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Viability Theorem for SPDE's Including HJM Framework

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Abstract. A viability theorem is proven for the mild solution of the stochastic differential equation in a Hilbert space of the form:

$$\begin{cases} dX^x(t) = AX^x(t)dt + b(X^x(t))dt + \sigma(X^x(t))dB(t), \\ X^x(0) = x. \end{cases}$$

It is driven by a Hilbert space-valued Wiener process B, with the infinitesimal generator A of a (C_0)-semigroup. This equation contains the stochastic partial differential equation within HJM framework in mathematical finance. Especially a viability theorem for "finite dimensional manifold" is proved, which is important for "consistency problems" in mathematical finance.

1. Introduction

In mathematical finance, the following stochastic partial differential equation (SPDE) is very important.

$$\begin{cases} dr(t,x) = \frac{\partial r}{\partial x}(t,x)\ dt + \left(\sum_{j\geq 1} \sigma_j(t,x) \int_0^x \sigma_j(t,u)du\right)\ dt \\ \qquad\qquad + \sum_{j\geq 1} \sigma_j(t,x)dB^j(t), \\ r(0,x) = r_0(x), \end{cases}$$

where $\sum_{j\geq 1}$ may be infinite sum, and generally $\sigma_j(t,x)$ may be stochastic. This SPDE is satisfied by the instantaneous forward rate at time $t+x$ observed at time t within HJM (Heath-Jarrow-Morton) framework. The path $x \mapsto r(t,x)$ is called forward curve observed at time t, and this is determined by the zero coupon yield curve. Practically zero coupon yield curves are often inferred by estimating finite number of parameters. So the

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set of forward curves forms a "finite dimensional manifold" rather than an infinite dimensional set. Taking into account that the zero coupon yield curve is estimated everyday, not only the initial forward curve r_0 but also forward curves generated by the SPDE should belong to the "finite dimensional manifold." This kind of problem is called "consistency problems." In the "consistency problem," the viability theorem for the "finite dimensional manifold" is very important. Björk [3] and Filipović [5] solved this kind of problem by differential geometry approach. In this paper, however, we shall prove the viability theorem by using the support theorem proved in Nakayama [7].

We deal with the following general framework containing the above SPDE as a special case.

Let H be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle_H$ and with its induced norm $\| \cdot \|_H$. We often abbreviate $\| \cdot \|_H$ to $\| \cdot \|$ for simplicity. Let A be the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t \geq 0}$ of bounded linear operators on H. Here $S(t)$, $t \geq 0$ are not necessarily shift operators.

Let us fix $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a right-continuous nondecreasing family $(\mathcal{F}(t))_{t \in [0,T]}$ of sub σ-fields of $\mathcal{F}$ such that each $\mathcal{F}(t)$ contains all P-null sets.

Let U be a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle_U$. Let Q be a nuclear strictly positive operator on U. We define a separable Hilbert space U_0 by $U_0 = Q^{1/2}(U)$ endowed with an inner product $\langle u, v \rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_U$, u, $v \in U_0$, and with its induced norm $\| \cdot \|_{U_0}$. Let $(B(t))_{t \in [0,T]}$ be a Q-Wiener process in $(\Omega, \mathcal{F}, P)$ having values in U with respect to $(\mathcal{F}(t))_{t \in [0,T]}$ in the sense of Da Prato and Zabczyk [4]. $(B(t))_{t \in [0,T]}$ can be characterized as a U-valued continuous $(\mathcal{F}(t))_{t \in [0,T]}$-adapted stochastic process such that

$$\lim_{n \to \infty} E\Big[\| B(t) - \sum_{j=1}^{n} B^j(t) g_j \|_U^2 \Big] = 0$$

for all $t \in [0, T]$, where $\{g_j \ ; \ j = 1, 2, \dots\}$ is a complete orthonormal system in U_0, and $(B^j(t))_{t \in [0,T]}$, $j = 1, 2, \dots$ are independent real-valued standard $(\mathcal{F}(t))_{t \in [0,T]}$-Brownian motions.

