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Support Theorem for Mild Solutions of SDE's in Hilbert Spaces

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Abstract. A support theorem is proven for the mild solution of the stochastic differential equation in a Hilbert space of the form:

$$dX(t) = AX(t)dt + b(X(t))dt + \sigma(X(t))dB(t).$$

It is driven by a Hilbert space-valued Wiener process B, with the infinitesimal generator A of a (C_0)-semigroup.

1. Introduction

Support theorem was first proved in Stroock and Varadhan [9] for SDE's in finite dimensional state spaces and with finite dimensional Wiener processes. Another method for these equations are found in Aida, Kusuoka and Stroock [2] and Millet and Sanz-Solé [6] which we use essentially. The extension of support theorem for SDE's in separable Hilbert spaces driven by separable Hilbert space-valued Wiener processes without infinitesimal generators was achieved in Aida [1] extending the method of Stroock and Varadhan [9]. Because our mild solutions are not necessarily expressed as strong solutions due to the existence of infinitesimal generators, our approach is different from Aida [1].

Let H be a separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle_H$ and with its induced norm $\| \cdot \|_H$. We often abbreviate $\| \cdot \|_H$ to $\| \cdot \|$ for simplicity. Let A be the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t \geq 0}$ of bounded linear operators on H.

Let us fix $T > 0$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a right-continuous nondecreasing family $(\mathcal{F}(t))_{t \in [0,T]}$ of sub σ-fields of $\mathcal{F}$ such that each $\mathcal{F}(t)$ contains all P-null sets.

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Let U be a separable Hilbert space with an inner product $\langle \cdot, \cdot \rangle_U$. Let Q be a nuclear strictly positive operator on U. We define a separable Hilbert space U_0 by $U_0 = Q^{1/2}(U)$ endowed with an inner product $\langle u, v\rangle_{U_0} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_U$, u, $v \in U_0$, and with its induced norm $\|\cdot\|_{U_0}$. Let $(B(t))_{t\in[0,T]}$ be a Q-Wiener process in $(\Omega, \mathcal{F}, P)$ having values in U with respect to $(\mathcal{F}(t))_{t\in[0,T]}$ in the sense of Da Prato and Zabczyk [4]. $(B(t))_{t\in[0,T]}$ can be characterized as a U-valued continuous $(\mathcal{F}(t))_{t\in[0,T]}$-adapted stochastic process such that

$$\lim_{n\to\infty} E\Big[\|B(t) - \sum_{j=1}^{n} B^j(t) g_j\|_U^2\Big] = 0$$

for all $t \in [0,T]$, where $\{g_j\ ;\ \ j = 1, 2, \dots\}$ is a complete orthonormal system in U_0, and $(B^j(t))_{t\in[0,T]}$, $j = 1, 2, \dots$ are independent real-valued standard $(\mathcal{F}(t))_{t\in[0,T]}$-Brownian motions.

Let $\sigma\colon H \to L_{(2)}(U_0; H)$ and $b\colon H \to H$ be Lipschitz continuous bounded mappings, that is, there exists a constant $C_1 > 0$ such that

$$\|\sigma(x)\|_{L_{(2)}(U_0;H)} \le C_1, \quad \|\sigma(x) - \sigma(y)\|_{L_{(2)}(U_0;H)} \le C_1\|x - y\|,$$
$$\|b(x)\| \le C_1 \quad \text{and} \quad \|b(x) - b(y)\| \le C_1\|x - y\|$$

for all $x, y \in H$, where $L_{(2)}(U_0; H)$ is the set of Hilbert-Schmidt operators from U_0 to H and $\|\cdot\|_{L_{(2)}(U_0;H)}$ denotes its norm. We define mappings $\sigma_j\colon H \to H$, $j = 1, 2, \dots,$ by

$$\sigma_j(x) = \sigma(x) g_j, \qquad x \in H.$$

We assume that σ_j, $j = 1, 2, \dots,$ are twice Fréchet differentiable and those Fréchet derivatives up to second order, denoted by $D\sigma_j$ and $D^2\sigma_j$, are bounded, i.e., $\sup\{\|D\sigma_j(x)h\|\,;\, h \in H, \|h\| \le 1, x \in H\} < \infty$ and $\sup\{\|D^2\sigma_j(x)(h_1, h_2)\|\,;\, h_1, h_2 \in H, \|h_1\| \le 1, \|h_2\| \le 1, x \in H\} < \infty$.

For each positive integer n, we define a mapping $\rho_n\colon H \to H$ by

$$\rho_n(x) = \frac{1}{2}\sum_{j=1}^{n} D\sigma_j(x)\sigma_j(x) \tag{1.1}$$

for $x \in H$. We assume that there exists a mapping $\rho\colon H \to H$ such that

$$\lim_{n\to\infty} \|\rho_n(x) - \rho(x)\| = 0$$

for all $x \in H$ and there exists a constant $C_2 > 0$ such that

$$\|\rho_n(x) - \rho_n(y)\| \leq C_2\|x - y\|$$

for all $x, y \in H$ and all $n \geq 1$. Therefore, it also holds that

$$\lim_{n\to\infty} \sup_{x\in K} \|\rho_n(x) - \rho(x)\| = 0$$

for each compact subset $K \subset H$.

Let $x_0 \in H$ be fixed and $(X(t))_{t\in[0,T]}$ be the H-valued continuous $(\mathcal{F}(t))_{t\in[0,T]}$-adapted stochastic process which is the unique mild solution of the stochastic differential equation

$$\begin{cases} dX(t) = AX(t)dt + b(X(t))dt + \sigma(X(t))dB(t), \\ X(0) = x_0, \end{cases} \tag{1.2}$$

that is, $(X(t))_{t\in[0,T]}$ satisfies the following stochastic integral equation

$$X(t) = S(t)x_0 + \int_0^t S(t-s)b(X(s))ds + \int_0^t S(t-s)\sigma(X(s))dB(s), \qquad t \in [0,T], \ P\text{-a.s.}$$

Let $h\colon [0,T] \to U_0$ be a continuous mapping which is piecewise continuously differentiable and satisfies $h(0) = 0$. We denote by $\xi(\,\cdot\,) = \xi(\,\cdot\,; h)\colon [0,T] \to H$ the unique mild solution of the following differential equation

$$\begin{cases} \dot{\xi}(t) = A\xi(t) + (b-\rho)(\xi(t)) + \sigma(\xi(t))\dot{h}(t), \\ \xi(0) = x_0. \end{cases}$$

That is, $\xi(\,\cdot\,) = \xi(\,\cdot\,; h)$ satisfies the integral equation

$$\xi(t) = S(t)x_0 + \int_0^t S(t-s)(b-\rho)(\xi(s))ds + \int_0^t S(t-s)\sigma(\xi(s))\dot{h}(s)ds, \qquad t \in [0,T].$$

Let

$$\mathcal{L} = \big\{\xi(\cdot;h)\ ;\ h\colon [0,T] \to U_0 \text{ is continuous and } \\ \text{piecewise continuously differentiable, } \ h(0)=0\big\} \subset C([0,T];H).$$

Our main theorem is the following.

Theorem 1.1.

$$\operatorname{supp} X(\cdot) = \bar{\mathcal{L}},$$

where $\bar{\mathcal{L}}$ denotes the closure of $\mathcal{L}$ in $C([0,T];H)$ and $\operatorname{supp} X(\cdot)$ denotes the support of the distribution $P \circ X^{-1}$. Here $P \circ X^{-1}$ is the image measure of the mapping $\omega \to X(\cdot,\omega)$ from Ω to $C([0,T];H)$.

Bally, Millet and Sanz-Solé [3], Millet and Sanz-Solé [7] and [8] proved support theorems for particular SPDE's. Further research is needed to answer the question of whether we can extend our support theorem to contain their results.

Our SDE's are motivated partly by SPDE's in forward interest rate models called Heath-Jarrow-Morton (HJM) models in mathematical finance. If we use our support theorem, we can prove an "invariance theorem", which is useful to determine "invariant manifolds" for HJM equations. This kind of topics shall be discussed in forthcoming papers.

To prove our support theorem, we have to prove mainly two theorems. First we prove "approximation theorem" in section 2. Secondly we prove "convergence of parameterized SDE's" from section 3 to 6. Using these two theorems, we shall prove our support theorem in Section 7. As a notational convention, constants denoted by C which will appear may be different from one part to another.

2. Approximation Theorem

In this Section, we consider the following situation. Let H be a separable Hilbert space endowed with a inner product $\langle\cdot,\cdot\rangle_H$. We denote its norm by $\|\cdot\|_H$. We often abbreviate $\|\cdot\|_H$ to $\|\cdot\|$ for simplicity. Let A be the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t\geq 0}$ on H.

Let us fix $T>0$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space equipped with a right-continuous nondecreasing family $(\mathcal{F}(t))_{t\in[0,T]}$ of sub σ-fields of $\mathcal{F}$ such that each $\mathcal{F}(t)$ contains all P-null sets.

Let r be a positive integer and $\sigma_j \colon H \to H$, $j = 1, 2, \ldots, r$, be twice Fréchet differentiable mappings such that $\sup_{x \in H} \|\sigma_j(x)\| < \infty$. We assume that those Fréchet derivatives up to second order $D\sigma_j$ and $D^2\sigma_j$ are bounded. Let $b \colon H \to H$ be a Lipschitz continuous mapping.

For each $m = 1, 2, \ldots$, let $\delta_m = T/m$ and

$$[t]_m^- = k\delta_m, \quad [t]_m^+ = (k+1)\delta_m$$

if $k\delta_m \leq t < (k+1)\delta_m$, $k \geq 0$. Let $(B^j(t))_{t \in [0,T]}$, $j = 1, 2, \ldots, r$, be independent real-valued standard $(\mathcal{F}(t))_{t \in [0,T]}$-Brownian motions. We define real-valued stochastic processes $(B_m^j(t))_{t \in [0,T]}$, $j = 1, 2, \ldots, r$, by

$$B_m^j(t) = B^j([t]_m^-) + \frac{t - [t]_m^-}{\delta_m}\big(B^j([t]_m^+) - B^j([t]_m^-)\big) \tag{2.1}$$

for $t \in [0, T]$ and $m \geq 1$.

Let $(X(t))_{t \in [0,T]}$ be the unique mild solution of the stochastic differential equation

$$\begin{cases} dX(t) = AX(t)dt + b(X(t))dt + \dfrac{1}{2}\displaystyle\sum_{j=1}^{r} D\sigma_j(X(t))\sigma_j(X(t))dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\sum_{j=1}^{r} \sigma_j(X(t))dB^j(t), \\ X(0) = x_0. \end{cases} \tag{2.2}$$

Let us denote by $\xi_m(\,\cdot\,) = \xi_m(\,\cdot\,, \omega) \colon [0, T] \to H$ the unique mild solution of the following differential equation

$$\begin{cases} \dot{\xi}_m(t) = A\xi_m(t) + b(\xi_m(t)) + \displaystyle\sum_{j=1}^{r} \sigma_j(\xi_m(t))\dot{B}_m^j(t), \\ \xi_m(0) = x_0 \end{cases}$$

for each ω.

We shall prove the following theorem in this section.

THEOREM 2.1. *For any* $p > 1$,

$$\lim_{m \to \infty} E\Big[\sup_{0 \leq t \leq T} \|\xi_m(t) - X(t)\|^{2p}\Big] = 0.$$

Before proving this theorem, we prepare some lemmas. For each $m \geq 1$, let

$$\bar{\xi}_m(t) = S(t - [t]_m^-)\xi_m([t]_m^-)$$

and

$$\bar{X}_m(t) = S(t - [t]_m^-)X([t]_m^-)$$

for $t \in [0, T]$

LEMMA 2.2. *For any $p > 1$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0 \leq t \leq T} \|\xi_m(t) - \bar{\xi}_m(t)\|^{2p}\Big] \leq C\delta_m^{p-1}.$$

PROOF. Note that

$$\begin{aligned} &\xi_m(t) - \bar{\xi}_m(t) \\ &= \int_{[t]_m^-}^t S(t-s)b(\xi_m(s))ds + \sum_{j=1}^r \int_{[t]_m^-}^t S(t-s)\sigma_j(\xi_m(s))\dot{B}_m^j(s)ds, \end{aligned} \tag{2.3}$$

and hence there exists a constant $C > 0$ such that

$$\|\xi_m(t) - \bar{\xi}_m(t)\| \leq C\Big(\delta_m + \sum_{j=1}^r |B^j([t]_m^+) - B^j([t]_m^-)|\Big) \tag{2.4}$$

for $t \in [0, T]$. Since

$$\begin{aligned} &E\Big[\sup_{0 \leq t \leq T} |B^j([t]_m^+) - B^j([t]_m^-)|^{2p}\Big] \\ &\qquad \leq \sum_{k=0}^{m-1} E\Big[|B^j((k+1)\delta_m) - B^j(k\delta_m)|^{2p}\Big] = (2p-1)!!\,T\delta_m^{p-1}, \end{aligned}$$

we obtain the lemma. □

REMARK 2.3. We shall use the following identity when arguments similar to Chapter 5 in Da Prato and Zabczyk [4] are needed.

$$\int_s^t (t-u)^{\alpha-1}(u-s)^{-\alpha}du = \frac{\pi}{\sin \pi\alpha}, \qquad 0 \leq s < t,\ 0 < \alpha < 1. \tag{2.5}$$

Lemma 2.4. *Let $p > 1$. Let $(B(t))_{t\geq 0}$ be a real-valued standard Brownian motion and $(\gamma(t))_{t\in[0,T]}$ be an H-valued predictable stochastic process that satisfies $\sup\big\{\|\gamma(t,\omega)\| \ ; \ 0 \leq t \leq T, \ \omega \in \Omega\big\} < \infty$. Then there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\leq t\leq T} \| \int_{[t]_m^-}^t S(t-s)\gamma(s)dB(s)\|^{2p}\Big] \leq C\delta_m^{p-1}.$$

Proof. We take $\alpha \in \left(\frac{1}{2p}, \frac{1}{2}\right)$ and fix it. By Equation (2.5) and the stochastic Fubini theorem, we have

$$\begin{aligned}&\int_{[t]_m^-}^t S(t-s)\gamma(s)dB(s)\\ &= \frac{\sin \pi\alpha}{\pi}\int_{[t]_m^-}^t (t-u)^{\alpha-1}S(t-u)\Big(\int_{[t]_m^-}^u (u-s)^{-\alpha}S(u-s)\gamma(s)dB(s)\Big)du.\end{aligned}$$

So we have

$$\begin{aligned}\| \int_{[t]_m^-}^t S(t-s)\gamma(s)dB(s)\| &\leq \frac{\sin \pi\alpha}{\pi}\sup_{0\leq u\leq T}\|S(u)\|\\ &\qquad \int_{[t]_m^-}^t (t-u)^{\alpha-1}\| \int_{[t]_m^-}^u (u-s)^{-\alpha}S(u-s)\gamma(s)dB(s)\|du.\end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned}&\| \int_{[t]_m^-}^t S(t-s)\gamma(s)dB(s)\|^{2p}\\ &\leq C\delta_m^{2p\alpha-1}\int_{[t]_m^-}^t \| \int_{[t]_m^-}^u (u-s)^{-\alpha}S(u-s)\gamma(s)dB(s)\|^{2p}du\\ &\leq C\delta_m^{2p\alpha-1}\sum_{k=0}^{m-1}\int_{k\delta_m}^{(k+1)\delta_m} \| \int_{k\delta_m}^u (u-s)^{-\alpha}S(u-s)\gamma(s)dB(s)\|^{2p}du.\end{aligned}$$

for all $t \in [0,T]$. Therefore we get

$$E\Big[\sup_{0\leq t\leq T} \| \int_{[t]_m^-}^t S(t-s)\gamma(s)dB(s)\|^{2p}\Big]$$

$$\leq C\delta_m^{2p\alpha-1}\sum_{k=0}^{m-1}\int_{k\delta_m}^{(k+1)\delta_m} E\Big[\Big(\int_{k\delta_m}^{u}(u-s)^{-2\alpha}\|S(u-s)\gamma(s)\|^2 ds\Big)^p\Big]du$$
$$\leq C\delta_m^{2p\alpha-1}\sum_{k=0}^{m-1}\int_{k\delta_m}^{(k+1)\delta_m}\Big(\frac{\delta_m^{1-2\alpha}}{1-2\alpha}\Big)^p du \leq C\delta_m^{p-1}. \ \square$$

LEMMA 2.5. *For any $p>1$, there exists a constant $C>0$ such that*

$$E\Big[\sup_{0\leq t\leq T}\|X(t)-\bar{X}_m(t)\|^{2p}\Big]\leq C\delta_m^{p-1}.$$

PROOF. Since we have

$$X(t)-\bar{X}_m(t)$$
$$=\int_{[t]_m^-}^{t} S(t-s)b(X(s))ds+\frac{1}{2}\sum_{j=1}^{r}\int_{[t]_m^-}^{t} S(t-s)D\sigma_j(X(s))\sigma_j(X(s))ds$$
$$+\sum_{j=1}^{r}\int_{[t]_m^-}^{t} S(t-s)\sigma_j(X(s))dB^j(s),$$

it holds that

$$\|X(t)-\bar{X}_m(t)\|\leq C\Big(\delta_m+\sum_{j=1}^{r}\|\int_{[t]_m^-}^{t} S(t-s)\sigma_j(X(s))dB^j(s)\|\Big)$$

for $t\in[0,T]$. Furthermore we have

$$E\Big[\sup_{0\leq t\leq T}\|\int_{[t]_m^-}^{t} S(t-s)\sigma_j(X(s))dB^j(s)\|^{2p}\Big]\leq C\delta_m^{p-1}$$

from Lemma 2.4, and therefore the lemma holds. $\square$

From Equation (2.3), we have

$$\sigma_j(\xi_m(t))=\sigma_j(\bar{\xi}_m(t))$$
$$+\int_0^1 D\sigma_j\big(\bar{\xi}_m(t)+v\ (\xi_m(t)-\bar{\xi}_m(t))\big)\big(\xi_m(t)-\bar{\xi}_m(t)\big)dv$$

$$= \gamma_{m,1}^{j}(t) + \sum_{l=1}^{r} \gamma_{m,2}^{j,l}(t) + \gamma_{m,3}^{j}(t) \tag{2.6}$$

for each $j = 1, 2, \ldots, r$, where

$$\gamma_{m,1}^{j}(t) = \sigma_j(\bar{\xi}_m(t)),$$

$$\gamma_{m,2}^{j,l}(t) = D\sigma_j(\bar{\xi}_m(t)) \int_{[t]_m^-}^{t} S(t-u)\sigma_l(\bar{\xi}_m(u))\dot{B}_m^l(u)du,$$

and

$$\begin{aligned}\gamma_{m,3}^{j}(t) = D\sigma_j(\bar{\xi}_m(t)) &\int_{[t]_m^-}^{t} S(t-u)b(\xi_m(u))du \\ &+ \sum_{l=1}^{r} D\sigma_j(\bar{\xi}_m(t)) \int_{[t]_m^-}^{t} S(t-u)\Big(\int_0^1 D\sigma_l(\bar{\xi}_m(u) \\ &+ v\ (\xi_m(u) - \bar{\xi}_m(u)))(\xi_m(u) - \bar{\xi}_m(u))dv\Big)\dot{B}_m^l(u)du \\ &+ \int_0^1 dv_1 \int_0^{v_1} D^2\sigma_j\big(\bar{\xi}_m(t) + v_2\ (\xi_m(t) - \bar{\xi}_m(t))\big) \\ &\qquad\qquad (\xi_m(t) - \bar{\xi}_m(t), \xi_m(t) - \bar{\xi}_m(t))dv_2.\end{aligned}$$

LEMMA 2.6. *For any $p > 1$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le T} \| \int_0^{[t]_m^-} S(t-s)\gamma_{m,3}^{j}(s)\dot{B}_m^j(s)ds\|^{2p}\Big] \le C\delta_m^p.$$

PROOF. From the inequality (2.4), we have

$$\begin{aligned}\|\gamma_{m,3}^{j}(t)\| &\le C\Big(\delta_m + \sum_{l=1}^{r} \int_{[t]_m^-}^{t} \|\xi_m(u) - \bar{\xi}_m(u)\||\dot{B}_m^l(u)|du + \|\xi_m(t) - \bar{\xi}_m(t)\|^2\Big) \\ &\le C\Big\{\delta_m + \sum_{l=1}^{r}\Big(\delta_m + \sum_{j=1}^{r} |B^j([t]_m^+) - B^j([t]_m^-)|\Big)|B^l([t]_m^+) - B^l([t]_m^-)| \\ &\qquad + \Big(\delta_m + \sum_{j=1}^{r} |B^j([t]_m^+) - B^j([t]_m^-)|\Big)^2\Big\}.\end{aligned}$$

This inequality and the following one concludes the lemma.

$$E\Big[\sup_{0\le t\le T}\|\int_0^{[t]_m^-} S(t-s)\gamma_{m,3}^j(s)\dot{B}_m^j(s)ds\|^{2p}\Big]$$
$$\le C\delta_m^{-2p}\int_0^T E\Big[\|\gamma_{m,3}^j(s)\|^{2p}\big(B^j([s]_m^+)-B^j([s]_m^-)\big)^{2p}\Big]ds. \square$$

Lemma 2.7.
(i)

$$\lim_{\delta\downarrow 0} E\Big[\sup_{\substack{u_1,u_2\in[0,T]\\ |u_1-u_2|\le\delta}}\|X(u_1)-X(u_2)\|^p\Big]=0$$

for any $p>1$.
(ii)

$$\lim_{\delta\downarrow 0}\sup_{\substack{u_1,u_2\in[0,T]\\ |u_1-u_2|\le\delta,\, x\in K}}\|(S(u_1)-S(u_2))x\|=0$$

for any compact subset $K\subset H$.

