{u}_s), e^{Ks} w_s \phi^2 \right) ds \\ &+ \int_0^t \int_{\mathbb{R}} e^{Ks} (u_s(x) - \bar{u}_s(x)) \phi(x)^2 (\eta - \bar{\eta})(dx, ds).\end{aligned}$$

Note that

$$\int_0^t \int_{\mathbb{R}} e^{Ks} (u_s(x) - \bar{u}_s(x)) \phi(x)^2 (\eta - \bar{\eta})(dx, ds) \le 0$$

and using uniform convexity condition, it is easy to obtain

$$\begin{aligned}\frac{1}{2} &\left(e^{Kt} \|w_t \phi\|^2 - \|w_0 \phi\|^2 - K \int_0^t e^{Ks} \|w_s \phi\|^2 \, ds \right) \\ &\le \frac{1}{2} \int_0^t \int_{\mathbb{R}} e^{Ks} w_s(x)^2 \left(\phi(x)^2 \right)'' dx \\ &\quad - \int_0^t e^{Ks} \left\| \frac{\partial w_s}{\partial x} \phi \right\|^2 ds - c \int_0^t e^{Ks} \|w_s \phi\|^2 \, ds.\end{aligned}$$

Now we approximate $\phi^2(x) \to e^{-2\lambda\chi(x)}$ and there exists a constant C_λ such that $(\phi(x)^2)'' \le C_\lambda e^{-2\lambda\chi(x)}$. Taking λ and K so that $K = 2c - C_\lambda > 0$, we have

$$|w_t|_\lambda^2 \le |w_0|_\lambda^2 e^{-Kt}. \ \square$$

THEOREM 5.4. *Assume that U fulfils the same conditions of Proposition 5.3. Then the tempered stationary measure for u is unique. In particular, the tempered reversible measure for u is also unique. Here, the stationarity means the condition for the reversibility, but for all $F \in \mathcal{F}C_b^\infty$ and $G \equiv 1$.*

PROOF. Let μ and $\tilde{\mu}$ be two stationary probability measures and let u and $\tilde{u}$ be the corresponding solutions with initial distributions μ and $\tilde{\mu}$, respectively. Then, for every $G \in \mathcal{F}C_b^\infty$, there exists a constant $K > 0$ such that $\left|E^\mu[G] - E^{\tilde{\mu}}[G]\right| \leq KE\left[|u_t - \tilde{u}_t|_\lambda\right] \to 0$ as $t \to \infty$, where E^μ denotes the expectation with respect to μ and we used Proposition 5.3. Thus the proof is completed. $\square$

Finally we obtained the following property.

COROLLARY 5.5. *Assume that U satisfies the same conditions of Proposition 5.3. Then the tempered Gibbs measure with hard-wall external potential is unique. Moreover, it coincides with the tempered reversible measure for the solution of the SPDE with reflection with the corresponding potential.*

Acknowledgement. The author would like to thank Prof. T. Funaki for valuable suggestions and kind encouragements.

References

[1] Bourbaki, N., Éléments de mathématique topologie générale, Hermann, Paris, 1958.
[2] Deuschel, J.-D. and G. Giacomin, Entropic repulsion for massless fields, Stochastic Processes Appl. **89** (2000), 333–354.
[3] Dobrushin, R. L., Description of a random field by means of conditional probabilities and conditions for its regularity, Theor. Probability Appl. **13** (1968), 201–229.
[4] Donati-Martin, C. and E. Pardoux, White noise driven SPDEs with reflection, Probab. Theory Relat. Fields **95** (1993), 1–24.
[5] Freidlin, M. I., Random perturbations of reaction-diffusion equations: The quasi-deterministic approximation, Trans. Amer. Math. Soc. **305** (1988), no. 2, 665–697.

[6] Funaki, T., Regularity properties for stochastic partial differential equations of parabolic type, Osaka J. Math. **28** (1991), 495–516.

[7] Funaki, T., The reversible measures of multi-dimensional Ginzburg–Landau type continuum model, Osaka J. Math. **28** (1991), 463–494.

[8] Funaki, T., Hydrodynamic limit for $\nabla\phi$-interface model on a wall, Probab. Theory Relat. Fields **126** (2003), no. 2, 155–183.

[9] Funaki, T. and S. Olla, Fluctuations for $\nabla\phi$ interface model on a wall, Stochastic Processes Appl. **94** (2001), 1–27.

[10] Funaki, T. and H. Spohn, Motion by mean curvature from the Ginzburg–Landau $\nabla\phi$ interface model, Comm. Math. Phys. **185** (1997), 1–36.

[11] Handa, K., Quasi-invariance and reversibility in the Fleming–Viot process, Probab. Theory Relat. Fields **122** (2002), no. 4, 545–566.

[12] Hariya, Y., Construction of Gibbs measures for 1-dimensional continuum fields, to appear, 2004.

[13] Hohenberg, P. C. and B. I. Halperin, Theory of dynamic critical phenomena, Rev. Modern Phys. **49** (1977), no. 3, 435–479.

[14] Lanford, O. E., III and D. Ruelle, Observables at infinity and states with short range correlations in statistical mechanics, Comm. Math. Phys. **13** (1969), 194–215.

[15] Ikeda, N. and S. Watanabe, Stochastic differential equations and diffusion processes, second ed., North Holland–Kodansha, Amsterdam, 1989.

[16] Itô, K., Foundations of stochastic differential equations in infinite dimensional spaces, Society for Industrial and Applied Mathematics, 1984.

[17] Iwata, K., Reversible measure of a $P(\phi)_1$-time evolution, Taniguchi Symp. PMMP Katata, 1985, pp. 195–209.

[18] Iwata, K., An infinite dimensional stochastic differential equation with state space $C(\mathbb{R})$, Probab. Theory Relat. Fields **74** (1987), 141–159.

[19] John, F., Partial differential equations, 4th ed., Springer-Verlag, 1982.

[20] Knight, F. B., Essentials of Brownian motion and diffusion, Mathematical Surveys, no. 18, American Mathematical Society, 1981.

[21] Lebowitz, G. F. and C. Maes, The effect of an external field on an interface, entropic repulsion, J. Statis. Phys. **46** (1987), 39–49.

[22] Nualart, D. and E. Pardoux, White noise driven quasilinear SPDEs with reflection, Probab. Theory Relat. Fields **93** (1992), 77–89.

[23] Otobe, Y., White noise driven stochastic diffusion equations defined on infinite interval with reflection, Master's thesis, The University of Tokyo, 1998.

[24] Otobe, Y., Stochastic reaction diffusion equations on an infinite interval with reflection, Stochastics Stochastics Rep. **74** (2002), 489–516.

[25] Parisi, G. and Y. S. Wu, Perturbation theory without gauge fixing, Sci. Sinica **24** (1981), 483–496.

[26] Shiga, T., Two contrasting properties of solutions for one-dimensional stochastic partial differential equations, Canad. J. Math. **46** (1994), no. 2, 415–437.

[27] Simon, B., Schrödinger semigroups, Bull. AMS **7** (1982), 447–526.
[28] Stroock, D. W. and S. R. S. Varadhan, Multidimensional diffusion process, Springer-Verlag, Berlin, 1979.
[29] Zambotti, L., A reflected stochastic heat equations as symmetric dynamics with respect to the 3-d Bessel bridge, J. Funct. Anal. **180** (2001), 195–209.

(Received April 16, 2004)
(Revised July 13, 2004)

Department of Mathematical Sciences
Faculty of Science
Shinshu University
3-1-1 Asahi, Matsumoto
Nagano 390-8621, Japan
E-mail: otobe@math.shinshu-u.ac.jp