Let $\sigma \colon H \to L_{(2)}(U_0; H)$ and $b \colon H \to H$ be Lipschitz continuous bounded mappings, that is, there exists a constant $C_1 > 0$ such that

$$\|\sigma(x)\|_{L_{(2)}(U_0;H)} \le C_1, \quad \|\sigma(x) - \sigma(y)\|_{L_{(2)}(U_0;H)} \le C_1\|x - y\|,$$
$$\|b(x)\| \le C_1 \quad \text{and} \quad \|b(x) - b(y)\| \le C_1\|x - y\|$$

for all $x, y \in H$, where $L_{(2)}(U_0; H)$ is the set of Hilbert-Schmidt operators from U_0 to H and $\|\cdot\|_{L_{(2)}(U_0;H)}$ denotes its norm. We define mappings $\sigma_j\colon H \to H$, $j = 1, 2, \dots,$ by

$$\sigma_j(x) = \sigma(x)g_j, \qquad x \in H.$$

We assume that σ_j, $j = 1, 2, \dots,$ are twice Fréchet differentiable and those Fréchet derivatives up to second order, denoted by $D\sigma_j$ and $D^2\sigma_j$, are bounded, i.e., $\sup\{\|D\sigma_j(x)h\|\,;\, h \in H, \|h\| \le 1, x \in H\} < \infty$ and $\sup\{\|D^2\sigma_j(x)(h_1, h_2)\|\,;\, h_1, h_2 \in H, \|h_1\| \le 1, \|h_2\| \le 1, x \in H\} < \infty$.

For each positive integer n, we define a mapping $\rho_n\colon H \to H$ by

$$\rho_n(x) = \frac{1}{2}\sum_{j=1}^{n} D\sigma_j(x)\sigma_j(x)$$

for $x \in H$. We assume that there exists a mapping $\rho\colon H \to H$ such that

$$\lim_{n\to\infty} \|\rho_n(x) - \rho(x)\| = 0$$

for all $x \in H$ and there exists a constant $C_2 > 0$ such that

$$\|\rho_n(x) - \rho_n(y)\| \le C_2\|x - y\|$$

for all $x, y \in H$ and all $n \ge 1$.

For any $x \in H$, let $(X^x(t))_{t\in[0,T]}$ be the unique continuous mild solution of the stochastic differential equation

$$\text{(1.1)} \qquad \begin{cases} dX^x(t) = AX^x(t)dt + b(X^x(t))dt + \sigma(X^x(t))dB(t), \\ X^x(0) = x, \end{cases}$$

that is, $(X^x(t))_{t\in[0,T]}$ satisfies the following stochastic integral equation

$$X^x(t) = S(t)x + \int_0^t S(t-s)b(X^x(s))ds$$
$$+ \int_0^t S(t-s)\sigma(X^x(s))dB(s), \qquad t \in [0,T],\ P\text{-a.s.}$$

Let

$$\mathcal{C}^1 = \big\{h\colon [0,T] \to U_0 \ ; \ \text{continuous} \\ \text{and piecewise continuously differentiable}, \ h(0)=0\big\}.$$

Notice that ρ is Lipschitz continuous because of the assumption on ρ_n. For any $h \in \mathcal{C}^1$, we denote by $\xi^x(\ \cdot\ ) = \xi^x(\ \cdot\ ; h)\colon [0,T] \to H$ the unique mild solution of the following differential equation

$$\begin{cases} \dot{\xi}^x(t) = A\xi^x(t) + (b-\rho)(\xi^x(t)) + \sigma(\xi^x(t))\dot{h}(t), \\ \xi^x(0) = x. \end{cases} \tag{1.2}$$

That is, $\xi^x(\ \cdot\ ) = \xi^x(\ \cdot\ ; h)$ satisfies the integral equation

$$\xi^x(t) = S(t)x + \int_0^t S(t-s)(b-\rho)(\xi^x(s))ds \\ + \int_0^t S(t-s)\sigma(\xi^x(s))\dot{h}(s)ds, \qquad t \in [0,T].$$

We shall prove the following proposition.

PROPOSITION 1.1. *Let $K \subset H$ be a closed subset. The following three conditions are equivalent.*

(1) For every $x \in K$, $P\big(X^x(t) \in K$ for all $t \in [0,T]\big) = 1$.

(2) For every $h \in \mathcal{C}^1$, $x \in K$ and $t \in [0,T]$, $\xi^x(t;h) \in K$.