Proof. We have $E\Big[\sup_{0\le t\le T}\|X(t)\|^q\Big]<\infty$ for any $q>2$ from Section 7 in Da Prato and Zabczyk [4]. Therefore, since $X(\cdot)$ is continuous a.s., (i) holds.

Since the function $f\colon [0,T]\times K\to H$ defined by $f(v,x)=S(v)x$ is uniformly continuous, we obtain (ii). $\square$

Lemma 2.8. *Let* $(Y(t))_{t\in[0,T]}$ *be a* $(\mathcal{F}(t))$*-adapted stochastic process with continuous paths a.s. Then there exists an nondecreasing sequence of compact subsets* $K_1\subset K_2\subset\cdots\subset H$ *such that*

$$\lim_{n\to\infty}P\big(Y(t)\in K_n \quad \textit{for all } t\in[0,T]\big)=1.$$

Proof. Let μ be the distribution of $\omega\mapsto Y(\,\cdot\,,\omega)$ on $C([0,T];H)$. Since $C([0,T];H)$ is the Polish space, there exist compact subsets $\tilde{K}_1\subset\tilde{K}_2\subset\cdots\subset C([0,T];H)$ such that $\lim_{n\to\infty}\mu(\tilde{K}_n)=1$. We define a continuous map $f\colon [0,T]\times C([0,T];H)\to H$ by $f(t,w)=w(t)$ for $(t,w)\in$

$[0,T] \times C([0,T];H)$. Then if we set $K_n = f([0,T] \times \tilde{K}_n)$, K_n is compact subset in H. Finally we have

$$P\big(Y(t) \in K_n \text{ for all } t \in [0,T]\big) \geq \mu(\tilde{K}_n) \to 1, \quad n \to \infty. \ \square$$

LEMMA 2.9. *Let*

$$I_m^j(t) = \int_0^{[t]_m^-} S(t-s)\gamma_{m,1}^j(s)\dot{B}_m^j(s)ds - \int_0^{[t]_m^-} S(t-s)\sigma_j(X(s))dB^j(s).$$

For any $p > 1$, there exist a constant $C > 0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n} = 0$ such that

$$E\Big[\sup_{0\leq t\leq v} \|I_m^j(t)\|^{2p}\Big] \leq C\int_0^v E\Big[\sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|^{2p}\Big]du + C_{m,n}$$

for all $v \in [0,T]$.

PROOF. We take $\alpha \in \left(\frac{1}{2p}, \frac{1}{2}\right)$ and fix it. By Equation (2.5) and the stochastic Fubini theorem, we have

$$\begin{aligned}
&\int_0^{[t]_m^-} S(t-s)\gamma_{m,1}^j(s)\dot{B}_m^j(s)ds \\
&= \sum_{k=0}^{[t/\delta_m]-1} \int_{k\delta_m}^{(k+1)\delta_m} S(t-s)\sigma_j(\bar{\xi}_m(s))\dot{B}_m^j(s)ds \\
&= \frac{1}{\delta_m}\sum_{k=0}^{[t/\delta_m]-1} \int_{k\delta_m}^{(k+1)\delta_m} dB^j(u)\int_{k\delta_m}^{(k+1)\delta_m} S(t-\tilde{u})\sigma_j(\bar{\xi}_m(\tilde{u}))d\tilde{u} \\
&= \int_0^{[t]_m^-}\Big(\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S(t-\tilde{u})\sigma_j(\bar{\xi}_m(\tilde{u}))d\tilde{u}\Big)dB^j(u) \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^{[t]_m^-}\Big(\int_u^{[t]_m^-}([t]_m^- - s)^{\alpha-1}(s-u)^{-\alpha}ds\Big) \\
&\qquad \times \Big(\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S(t-\tilde{u})\sigma_j(\bar{\xi}_m(\tilde{u}))d\tilde{u}\Big)dB^j(u) \\
&= \frac{\sin\pi\alpha}{\pi}\int_0^{[t]_m^-}([t]_m^- - s)^{\alpha-1}S(t-[s]_m^+)
\end{aligned}$$

$$\Big\{\int_0^s (s-u)^{-\alpha}\Big(\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S([s]_m^+-\tilde{u})\sigma_j(\bar{\xi}_m(\tilde{u}))d\tilde{u}\Big)dB^j(u)\Big\}ds.$$

Hence

$$I_m^j(t)=\frac{\sin\pi\alpha}{\pi}\int_0^{[t]_m^-}([t]_m^- - s)^{\alpha-1}S(t-[s]_m^+)Y_m^j(s)ds,$$

where

$$Y_m^j(s)=\int_0^s (s-u)^{-\alpha}S([s]_m^+-[u]_m^+)\tilde{Y}_m^j(u)dB^j(u)$$

and

$$\tilde{Y}_m^j(u)=\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S([u]_m^+-\tilde{u})\sigma_j(\bar{\xi}_m(\tilde{u}))d\tilde{u}-S([u]_m^+-u)\sigma_j(X(u)).$$

Therefore, by using the Hölder inequality, we have

$$\begin{aligned}\|I_m^j(t)\|^{2p}\le\Big(\frac{\sin\pi\alpha}{\pi}\sup_{0\le u\le T}\|S(u)\|\Big)^{2p}T^{2p\alpha-1}\left(\frac{2p-1}{2p\alpha-1}\right)^{2p-1}\\ \times\int_0^{[t]_m^-}\|Y_m^j(s)\|^{2p}ds.\end{aligned}$$

Taking the expectation, we get

$$\begin{aligned}E\Big[\sup_{0\le t\le v}\|I_m^j(t)\|^{2p}\Big]&\le C\int_0^{[v]_m^-}E\Big[\|Y_m^j(s)\|^{2p}\Big]ds\\ &\le C\int_0^{[v]_m^-}E\Big[\Big(\int_0^s (s-u)^{-2\alpha}\|\tilde{Y}_m^j(u)\|^2du\Big)^p\Big]ds\\ &\le C\Big(\frac{T^{1-2\alpha}}{1-2\alpha}\Big)^pE\Big[\int_0^{[v]_m^-}\|\tilde{Y}_m^j(u)\|^{2p}du\Big],\qquad v\in[0,T].\end{aligned}$$

Here we used Young's inequality.

Now we have

$$\begin{aligned}&\tilde{Y}_m^j(u)\\ &=\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+}S([u]_m^+-\tilde{u})\Big(\sigma_j(\bar{\xi}_m(\tilde{u}))-\sigma_j(S(\tilde{u}-[u]_m^-)X([u]_m^-))\Big)d\tilde{u}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S([u]_m^+-\tilde{u}) \\
&\qquad\qquad \Big(\sigma_j(S(\tilde{u}-[u]_m^-)X([u]_m^-))-\sigma_j(S(\tilde{u}-[u]_m^-)X(u))\Big)d\tilde{u} \\
&+\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} S([u]_m^+-\tilde{u})\Big(\sigma_j(S(\tilde{u}-[u]_m^-)X(u))-\sigma_j(X(u))\Big)d\tilde{u} \\
&+\frac{1}{\delta_m}\int_{[u]_m^-}^{[u]_m^+} \Big(S([u]_m^+-\tilde{u})-S([u]_m^+-u)\Big)\sigma_j(X(u))d\tilde{u},
\end{aligned}
$$

and so

$$
\begin{aligned}
&\|\tilde{Y}_m^j(u)\| \le C\|\xi_m([u]_m^-)-X([u]_m^-)\|+C\|X([u]_m^-)-X(u)\| \\
&+C\sup_{0\le\tilde{u}\le\delta_m}\|(S(\tilde{u})-I)X(u)\|+C\sup_{\substack{u_1,u_2\in[0,T]\\|u_1-u_2|\le\delta_m}}\|(S(u_1)-S(u_2))\sigma_j(X(u))\|
\end{aligned}
$$

for $u\le T$. From Lemma 2.8, we can choose an nondecreasing sequence of compact subsets $K_1\subset K_2\subset\cdots\subset H$ such that $\lim_{n\to\infty}P(\Omega_n)=1$, where

$$
\Omega_n=\big\{X(t)\in K_n \quad \text{for all } t\in[0,T]\big\}.
$$

Noting that by the definition of $\tilde{Y}_m^j(u)$, there exists a constant $C>0$ such that $\|\tilde{Y}_m^j(u)\|\le C$, we have

$$
\begin{aligned}
E\Big[\|\tilde{Y}_m^j(u)\|^{2p}\Big] &= E\Big[\|\tilde{Y}_m^j(u)\|^{2p}\,;\,\Omega_n\Big]+E\Big[\|\tilde{Y}_m^j(u)\|^{2p}\,;\,\Omega\setminus\Omega_n\Big] \\
&\le C\Big\{E\Big[\sup_{0\le s\le u}\|\xi_m(s)-X(s)\|^{2p}\Big]+E\Big[\sup_{\substack{u_1,u_2\in[0,T]\\|u_1-u_2|\le\delta_m}}\|X(u_1)-X(u_2)\|^{2p}\Big] \\
&\qquad\qquad +\sup_{\substack{0\le\tilde{u}\le\delta_m\\x\in K_n}}\|(S(\tilde{u})-I)x\|^{2p} \\
&\qquad +\sup_{\substack{u_1,u_2\in[0,T]\\|u_1-u_2|\le\delta_m,x\in K_n}}\|(S(u_1)-S(u_2))\sigma_j(x)\|^{2p}+P(\Omega\setminus\Omega_n)\Big\}.
\end{aligned}
$$

Therefore Lemma 2.7 completes the proof. □

The next lemma is an extension of Burkholder's inequality to Hilbert space-valued martingales.

LEMMA 2.10. *Let $p > 1$ and $(\mathcal{A}_n)_{n=0,1,\dots}$ be an nondecreasing family of sub σ-fields of $\mathcal{F}$. For any H-valued $(\mathcal{A}_n)_{n=0,1,\dots}$-martingale $(M_n)_{n=0,1,\dots}$ with $M_0 = 0$, there exists a constant $C_p > 0$ depending only on p such that*

$$E\Big[\max_{1\le k\le n} \|M_k\|^{2p}\Big] \le C_p E\Big[\Big(\sum_{k=1}^n \|M_k - M_{k-1}\|^2\Big)^p\Big]$$

for all $n \ge 1$.

PROOF. Let

$$I_1 = \Big\{E\Big[\max_{1\le k\le n} \|M_k\|^{2p}\Big]\Big\}^{1/2}, \qquad I_2 = \Big\{E\Big[\Big(\sum_{k=1}^n \|M_k - M_{k-1}\|^2\Big)^p\Big]\Big\}^{1/2}.$$

Letting $\eta_j = M_j - M_{j-1}$, $j = 1, 2, \dots$, we have $M_k = \sum_{j=1}^k \eta_j$. Since

$$\|M_k\|^2 = 2\sum_{j=1}^k < M_{j-1}, \eta_j >_H + \sum_{j=1}^k \|\eta_j\|^2,$$

we have

$$\|M_k\|^{2p} \le 2^{p-1}\Big\{2^p\Big|\sum_{j=1}^k < M_{j-1}, \eta_j >_H\Big|^p + \Big(\sum_{j=1}^k \|\eta_j\|^2\Big)^p\Big\}.$$

Hence we get

$$\max_{1\le k\le n} \|M_k\|^{2p} \le 2^{p-1}\Big\{2^p \max_{1\le k\le n}\Big|\sum_{j=1}^k < M_{j-1}, \eta_j >_H\Big|^p + \Big(\sum_{j=1}^n \|\eta_j\|^2\Big)^p\Big\}.$$

From Burkholder's inequality for real-valued martingales, there exists a constant C_p depending only on p such that

$$E\Big[\max_{1\le k\le n}\Big|\sum_{j=1}^k < M_{j-1}, \eta_j >_H\Big|^p\Big] \le C_p E\Big[\Big(\sum_{j=1}^n < M_{j-1}, \eta_j >_H^2\Big)^{p/2}\Big].$$

Therefore we have

$$I_1^2 \le 2^{p-1}\Big\{2^p C_p E\Big[\Big(\sum_{j=1}^n \|M_{j-1}\|^2 \|\eta_j\|^2\Big)^{p/2}\Big] + I_2^2\Big\}$$

$$
\begin{aligned}
&\leq 2^{p-1}\Big\{2^p C_p E\Big[\max_{1\leq k\leq n}\|M_k\|^p\Big(\sum_{j=1}^n\|\eta_j\|^2\Big)^{p/2}\Big]+I_2^2\Big\}\\
&\leq 2^{p-1}\Big\{2^p C_p I_1 I_2+I_2^2\Big\}.
\end{aligned}
$$

This yields

$$
\begin{aligned}
I_1 &\leq 2^{2(p-1)}C_p I_2+\sqrt{2^{4(p-1)}C_p^2 I_2^2+2^{p-1}I_2^2}\\
&=\Big(2^{(p-1)}C_p+\sqrt{2^{4(p-1)}C_p^2+2^{p-1}}\Big)I_2. \ \square
\end{aligned}
$$

LEMMA 2.11. *Let $p>1$, $\alpha\in\left(\frac{1}{2p},\frac{1}{2}\right)$ and fix them. Then, there exist a constant $C>0$ such that*

$$
\begin{aligned}
&E\Big[\sup_{0\leq t\leq T}\|\int_0^{[t]_m^-}S(t-s)\gamma_{m,2}^{j,j}(s)\dot{B}_m^j(s)ds\\
&\quad-\frac{1}{\delta_m}\int_0^{[t]_m^-}S(t-s)D\sigma_j(\bar{\xi}_m(s))\Big(\int_{[s]_m^-}^s S(s-u)\sigma_j(\bar{\xi}_m(u))du\Big)ds\|^{2p}\Big]\\
&\qquad\qquad\leq C\delta_m^{p(1-2\alpha)}.
\end{aligned}
$$

If $j\neq l$, there exists a constant $C>0$ such that

$$
E\Big[\sup_{0\leq t\leq T}\|\int_0^{[t]_m^-}S(t-s)\gamma_{m,2}^{j,l}(s)\dot{B}_m^j(s)ds\|^{2p}\Big]\leq C\delta_m^{p(1-2\alpha)}.
$$

PROOF. Let

$$
I_m^{j,l}(t)=\delta_m^{-2}\int_0^{[t]_m^-}S(t-s)L_m^{j,l}(s)K_m^{j,l}(s)ds,
$$

where

$$
L_m^{j,l}(s)=D\sigma_j(\bar{\xi}_m(s))\int_{[s]_m^-}^s S(s-u)\sigma_l(\bar{\xi}_m(u))du
$$

and

$$
K_m^{j,l}(s)=\begin{cases}\big((B^j([s]_m^+)-B^j([s]_m^-)\big)^2-\delta_m, & j=l\\ \big(B^j([s]_m^+)-B^j([s]_m^-)\big)\big(B^l([s]_m^+)-B^l([s]_m^-)\big), & j\neq l.\end{cases}
$$

Then it has to be shown that

$$E\Big[\sup_{0\le t\le T}\|I_m^{j,l}(t)\|^{2p}\Big]\le \delta_m^{p(1-2\alpha)}.$$

We immediately have

$$E\Big[\|K_m^{j,l}(s)\|^{2p}\Big]\le C\delta_m^{2p}.$$

By Equation (2.5) and the Fubini theorem,

$$\begin{aligned}
&\|I_m^{j,l}(t)\|^{2p}\le C\delta_m^{-4p}\Big\|\int_0^{[t]_m^-}S([t]_m^- - s)L_m^{j,l}(s)K_m^{j,l}(s)ds\Big\|^{2p}\\
&\le C\delta_m^{-4p}\Big\|\int_0^{[t]_m^-}([t]_m^- - u)^{\alpha-1}S([t]_m^- - u)Y_m^{j,l}(u)du\Big\|^{2p}\\
&\le C\delta_m^{-4p}\int_0^{[t]_m^-}\|Y_m^{j,l}(u)\|^{2p}du,
\end{aligned}$$

where

$$Y_m^{j,l}(u)=\int_0^u (u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)K_m^{j,l}(s)ds.$$

Hence

$$E\Big[\sup_{0\le t\le T}\|I_m^{j,l}(t)\|^{2p}\Big]\le C\delta_m^{-4p}\int_0^T E\Big[\|Y_m^{j,l}(u)\|^{2p}\Big]du.$$

Now we have

$$Y_m^{j,l}(u)=Y_{m,1}^{j,l}(u)+Y_{m,2}^{j,l}(u),$$

where

$$Y_{m,1}^{j,l}(u)=\int_0^{[u]_m^-}(u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)K_m^{j,l}(s)ds$$

and

$$Y_{m,2}^{j,l}(u)=\int_{[u]_m^-}^{u}(u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)K_m^{j,l}(s)ds.$$

From Lemma 2.10, we have

$$\begin{aligned}
&E\Big[\|Y_{m,1}^{j,l}(u)\|^{2p}\Big]\\
&=E\Big[\Big\|\sum_{k=0}^{\delta_m^{-1}[u]_m^- -1}K_m^{j,l}(k\delta_m)\int_{k\delta_m}^{(k+1)\delta_m}(u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)ds\Big\|^{2p}\Big]
\end{aligned}$$

$$\leq CE\Big[\Big(\sum_{k=0}^{\delta_m^{-1}[u]_m^- -1}\big(K_m^{j,l}(k\delta_m)\big)^2\|\int_{k\delta_m}^{(k+1)\delta_m}(u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)ds\|^2\Big)^p\Big].$$

Since we have

$$\|\int_{k\delta_m}^{(k+1)\delta_m}(u-s)^{-\alpha}S(u-s)L_m^{j,l}(s)ds\|$$
$$\leq C\delta_m\int_{k\delta_m}^{(k+1)\delta_m}(u-s)^{-\alpha}ds\leq C\frac{1}{1-\alpha}\delta_m^{2-\alpha},$$

it follows that

$$E\Big[\|Y_{m,1}^{j,l}(u)\|^{2p}\Big]\leq C\delta_m^{2p(2-\alpha)}E\Big[\Big(\sum_{k=0}^{\delta_m^{-1}[u]_m^- -1}\big(K_m^{j,l}(k\delta_m)\big)^2\Big)^p\Big]$$
$$\leq C\delta_m^{2p(2-\alpha)}(\delta_m^{-1}[u]_m^-)^{p-1}\sum_{k=0}^{\delta_m^{-1}[u]_m^- -1}E\Big[\big(K_m^{j,l}(k\delta_m)\big)^{2p}\Big]\leq C\delta_m^{p(5-2\alpha)}.$$

Next we have

$$E\Big[\|Y_{m,2}^{j,l}(u)\|^{2p}\Big]\leq C\delta_m^{2p}E\Big[\Big(\int_{[u]_m^-}^u(u-s)^{-\alpha}|K_m^{j,l}(s)|ds\Big)^{2p}\Big]$$
$$\leq C\delta_m^{2p}\Big(\int_{[u]_m^-}^u(u-s)^{-\alpha\frac{2p}{2p-1}}ds\Big)^{2p-1}\int_{[u]_m^-}^uE\Big[|K_m^{j,l}(s)|^{2p}\Big]ds$$
$$\leq C\delta_m^{2p}\delta_m^{2p(1-\alpha)-1}\delta_m^{2p+1}=C\delta_m^{2p(3-\alpha)}.$$

Therefore we get

$$E\Big[\sup_{0\leq t\leq T}\|I_m^{j,l}(t)\|^{2p}\Big]\leq C\delta_m^{-4p}(\delta_m^{p(5-2\alpha)}+\delta_m^{2p(3-\alpha)})\leq C\delta_m^{p(1-2\alpha)}.\ \square$$

LEMMA 2.12. *Let*

$$I_m^j(t)=\int_0^{[t]_m^-}S(t-s)$$
$$\Big(\frac{1}{\delta_m}D\sigma_j(\bar{\xi}_m(s))\int_{[s]_m^-}^sS(s-u)\sigma_j(\bar{\xi}_m(u))du-\frac{1}{2}D\sigma_j(X(s))\sigma_j(X(s))\Big)ds.$$

For any $p > 1$, there exist a constant $C > 0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n} = 0$ such that

$$E\Big[\sup_{0\le t\le v}\|I_m^j(t)\|^{2p}\Big] \le C\int_0^v E\Big[\sup_{0\le t\le u}\|\xi_m(t)-X(t)\|^{2p}\Big]du + C_{m,n}$$

for all $v \in [0,T]$.

PROOF. Note that

$$I_m^j(t) = I_{m,1}^j(t) + I_{m,2}^j(t) + I_{m,3}^j(t) + I_{m,4}^j(t), \tag{2.7}$$

where

$$I_{m,1}^j(t) = \frac{1}{\delta_m}\int_0^{[t]_m^-} dsS(t-s)D\sigma_j(\bar{\xi}_m(s))\int_{[s]_m^-}^s S(s-u)\sigma_j(\bar{\xi}_m(u))du$$
$$-\frac{1}{\delta_m}\int_0^{[t]_m^-} dsS(t-s)D\sigma_j(\bar{X}_m(s))\int_{[s]_m^-}^s S(s-u)\sigma_j(\bar{X}_m(u))du,$$

$$I_{m,2}^j(t) = \frac{1}{\delta_m}\int_0^{[t]_m^-} dsS(t-s)D\sigma_j(\bar{X}_m(s))\int_{[s]_m^-}^s S(s-u)\sigma_j(\bar{X}_m(u))du$$
$$-\frac{1}{\delta_m}\int_0^{[t]_m^-} dsS(t-s)D\sigma_j(X([s]_m^-))\int_{[s]_m^-}^s S(s-u)\sigma_j(X([u]_m^-))du,$$

$$I_{m,3}^j(t) = \frac{1}{\delta_m}\int_0^{[t]_m^-} dsS(t-s)D\sigma_j(X([s]_m^-))\int_{[s]_m^-}^s S(s-u)\sigma_j(X([u]_m^-))du$$
$$-\frac{1}{\delta_m}\int_0^{[t]_m^-}(s-[s]_m^-)S(t-[s]_m^-))D\sigma_j(X([s]_m^-))\sigma_j(X([s]_m^-)ds$$

and

$$I_{m,4}^j(t) = \frac{1}{\delta_m}\int_0^{[t]_m^-}(s-[s]_m^-)S(t-[s]_m^-)D\sigma_j(X([s]_m^-))\sigma_j(X([s]_m^-))ds$$
$$-\frac{1}{2}\int_0^{[t]_m^-} S(t-s)D\sigma_j(X(s))\sigma_j(X(s))ds.$$

As for $I^j_{m,1}(t)$ we have

$$\|I^j_{m,1}(t)\| \leq \frac{1}{\delta_m}\| \int_0^{[t]^-_m} dsS(t-s)D\sigma_j(\bar{\xi}_m(s))$$
$$\int_{[s]^-_m}^s S(s-u)(\sigma_j(\bar{\xi}_m(u)) - \sigma_j(\bar{X}_m(u)))du\|$$
$$+ \frac{1}{\delta_m}\| \int_0^{[t]^-_m} dsS(t-s)$$
$$(D\sigma_j(\bar{\xi}_m(s)) - D\sigma_j(\bar{X}_m(s))) \int_{[s]^-_m}^s S(s-u)\sigma_j(\bar{X}_m(u))du\|$$
$$\leq C\Big\{ \frac{1}{\delta_m}\int_0^{[t]^-_m} ds \int_{[s]^-_m}^s \|\bar{\xi}_m(u) - \bar{X}_m(u)\|du + \int_0^{[t]^-_m} \|\bar{\xi}_m(s) - \bar{X}_m(s)\|ds\Big\}$$
$$\leq C\int_0^{[v]^-_m} \|\xi_m([s]^-_m) - X([s]^-_m)\|ds \leq C\int_0^{[v]^-_m} \sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|du$$

for $0 \leq t \leq v \leq T$. Therefore we get

$$E\Big[\sup_{0\leq t\leq v} \|I^j_{m,1}(t)\|^{2p}\Big] \leq C\int_0^v E\Big[\sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|^{2p}\Big]du.$$

From Lemma 2.8, we can choose an nondecreasing sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset H$ such that $\lim_{n\to\infty} P(\Omega_n) = 1$, where

$$\Omega_n = \big\{X(t) \in K_n \quad \text{for all } t \in [0,T]\big\}.$$

Since we have

$$\|I^j_{m,2}(t)\| \leq \frac{1}{\delta_m}\| \int_0^{[t]^-_m} dsS(t-s)D\sigma_j(\bar{X}_m(s))$$
$$\int_{[s]^-_m}^s S(s-u)\big(\sigma_j(\bar{X}_m(u)) - \sigma_j(X([s]^-_m)\big)du\|$$
$$+ C\frac{1}{\delta_m}\| \int_0^{[t]^-_m} dsS(t-s)\big(D\sigma_j(\bar{X}_m(s)) - D\sigma_j(X([s]^-_m))\big)$$
$$\int_{[s]^-_m}^s S(s-u)\sigma_j(X([s]^-_m)du\|$$
$$\leq C\Big(\frac{1}{\delta_m}\int_0^{[t]^-_m} ds \int_{[s]^-_m}^s \|\big(S(u-[s]^-_m) - I\big)X([s]^-_m)\|du$$

$$+\int_0^{[t]_m^-} \|\big(S(s-[s]_m^-)-I\big)X([s]_m^-)\|ds\Big),$$

it holds that

$$\|I_{m,2}^j(t)\| \le C \sup_{\substack{0\le u\le \delta_m \\ x\in K_n}} \|(S(u)-I)x\|, \quad t\in[0,T],\ \omega\in\Omega_n.$$

Since there exists a constant $C>0$ such that

$$\sup_{0\le t\le T}\|I_{m,2}^j(t)\| \le C \qquad \text{a.s.},$$

we get

$$\begin{aligned}
E\Big[\sup_{0\le t\le T}\|I_{m,2}^j(t)\|^{2p}\Big] &= E\Big[\sup_{0\le t\le T}\|I_{m,2}^j(t)\|^{2p}\,;\,\Omega_n\Big] \\
&\quad + E\Big[\sup_{0\le t\le T}\|I_{m,2}^j(t)\|^{2p}\,;\,\Omega\setminus\Omega_n\Big] \\
&\le C\Big(\sup_{\substack{0\le u\le \delta_m \\ x\in K_n}} \|(S(u)-I)x\|^{2p} + P(\Omega\setminus\Omega_n)\Big).
\end{aligned}$$

Since we have

$$\begin{aligned}
&\|I_{m,3}^j(t)\| \\
&\le \frac{1}{\delta_m}\|\int_0^{[t]_m^-} ds S(t-s)D\sigma_j(X([s]_m^-))\int_{[s]_m^-}^s (S(s-u)-I)\sigma_j(X([s]_m^-))du\| \\
&+\frac{1}{\delta_m}\|\int_0^{[t]_m^-} (s-[s]_m^-)(S(t-s)-S(t-[s]_m^-)) \\
&\qquad\qquad \times D\sigma_j(X([s]_m^-))\sigma_j(X([s]_m^-))ds\| \\
&\le C\Big(\frac{1}{\delta_m}\int_0^{[v]_m^-} ds\int_{[s]_m^-}^s \|(S(s-u)-I)\sigma_j(X([s]_m^-))\|du \\
&+C\int_0^{[v]_m^-} \|(S(t-s)-S(t-[s]_m^-))D\sigma_j(X([s]_m^-))\sigma_j(X([s]_m^-))\|ds\Big),
\end{aligned}$$

it holds that

$$\|I_{m,3}^j(t)\| \le C\Big(\sup_{\substack{0\le u\le \delta_m \\ x\in K_n}} \|(S(u)-I)\sigma_j(x)\| + \sup_{\substack{0\le u\le \delta_m \\ x\in K_n}} \|(S(u)-I)D\sigma_j(x)\sigma_j(x)\|\Big)$$

for all $t \in [0,T]$ and $\omega \in \Omega_n$. Since there exists a constant $C > 0$ such that

$$\sup_{0\le t\le T} \|I^j_{m,3}(t)\| \le C \qquad \text{a.s.},$$

we get

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T} \|I^j_{m,3}(t)\|^{2p}\Big] \\
&= E\Big[\sup_{0\le t\le T} \|I^j_{m,3}(t)\|^{2p}\,;\,\Omega_n\Big] + E\Big[\sup_{0\le t\le T} \|I^j_{m,3}(t)\|^{2p}\,;\,\Omega\setminus\Omega_n\Big] \\
&\le C\Big(\sup_{\substack{0\le u\le\delta_m\\ x\in K_n}} \|(S(u)-I)\sigma_j(x)\|^{2p} + \sup_{\substack{0\le u\le\delta_m\\ x\in K_n}} \|(S(u)-I)D\sigma_j(x)\sigma_j(x)\|^{2p} \\
&\qquad\qquad + P(\Omega\setminus\Omega_n)\Big).
\end{aligned}$$

Finally we estimate $I^j_{m,4}(t)$. Since

$$\begin{aligned}
&\frac{1}{\delta_m}\int_0^{[t]^-_m} (s-[s]^-_m)S(t-[s]^-_m)D\sigma_j(X([s]^-_m))\sigma_j(X([s]^-_m))ds \\
&= \frac{1}{\delta_m}\sum_{k=0}^{\delta_m^{-1}[t]^-_m-1}\int_{k\delta_m}^{(k+1)\delta_m} (s-k\delta_m)dsS(t-k\delta_m)D\sigma_j(X(k\delta_m))\sigma_j(X(k\delta_m)) \\
&= \frac{\delta_m}{2}\sum_{k=0}^{\delta_m^{-1}[t]^-_m-1} S(t-k\delta_m)D\sigma_j(X(k\delta_m))\sigma_j(X(k\delta_m)) \\
&= \frac{1}{2}\int_0^{[t]^-_m} S(t-[s]^-_m)D\sigma_j(X([s]^-_m))\sigma_j(X([s]^-_m))ds,
\end{aligned}$$

we have the following.

$$\begin{aligned}
\|I^j_{m,4}(t)\| &\le \frac{1}{2}\Big\|\int_0^{[t]^-_m} S(t-[s]^-_m) \\
&\qquad \big(D\sigma_j(X([s]^-_m))\sigma_j(X([s]^-_m)) - D\sigma_j(X(s))\sigma_j(X(s))\big)ds\Big\| \\
&+ \frac{1}{2}\Big\|\int_0^{[t]^-_m} \big(S(t-[s]^-_m)-S(t-s)\big)D\sigma_j(X(s))\sigma_j(X(s))ds\Big\| \\
&\le C\Big(\int_0^{[v]^-_m} \|X([s]^-_m)-X(s)\|ds
\end{aligned}$$

$$+\int_0^{[v]_m^-}\|(S(t-[s]_m^-)-S(t-s))D\sigma_j(X(s))\sigma_j(X(s))\|ds\Big).$$

Therefore

$$\|I_{m,4}^j(t)\|\le C\Big(\sup_{\substack{u_1,u_2\in[0,T]\\|u_1-u_2|\le\delta_m}}\|X(u_1)-X(u_2)\|+\sup_{\substack{0\le u\le\delta_m\\x\in K_n}}\|(S(u)-I)D\sigma_j(x)\sigma_j(x)\|\Big)$$

for all $t\in[0,T]$ and $\omega\in\Omega_n$. Since there exists a constant $C>0$ such that

$$\sup_{0\le t\le T}\|I_{m,4}^j(t)\|\le C \qquad a.s.,$$

we get

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T}\|I_{m,4}^j(t)\|^{2p}\Big]\\
&=E\Big[\sup_{0\le t\le T}\|I_{m,4}^j(t)\|^{2p}\,;\,\Omega_n\Big]+E\Big[\sup_{0\le t\le T}\|I_{m,4}^j(t)\|^{2p}\,;\,\Omega\setminus\Omega_n\Big]\\
&\le C\Big(E\Big[\sup_{\substack{u_1,u_2\in[0,T]\\|u_1-u_2|\le\delta_m}}\|X(u_1)-X(u_2)\|^{2p}\Big]\\
&+\sup_{\substack{0\le u\le\delta_m\\x\in K_n}}\|(S(u)-I)D\sigma_j(x)\sigma_j(x)\|^{2p}+P(\Omega\setminus\Omega_n)\Big).
\end{aligned}$$

Thus the result follows from Lemma 2.7. □

Combining Lemmas 2.6, 2.9, 2.11, 2.12 and Equation (2.6), we obtain the following lemma.

Lemma 2.13. *For any $p>1$, there exist a constant $C>0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty}\lim_{m\to\infty}C_{m,n}=0$ such that*

$$E\Big[\sup_{0\le t\le v}\|\int_0^{[t]_m^-}S(t-s)\sigma_j(\xi_m(s))\dot{B}_m^j(s)ds-\int_0^{[t]_m^-}S(t-s)\sigma_j(X(s))dB^j(s)$$

$$- \frac{1}{2} \int_0^{[t]_m^-} S(t-s) D\sigma_j(X(s))\sigma_j(X(s)) ds\|^{2p}\Big]$$
$$\leq C \int_0^v E\Big[\sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|^{2p}\Big] du + C_{m,n}.$$

From the Lipschitz continuity of b, we have the following lemma.

LEMMA 2.14. *For any $p > 1$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\leq t\leq v} \| \int_0^{[t]_m^-} S(t-s)\big(b(\xi_m(s)) - b(X(s))\big) ds\|^{2p}\Big]$$
$$\leq C \int_0^v E\Big[\sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|^{2p}\Big] du.$$

Note that

$$\begin{aligned} \bar{\xi}_m(t) - \bar{X}_m(t) &= \int_0^{[t]_m^-} S(t-s)\big(b(\xi_m(s)) - b(X(s))\big) ds \\ &+ \sum_{j=1}^r \Big\{ \int_0^{[t]_m^-} S(t-s)\sigma_j(\xi_m(s)) \dot{B}_m^j(s) ds \\ &- \int_0^{[t]_m^-} S(t-s)\sigma_j(X(s)) dB^j(s) \\ &- \frac{1}{2} \int_0^{[t]_m^-} S(t-s) D\sigma_j(X(s))\sigma_j(X(s)) ds \Big\} \end{aligned} \tag{2.8}$$

for $t \in [0, T]$. Therefore combining Lemmas 2.2, 2.5, 2.13 and 2.14 we obtain the following lemma.

LEMMA 2.15. *For any $p > 1$, there exist a constant $C > 0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty} \lim_{m\to\infty} C_{m,n} = 0$ such that*

$$E\Big[\sup_{0\leq t\leq v} \|\xi_m(t) - X(t)\|^{2p}\Big] \leq C \int_0^v E\Big[\sup_{0\leq t\leq u} \|\xi_m(t) - X(t)\|^{2p}\Big] du + C_{m,n}. \tag{2.9}$$

Applying the Gronwall inequality to Equation (2.9), we obtain

$$E\Big[\sup_{0\le t\le T}\|\xi_m(t)-X(t)\|^{2p}\Big]\le C_{m,n}e^{CT}.$$

This proves Theorem 2.1.

3. A Parameterized SDE

In sections 3, 4, 5 and 6, we consider another situation. Let H and U be separable Hilbert spaces endowed with inner products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_U$, respectively. We denote their norms by $\|\cdot\|_H$ and $\|\cdot\|_U$ respectively. We often abbreviate $\|\cdot\|_H$ to $\|\cdot\|$ for simplicity. Let A be the infinitesimal generator of a (C_0)-semigroup $(S(t))_{t\ge0}$ on H.

Let Q be a nuclear strictly positive operator on U. We define a separable Hilbert space U_0 by $U_0=Q^{1/2}(U)$ endowed with a inner product $\langle u,v\rangle_{U_0}=\langle Q^{-1/2}u,Q^{-1/2}v\rangle_U$, $u,\ v\in U_0$, and with its induced norm $\|\cdot\|_{U_0}$. Let us fix $T>0$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space equipped with a right-continuous nondecreasing family $(\mathcal{F}(t))_{t\in[0,T]}$ of sub σ-fields of $\mathcal{F}$ such that each $\mathcal{F}(t)$ contains all P-null sets.

Let $\sigma_j\colon H\to H$, $j=1,2,\dots,r$, be twice Fréchet differentiable mappings such that $\sup_{x\in H}\|\sigma_j(x)\|<\infty$. We assume that its Fréchet derivatives up to second order $D\sigma_j$ and $D^2\sigma_j$ are bounded. Let $V\colon H\to L_{(2)}(U_0;H)$ and $b\colon H\to H$ be Lipschitz continuous bounded mappings.

Let $(B^j(t))_{t\in[0,T]}$, $j=1,2,\dots,r$, be independent real-valued standard $(\mathcal{F}(t))_{t\in[0,T]}$-Brownian motions and set $\delta_m=T/m$ for positive integers m. We define $\mathbf{R}$-valued stochastic processes $(Z^j_m(t))_{t\in[0,T]}$, $j=1,2,\dots,r$, by

$$Z^j_m(t)=B^j(t)-B^j(k\delta_m)-\frac{t-k\delta_m}{\delta_m}(B^j((k+1)\delta_m)-B^j(k\delta_m))$$

if $t\in[k\delta_m,(k+1)\delta_m]$, $k=0,1,\dots,m-1$. Then the processes

$$(Z^j_m(t))_{t\in[k\delta_m,(k+1)\delta_m]}$$

are pinned Brownian motions from 0 to 0 on each intervals $[k\delta_m,(k+1)\delta_m]$, $k=0,1,2,\dots,m-1$.