(3) For every $x \in K$ and $u \in U_0$,

$$\varliminf_{t\downarrow 0} \frac{1}{t} \operatorname{dis}\big(S(t)x + t(b(x) - \rho(x) + \sigma(x)u), \ K\big) = 0,$$

where $\operatorname{dis}(y,K) = \inf\{\|y-\hat{y}\|_H \ ; \ \hat{y} \in K\}$ *for any* $y \in H$.

We say that K is invariant for (1.1) if (1) is satisfied, and K is invariant for (1.2) if (2) is satisfied. Proposition 1.1 says that K is invariant for (1.1) if and only if K is invariant for (1.2). And these are also equivalent to the condition (3), which is called *semigroup Nagumo's condition.*

Denote the domain of A by $D(A)$. If $K \subset D(A)$, the condition (3) in Proposition 1.1 is equivalent to the condition

$$\lim_{t \downarrow 0} \frac{1}{t} \operatorname{dis}\bigl(x + t(Ax + b(x) - \rho(x) + \sigma(x)u),\ K\bigr) = 0, \qquad u \in U_0,\ x \in K. \tag{1.3}$$

This is called *Nagumo's condition.*

The equivalence of (2) and (3) comes essentially from Jachimiak [6], and Zabczyk [8] proved the inclusion (3) $\Rightarrow$ (1) under a specific assumption.

Now we want to state our main theorem for an application to mathematical finance. Before that we prepare the definition of "finite dimensional submanifold."

Let n be a positive integer. A subset $M \subset H$ is called an n-dimensional C^1 submanifold if for any $x \in M$ there exist an open subset $O \subset H$ which satisfies $x \in O$, an open subset $\mathcal{Z} \subset \mathbf{R}^n$ and a homeomorphism $\phi \colon \mathcal{Z} \to O \cap M$ which has an injective Fréchet derivative $D\phi(z) \colon \mathbf{R}^n \to H$ at every $z \in \mathcal{Z}$. The linear space $D\phi(\phi^{-1}(x))\mathbf{R}^n$ is called the tangent space to M at $x \in M$ and denoted by $T_x M$. The tangent space is defined independently of the choice of homeomorphism ϕ.

THEOREM 1.2. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. Then M is invariant for (1.1) if and only if $M \subset D(A)$ and*

$$Ax + b(x) - \rho(x) \in T_x M,$$
$$\sigma(x) U_0 \subset T_x M$$

hold for all $x \in M$.

2. Proofs of Proposition 1.1 and Theorem 1.2

First we prove Proposition 1.1. Let

$$\mathcal{L}^x = \bigl\{\xi^x(\,\cdot\,; h)\ ;\ h \in \mathcal{C}^1\bigr\} \subset C([0,T]; H).$$

From Nakayama [7], we have the following support theorem.

Theorem 2.1.

$$\operatorname{supp} X^x(\cdot) = \bar{\mathcal{L}}^x,$$

where $\bar{\mathcal{L}}^x$ *is the closure of* $\mathcal{L}^x$ *in* $C([0,T];H)$ *and* $\operatorname{supp} X^x(\cdot)$ *is the support of the law of* X^x.

Let $\tilde{K} = C([0,T];K)$.
(1) $\Rightarrow$ (2): It holds that $P(X^x(\cdot) \in \tilde{K}) = 1$ for all $x \in K$. Therefore we have $\bar{\mathcal{L}}^x = \operatorname{supp} X^x(\cdot) \subset \tilde{K}$, which implies (2).
(2) $\Rightarrow$ (1): We have $\mathcal{L}^x \subset \tilde{K}$ for all $x \in K$. Since the set $\tilde{K} \subset C([0,T];H)$ is closed, $\operatorname{supp} X^x(\cdot) = \bar{\mathcal{L}}^x \subset \tilde{K}$. Therefore we get $P(X^x(\cdot) \in \tilde{K}) = 1$, which is equivalent to (1).