We note that $(Z_m^j(t))_{t\in[0,T]}$, $j = 1, 2, \ldots, r$, are not $(\mathcal{F}(t))_{t\in[0,T]}$-adapted. Let $(\mathcal{G}_m(t))_{t\in[0,T]}$ be a filtration defined by $\mathcal{G}_m(t) = \mathcal{F}(t) \vee \sigma\{B^j(k\delta_m)\ ;\ k = 1, 2, \ldots m,\ j = 1, 2, \ldots, r\}$. Then $(Z_m^j(t))_{t\in[0,T]}$ are $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted processes.

For each $y = (y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$, we define a mapping

$$\varphi_m(\,\cdot\,; y)\colon [0, T] \to \mathbf{R}$$

by

$$\varphi_m(t; y) = y_k + \frac{t - k\delta_m}{\delta_m}(y_{k+1} - y_k)$$

for $t \in [k\delta_m, (k+1)\delta_m]$, $k = 0, 1, \ldots, m-1$, where we have set $y_0 = 0$.

Let $(W(t))_{t\in[0,T]}$ be a Q-Wiener process in $(\Omega, \mathcal{F}, P)$ having values in U with respect to $(\mathcal{F}(t))_{t\in[0,T]}$. We assume that $(W(t))_{t\in[0,T]}$ and $(B^j(t))_{t\in[0,T]}$, $j = 1, 2, \ldots, r$, are independent, and so $(W(t))_{t\in[0,T]}$ is also $(\mathcal{G}_m(t))_{t\in[0,T]}$- Wiener process.

Let $\theta = (\theta^1, \ldots, \theta^r)$ be $(\mathbf{R}^m)^r$-valued $\mathcal{G}_m(0)$-measurable random variable and $x_0 \in H$ be fixed. An H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted stochastic process $(X_m(t;\theta))_{t\in[0,T]}$ is said to be a mild solution of the stochastic differential equation

$$\begin{cases} dX_m(t;\theta) = AX_m(t;\theta)dt + b(X_m(t;\theta))dt + V(X_m(t;\theta))dW(t) \\ \qquad + \sum_{j=1}^r \sigma_j(X_m(t;\theta))dZ_m^j(t) \\ \qquad + \sum_{j=1}^r \sigma_j(X_m(t;\theta))\dot{\varphi}_m(t;\theta^j)dt, \\ X_m(0;\theta) = x_0 \end{cases} \tag{3.1}$$

if the following stochastic integral equation holds.

$$\begin{aligned} X_m(t;\theta) = S(t)x_0 &+ \int_0^t S(t-s)b(X_m(s;\theta))ds \\ &+ \int_0^t S(t-s)V(X_m(s;\theta))dW(s) \\ &+ \sum_{j=1}^r \int_0^t S(t-s)\sigma_j(X_m(s;\theta))dZ_m^j(s) \\ &+ \sum_{j=1}^r \int_0^t S(t-s)\sigma_j(X_m(s;\theta))\dot{\varphi}_m(s;\theta^j)ds, \\ &\qquad t \in [0,T],\ P\text{-a.s.} \end{aligned}$$

Here $\dot{\varphi}_m(\cdot\ ;\ \theta^j)$ denotes its derivative for the mapping $t \mapsto \varphi_m(t;\ \theta^j)$.

We shall prove following theorems.

Theorem 3.1. *The stochastic differential equation (3.1) has the unique H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted mild solution, which has continuous paths a.s.*

Theorem 3.2. *Let $\theta_n = (\theta_n^1, \dots, \theta_n^r)$, $\theta = (\theta^1, \dots, \theta^r) \in (\mathbf{R}^m)^r$ such that $\lim_{n\to\infty} |\theta_n - \theta|_{(\mathbf{R}^m)^r} = 0$, where $|\cdot|_{(\mathbf{R}^m)^r}$ means Euclidean norm. Then the sequence*

$$\sup_{0\le t\le T} \|X_m(t;\theta_n) - X_m(t;\theta)\|$$

converges to 0 in probability as $n \to \infty$.

Let $h\colon [0,T] \to \mathbf{R}^r$ be infinitely continuously differentiable functions such that $h(0) = 0$. We denote $X(\ \cdot\ ;\theta)$ by $X(\ \cdot\ ;h)$ for

$$\theta = \big((h^1(\delta), h^1(2\delta), \dots, h^1(T)), \dots, (h^r(\delta), h^r(2\delta), \dots, h^r(T))\big),$$

where h^j denote j-th components of h. Let $(\eta(t;h))_{t\in[0,T]}$ be the H-valued continuous $(\mathcal{F}(t))_{t\ge 0}$-adapted stochastic process which is the unique mild solution of the stochastic differential equation

$$\begin{cases} d\eta(t;h) = A\eta(t;h)dt + b(\eta(t;h))dt + V(\eta(t;h))dW(t) \\ \qquad -\dfrac{1}{2}\displaystyle\sum_{j=1}^{r} D\sigma_j(\eta(t;h))\sigma_j(\eta(t;h))dt + \sum_{j=1}^{r} \sigma_j(\eta(t;h))\dot{h}^j(t)dt, \\ \eta(0;h) = x_0. \end{cases}$$

We shall prove the following theorem.

Theorem 3.3. *For any $p > 1$,*

$$\lim_{m\to\infty} E\Big[\sup_{0\le t\le T} \|X_m(t;h) - \eta(t;h)\|^{2p}\Big] = 0.$$

4. Some Lemmas under the Situation in Section 3

In this section, we prepare some lemmas. The following lemma is due to Chapter 7 in Da Prato and Zabczyk [4].

LEMMA 4.1. *Let $p > 1$ and $(\gamma(t))_{t\in[0,T]}$ be a $L_{(2)}(U_0;H)$-valued $(\mathcal{F}(t))_{t\in[0,T]}$-predictable process that satisfies*

$$E\Big[\int_0^T \|\gamma(u)\|^{2p}_{L_{(2)}(U_0;H)}du\Big] < \infty.$$

Then

$$E\Big[\sup_{0\le t\le T}\|\int_0^t S(t-u)\gamma(u)dW(u)\|^{2p}\Big] \le C_{p,\alpha}E\Big[\int_0^T \|\gamma(u)\|^{2p}_{L_{(2)}(U_0;H)}du\Big],$$

where

$$C_{p,\alpha} = \Big(\frac{\sin\pi\alpha}{\pi}\Big)^{2p}\sup_{0\le t\le T}\|S(t)\|^{4p}\frac{T^{p-1}}{\Big(\frac{2p\alpha-1}{2p-1}\Big)^{2p-1}} \\ \times\Big(\frac{p(2p-1)}{1-2\alpha}\Big)^{p}\Big(\frac{2p}{2p-1}\Big)^{2p(p-1)}$$

for any constant $\alpha\in\big(\frac{1}{2p},\frac{1}{2}\big)$.

From the argument in Chapter IV in Ikeda and Watanabe [5], we have the following lemma.

LEMMA 4.2. *For each $j = 1, 2, \dots, r$, we define a real-valued stochastic process $(\beta^j_m(t))_{t\in[0,T]}$ by*

$$\beta^j_m(t) = \beta^j_m(k\delta_m) + Z^j_m(t) + \int_{k\delta_m}^t \frac{Z^j_m(s)}{(k+1)\delta_m - s}ds$$

for $t\in(k\delta_m,(k+1)\delta_m]$, $k = 0, 1, \dots, m-1$, with $\beta^j_m(0) = 0$. This process is a standard $(\mathcal{G}_m(t))_{t\in[0,T]}$-Brownian motion.

LEMMA 4.3. *Let $p > 3$. Let $(\gamma(t))_{t\in[0,T]}$ be an H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-predictable process such that*

$$E\Big[\int_0^T \|\gamma(t)\|^{4p}dt\Big] < \infty.$$

(i) *Letting* $\alpha \in \left(\frac{1}{2p}, \frac{1}{6}\right)$ *be fixed, there exists a constant* $C > 0$ *such that*

$$E\Big[\| \int_{k\delta_m}^t (t-s)^{-\alpha} S(t-s)\gamma(s) dZ_m^j(s)\|^{2p}\Big] \leq C\delta_m^{-p}\Big(\int_{k\delta_m}^t E\Big[\|\gamma(s)\|^{4p}\Big]ds\Big)^{1/2}$$

for $k = 0, 1, \ldots, m-1$ *and* $t \in [k\delta_m, T]$, *and*

$$E\Big[\| \int_{k\delta_m}^t (t-s)^{-\alpha} S(t-s)\gamma(s) dZ_m^j(s)\|^{2p}\Big] \leq C\delta_m^{p(1-2\alpha)-\frac{1}{2}}\Big(\int_{k\delta_m}^t E\Big[\|\gamma(s)\|^{4p}\Big]ds\Big)^{1/2}$$

for $k = 0, 1, \ldots, m-1$ *and* $t \in [k\delta_m, (k+1)\delta_m]$.
(ii) *There exists a constant* $C > 0$ *such that*

$$E\Big[\sup_{k\delta_m \leq t \leq v} \| \int_{k\delta_m}^t S(t-s)\gamma(s) dZ_m^j(s)\|^{2p}\Big] \leq C\delta_m^{p-\frac{1}{2}}\Big(\int_{k\delta_m}^v E\Big[\|\gamma(s)\|^{4p}\Big]ds\Big)^{1/2}$$

for $k = 0, 1, \ldots, m-1$ *and* $v \in [k\delta_m, (k+1)\delta_m]$.

PROOF. Under the notation of Lemmas 4.2, we have

$$\int_{k\delta_m}^t (t-s)^{-\alpha} S(t-s)\gamma(s) dZ_m^j(s) = I_{m,1}^j(t) - I_{m,2}^j(t),$$

where

$$I_{m,1}^j(t) = \int_{k\delta_m}^t (t-s)^{-\alpha} S(t-s)\gamma(s) d\beta_m^j(s)$$

and

$$I_{m,2}^j(t) = \int_{k\delta_m}^t (t-s)^{-\alpha} S(t-s)\gamma(s) \frac{Z_m^j(s)}{[s]_m^+ - s} ds,$$

where $[s]_m^+ = (k+1)\delta_m$ if $k\delta_m \leq s < (k+1)\delta_m$, $k \geq 0$.

From the Hölder inequality, we have

$$E\Big[\|I_{m,1}^j(t)\|^{2p}\Big] \leq CE\Big[\Big(\int_{k\delta_m}^t (t-s)^{-2\alpha}\|\gamma(s)\|^2 ds\Big)^p\Big]$$

$$\leq C\Big(\int_{k\delta_m}^t (t-s)^{-6\alpha} ds\Big)^{\frac{p}{3}} E\Big[\Big(\int_{k\delta_m}^t \|\gamma(s)\|^3 ds\Big)^{\frac{2p}{3}}\Big]$$

$$\leq C(t-k\delta_m)^{\frac{p}{3}-2p\alpha}(t-k\delta_m)^{\frac{2p}{3}-1} E\Big[\int_{k\delta_m}^t \|\gamma(s)\|^{2p} ds\Big]$$

$$\leq C(t-k\delta_m)^{p(1-2\alpha)-\frac{1}{2}}\Big(\int_{k\delta_m}^t E\Big[\|\gamma(s)\|^{4p}\Big] ds\Big)^{1/2}.$$

Let us fix $\epsilon \in (0, \frac{1}{6})$. From the Hölder inequality, we have

$$E\Big[\|I_{m,2}^j(t)\|^{2p}\Big] \leq C\Big(\int_{k\delta_m}^t ([s]_m^+ - s)^{-\frac{3}{2}(\frac{1}{2}+\epsilon)} ds\Big)^{\frac{4p}{3}}$$

$$\times E\Big[\Big\{\int_{k\delta_m}^t (t-s)^{-3\alpha}\|\gamma(s)\|^3 \Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^3 ds\Big\}^{\frac{2p}{3}}\Big].$$

Furthermore

$$E\Big[\Big\{\int_{k\delta_m}^t (t-s)^{-3\alpha}\|\gamma(s)\|^3 \Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^3 ds\Big\}^{\frac{2p}{3}}\Big]$$

$$\leq \Big(\int_{k\delta_m}^t (t-s)^{-6\alpha} ds\Big)^{\frac{p}{3}} E\Big[\Big\{\int_{k\delta_m}^t \|\gamma(s)\|^6 \Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^6 ds\Big\}^{\frac{p}{3}}\Big]$$

$$\leq (1-6\alpha)^{-\frac{p}{3}}(t-k\delta_m)^{\frac{2p}{3}-1-2p\alpha} E\Big[\int_{k\delta_m}^t \|\gamma(s)\|^{2p} \Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^{2p} ds\Big]$$

$$\leq (1-6\alpha)^{-\frac{p}{3}}(t-k\delta_m)^{\frac{2p}{3}-1-2p\alpha}$$

$$\int_{k\delta_m}^t \Big(E\Big[\|\gamma(s)\|^{4p}\Big]\Big)^{1/2} \Big(E\Big[\Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^{4p}\Big]\Big)^{1/2} ds.$$

Since

$$E\Big[\Big(\frac{|Z_m^j(s)|}{([s]_m^+ - s)^{\frac{1}{2}-\epsilon}}\Big)^{4p}\Big] \leq (4p-1)!!\ \delta_m^{4p\epsilon},$$

it holds that

$$E\Big[\|I_{m,2}^j(t)\|^{2p}\Big] \leq C\Big(\int_{k\delta_m}^t ([s]_m^+ - s)^{-\frac{3}{2}(\frac{1}{2}+\epsilon)} ds\Big)^{\frac{4p}{3}}$$

$$\times (t-k\delta_m)^{\frac{2p}{3}-\frac{1}{2}-2p\alpha} \delta_m^{2p\epsilon} \Big(\int_{k\delta_m}^t E\Big[\|\gamma(s)\|^{4p}\Big] ds\Big)^{1/2}.$$

Therefore we obtain (i). From Equation (2.5) and the stochastic Fubini theorem, using the result (i), we also obtain (ii). □

LEMMA 4.4. *Let $p > 3$. Let $(\gamma(t))_{t\in[0,T]}$ be an H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-predictable process such that*

$$E\Big[\int_0^T \|\gamma(t)\|^{4p}dt\Big] < \infty,$$

and set

$$I_m^j(t) = \int_0^t S(t-s)\gamma(s)dZ_m^j(s).$$

Then there exists a constant $C > 0$ such that

$$E\Big[\sup_{0\le t\le v}\|I_m^j(t)\|^{2p}\Big] \le C\delta_m^{-p}\Big(E\Big[\int_0^v \|\gamma(s)\|^{4p}ds\Big]\Big)^{1/2}, \qquad v\in[0,T].$$

PROOF. We take $\alpha \in \left(\frac{1}{2p}, \frac{1}{6}\right)$ and fix it. By Equation (2.5) and the stochastic Fubini theorem, we have

$$I_m^j(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t du(t-u)^{\alpha-1}S(t-u)\int_0^u (u-s)^{-\alpha}S(u-s)\gamma(s)dZ_m^j(s),$$

and therefore

$$\|I_m^j(t)\|^{2p} \le CT^{2p\alpha-1}\int_0^t \|\int_0^u (u-s)^{-\alpha}S(u-s)\gamma(s)dZ_m^j(s)\|^{2p}du.$$

Therefore, from Lemma 4.3 (i), we have

$$\begin{aligned}
&E\Big[\sup_{0\le t\le v}\|I_m^j(t)\|^{2p}\Big]\\
&\le CT^{2p\alpha-1}\int_0^v E\Big[\|\int_0^u (u-s)^{-\alpha}S(u-s)\gamma(s)dZ_m^j(s)\|^{2p}\Big]du\\
&\le CT^{2p\alpha-1}\delta_m^{-p}\int_0^v \Big(E\Big[\int_0^u \|\gamma(s)\|^{4p}ds\Big]\Big)^{1/2}du\\
&\le CT^{2p\alpha}\delta_m^{-p}\Big(E\Big[\int_0^v \|\gamma(s)\|^{4p}ds\Big]\Big)^{1/2}.
\end{aligned}$$

This completes the proof. □

Proposition 4.5. *Let $p > 3$ and $\alpha \in \left(\frac{1}{2p}, \frac{1}{6}\right)$ be fixed. Let $(\gamma(t))_{t\in[0,T]}$ be an H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-predictable process.*
(i) *Assume that there exists a constant $C_0 > 0$ such that*

$$E\Big[\|\gamma(t)\|^{4p}\Big] \le C_0\delta_m^{2p}, \qquad t \in [0,T].$$

Let

$$\chi^j_{m,k}(t) = \int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)\gamma(s)dZ^j_m(s) \tag{4.1}$$

for $t \in [0,T]$ and $k = 0,1,\dots,m-1$. Then there exists a constant $C > 0$ such that

$$E\Big[\sup_{0\le t\le T}\|\sum_{k=0}^{m-1}\big(\chi^j_{m,k}(t) - E\big[\chi^j_{m,k}(t)\,|\,\mathcal{G}_m(k\delta_m)\big]\big)\|^{2p}\Big] \le C\delta_m^{p(1-2\alpha)}.$$

(ii) *Assume that $\gamma(t)$ is $\mathcal{G}_m(k\delta_m)$-measurable for $t \in [k\delta_m,(k+1)\delta_m)$, $k = 0,1,\dots,m-1$, and assume*

$$E\Big[\sup_{0\le t\le T}\|\gamma(t)\|^{2p}\Big] < \infty.$$

Then there exists a constant $C > 0$ such that

$$E\Big[\sup_{0\le t\le v}\|\int_0^t S(t-s)\gamma(s)dZ^j_m(s)\|^{2p}\Big] \le C\int_0^v E\Big[\sup_{0\le t\le s}\|\gamma(t)\|^{2p}\Big]ds.$$

for all $v \in [0,T]$.

Proof. Assume that $(\gamma(t))_{t\in[0,T]}$ satisfies the condition of (i). Let

$$\tilde{\chi}^j_{m,k}(t) = \int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t}(t-s)^{-\alpha}S(t-s)\gamma(s)dZ^j_m(s)$$

for $t \in [0,T]$ and $k = 0,1,\dots,m-1$. By Equation (2.5) and the stochastic Fubini theorem, we have

$$\chi^j_{m,k}(t) = \int_0^t S(t-s)1_{[k\delta_m\wedge t,(k+1)\delta_m\wedge t]}(s)\gamma(s)dZ^j_m(s)$$

$$= \frac{\sin \pi\alpha}{\pi} \int_0^t (t-u)^{\alpha-1} S(t-u) \\ \Big\{ \int_0^u (u-s)^{-\alpha} S(u-s) 1_{[k\delta_m \wedge t, (k+1)\delta_m \wedge t]}(s) \gamma(s) dZ_m^j(s) \Big\} du \\ = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-u)^{\alpha-1} S(t-u) \tilde{\chi}_{m,k}^j(u) du.$$

Therefore

$$\sum_{k=0}^{m-1} \big(\chi_{m,k}^j(t) - E\big[\chi_{m,k}^j(t) | \mathcal{G}_m(k\delta_m)\big]\big) \\ = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-u)^{\alpha-1} S(t-u) \sum_{k=0}^{m-1} \big(\tilde{\chi}_{m,k}^j(u) - E\big[\tilde{\chi}_{m,k}^j(u) | \mathcal{G}_m(k\delta_m)\big]\big) du.$$

From the Hölder inequality, we have

$$E\Big[\sup_{0 \le t \le T} \| \sum_{k=0}^{m-1} \big(\chi_{m,k}^j(t) - E\big[\chi_{m,k}^j(t) | \mathcal{G}_m(k\delta_m)\big]\big) \|^{2p} \Big] \\ \le C \int_0^T E\Big[\| \sum_{k=0}^{m-1} \big(\tilde{\chi}_{m,k}^j(u) - E\big[\tilde{\chi}_{m,k}^j(u) | \mathcal{G}_m(k\delta_m)\big]\big) \|^{2p} \Big] du.$$

By Lemma 2.10 and the Hölder inequality, we obtain

$$E\Big[\| \sum_{k=0}^{m-1} \big(\tilde{\chi}_{m,k}^j(u) - E\big[\tilde{\chi}_{m,k}^j(u) | \mathcal{G}_m(k\delta_m)\big]\big) \|^{2p} \Big] \\ \le CE\Big[\Big(\sum_{k=0}^{m-1} \| \tilde{\chi}_{m,k}^j(u) - E\big[\tilde{\chi}_{m,k}^j(u) | \mathcal{G}_m(k\delta_m)\big] \|^2 \Big)^p \Big] \\ \le Cm^{p-1} E\Big[\sum_{k=0}^{m-1} \| \tilde{\chi}_{m,k}^j(u) - E\big[\tilde{\chi}_{m,k}^j(u) | \mathcal{G}_m(k\delta_m)\big] \|^{2p} \Big] \\ \le Cm^{p-1} \sum_{k=0}^{m-1} E\Big[\| \tilde{\chi}_{m,k}^j(u) \|^{2p} \Big].$$

Furthermore, from Lemma 4.3 (i), we have

$$E\Big[\| \tilde{\chi}_{m,k}^j(t) \|^{2p} \Big] \le C \delta_m^{p(1-2\alpha)-\frac{1}{2}} \Big(\int_{k\delta_m \wedge t}^{(k+1)\delta_m \wedge t} E\Big[\| \gamma(s) \|^{4p} \Big] ds \Big)^{1/2}.$$

This completes the proof of (i).

Now let $(\gamma(t))_{t\in[0,T]}$ satisfy the condition of (ii). From the above argument, we have

$$E\Big[\sup_{0\le t\le v}\|\int_0^t S(t-s)\gamma(s)dZ_m^j(s)\|^{2p}\Big]\le C\int_0^v E\Big[\Big(\sum_{k=0}^{m-1}\|\tilde{\chi}_{m,k}^j(u)\|^2\Big)^p\Big]du$$

Letting

$$q_{m,k}(t)=\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t}(t-s)^{-2\alpha}ds,$$
$$q(t)=\sum_{k=0}^{m-1}q_{m,k}(t)=\frac{t^{1-2\alpha}}{1-2\alpha},\qquad t\in[0,T],$$

we have

$$E\Big[\Big(\sum_{k=0}^{m-1}\|\tilde{\chi}_{m,k}^j(u)\|^2\Big)^p\Big]\le q(u)^{p-1}\sum_{k=0}^{m-1}q_{m,k}(u)^{1-p}E\Big[\|\tilde{\chi}_{m,k}^j(u)\|^{2p}\Big]$$
$$\le q(u)^{p-1}\sum_{k=0}^{m-1}q_{m,k}(u)^{1-p}E\Big[\Big(\int_{k\delta_m\wedge u}^{(k+1)\delta_m\wedge u}(u-s)^{-2\alpha}\|S(u-s)\gamma(s)\|^2ds\Big)^p\Big]$$
$$\le q(T)^pE\Big[\sup_{0\le t\le u}\|\gamma(t)\|^{2p}\Big].$$

This proves (ii). □

5. Proofs for Theorem 3.1 and 3.2

In this section we fix a positive integer m under the Situation in Section 3. Let us fix a $(\mathbf{R}^m)^r$-valued $\mathcal{G}_m(0)$-measurable random variable $\theta=(\theta^1,\dots,\theta^r)$. Let us fix $\epsilon\in(0,\frac{1}{3})$. For each $R>0$, we define a $(\mathcal{G}_m(t))_{t\in[0,T]}$-stopping time by

$$\tau^R=\inf\Big\{t\in[0,T]\ ;\sup\Big\{\frac{|Z_m^j(s)|}{([s]_m^+-s)^{\frac{1}{2}-\epsilon}}\ ;$$
$$0\le s\le t,\ j=1,2,\dots,r\Big\}\vee|\theta|_{(\mathbf{R}^m)^r}>R\Big\}$$

where we have set $\tau^R = T$ if this set is empty and $[s]_m^+ = (k+1)\delta_m$ if $k\delta_m \le s < (k+1)\delta_m$, $k \ge 0$.

Lemma 5.1. *For a.e. $\omega \in \Omega$, there exists a positive number $R_0(\omega)$ such that $\tau^R(\omega) = T$ for all $R \ge R_0(\omega)$.*

Proof. By definition, we have

$$Z_m^j(s) = \frac{(k+1)\delta_m - s}{\delta_m}\big(B^j((k+1)\delta_m) - B^j(k\delta_m)\big) - \big(B^j((k+1)\delta_m) - B^j(s)\big)$$

for $s \in [k\delta_m, (k+1)\delta_m]$, $k = 0, 1, 2, \dots, m-1$. Therefore we have

$$\frac{|Z_m^j(s)|}{((k+1)\delta_m - s)^{\frac{1}{2}-\epsilon}} = \frac{((k+1)\delta_m - s)^{\frac{1}{2}+\epsilon}}{\delta_m}\big(B^j((k+1)\delta_m) - B^j(k\delta_m)\big) - \frac{B^j((k+1)\delta_m) - B^j(s)}{((k+1)\delta_m - s)^{\frac{1}{2}-\epsilon}}.$$

Since the Brownian motion $(B^j(t))_{t\in[0,T]}$ has locally Hölder-continuous path a.s. with any exponent $\gamma \in (0, \frac{1}{2})$, we have the desired result. □

Let $p > 3$ and fix it in this section. Let $\mathcal{E}$ be the set of H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted stochastic processes $(X(t))_{t\in[0,T]}$ such that

$$E\Big[\sup_{0\le t\le T} \|X(t)\|^{2p}\Big] < \infty.$$

We define a mapping $\phi^R \colon \mathcal{E} \to \mathcal{E}$ by

$$\phi^R(X)(t) = S(t\wedge\tau^R)x_0 + \phi_1(X)(t\wedge\tau^R) + \phi_2(X)(t\wedge\tau^R) + \sum_{j=1}^{r}\phi_3^j(X)(t\wedge\tau^R) + \sum_{j=1}^{r}\phi_4^j(X)(t\wedge\tau^R)$$

for $X \in \mathcal{E}$, where $\phi_1, \phi_2, \phi_3^j, \phi_4^j \colon \mathcal{E} \to \mathcal{E}$ are defined by

$$\phi_1(X)(t) = \int_0^t S(t-s)b(X(s))ds,$$

$$\phi_2(X)(t) = \int_0^t S(t-s)V(X(s))dW(s),$$

$$\phi_3^j(X)(t) = \int_0^t S(t-s)\sigma_j(X(s))dZ_m^j(s)$$

and

$$\phi_4^j(X)(t) = \int_0^t S(t-s)\sigma_j(X(s))\dot{\varphi}_m(s;\theta^j)ds.$$

This is well defined by virtue of the next lemma which comes from Lemmas 4.1 and 4.4.

LEMMA 5.2. *For $X \in \mathcal{E}$, it holds that*

$$E\Big[\sup_{0\le t\le T}\|\phi^R(X)(t)\|^{2p}\Big] < \infty.$$

LEMMA 5.3. *Let $(\gamma(t))_{t\in[0,T]}$ be an H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-predictable process such that*

$$E\Big[\int_0^T \|\gamma(t)\|^{2p}dt\Big] < \infty,$$

and set

$$I^j(t) = \int_0^t S(t-s)\gamma(s)dZ_m^j(s).$$

Then there exists a constant $C > 0$ such that

$$E\Big[\sup_{0\le t\le v}\|I^j(t\wedge\tau^R)\|^{2p}\Big] \le C(1+R^{2p}\delta_m^{-p(1+2\epsilon)})E\Big[\int_0^{v\wedge\tau^R}\|\gamma(s)\|^{2p}ds\Big]$$

for $v \in [0,T]$.

PROOF. We use the real-valued $(\mathcal{G}_m(t))_{t\ge0}$-Brownian motion $(\beta_m^j(t))_{t\ge0}$ in lemma 4.2. Then

$$I^j(t) = I_1^j(t) - I_2^j(t),$$

where

$$I_1^j(t) = \int_0^t S(t-s)\gamma(s)d\beta_m^j(s)$$

and

$$I_2^j(t) = \int_0^t S(t-s)\gamma(s)\frac{Z_m^j(s)}{[s]_m^+ - s}ds.$$

From Lemma 4.1, we have

$$E\Big[\sup_{0\le t\le v} \|I_1^j(t\wedge\tau^R)\|^{2p}\Big] \le CE\Big[\int_0^{v\wedge\tau^R} \|\gamma(s)\|^{2p}ds\Big].$$

From, the Hölder inequality, we have

$$\begin{aligned}
&E\Big[\sup_{0\le t\le v} \|I_2^j(t\wedge\tau^R)\|^{2p}\Big] \\
&\le CR^{2p}E\Big[\Big(\int_0^{v\wedge\tau^R} \|\gamma(s)\|([s]_m^+ - s)^{-\frac{1}{2}-\epsilon}ds\Big)^{2p}\Big] \\
&\le CR^{2p}\Big(\int_0^T ([s]_m^+ - s)^{-\left(\frac{1}{2}+\epsilon\right)\frac{2p}{2p-1}}ds\Big)^{2p-1}E\Big[\int_0^{v\wedge\tau^R} \|\gamma(s)\|^{2p}ds\Big]
\end{aligned}$$

By observing

$$\int_0^T ([s]_m^+ - s)^{-\left(\frac{1}{2}+\epsilon\right)\frac{2p}{2p-1}}ds = \frac{2p-1}{(1-2\epsilon)p-1}T\delta_m^{-\left(\frac{1}{2}+\epsilon\right)\frac{2p}{2p-1}},$$

the proof is completed. □

LEMMA 5.4. *There exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le v} \|\phi^R(X)(t) - \phi^R(X')(t)\|^{2p}\Big] \le CE\Big[\int_0^{v\wedge\tau^R} \|X(t) - X'(t)\|^{2p}dt\Big]$$

for $v \in [0,T]$ and $X, X' \in \mathcal{E}$.

PROOF. From the Hölder inequality, we have

$$\begin{aligned}
E\Big[\sup_{0\le t\le v} &\|\phi_1(X)(t\wedge\tau^R) - \phi_1(X')(t\wedge\tau^R)\|^{2p}\Big] \\
&\le CE\Big[\Big(\int_0^{v\wedge\tau^R} \|X(s) - X'(s)\|ds\Big)^{2p}\Big] \\
&\le CT^{2p-1}E\Big[\int_0^{v\wedge\tau^R} \|X(s) - X'(s)\|^{2p}ds\Big].
\end{aligned}$$

From Lemma 4.1, we have

$$E\Big[\sup_{0\le t\le v}\|\phi_2(X)(t\wedge\tau^R)-\phi_2(X')(t\wedge\tau^R)\|^{2p}\Big]$$
$$\le CE\Big[\int_0^{v\wedge\tau^R}\|V(X(s))-V(X'(s))\|^{2p}ds\Big]$$
$$\le CE\Big[\int_0^{v\wedge\tau^R}\|X(s)-X'(s)\|^{2p}ds\Big].$$

By Lemma 5.3, we have

$$E\Big[\sup_{0\le t\le v}\|\phi_3^j(X)(t\wedge\tau^R)-\phi_3^j(X')(t\wedge\tau^R)\|^{2p}\Big]$$
$$\le C(1+R^{2p}\delta_m^{-p(1+2\epsilon)})E\Big[\int_0^{v\wedge\tau^R}\|X(s)-X'(s)\|^{2p}ds\Big].$$

Finally we have

$$E\Big[\sup_{0\le t\le v}\|\phi_4^j(X)(t\wedge\tau^R)-\phi_4^j(X')(t\wedge\tau^R)\|^{2p}\Big]$$
$$\le C\left(\frac{R}{\delta_m}\right)^{2p}E\Big[\Big(\int_0^{v\wedge\tau^R}\|X(s)-X'(s)\|ds\Big)^{2p}\Big]$$
$$\le CT^{2p-1}\left(\frac{R}{\delta_m}\right)^{2p}E\Big[\int_0^{v\wedge\tau^R}\|X(s)-X'(s)\|^{2p}ds\Big].\ \Box$$

Proof of Theorem 3.1. We consider the sequence defined by

$$\begin{cases} X_0^R(t)\equiv x_0, \\ X_n^R=\phi^R(X_{n-1}^R), \quad n=1,2,\dots. \end{cases} \tag{5.1}$$

From Lemma 5.4 and Lemma 5.2, there exist constants $C_0>0$ and $C_1>0$ which are independent of n such that

$$\begin{aligned} &E\Big[\sup_{0\le t\le T}\|X_{n+1}^R(t)-X_n^R(t)\|^{2p}\Big] \\ &\le C_0^n\int_0^T ds_n\int_0^{s_n}ds_{n-1}\cdots\int_0^{s_2}ds_1E\Big[\|X_1^R(s_1)-X_0^R(s_1)\|^{2p}\Big] \\ &\le C_1\frac{(C_0T)^n}{n!}. \end{aligned} \tag{5.2}$$

From this we have

$$P\Big(\sup_{0\le t\le T}\|X^R_{n+1}(t)-X^R_n(t)\|>\frac{1}{2^n}\Big)$$
$$\le 2^{2pn}E\Big[\sup_{0\le t\le T}\|X^R_{n+1}(t)-X^R_n(t)\|^{2p}\Big]\le C_1\frac{(2^pC_0T)^n}{n!},$$

and hence

$$\sum_{n=0}^{\infty}P\Big(\sup_{0\le t\le T}\|X^R_{n+1}(t)-X^R_n(t)\|>\frac{1}{2^n}\Big)<\infty.$$

By applying the Borel-Cantelli Lemma, we get

$$P\Big(\varliminf_{n\to\infty}\Big\{\sup_{0\le t\le T}\|X^R_{n+1}(t)-X^R_n(t)\|\le\frac{1}{2^n}\Big\}\Big)=1.$$

This implies that $(X^R_n(\cdot))_n$ is the Cauchy sequence in $C([0,T];H)$ a.s., therefore there exists a $C([0,T];H)$-valued random variable $X^R(\cdot)$ such that

$$\lim_{n\to\infty}\sup_{0\le t\le T}\|X^R_n(t)-X^R(t)\|=0 \qquad \text{a.s.}$$

By applying the Hölder inequality and from the inequality (5.2), we have

$$\sum_{l=0}^{n-1}\Big(E\Big[\sup_{0\le t\le T}\|X^R_{l+1}(t)-X^R_l(t)\|^{2p}\Big]\Big)^{\frac{1}{2p}}$$
$$\le\Big(\sum_{l=0}^{n-1}2^{-\frac{l}{(2p-1)}}\Big)^{\frac{2p-1}{2p}}\Big(\sum_{l=0}^{n-1}2^lE\Big[\sup_{0\le t\le T}\|X^R_{l+1}(t)-X^R_l(t)\|^{2p}\Big]\Big)^{\frac{1}{2p}}$$
$$\le\Big(\sum_{l=0}^{\infty}2^{-\frac{l}{(2p-1)}}\Big)^{\frac{2p-1}{2p}}\Big(\sum_{l=0}^{\infty}C_1\frac{(2C_0T)^l}{l!}\Big)^{\frac{1}{2p}}=\big(1-2^{\frac{-1}{2p-1}}\big)^{-\frac{2p-1}{2p}}\big(C_1e^{2C_0T}\big)^{\frac{1}{2p}}$$

for all $n\ge 1$, and so we get

$$\Big(E\Big[\sup_{0\le t\le T}\|X^R_n(t)\|^{2p}\Big]\Big)^{\frac{1}{2p}}$$
$$\le\Big(E\Big[\sup_{0\le t\le T}\|X^R_0(t)\|^{2p}\Big]\Big)^{\frac{1}{2p}}+\sum_{l=0}^{n-1}\Big(E\Big[\sup_{0\le t\le T}\|X^R_{l+1}(t)-X^R_l(t)\|^{2p}\Big]\Big)^{\frac{1}{2p}}$$

$$\le \|x_0\| + \left(1 - 2^{\frac{-1}{2p-1}}\right)^{-\frac{2p-1}{2p}} \left(C_1 e^{2C_0T}\right)^{\frac{1}{2p}}.$$

Moreover, by the Fatou's lemma, we have

$$E\Big[\sup_{0\le t\le T} \|X^R(t)\|^{2p}\Big] \le \varliminf_{n\to\infty} E\Big[\sup_{0\le t\le T} \|X_n^R(t)\|^{2p}\Big]$$
$$\le \Big(\|x_0\| + \left(1 - 2^{\frac{-1}{2p-1}}\right)^{-\frac{2p-1}{2p}} \left(C_1 e^{2C_0T}\right)^{\frac{1}{2p}}\Big)^{2p}.$$

Therefore we obtain

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} \|X_n^R(t) - X^R(t)\|^{2p}\Big] = 0. \tag{5.3}$$

From Lemma 5.4, Equation (5.3) and the inequality (5.2), it follows that

$$E\Big[\sup_{0\le t\le T} \|\phi^R(X^R)(t) - X^R(t)\|^{2p}\Big]$$
$$\le \lim_{n\to\infty} E\Big[\sup_{0\le t\le T} \|\phi^R(X_n^R)(t) - X_n^R(t)\|^{2p}\Big]$$
$$= \lim_{n\to\infty} E\Big[\sup_{0\le t\le T} \|X_{n+1}^R(t) - X_n^R(t)\|^{2p}\Big] \le C_1 \lim_{n\to\infty} \frac{(C_0T)^n}{n!} = 0,$$

and therefore

$$\phi^R(X^R)(t) = X^R(t)$$

for all $t \in [0,T]$ a.s.

Let positive numbers R and R' satisfy $R < R'$. Then from Lemma 5.4,

$$E\Big[\sup_{0\le t\le v} \|X^R(t\wedge\tau^R) - X^{R'}(t\wedge\tau^R)\|^{2p}\Big]$$
$$= E\Big[\sup_{0\le t\le v} \|\phi^R(X^R)(t\wedge\tau^R) - \phi^R(X^{R'})(t\wedge\tau^R)\|^{2p}\Big]$$
$$\le C\int_0^v E\Big[\|X^R(t\wedge\tau^R) - X^{R'}(t\wedge\tau^R)\|^{2p}\Big]dt$$

for $v \in [0,T]$. Therefore, from the Gronwall inequality, we get

$$E\Big[\sup_{0\le t\le T} \|X^R(t\wedge\tau^R) - X^{R'}(t\wedge\tau^R)\|^{2p}\Big] = 0$$

and hence

$$P\Big(\sup_{0\le t\le \tau^R}\|X^R(t)-X^{R'}(t)\|>0\Big)=0. \tag{5.4}$$

Now we define the stochastic process $(X_m(t;\theta))_{t\in[0,T]}$ by $X_m(t;\theta)=X^R(t)$ for $t\in[0,\tau^R]$. From Equation (5.4) and Lemma 5.1, this process is well defined and is the unique H-valued $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted mild solution of the stochastic differential equation (3.1), which has continuous paths a.s. □

Proof of Theorem 3.2. We abbreviate $X_m(\,\cdot\,;\theta)$ to $X(\,\cdot\,;\theta)$ in this proof. Let $R>0$. We have

$$\begin{aligned}&X(t\wedge\tau^R;\theta_n)-X(t\wedge\tau^R;\theta)\\&\quad=I_{1,n}(t\wedge\tau^R)+I_{2,n}(t\wedge\tau^R)+\sum_{j=1}^r I_{3,n}^j(t\wedge\tau^R)+\sum_{j=1}^r I_{4,n}^j(t\wedge\tau^R),\end{aligned}$$

where

$$\begin{aligned}I_{1,n}(t)&=\int_0^t S(t-s)\big(b(X(s;\theta_n))-b(X(s;\theta))\big)ds,\\I_{2,n}(t)&=\int_0^t S(t-s)\big(V(X(s;\theta_n))-V(X(s;\theta))\big)dW(s),\\I_{3,n}^j(t)&=\int_0^t S(t-s)\big(\sigma_j(X(s;\theta_n))-\sigma_j(X(s;\theta))\big)dZ_m^j(s)\end{aligned}$$

and

$$I_{4,n}^j(t)=\int_0^t S(t-s)\big(\sigma_j(X(s;\theta_n))\dot\varphi_m(s;\theta_n^j)-\sigma_j(X(s;\theta))\dot\varphi_m(s;\theta^j)\big)ds$$

for $t\in[0,T]$.