Now we prove the equivalence of (2) and (3). For each $u \in U_0$, we define a mapping $f_u \colon [0,T] \to U_0$ by

$$f_u(t) = tu, \qquad t \in [0,T].$$

From Theorem 2 in Jachimiak [6], the condition

$$\xi^x(t; f_u) \in K, \qquad t \in [0,T],\ x \in K,\ u \in U_0 \tag{2.1}$$

is equivalent to (3). So we have only to prove that the condition (2.1) implies (2). However, (2.1) implies

$$\xi^x(t; h) \in K, \qquad t \in [0,T],\ x \in K$$

for all mappings $h \colon [0,T] \to U_0$ which are piecewise linear and satisfy $h(0) = 0$ because we have

$$\xi^{\xi^x(s;h)}(t-s; h(s+\ \cdot\ )) = \xi^x(t;h), \qquad 0 \le s \le t \le T.$$

For any $h \in \mathcal{C}^1$, we can take mappings $h_n \colon [0,T] \to U_0$ which are piecewise linear and satisfy $h_n(0) = 0$ such that

$$\sup_{0 \le t \le T} \|h_n(t) - h(t)\|_{U_0} + \operatorname*{ess.sup}_{0 \le t \le T} \|\dot{h}_n(t) - \dot{h}(t)\|_{U_0} \to 0$$

as $n \to \infty$. Since we have

$$\begin{aligned}
&\xi^x(t;h_n) - \xi^x(t;h) \\
&= \int_0^t S(t-s)\big((b-\rho)(\xi^x(s;h_n)) - (b-\rho)(\xi^x(s;h))\big)ds \\
&+ \int_0^t S(t-s)\sigma(\xi^x(s;h_n))(\dot{h}_n(s) - \dot{h}(s))ds \\
&+ \int_0^t S(t-s)\big(\sigma^x(\xi(s;h_n)) - \sigma(\xi^x(s;h))\big)\dot{h}(s)ds,
\end{aligned}$$

it holds that

$$\begin{aligned}
&\|\xi^x(t;h_n) - \xi^x(t;h)\| \\
&\leq C\big(1 + \operatorname*{ess.\,sup}_{0\leq t\leq T} \|\dot{h}(t)\|_{U_0}\big) \int_0^t \|\xi^x(s;h_n) - \xi^x(s;h)\|ds \\
&+ C \sup_{0\leq t\leq T} \|\dot{h}_n(t) - \dot{h}(t)\|_{U_0}.
\end{aligned}$$

Therefore, from Gronwall inequality, we get

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \|\xi^x(t;h_n) - \xi^x(t;h)\| = 0$$

for all $x \in K$. $\square$

We prepare some lemmas to prove Theorem 1.2.

Lemma 2.2. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. Let $x \in M$ and $\{x_n\}_{n\geq 1} \subset M \setminus \{x\}$ be a sequence which satisfies $\lim_{n\to\infty} \|x_n - x\| = 0$. Then there exists an element $v \in T_xM$ and a subsequence $\{x_{n_k}\}_{k\geq 1} \subset \{x_n\}_{n\geq 1}$ such that*

$$\lim_{k\to\infty} \|\frac{1}{\|x_{n_k} - x\|}(x_{n_k} - x) - v\| = 0.$$

Proof. Let O, $\mathcal{Z}$ and ϕ be such as lines above Theorem 1.2. Here we may assume $x_n \in O$, $n \geq 1$. There exist z and z_n in $\mathcal{Z}$ such that $x = \phi(z)$ and $x_n = \phi(z_n)$ for $n \geq 1$. Since $\lim_{n\to\infty} \|x_n - x\|_H = 0$ and ϕ

is homeomorphic, it also holds that $\lim_{n\to\infty}\|z_n - z\|_{\mathbf{R}^n} = 0$. Since z and z_n are elements of the finite dimensional space, there exist a subsequence $\{z_{n_k}\}_{k\geq 1} \subset \{z_n\}_{n\geq 1}$ and an element $\lambda \in \mathbf{R}^n$ such that

$$\lim_{k\to\infty} \| \frac{1}{\|z_{n_k} - z\|_{\mathbf{R}^n}}(z_{n_k} - z) - \lambda\|_{\mathbf{R}^n} = 0.$$

Then we have

$$\lim_{k\to\infty} \| \frac{1}{\|z_{n_k} - z\|_{\mathbf{R}^n}}(\phi(z_{n_k}) - \phi(z)) - D\phi(z)\lambda\|_H = 0.$$