We have from the Hölder inequality

$$E\Big[\sup_{0\le t\le v}\|I_{1,n}(t\wedge\tau^R)\|^{2p}\Big]\le CE\Big[\int_0^{v\wedge\tau^R}\|X(s;\theta_n)-X(s;\theta)\|^{2p}ds\Big]$$

and by using Lemma 4.1

$$E\Big[\sup_{0\le t\le v}\|I_{2,n}(t\wedge\tau^R)\|^{2p}\Big]\le CE\Big[\int_0^{v\wedge\tau^R}\|X(s;\theta_n)-X(s;\theta))\|^{2p}ds\Big].$$

From Lemma 5.3, we get

$$E\Big[\sup_{0\le t\le v}\|I_{3,n}(t\wedge\tau^R)\|^{2p}\Big] \le C(1+R^{2p}\delta_m^{-p(1+2\epsilon)})E\Big[\int_0^{v\wedge\tau^R}\|X(s;\theta_n)-X(s;\theta)\|^{2p}ds\Big].$$

As for $I_{4,n}^j(t\wedge\tau^R)$, we have

$$E\Big[\sup_{0\le t\le v}\|I_{4,n}^j(t\wedge\tau^R)\|^{2p}\Big] \le CE\Big[\Big(\int_0^{v\wedge\tau^R}\|\sigma_j(X(s;\theta_n))\dot\varphi_m(s;\theta_n^j)-\sigma_j(X(s;\theta))\dot\varphi_m(s;\theta^j)\|ds\Big)^{2p}\Big].$$

Since we have

$$\begin{aligned}&\|\sigma_j(X(s;\theta_n))\dot\varphi_m(s;\theta_n^j)-\sigma_j(X(s;\theta))\dot\varphi_m(s;\theta^j)\|\\ &\le \|\sigma_j(X(s;\theta_n))\dot\varphi_m(s;\theta_n^j-\theta^j)\|+\|\big(\sigma_j(X(s;\theta_n))-\sigma_j(X(s;\theta))\big)\dot\varphi_m(s;\theta^j)\|\\ &\le C\delta_m^{-1}\big(|\theta_n-\theta|+|\theta|\|X(s;\theta_n)-X(s;\theta)\|\big),\end{aligned}$$

it follows that

$$\begin{aligned}&E\Big[\sup_{0\le t\le v}\|I_{4,n}^j(t\wedge\tau^R)\|^{2p}\Big]\\ &\le C\delta_m^{-2p}\Big\{T^{2p}|\theta_n-\theta|^{2p}+|\theta|^{2p}E\Big[\Big(\int_0^{v\wedge\tau^R}\|X(s;\theta_n)-X(s;\theta)\|ds\Big)^{2p}\Big]\Big\}\\ &\le C\delta_m^{-2p}\Big\{T^{2p}|\theta_n-\theta|^{2p}+T^{2p-1}|\theta|^{2p}E\Big[\int_0^{v\wedge\tau^R}\|X(s;\theta_n)-X(s;\theta)\|^{2p}ds\Big]\Big\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&E\Big[\sup_{0\le t\le v}\|X(s\wedge\tau^R;\theta_n)-X(s\wedge\tau^R;\theta)\|^{2p}\Big]\\ &\qquad\le C\Big(1+R^{2p}\delta_m^{-p(1+2\epsilon)}+\delta_m^{-2p}|\theta|^{2p}\Big)\\ &\qquad\times\int_0^v E\Big[\|X(s\wedge\tau^R;\theta_n)-X(s\wedge\tau^R;\theta)\|^{2p}\Big]ds+C\delta_m^{-2p}|\theta_n-\theta|^{2p}.\end{aligned}$$

By using the Gronwall inequality, we get

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} \|X(t\wedge\tau^R;\theta_n) - X(t\wedge\tau^R;\theta)\|^{2p}\Big] = 0.$$

Now we see

$$P\Big(\sup_{0\le t\le T} \|X(t;\theta_n) - X(t;\theta)\| > \epsilon\Big)$$
$$\le P\Big(\sup_{0\le t\le T} \|X(t\wedge\tau^R;\theta_n) - X(t\wedge\tau^R;\theta)\| > \epsilon\Big) + P(\tau^R < T).$$

If we recall Lemma 5.1, the proof is completed. □

6. Proof for Theorem 3.3

In this section, we also use notation in Section 3.

We denote $\varphi_m(\,\cdot\,;(h^j(\delta), h^j(2\delta),\dots,h^j(T)))$ by $\varphi_m(\,\cdot\,;h^j)$ for $j = 1,2,\dots,r$. Let $[t]_m^- = k\delta_m$ if $k\delta_m \le t < (k+1)\delta_m$, and

$$\bar X_m(t;h) = S(t-[t]_m^-)X_m([t]_m^-;h), \quad t\in[0,T].$$

We shall first prove the following proposition.

PROPOSITION 6.1. *For any $p>3$, there exist a constant $C>0$ and a sequence $\{C_m\}$ which satisfies $\lim_{m\to\infty} C_m = 0$ such that*

$$E\Big[\sup_{0\le t\le v} \Big\| \int_0^t S(t-s)$$
$$\Big(D\sigma_j(\bar X_m(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(\bar X_m(v;h))dZ_m^j(v)\Big)dZ_m^j(s)$$
$$+\frac{1}{2}\int_0^t S(t-s)D\sigma_j(\eta(s;h))\sigma_j(\eta(s;h))ds\Big\|^{2p}\Big]$$
$$\le C\int_0^v E\Big[\sup_{0\le t\le s}\|X_m(t;h)-\eta(t;h)\|^{2p}\Big]ds + C_m$$

for all $v\in[0,T]$.

To prove this proposition, we have to prove some lemmas.

LEMMA 6.2. *Let $p > 3$. There exists a constant $C > 0$ such that*

$$E\Big[\|X_m(t;h) - \bar{X}_m(t;h)\|^{2p}\Big] \le C\delta_m^{p-1}$$

for all $t \in [0,T]$.

PROOF. Note that

$$\begin{aligned} X_m(t;h) - \bar{X}_m(t;h) &= \int_{[t]_m^-}^t S(t-s)V(X_m(s;h))dW(s) \\ &+ \sum_{j=1}^r \int_{[t]_m^-}^t S(t-s)\sigma_j(X_m(s;h))dZ_m^j(s) \\ &+ \int_{[t]_m^-}^t S(t-s)b(X_m(s;h))ds \\ &+ \sum_{j=1}^r \int_{[t]_m^-}^t S(t-s)\sigma_j(X_m(s;h))\dot{\varphi}_m(s;h^j)ds. \end{aligned} \tag{6.1}$$

By the same argument as the proof of Lemma 2.4, we get

$$E\Big[\sup_{0\le t\le T} \| \int_{[t]_m^-}^t S(t-s)V(X_m(s;h))dW(s)\|^{2p}\Big] \le C\delta_m^{p-1}.$$

By using Lemma 4.3 (ii), we get

$$\begin{aligned} &E\Big[\sup_{0\le t\le T} \| \int_{[t]_m^-}^t S(t-s)\sigma_j(X_m(s;h))dZ_m^j(s)\|^{2p}\Big] \\ &\le \sum_{k=0}^{m-1} E\Big[\sup_{k\delta_m\le t\le (k+1)\delta_m} \| \int_{k\delta_m}^t S(t-s)\sigma_j(X_m(s;h))dZ_m^j(s)\|^{2p}\Big] \\ &\le C\sum_{k=0}^{m-1} \delta_m^{p-\frac{1}{2}} \Big(\int_{k\delta_m}^{(k+1)\delta_m} E\Big[\|\sigma_j(X_m(s;h))\|^{4p}\Big]ds\Big)^{1/2} \le C\delta_m^{p-1}. \end{aligned}$$

Obviously we have

$$E\Big[\sup_{0\le t\le T} \| \int_{[t]_m^-}^t S(t-s)b(X_m(s;h))ds\|^{2p}\Big] \le C\delta_m^{2p}$$

and

$$E\Big[\sup_{0\le t\le T}\|\int_{[t]_m^-}^t S(t-s)\sigma_j(X_m(s;h))\dot{\varphi}_m(s;h^j)ds\|^{2p}\Big] \le C\sup_{0\le t\le T}\|\dot{h}(t)\|_{U_0}^{2p}\ \delta_m^{2p}. \square$$

Let

$$\chi_{m,k}^j(t)=\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s) \Big\{D\sigma_j(\bar{X}_m(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dZ_m^j(v)\Big\}dZ_m^j(s)$$

for $t\in[0,T]$ and $k=0,1,\dots,m-1$.

LEMMA 6.3.

$$\sum_{k=0}^{m-1}E\big[\chi_{m,k}^j(t)|\mathcal{G}_m(k\delta_m)\big] = -\frac{1}{\delta_m}\int_0^t S(t-s)D\sigma_j(\bar{X}_m(s;h))\Big(\int_{[s]_m^-}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dv\Big)ds$$

for $t\in[0,T]$.

PROOF. By using the real-valued $(\mathcal{G}_m(t))_{t\ge0}$-Brownian motion $(\beta_m^j(t))_{t\ge0}$ in lemma 4.2, we can write

$$\chi_{m,k}^j(t)=I_{m,k,1}^j(t)+I_{m,k,2}^j(t)+I_{m,k,3}^j(t),$$

where

$$I_{m,k,1}^j(t)=\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h)) \Big(\int_{k\delta_m}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dZ_m^j(v)\Big)d\beta_m^j(s),$$

$$I^j_{m,k,2}(t) = -\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h)) \Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))d\beta^j_m(v)\Big)\frac{Z^j_m(s)}{(k+1)\delta_m - s}ds$$

and

$$I^j_{m,k,3}(t) = \int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h)) \Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))\frac{Z^j_m(v)}{(k+1)\delta_m - v}dv\Big)\frac{Z^j_m(s)}{(k+1)\delta_m - s}ds.$$

We immediately have

$$E\Big[I^j_{m,k,1}(t)|\mathcal{G}_m(k\delta_m)\Big] = 0.$$

As for $I^j_{m,k,2}(t)$, it holds that

$$\begin{aligned} &E\Big[I^j_{m,k,2}(t)|\mathcal{G}_m(k\delta_m)\Big] \\ &= -\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h))E \\ &\qquad \Big[\Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))d\beta^j_m(v)\Big) \\ &\qquad \Big(\int_{k\delta_m}^{s} \frac{d\beta^j_m(v)}{(k+1)\delta_m - v}\Big)|\mathcal{G}_m(k\delta_m)\Big]ds \\ &= -\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h)) \\ &\qquad \Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))\frac{dv}{(k+1)\delta_m - v}\Big)ds. \end{aligned}$$

Finally for $I^j_{m,k,3}(t)$ we have

$$\begin{aligned} &E\Big[I^j_{m,k,3}(t)|\mathcal{G}_m(k\delta_m)\Big] \\ &= \int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h))\Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h)) \end{aligned}$$

$$\times E\Big[Z_m^j(v)Z_m^j(s)|\mathcal{G}_m(k\delta_m)\Big]\frac{dv}{(k+1)\delta_m - v}\Big)\frac{ds}{(k+1)\delta_m - s}$$

$$= \int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)D\sigma_j(\bar{X}_m(s;h))$$

$$\Big(\int_{k\delta_m}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))\frac{v-k\delta_m}{\delta_m}\frac{dv}{(k+1)\delta_m - v}\Big)ds.$$

Hence the proof is completed. □

From Lemma 6.3, we have

$$\sum_{k=0}^{m-1} E\big[\chi_{m,k}^j(t)|\mathcal{G}_m(k\delta_m)\big] + \frac{1}{2}\int_0^t S(t-s)D\sigma_j(\eta(s;h))\sigma_j(\eta(s;h))ds$$
$$= -\big(J_{m,1}^j(t) + J_{m,2}^j(t) + J_{m,3}^j(t)\big),$$

where

$$J_{m,1}^j(t) = \frac{1}{\delta_m}\int_0^t S(t-s)\Big(D\sigma_j(\bar{X}_m(s;h))\int_{[s]_m^-}^{s} S(s-v)\sigma_j(\bar{X}_m(v;h))dv$$
$$- D\sigma_j(X_m(s;h))\int_{[s]_m^-}^{s} S(s-v)\sigma_j(X_m(v;h))dv\Big)ds,$$

$$J_{m,2}^j(t) = \frac{1}{\delta_m}\int_0^t S(t-s)\Big(D\sigma_j(X_m(s;h))\int_{[s]_m^-}^{s} S(s-v)\sigma_j(X_m(v;h))dv$$
$$- D\sigma_j(\eta(s;h))\int_{[s]_m^-}^{s} S(s-v)\sigma_j(\eta(v;h))dv\Big)ds$$

and

$$J_{m,3}^j(t) = \frac{1}{\delta_m}\int_0^t S(t-s)\Big(D\sigma_j(\eta(s;h))\int_{[s]_m^-}^{s} S(s-v)\sigma_j(\eta(v;h))dv\Big)ds$$
$$- \frac{1}{2}\int_0^t S(t-s)D\sigma_j(\eta(s;h))\sigma_j(\eta(s;h))ds.$$

Lemma 6.4. *For any $p > 3$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le T}\|J_{m,1}^j(t)\|^{2p}\Big] \le C\delta_m^{p-1}.$$

PROOF. We see that

$$\begin{aligned}
&\|J^j_{m,1}(t)\| \\
&\le C\frac{1}{\delta_m}\int_0^t \|D\sigma_j(\bar{X}_m(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dv \\
&- D\sigma_j(X_m(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(X_m(v;h))dv\|ds \\
&\le C\frac{1}{\delta_m}\int_0^t \|\big(D\sigma_j(\bar{X}_m(s;h)) - D\sigma_j(X_m(s;h))\big) \\
&\qquad\qquad \int_{[s]_m^-}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dv\|ds \\
&+ C\frac{1}{\delta_m}\int_0^t \|D\sigma_j(X_m(s;h)) \\
&\qquad\qquad \int_{[s]_m^-}^s S(s-v)\big(\sigma_j(\bar{X}_m(v;h)) - \sigma_j(X_m(v;h))\big)dv\|ds \\
&\le C\Big(\int_0^t \|\bar{X}_m(s;h) - X_m(s;h)\|ds \\
&\qquad\qquad + \frac{1}{\delta_m}\int_0^t ds\int_{[s]_m^-}^s \|\bar{X}_m(v;h) - X_m(v;h)\|dv\Big).
\end{aligned}$$

Hence, by the Hölder inequality, we get

$$\begin{aligned}
E\Big[\sup_{0\le t\le T}\|J^j_{m,1}(t)\|^{2p}\Big] &\le C\Big(\int_0^T E\Big[\|\bar{X}_m(s;h) - X_m(s;h)\|^{2p}\Big]ds \\
&\qquad + \frac{1}{\delta_m}\int_0^T ds\int_{[s]_m^-}^s E\Big[\|\bar{X}_m(v;h) - X_m(v;h)\|^{2p}\Big]dv\Big).
\end{aligned}$$

Lemma 6.2 completes the proof. □

LEMMA 6.5. *For any $p > 3$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le v}\|J^j_{m,2}(t)\|^{2p}\Big] \le C\int_0^v E\Big[\sup_{0\le t\le s}\|X_m(t;h) - \eta(t;h)\|^{2p}\Big]ds$$

for all $v \in [0,T]$.

PROOF. We see that

$$
\begin{aligned}
&\|J^j_{m,2}(t)\| \\
&\le C\frac{1}{\delta_m}\int_0^t \|D\sigma_j(X_m(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(X_m(v;h))dv \\
&\qquad\qquad - D\sigma_j(\eta(s;h))\int_{[s]_m^-}^s S(s-v)\sigma_j(\eta(v;h))dv\|ds \\
&\le C\frac{1}{\delta_m}\int_0^t \|\big(D\sigma_j(X_m(s;h)) - D\sigma_j(\eta(s;h))\big) \\
&\qquad\qquad\qquad \int_{[s]_m^-}^s S(s-v)\sigma_j(X_m(v;h))dv\|ds \\
&+ C\frac{1}{\delta_m}\int_0^t \|D\sigma_j(\eta(s;h)) \\
&\qquad\qquad \int_{[s]_m^-}^s S(s-v)\big(\sigma_j(X_m(v;h)) - \sigma_j(\eta(v;h))\big)dv\|ds \\
&\le C\Big(\int_0^t \|X_m(s;h)-\eta(s;h)\|ds + \frac{1}{\delta_m}\int_0^t ds\int_{[s]_m^-}^s \|X_m(v;h)-\eta(v;h)\|dv\Big) \\
&\le C\int_0^t \sup_{0\le u\le s}\|X_m(u;h)-\eta(u;h)\|ds.
\end{aligned}
$$

Hence, by the Hölder inequality, we obtain the lemma. □

LEMMA 6.6. *For any* $p > 3$,

$$
\lim_{m\to\infty} E\Big[\sup_{0\le t\le T}\|J^j_{m,3}(t)\|^{2p}\Big] = 0.
$$

PROOF. If we set

$$
f^j(s,v) = D\sigma_j(\eta(s;h))S(s-v)\sigma_j(\eta(v;h)), \qquad 0\le v\le s\le T,
$$

then

$$
J^j_{m,3}(t) = \frac{1}{\delta_m}\int_0^t S(t-s)\Big(\int_{[s]_m^-}^s f^j(s,v)dv\Big)ds - \frac{1}{2}\int_0^t S(t-s)f^j(s,s)ds.
$$

Furthermore we can write as

$$
J^j_{m,3}(t) = I^j_{m,1}(t) + I^j_{m,2}(t) + I^j_{m,3}(t),
$$

where

$$I^j_{m,1}(t) = \frac{1}{\delta_m}\int_0^t S(t-s)\Big(\int_{[s]^-_m}^s f^j(s,v)dv\Big)ds \\ - \frac{1}{\delta_m}\int_0^t \big(s-[s]^-_m\big)S(t-[s]^-_m)f^j([s]^-_m,[s]^-_m)ds,$$

$$I^j_{m,2}(t) = \int_0^t \Big(\frac{s-[s]^-_m}{\delta_m}-\frac{1}{2}\Big)S(t-[s]^-_m)f^j([s]^-_m,[s]^-_m)ds \\ = \int_{[t]^-_m}^t \Big(\frac{s-[t]^-_m}{\delta_m}-\frac{1}{2}\Big)dsS(t-[t]^-_m)f^j([t]^-_m,[t]^-_m)$$

and

$$I^j_{m,3}(t) = \frac{1}{2}\int_0^t S(t-[s]^-_m)f^j([s]^-_m,[s]^-_m)ds - \frac{1}{2}\int_0^t S(t-s)f^j(s,s)ds.$$

As for $I^j_{m,1}(t)$, we have

$$\|I^j_{m,1}(t)\| \\ \le \frac{1}{\delta_m}\int_0^t\Big(\int_{[s]^-_m}^s \|S(t-s)f^j(s,v)-S(t-[s]^-_m)f^j([s]^-_m,[s]^-_m)\|dv\Big)ds \\ \le T\ g^j_m(t),$$

where

$$g^j_m(t) = \sup\big\{\|S(t-s_1)f^j(s_1,v_1)-S(t-s_2)f^j(s_2,v_2)\|\ ; \\ 0\le v_1\le s_1\le t, 0\le v_2\le s_2\le t, 0\le s_1-s_2\le\delta_m, 0\le v_1-v_2\le\delta_m\big\}.$$

We immediately get $\|I^j_{m,2}(t)\| \le C\delta_m$ for some constant $C>0$ and $\|I^j_{m,3}(t)\| \le \frac{T}{2}g^j_m(t)$. Therefore we have

$$\|J^j_{m,3}(t)\| \le \frac{3T}{2}g^j_m(t) + C\delta_m, \quad t\in[0,T].$$

Hence it suffices to show

$$\lim_{m\to\infty} E\Big[\sup_{0\le t\le T} g^j_m(t)^{2p}\Big] = 0.$$

From Lemma 2.8, there exists an nondecreasing sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset H$ such that $\lim_{n\to\infty} P(\Omega_n) = 1$, where

$$\Omega_n = \big\{\eta(t;h) \in K_n \quad \text{for all } t \in [0,T]\big\}.$$

We note that the set $\tilde{K}_n$ defined by

$$\tilde{K}_n = \big\{D\sigma_j(y_1)S(s-v)\sigma_j(y_2)\ ;\ 0 \le v \le s \le T,\ y_1, y_2 \in K_n\big\} \subset H$$

is compact. We have the following estimate.

$$\begin{aligned}
&\|S(t-s_1)f^j(s_1,v_1) - S(t-s_2)f^j(s_2,v_2)\| \\
&\le C\big(\|\big(S(t-s_1)-S(t-s_2)\big)D\sigma_j(\eta(s_1;h))S(s_1-v_1)\sigma_j(\eta(v_1;h))\| \\
&+ \|D\sigma_j(\eta(s_1;h)) - D\sigma_j(\eta(s_2;h))\| \\
&+ \|\big(S(s_1-v_1)-S(s_2-v_2)\big)\sigma_j(\eta(v_1;h))\| \\
&+ \|\sigma_j(\eta(v_1;h)) - \sigma_j(\eta(v_2;h))\|\big) \\
&\le C\big(\sup_{\tilde{y}\in\tilde{K}_n} \|\big(S(t-s_1)-S(t-s_2)\big)\tilde{y}\| + \|\eta(s_1;h)) - \eta(s_2;h))\| \\
&+ \sup_{y\in K_n} \|\big(S(s_1-v_1)-S(s_2-v_2)\big)\sigma_j(y)\| + \|\eta(v_1;h)) - \eta(v_2;h)\|\big)
\end{aligned}$$

for $0 \le v_1 \le s_1 \le t$, $0 \le v_2 \le s_2 \le t$, $0 \le s_1 - s_2 \le \delta_m$, $0 \le v_1 - v_2 \le \delta_m$ and $\omega \in \Omega_n$. Furthermore there exists a constant $C > 0$ such that $\sup_{0\le t\le T} g_m^j(t) \le C$. Therefore we have

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T} g_m^j(t)^{2p}\Big] = E\Big[\sup_{0\le t\le T} g_m^j(t)^{2p}\ ;\ \Omega_n\Big] + E\Big[\sup_{0\le t\le T} g_m^j(t)^{2p}\ ;\ \Omega\setminus\Omega_n\Big] \\
&\le C\Big(\ \sup\big\{\|\big(S(u_1)-S(u_2)\big)\tilde{y}\|^{2p} ; u_1,u_2 \in [0,T], |u_1-u_2| \le \delta_m, \tilde{y} \in \tilde{K}_n\big\} \\
&+ E\Big[\sup\big\{\|\eta(u_1;h)) - \eta(u_2;h))\|^{2p} ; u_1,u_2 \in [0,T], |u_1-u_2| \le \delta_m\big\}\Big] \\
&+ \sup\big\{\|\big(S(u_1)-S(u_2)\big)\sigma_j(y)\|^{2p} ; u_1,u_2 \in [0,T], |u_1-u_2| \le 2\delta_m, y \in K_n\big\} \\
&+ P(\Omega\setminus\Omega_n)\ \Big).
\end{aligned}$$

Since the stochastic process $(\eta(t;h))_{t\in[0,T]}$ has continuous path a.s. and the following holds

$$E\Big[\sup_{0\le t\le t} \|\eta(t;h)\|^{2p}\Big] < \infty,$$

from dominated convergence theorem, we have

$$\sup_{m\to\infty} E\Big[\sup\big\{\|\eta(u_1;h)) - \eta(u_2;h))\|^{2p}\,; \\ u_1, u_2 \in [0,T], |u_1-u_2| \le \delta_m\big\}\Big] = 0.$$

Furthermore if we recall Lemma 2.7 (ii), the proof is completed. □

Therefore, from Propositions 4.3 (ii), 4.5 (i), Lemma 6.4, 6.5 and 6.6, we obtain Proposition 6.1.

From Equation (6.1), we have

$$\begin{aligned} \sigma_j(X_m(s;h)) &= \sigma_j(\bar{X}_m(s;h)) \\ &+ \int_0^1 D\sigma_j\big(\bar{X}_m(s;h) + v(X_m(s;h) - \bar{X}_m(s;h)\big) \\ &\qquad \big(X_m(s;h) - \bar{X}_m(s;h)\big)dv \\ &= D\sigma_j(\bar{X}_m(s;h)) \int_{[s]_m^-}^s S(s-v)\sigma_j(\bar{X}_m(v;h))dZ_m^j(v) \\ &\qquad + \gamma_{m,1}^j(s) + \gamma_{m,2}^j(s) + \gamma_m^j(s) \end{aligned} \tag{6.2}$$

where

$$\gamma_{m,1}^j(s) = \sum_{1\le l\le r,\ l\ne j} D\sigma_j(\bar{X}_m(s;h)) \int_{[s]_m^-}^s S(s-v)\sigma_l(\bar{X}_m(v;h))dZ_m^l(v) \\ + D\sigma_j(\bar{X}_m(s;h)) \int_{[s]_m^-}^s S(s-v)V(\bar{X}_m(v;h))dW(v),$$

$$\gamma_{m,2}^j(s) = \sigma_j(\bar{X}_m(s;h)),$$

$$\gamma_{m,3}^{j,l}(s) = D\sigma_j(\bar{X}_m(s;h)) \int_{[s]_m^-}^s S(s-v) \\ \big(\sigma_l(X_m(v;h)) - \sigma_l(\bar{X}_m(v;h))\big)dZ_m^l(v),$$

$$\gamma_{m,4}^j(s) = D\sigma_j(\bar{X}_m(s;h)) \int_{[s]_m^-}^s S(s-v) \\ \big(V(X_m(v;h)) - V(\bar{X}_m(v;h))\big)dW(v),$$

$$\gamma^j_{m,5}(s) = D\sigma_j(\bar{X}_m(s;h)) \int_{[s]^-_m}^s S(s-v)b(X_m(v;h))dv,$$

$$\gamma^{j,l}_{m,6}(s) = D\sigma_j(\bar{X}_m(s;h)) \int_{[s]^-_m}^s S(s-v)\sigma_l(X_m(v;h))\dot{\varphi}_m(v;h^l)dv,$$

$$\gamma^j_{m,7}(s) = \int_0^1 dv_1 \int_0^{v_1} D^2\sigma_j\big(\bar{X}_m(s;h) + v_2\left(X_m(s;h) - \bar{X}_m(s;h)\right)\big) \\ \big(X_m(s;h) - \bar{X}_m(s;h), X_m(s;h) - \bar{X}_m(s;h)\big)dv_2,$$

and

$$\gamma^j_m(s) = \sum_{l=1}^r \gamma^{j,l}_{m,3}(s) + \gamma^j_{m,4}(s) + \gamma^j_{m,5}(s) + \sum_{l=1}^r \gamma^{j,l}_{m,6}(s) + \gamma^j_{m,7}(s)$$

for $s \in [0,T]$.

Lemma 6.7. *For any $p > 3$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le v} \| \int_0^t S(t-s)\gamma^j_{m,1}(s)dZ^j_m(s)\|^{2p}\Big] \le C\delta_m^{p(1-2\alpha)}.$$

for $v \in [0,T]$.

Proof. Since Z^l_m and Z^j_m, $l \neq j$, are independent, and W and Z^j_m are also independent, we have

$$E\Big[\int_{k\delta_m\wedge t}^{(k+1)\delta_m\wedge t} S(t-s)\gamma^j_{m,1}(s)dZ^j_m(s) \,|\, \mathcal{G}_m(k\delta_m)\Big] = 0.$$

Therefore, by using Proposition 4.5 (i), the proof is completed. □

Lemma 6.8. *For any $p > 3$, there exists a constant $C > 0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n} = 0$ such that*

$$E\Big[\sup_{0\le t\le v} \| \int_0^t S(t-s)\gamma^j_{m,2}(s)dZ^j_m(s)\|^{2p}\Big] \\ \le C\int_0^v E\Big[\sup_{0\le t\le s} \|X_m(t;h) - \eta(t;h)\|^{2p}\Big]ds + C_{m,n}$$

for $v \in [0, T]$.

PROOF. Note that

$$\begin{aligned}
&\int_0^t S(t-s)\gamma_{m,2}^j(s)dZ_m^j(s) \\
&\quad = \int_0^t S(t-s)\big(\sigma_j(\bar{X}_m(s;h)) - \sigma_j(S(s-[s]_m^-)\eta([s]_m^-;h))\big)dZ_m^j(s) \\
&\quad + \int_0^t S(t-s)\big(\sigma_j(S(s-[s]_m^-)\eta([s]_m^-;h)) - \sigma_j(\eta([s]_m^-;h))\big)dZ_m^j(s) \\
&\qquad + \int_0^t S(t-s)\big(I - S(s-[s]_m^-)\big)\sigma_j(\eta([s]_m^-;h))dZ_m^j(s) \\
&\qquad\qquad + S(t-[t]_m^-)\sigma_j(\eta([t]_m^-;h))Z_m^j(t).
\end{aligned}$$

Since $\bar{X}_m(s;h)$ and $\eta([s]_m^-;h)$ are $\mathcal{G}_m([s]_m^-)$-measurable for all $s \in [0,T]$, we can apply Proposition 4.5 (ii) to get

$$\begin{aligned}
&E\Big[\sup_{0\le t\le v} \| \int_0^t S(t-s)\gamma_{m,2}^j(s)dZ_m^j(s)\|^{2p}\Big] \\
&\qquad \le CE\Big[\int_0^v \sup_{0\le t\le s} \|X_m([t]_m^-;h) - \eta([t]_m^-;h)\|^{2p}ds \\
&\qquad + \int_0^v \sup_{0\le t\le s} \|\big(S(t-[t]_m^-) - I\big)\sigma_j(\eta([t]_m^-;h))\|^{2p}ds \\
&\qquad + \int_0^v \sup_{0\le t\le s} \|\big(I - S(t-[t]_m^-)\big)\sigma_j(\eta([t]_m^-;h))\|^{2p}ds\Big] + C\delta_m^{p-1}.
\end{aligned}$$

From Lemma 2.8, we can choose an nondecreasing sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset H$ such that $\lim_{n\to\infty} P(\Omega_n) = 1$, where

$$\Omega_n = \big\{\eta(t;h) \in K_n \quad \text{for all } t \in [0,T]\big\}.$$

Since σ_j are bounded,

$$\begin{aligned}
&E\Big[\int_0^v \sup_{0\le t\le s} \|\big(S(t-[t]_m^-) - I\big)\sigma_j(\eta([t]_m^-;h))\|^{2p}ds \\
&\qquad + \int_0^v \sup_{0\le t\le s} \|\big(I - S(t-[t]_m^-)\big)\sigma_j(\eta([t]_m^-;h))\|^{2p}ds\Big]
\end{aligned}$$

$$\leq C\Big(\sup_{\substack{0\leq u\leq\delta_m\\ x\in K_n}}\|(S(u)-I)\sigma_j(x)\|^{2p}+P(\Omega\setminus\Omega_n)\Big).$$

Thus the result follows from Lemma 2.7 (ii). □

LEMMA 6.9. *For any $p>3$, there exists a constant $C>0$ such that*

$$E\Big[\sup_{0\leq t\leq T}\|\int_0^t S(t-s)\gamma_m^j(s)dZ_m^j(s)\|^{2p}\Big]\leq C\delta_m^{p-\frac{1}{2}}.$$

PROOF. From Lemma 4.4, there exists a constant $C>0$ such that

$$E\Big[\sup_{0\leq t\leq T}\|\int_0^t S(t-s)\gamma_m^j(s)dZ_m^j(s)\|^{2p}\Big]\leq C\delta_m^{-p}\Big(\int_0^T E\Big[\|\gamma_m^j(s)\|^{4p}\Big]ds\Big)^{1/2}.$$

From Lemma 4.3 (ii), we have

$$\begin{aligned}
&E\Big[\|\gamma_{m,3}^{j,l}(s)\|^{4p}\Big]\\
&\leq CE\Big[\|\int_{[s]_m^-}^s S(s-v)\big(\sigma_j(X_m(v;h))-\sigma_j(\bar{X}_m(v;h))\big)dZ_m^l(v)\|^{4p}\Big]\\
&\leq C\delta_m^{2p-\frac{1}{2}}\Big(\int_{[s]_m^-}^s E\Big[\|X_m(v;h)-\bar{X}_m(v;h)\|^{8p}\Big]dv\Big)^{1/2}\leq C\delta_m^{4p-\frac{1}{2}}.
\end{aligned}$$

From Lemma 6.2, we have

$$\begin{aligned}
&E\Big[\|\gamma_{m,4}^{j}(s)\|^{4p}\Big]\\
&\leq CE\Big[\|\int_{[s]_m^-}^s S(s-v)\big(V(X_m(v;h))-V(\bar{X}_m(v;h))\big)dW(v)\|^{4p}\Big]\\
&\leq CE\Big[\Big(\int_{[s]_m^-}^s\|X_m(v;h)-\bar{X}_m(v;h)\|^2dv\Big)^{2p}\Big]\\
&\leq C\delta_m^{2p-1}\int_{[s]_m^-}^s E\Big[\|X_m(v;h)-\bar{X}_m(v;h)\|^{4p}\Big]dv\leq C\delta_m^{4p-1}.
\end{aligned}$$

$$E\Big[\|\gamma_{m,5}^{j}(s)\|^{4p}\Big]\leq CE\Big[\|\int_{[s]_m^-}^s S(s-v)b(X_m(v;h))dv\|^{4p}\Big]\leq C\delta_m^{4p}.$$

$$E\Big[\|\gamma_{m,6}^{j,l}(s)\|^{4p}\Big] \le CE\Big[\|\int_{[s]_m^-}^{s} S(s-v)\sigma_l(X_m(v;h))\dot{\varphi}_m(v;h^l)dv\|^{4p}\Big]$$
$$\le C \sup_{0\le t\le T} |\dot{h}^l(t)|^{4p}\ \delta_m^{4p}.$$

From Lemma 6.2, we have

$$E\Big[\|\gamma_{m,7}^{j}(s)\|^{4p}\Big] \le CE\Big[\|X_m(s;h) - \bar{X}_m(s;h)\|^{8p}\Big] \le C\delta_m^{4p-1}. \ \Box$$

Combining Equation (6.2), Proposition 6.1, Lemmas 6.7, 6.8 and 6.9, we obtain the following Lemma.

Lemma 6.10. *For any $p > 3$, there exist a constant $C > 0$ and a double sequence $\{C_{m,n}\}$ which satisfies $\lim_{n\to\infty}\lim_{m\to\infty} C_{m,n} = 0$ such that*

$$E\Big[\sup_{0\le t\le v} \|\int_0^t S(t-s)\sigma_j(X_m(s;h))dZ_m^j(s)$$
$$+ \frac{1}{2}\int_0^t S(t-s)D\sigma_j(\eta(s;h))\sigma_j(\eta(s;h))ds\|^{2p}\Big]$$
$$\le C\int_0^v E\Big[\sup_{0\le t\le s}\|X_m(t;h) - \eta(t;h)\|^{2p}\Big]ds + C_{m,n}$$

for $v \in [0,T]$.

From Lemma 4.1, we get the following Lemma.

Lemma 6.11. *For any $p > 3$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le v} \|\int_0^t S(t-s)\big(V(X_m(s;h)) - V(\eta(s;h))\big)dW(s)\|^{2p}\Big]$$
$$\le C\int_0^v E\Big[\|X_m(s;h) - \eta(s;h)\|^{2p}\Big]ds$$

for $v \in [0,T]$.

By the Lipschitz continuity of b, we have the following Lemma.

Lemma 6.12. *For any $p > 3$, there exists a constant $C > 0$ such that*

$$E\Big[\sup_{0\le t\le v}\|\int_0^t S(t-s)\big(b(X_m(s;h)) - b(\eta(s;h))\big)ds\|^{2p}\Big]$$
$$\le C\int_0^v E\Big[\|X_m(s;h) - \eta(s;h)\|^{2p}\Big]ds$$

for $v \in [0,T]$.

Lemma 6.13. *Let*

$$I_m^j(t) = \int_0^t S(t-s)\Big\{\sigma_j(X_m(s;h))\dot{\varphi}_m(s;h^j) - \sigma_j(\eta(s;h))\dot{h}^j(s)\Big\}ds$$

for $t \in [0,T]$. For any $p > 3$, there exist a constant $C > 0$ and a sequence $\{C_m\}$ which satisfies $\lim_{m\to\infty} C_m = 0$ such that

$$E\Big[\sup_{0\le t\le v}\|I_m^j(t)\|^{2p}\Big] \le C\int_0^v E\Big[\|X_m(s;h) - \eta(s;h)\|^{2p}\Big]ds + C_m$$

for $v \in [0,T]$.

Proof. We see that

$$\sup_{0\le t\le v}\|I_m^j(t)\| \le \int_0^v \|\sigma_j(X_m(s;h))\dot{\varphi}_m(s;h^j) - \sigma_j(\eta(s;h))\dot{h}^j(s)\|ds$$
$$\le \int_0^v \|\big(\sigma_j(X_m(s;h)) - \sigma_j(\eta(s;h))\big)\dot{\varphi}_m(s;h^j)\|ds$$
$$+ \int_0^v \|\sigma_j(\eta(s;h))\big(\dot{\varphi}_m(s;h^j) - \dot{h}^j(s)\big)\|ds$$
$$\le C\Big\{\sup_{0\le t\le T}|\dot{h}^j(t)|\int_0^v \|X_m(s;h) - \eta(s;h)\|ds$$
$$+ \int_0^T \|\dot{\varphi}_m(s;h^j) - \dot{h}^j(s)\|ds\Big\}. \square$$

Note that

$$
\begin{aligned}
X_m(t;h) - \eta(t;h) &= \int_0^t S(t-s)\big(b(X_m(s;h)) - b(\eta(s;h))\big)ds \\
&+ \int_0^t S(t-s)\big(V(X_m(s;h)) - V(\eta(s;h))\big)dW(s) \\
&+ \sum_{j=1}^r \Big\{ \int_0^t S(t-s)\sigma_j(X_m(s;h))dZ_m^j(s) \\
&\qquad + \frac{1}{2}\int_0^t S(t-s)D\sigma_j(\eta(s;h))\sigma_j(\eta(s;h))ds \Big\} \\
&+ \sum_{j=1}^r \int_0^t S(t-s)\Big\{\sigma_j(X_m(s;h))\dot{\varphi}_m(s;h^j) - \sigma_j(\eta(s;h))\dot{h}^j(s)\Big\}ds.
\end{aligned}
\tag{6.3}
$$

From Lemmas 6.10, 6.11, 6.12 and 6.13, we get Theorem 3.3 by using the Gronwall inequality.