Therefore it holds that

$$\frac{1}{\|x_{n_k} - x\|_H}(x_{n_k} - x) = \Big(\frac{\|\phi(z_{n_k}) - \phi(z)\|_H}{\|z_{n_k} - z\|_{\mathbf{R}^n}}\Big)^{-1} \times \frac{1}{\|z_{n_k} - z\|_{\mathbf{R}^n}}(\phi(z_{n_k}) - \phi(z)) \to \|D\phi(z)\lambda\|_H^{-1} D\phi(z)\lambda.$$

This completes the proof. □

LEMMA 2.3. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. If M is invariant for (1.1), then $M \subset D(A)$ holds.*

PROOF. Let us fix an arbitrary $x \in M$. We prove $x \in D(A)$. Since the condition (3) in Proposition 1.1 holds, we can choose $t_n > 0$ and $y(t_n) \in M$, $n = 1, 2, \ldots$ such that $t_n \downarrow 0$ and

$$\lim_{n\to\infty} \frac{1}{t_n}\|S(t_n)x + t_n(b(x) - \rho(x) + \sigma(x)u) - y(t_n)\| = 0. \tag{2.2}$$

This implies $\|S(t_n)x - y(t_n)\| \to 0$, and therefore $\|y(t_n) - x\| \to 0$. If $y(t_n) = x$ takes place infinitely often, then (2.2) implies $x \in D(A)$. So we have only to consider the case $y(t_n) \neq x$ for all $n \geq 1$. From Lemma 2.2, there exists a subsequence $\{t'_n\} \subset \{t_n\}$ and $v \in T_pM$, $\|v\| = 1$ such that

$$\lim_{n\to\infty} \| \frac{1}{\|y(t'_n) - x\|}(y(t'_n) - x) - v\| = 0.$$

If $\frac{1}{t'_n}\|y(t'_n) - x\|$, $n \geq 1$ is not bounded, then there exists a subsequence $\{t''_n\} \subset \{t'_n\}$ such that $\frac{\epsilon_n}{t''_n} \to \infty$, where $\epsilon_n = \|y(t''_n) - x\|$, $n \geq 1$. From (2.2), the sequence $\left\{\frac{1}{t''_n}(S(t''_n)x - y(t''_n))\right\}_n$ is convergent, and therefore we have

$$\epsilon_n^{-1}(S(t''_n)x - x) - v = \frac{t''_n}{\epsilon_n}\frac{1}{t''_n}(S(t''_n)x - y(t''_n)) + \left(\frac{1}{\epsilon_n}(y(t''_n) - x) - v\right) \to 0$$

as $n \to \infty$. Let us fix any $t \in [0,T]$ and set $N_n = [t/t''_n]$, where $[a]$ is the largest integer which is not greater than a. Letting $n \to \infty$ in the identity

$$\begin{aligned}&\frac{t''_n}{\epsilon_n}(S(N_n t''_n)x - x)\\ &\qquad = t''_n \sum_{k=1}^{N_n} S((k-1)t''_n)v + t''_n \sum_{k=1}^{N_n} S((k-1)t''_n)\left\{\frac{1}{\epsilon_n}(S(t''_n)x - x) - v\right\},\end{aligned}$$

we get

$$0 = \int_0^t S(s)v ds, \quad t \in [0,T],$$

which contradicts $v \neq 0$. Therefore we can conclude that $\frac{1}{t'_n}\|y(t'_n) - x\|$, $n \geq 1$ is bounded. This implies that there exists a subsequence $\{s_n\} \subset \{t'_n\}$ such that the convergence

$$\frac{1}{s_n}\|y(s_n) - x\| \to \lambda$$

holds for some constant $\lambda \in \mathbf{R}$. Then we have

$$\frac{1}{s_n}\big(y(s_n) - x\big) \to \lambda v,$$

which yields $x \in D(A)$. □

LEMMA 2.4. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. If M is invariant for (1.1), then mild solutions $(X^x(t))_{t\in[0,T]}$ and $\xi^x(\,\cdot\,;h)$ are also strong solutions of the equations (1.1) and (1.2) respectively for all $x \in M$ and $h \in \mathcal{C}^1$. This means that the following equations hold.*

$$X^x(t) = x + \int_0^t \big(AX^x(s) + b(X^x(s))\big)ds + \sigma(X^x(s))dB(s), \quad t \in [0,T], \ a.s.$$

and

$$\xi(t;h) = x + \int_0^t \big(A\xi(s;h) + (b-\rho)(\xi(s;h)) + \sigma(\xi(s;h))\dot{h}(s)\big)ds, \quad t \in [0,T].$$

PROOF. Let $a \in D(A^*)$, where A^* denotes the adjoint operator of A. From the stochastic Fubini theorem, we can calculate as follows.

$$\begin{aligned}\int_0^t \Big\langle A^*a, \int_0^s S(s-u)\sigma(X^x(u))dW(u)\Big\rangle ds \\ = \Big\langle a, \int_0^t \big(S(t-u)-I\big)\sigma(X^x(u))dW(u)\Big\rangle.\end{aligned}$$

$$\begin{aligned}\int_0^t \Big\langle A^*a, \int_0^s S(s-u)b(X^x(u))du\Big\rangle ds \\ = \Big\langle a, \int_0^t \big(S(t-u)-I\big)b(X^x(u))du\Big\rangle.\end{aligned}$$

$$\int_0^t \big\langle A^*a, S(s)x\big\rangle ds = \big\langle a, \big(S(t)-I\big)x\big\rangle.$$

Therefore we obtain

$$\begin{aligned}\langle a, X^x(t)\rangle &= \langle a, x\rangle + \int_0^t \langle A^*a, X^x(s)\rangle ds \\ &\quad + \int_0^t \langle a, b(X^x(s))\rangle ds + \int_0^t \langle a, \sigma(X^x(s))dW(s)\rangle, \quad t \in [0,T], \text{ a.s.}\end{aligned}$$

for all $a \in D(A^*)$. Furthermore, from lemma 2.3, we have

$$P(X^x(t) \in D(A) \quad \text{for all } t \in [0,T]) = 1.$$

Therefore $(X^x(t))_{t\in[0,T]}$ is the strong solution of the equation (1.1). As for $\xi^x(\,\cdot\,;h)$, we can conclude by the same way. $\square$

From Lemma 2.4, we have

$$\dot{\xi}(0;h) = Ax + (b-\rho)(x) + \sigma(x)\dot{h}(0) \in T_xM.$$

This implies the following lemma.

LEMMA 2.5. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. If M is invariant for (1.1), then*

$$Ax + b(x) - \rho(x) \in T_x M,$$
$$\sigma(x)U_0 \subset T_x M$$

holds for all $x \in M$.

LEMMA 2.6. *Let $M \subset H$ be an n-dimensional C^1 submanifold which is closed as a subset. We assume that $M \subset D(A)$ and*

$$Ax + b(x) - \rho(x) \in T_x M,$$
$$\sigma(x)U_0 \subset T_x M$$

holds for all $x \in M$. Then M is invariant for (1.1).

PROOF. Let $u \in U_0$ and $x \in M$. It is sufficient to prove the condition (3) in Proposition 1.1. Let us fix $u \in U_0$ and $x \in M$, and set

$$v = Ax + b(x) - \rho(x) + \sigma(x)u.$$

Let $\mathcal{Z}$ and ϕ be such as lines above Theorem 1.2. From the assumption, we have $v \in T_x M$, and therefore there exist $z_0 \in \mathcal{Z}$ and $\lambda \in \mathbf{R}^n$ such that $\phi(z_0) = x$ and $D\phi(z_0)\lambda = v$. Take $\epsilon > 0$ such that $\big\{z_0 + t\lambda\ ;\ t \in (-\epsilon, \epsilon)\big\} \subset \mathcal{Z}$, and we define the mapping $g\colon (\epsilon, \epsilon) \to \mathcal{Z}$ by $g(t) = z_0 + t\lambda$. Then we have

$$\lim_{t \to 0} \| \frac{1}{t}\big(\phi(g(t)) - x\big) - v\| = 0,$$

and therefore

$$\varliminf_{t \downarrow 0} \frac{1}{t} \operatorname{dis}\big(x + tv, M\big) = 0.$$

It completes the proof. □

From Lemma 2.3, 2.5 and 2.6, we obtain Theorem 1.2.

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