7. Proof of Support Theorem

In this section we use the same notation as in Section 1. Let

$$
\mathcal{L}_\infty = \big\{\xi(\cdot;h)\ ;\ h\colon [0,T] \to U_0 \text{ is infinitely continuously differentiable, } h(0)=0\big\} \subset C([0,T];H)
$$

Lemma 7.1.

$$
\bar{\mathcal{L}}_\infty = \bar{\mathcal{L}},
$$

where $\bar{\mathcal{L}}_\infty$ and $\bar{\mathcal{L}}$ mean the closure of $\mathcal{L}_\infty$ and $\mathcal{L}$ in $C([0,T];H)$, respectively.

Proof. Let $h\colon [0,T] \to U_0$ be a continuous mapping which is piecewise continuously differentiable and satisfies $h(0)=0$. Then h has an extension to $\mathbf{R}$, say $h_1 \in C(\mathbf{R};U_0)$, such that it is piecewise continuously differentiable and its support is contained in the closed interval $[0,T+1]$. For $\epsilon \in (0,1)$, we define a infinitely continuously differentiable mapping $h_{(\epsilon)}\colon [0,T] \to U_0$ by

$$
h_{(\epsilon)}(t) = \int_{\mathbf{R}} f_{(\epsilon)}(t-s)h_1(s)ds = \int_{-1}^1 f(u)h_1(t-\epsilon u)du,
$$

where $f\colon \mathbf{R} \to \mathbf{R}$ is a infinitely continuously differentiable non-negative function with its support $\operatorname{supp} f \subset [-1,1]$ such that $\int_{-1}^{1} f(t)dt = 1$, and $f_{(\epsilon)}\colon \mathbf{R} \to \mathbf{R}$ is defined by $f_{(\epsilon)}(t) = \epsilon^{-1} f(\epsilon^{-1}t)$. Then we have

$$\begin{aligned}\dot{h}_{(\epsilon)}(t) &= \frac{1}{\epsilon^2}\int_{\mathbf{R}} \dot{f}\Big(\frac{t-s}{\epsilon}\Big) h_1(s)ds = \frac{1}{\epsilon}\int_{-1}^{1} \dot{f}(u) h_1(t-\epsilon u)du \\ &= \int_{-1}^{1} f(u)\dot{h}_1(t-\epsilon u)du,\end{aligned}$$

therefore

$$\dot{h}_{(\epsilon)}(t) - \dot{h}(t) = \int_{-1}^{1} f(u)\big(\dot{h}_1(t-\epsilon u) - \dot{h}(t)\big)du$$

for $t \in [0,T]$. Hence we get

$$\int_0^T \|\dot{h}_{(\epsilon)}(s) - \dot{h}(s)\|ds \le \int_{-1}^{1} f(u)\Big(\int_0^T \|\dot{h}_1(s-\epsilon u) - \dot{h}(s)\|ds\Big)du.$$

Therefore, from dominated convergence theorem,

$$\lim_{\epsilon \downarrow 0} \int_0^T \|\dot{h}_{(\epsilon)}(s) - \dot{h}(s)\|ds = 0. \tag{7.1}$$

Now we estimate

$$\begin{aligned}\xi(t;h_{(\epsilon)}) - \xi(t;h) &= \int_0^t S(t-s)\big((b-\rho)(\xi(s;h_{(\epsilon)})) - (b-\rho)(\xi(s;h))\big)ds \\ &\quad + \int_0^t S(t-s)\sigma(\xi(s;h_{(\epsilon)}))\big(\dot{h}_{(\epsilon)}(s) - \dot{h}(s)\big)ds \\ &\qquad + \int_0^t S(t-s)\big(\sigma(\xi(s;h_{(\epsilon)})) - \sigma(\xi(s;h))\big)\dot{h}(s)ds.\end{aligned}$$

Therefore there exists a constant $C > 0$ independent of ϵ such that

$$\begin{aligned}&\sup_{0\le t\le v} \|\xi(t;h_{(\epsilon)}) - \xi(t;h)\| \\ &\le C\Big\{(1 + \operatorname*{ess.sup}_{0\le t\le T} \|\dot{h}(t)\|_{U_0}) \int_0^v \|\xi(s;h_{(\epsilon)}) - \xi(s;h)\|ds \\ &\qquad\qquad + \int_0^T \|\dot{h}_{(\epsilon)}(s) - \dot{h}(s)\|ds\Big\}\end{aligned}$$

for any $v \in [0,T]$. Applying the Gronwall inequality, we get

$$\sup_{0\le t\le T} \|\xi(t;h_{(\epsilon)}) - \xi(t;h)\|$$
$$\le Ce^{C(1+\text{ess.sup}_{0\le t\le T}\|\dot{h}(t)\|_{U_0})T} \int_0^T \|\dot{h}_{(\epsilon)}(s) - \dot{h}(s)\|ds.$$

Using Equation (7.1), we see

$$\lim_{\epsilon\downarrow 0} \sup_{0\le t\le T} \|\xi(t;h_{(\epsilon)}) - \xi(t;h)\| = 0. \square$$

Let $\delta_m = T/m$ for $m = 1,2,\ldots$. We define real-valued stochastic processes $(B^j_m(t))_{t\in[0,T]}$, $j \ge 1$, $m \ge 1$ by

$$B^j_m(t) = B^j(k\delta_m) + \frac{t-k\delta_m}{\delta_m}\big(B^j((k+1)\delta_m) - B^j(k\delta_m)\big) \tag{7.2}$$

if $k\delta_m \le t < (k+1)\delta_m$, $k = 0,1,\ldots,m-1$. For each $m \ge 1$ and $r \ge 1$, let $(B_{m,r}(t))_{t\in[0,T]}$ be the U_0-valued stochastic process defined by

$$B_{m,r}(t) = \sum_{j=1}^r B^j_m(t)g_j$$

for $t \in [0,T]$.

For each $r \ge 1$, let $(X_r(t))_{t\in[0,T]}$ be the H-valued continuous $(\mathcal{F}(t))_{t\in[0,T]}$-adapted stochastic process which is the unique mild solution of the stochastic differential equation

$$\begin{cases} dX_r(t) = AX_r(t)dt + (b-\rho+\rho_r)(X_r(t))dt \\ \qquad\qquad + \sum_{j=1}^r \sigma_j(X_r(t))dB^j(t), \\ X_r(0) = x_0. \end{cases} \tag{7.3}$$

From theorem 2.1, we have the following proposition.

PROPOSITION 7.2. *For any $p > 1$,*

$$\lim_{m\to\infty} E\Big[\sup_{0\le t\le T} \|\xi(t;B_{m,r}) - X_r(t)\|^{2p}\Big] = 0$$

for all $r = 1, 2, \ldots$.

Proposition 7.3. *For any* $p > 1$,

$$\lim_{r\to\infty} E\Big[\sup_{0\le t\le T} \|X_r(t) - X(t)\|^{2p}\Big] = 0.$$

Proof. We have

$$X_r(t) - X(t) = J_{1,r}(t) + J_{2,r}(t) + J_{3,r}(t) + J_{4,r}(t),$$

where

$$J_{1,r}(t) = \int_0^t S(t-s)\big(b(X_r(s)) - b(X(s))\big)ds,$$

$$J_{2,r}(t) = \int_0^t S(t-s)\big(\sigma(X_r(s)) - \sigma(X(s))\big)P_r dB(s),$$

$$J_{3,r}(t) = -\int_0^t S(t-s)(\rho - \rho_r)(X_r(s))ds,$$

$$J_{4,r}(t) = -\int_0^t S(t-s)\sigma(X(s))(I - P_r)dB(s)$$

and $P_r\colon U_0 \to U_0$ is a orthogonal projection to the subspace spanned by $\{g_1, g_2, \ldots, g_r\}$.

By the Lipschitz continuity of b, we have

$$E\Big[\sup_{0\le t\le v} \|J_{1,r}(t)\|^{2p}\Big] \le CE\Big[\int_0^v \|X_r(s) - X(s)\|^{2p} ds\Big].$$

From Lemma 4.1, we have

$$\begin{aligned} E\Big[\sup_{0\le t\le v} \|J_{2,r}(t)\|^{2p}\Big] &\le CE\Big[\int_0^v \|\sigma(X_r(s)) - \sigma(X(s))\|^{2p}_{L_{(2)}(U_0;H)} ds\Big] \\ &\le C\int_0^v E\Big[\|X_r(s) - X(s)\|^{2p}\Big] ds. \end{aligned}$$

From our assumption, there exists a constant $C > 0$ which is independent of k such that

$$\|(b - \rho + \rho_k)(x)\| \le C(1 + \|x\|)$$

for all $k = 1, 2, \dots$ and all $x \in H$. Therefore, from Section 7 in Da Prato and Zabczyk [4], we have

$$\sup_{k1,2,\dots} E\Big[\sup_{0\le t\le T} \|X_k(t)\|^{4p}\Big] < \infty.$$

From Lemma 2.8, we can choose an nondecreasing sequence of compact subsets $K_1 \subset K_2 \subset \cdots \subset H$ such that $\lim_{n\to\infty} P(\Omega_n) = 1$, where

$$\Omega_n = \big\{X(t) \in K_n \quad \text{for all } t \in [0, T]\big\}.$$

$$\begin{aligned}
E\Big[\sup_{0\le t\le T} \|J_{3,r}(t)\|^{2p}\Big] &\le CE\Big[\int_0^T \|(\rho-\rho_r)(X_r(s))\|^{2p} ds\Big] \\
\le C\Big\{ &\sup_{x\in K_n} \|(\rho-\rho_r)(x)\|^{2p} \\
&+ \Big(E\Big[\Big(\int_0^T (1+\|X_r(s)\|)^{2p} ds\Big)^2\Big]\Big)^{1/2} P(\Omega\setminus\Omega_n)^{1/2}\Big\} \\
\le C\Big\{ &\sup_{x\in K_n} \|(\rho-\rho_r)(x)\|^{2p} \\
&+ \Big(1 + \sup_{k=1,2,\dots} E\Big[\sup_{0\le t\le T} \|X_k(t)\|^{4p}\Big]\Big)^{1/2} P(\Omega\setminus\Omega_n)^{1/2}\Big\}.
\end{aligned}$$

We also have

$$E\Big[\sup_{0\le t\le T} \|J_{4,r}(t)\|^{2p}\Big] \le CE\Big[\int_0^T \|\sigma(X(s))(I-P_r)\|^{2p}_{L_{(2)}(U_0;H)} ds\Big].$$

Hence Gronwall inequality completes the Proof. □

Now we obtain the following proposition.

PROPOSITION 7.4. $\bar{\mathcal{L}} \supset \operatorname{supp} X(\cdot)$

PROOF. By Proposition 7.2 and Proposition 7.3, there exist sequences $(m_k)_k$ and $(r_k)_k$ such that

$$\sup_{0\le t\le T} \|\xi(t; B_{m_k,r_k}) - X(t)\| \to 0$$

in probability as $k \to \infty$. Then we get

$$P(X(\cdot) \in \bar{\mathcal{L}}) \geq \overline{\lim}_{k\to\infty} P(\xi(\,\cdot\,; B_{m_k,r_k}) \in \bar{\mathcal{L}}) = 1. \ \square$$

For each $y = (y_1, y_2, \dots, y_m) \in \mathbf{R}^m$, we define a mapping

$$\varphi_m(\,\cdot\,; y)\colon [0,T] \to \mathbf{R}$$

by

$$\varphi_m(t; y) = y_k + \frac{t - k\delta_m}{\delta_m}(y_{k+1} - y_k)$$

for $t \in [k\delta_m, (k+1)\delta_m]$, $k = 0, 1, \dots, m-1$, where we have set $y_0 = 0$.

We define a mapping $I_m\colon C([0,T]; \mathbf{R}) \to \mathbf{R}^m$ by

$$I_m f = \big(f(\delta_m), f(2\delta_m), \dots, f(T)\big)$$

for $f \in C([0,T]; \mathbf{R})$.

For $j = 1, 2, \dots, r$, we define a $\mathbf{R}$-valued stochastic process $(Z_m^j(t))_{t\geq 0}$ by

$$Z_m^j(t) = B^j(t) - \varphi_m(t; I_m B^j)$$

for $t \in [k\delta_m, (k+1)\delta_m]$, $k = 0, 1, \dots, m-1$. Then the processes

$$(Z_m^j(t))_{t\in[k\delta_m,(k+1)\delta_m]}$$

are pinned Brownian motions from 0 to 0 on each intervals $[k\delta_m, (k+1)\delta_m]$, $k = 0, 1, 2, \dots, m-1$.

We note that $(Z_m^j(t))_{t\geq 0}$, $j = 1, 2, \dots, r$, are not $(\mathcal{F}(t))_{t\geq 0}$-adapted. If we let $(\mathcal{G}_m(t))_{t\geq 0}$ be a filtration defined by $\mathcal{G}_m(t) = \mathcal{F}(t) \vee \sigma\{B(k\delta_m)\ ;\ k = 1, 2, \dots m\}$. Then $(Z_m^j(t))_{t\geq 0}$ are $(\mathcal{G}_m(t))_{t\geq 0}$-adapted processes.

Let

$$W_r(t) = B(t) - \sum_{j=1}^{r} B^j(t) g_j.$$

for $t \in [0,T]$.

Let $\theta = (\theta^1, \dots, \theta^r)$ be $(\mathbf{R}^m)^r$-valued $\mathcal{G}_m(0)$-measurable random variable. Let $(\hat{X}_{m,r}(t;\theta))_{t\in[0,T]}$ be the H-valued continuous $(\mathcal{G}_m(t))_{t\geq 0}$-adapted stochastic process which is the unique mild solution of the following stochastic differential equation

$$(7.4)\quad \begin{cases} d\hat{X}_{m,r}(t;\theta) = A\hat{X}_{m,r}(t;\theta)dt + b(\hat{X}_{m,r}(t;\theta))dt \\ \qquad +\sigma(\hat{X}_{m,r}(t;\theta))dW_r(t) \\ \qquad +\sum_{j=1}^{r}\sigma_j(\hat{X}_{m,r}(t;\theta))dZ_m^j(t) + \sum_{j=1}^{r}\sigma_j(\hat{X}_{m,r}(t;\theta))\dot{\varphi}_m(t;\theta^j)dt, \\ \hat{X}_{m,r}(0;\theta) = x_0. \end{cases}$$

REMARK 7.5. Since the mild solution $(X(t))_{t\in[0,T]}$ of Equation (1.2) is $(\mathcal{G}_m(t))_{t\in[0,T]}$-adapted, $(X(t))_{t\in[0,T]}$ is also the mild solution of Equation (7.4) for $\theta = (I_mB^1, I_mB^2, \dots, I_mB^r)$. Therefore, from the uniqueness, we have

$$X(t) = \hat{X}_{m,r}(t;(I_mB^1, I_mB^2, \dots, I_mB^r))$$

for all $t \in [0,T]$ a.s.

PROPOSITION 7.6. *For any $m \geq 1$, $r \geq 1$ and $\theta \in (\mathbf{R}^m)^r$, it holds that*

$$\operatorname{supp} \hat{X}_{m,r}(\,\cdot\,;\theta) \subset \operatorname{supp} X(\,\cdot\,).$$

PROOF. Let $w \in C([0,T];H) \setminus \operatorname{supp} X(\cdot)$. Then there exists an $\epsilon > 0$ such that

$$P(\sup_{0\le t\le T} \|X(t) - w(t)\| < \epsilon) = 0.$$

From the remark 7.5, the left-hand side is equals to

$$\int_{(\mathbf{R}^m)^r} P\big(\sup_{0\le t\le T} \|\hat{X}_{m,r}(t,\theta) - w(t)\| < \epsilon)P((I_mB^1, I_mB^2, \dots, I_mB^r) \in d\theta\big).$$

Therefore we have

$$P(\sup_{0\le t\le T} \|\hat{X}_{m,r}(t,\theta) - w(t)\| < \epsilon) = 0$$

for a.e. $\theta \in (\mathbf{R}^m)^r$ with respect to the Lebesgue measure. For any $\theta \in (\mathbf{R}^m)^r$, there exists a sequence $(\theta_n)_{n\in\{1,2,\dots\}}$ in $(\mathbf{R}^m)^r$ such that

$$\lim_{n\to\infty} |\theta_n - \theta|_{(\mathbf{R}^m)^r} = 0$$

and

$$P(\sup_{0\le t\le T}\|\hat{X}_{m,r}(t,\theta_n)-w(t)\|<\epsilon)=0$$

for all n. Then by using Theorem 3.2,

$$\begin{aligned}&P(\sup_{0\le t\le T}\|\hat{X}_{m,r}(t,\theta)-w(t)\|<\epsilon)\\ &\le \varliminf_{n\to\infty} P(\sup_{0\le t\le T}\|\hat{X}_{m,r}(t,\theta_n)-w(t)\|<\epsilon)=0.\end{aligned}$$

This means $w\notin \operatorname{supp}\hat{X}_{m,r}(\,\cdot\,;\theta)$. $\square$

Let $h\colon [0,T]\to U_0$, be infinitely continuously differentiable mapping such that $h(0)=0$, which has an expansion $h(t)=\sum_{j=1}^{\infty}h^j(t)g_j$. Let $(\eta_r(t;h))_{t\in[0,T]}$ be an H-valued continuous $(\mathcal{F}(t))_{t\ge 0}$-adapted stochastic process which is the unique mild solution of the stochastic differential equation

$$\begin{cases} d\eta_r(t;h)=A\eta_r(t;h)dt+b(\eta_r(t;h))dt+\sigma(\eta_r(t;h))dW_r(t)\\ \qquad -\rho_r(\eta_r(t;h))dt+\displaystyle\sum_{j=1}^{r}\sigma_j(\eta_r(t;h))\dot{h}^j(t)dt,\\ \eta_r(0;h)=x_0.\end{cases}$$

Let

$$X_{m,r}(t;h)=\hat{X}_{m,r}(t;(I_m h^1,I_m h^2,\dots,I_m h^r))$$

for $t\in[0,T]$.

From Theorem 3.3, we have the following proposition.

PROPOSITION 7.7. *For any $p>1$, we have*

$$\lim_{m\to\infty}E\Big[\sup_{0\le t\le T}\|X_{m,r}(t;h)-\eta_r(t;h)\|^{2p}\Big]=0.$$

PROPOSITION 7.8.

$$\operatorname{supp}\eta_r(\,\cdot\,;h)\subset\operatorname{supp}X(\,\cdot\,).$$

Proof. From Propositions 7.6 and 7.7, we have

$$P\big(\eta_r(\,\cdot\,;h) \in \operatorname{supp} X(\,\cdot\,)\big) \geq \overline{\lim}_{m\to\infty} P\big(X_{m,r}(\,\cdot\,;h) \in \operatorname{supp} X(\,\cdot\,)\big) = 1. \ \square$$

Proposition 7.9. *For any $p > 1$, we have*

$$\lim_{r\to\infty} E\Big[\sup_{0\leq t\leq T} \|\eta_r(t;h) - \xi(t;h)\|^{2p}\Big] = 0.$$

Proof. We write

$$\eta_r(t;h) - \xi(t;h) = I_{1,r}(t) + I_{2,r}(t) + I_{3,r}(t) + I_{4,r}(t),$$

where

$$\begin{aligned} I_{1,r}(t) = \int_0^t & S(t-s)\big(b(\eta_r(s;h)) - b(\xi(s;h))\big)ds \\ & - \int_0^t S(t-s)\big(\rho_r(\eta_r(s;h)) - \rho_r(\xi(s;h))\big)ds \\ & \quad + \sum_{j=1}^r \int_0^t S(t-s)\big(\sigma_j(\eta_r(s;h)) - \sigma_j(\xi(s;h))\big)\dot{h}^j(s)ds, \end{aligned}$$

$$I_{2,r}(t) = \int_0^t S(t-s)(\rho - \rho_r)(\xi(s;h))ds,$$

$$I_{3,r}(t) = \int_0^t S(t-s)\sigma(\eta_r(s;h))(I - P_r)dB(s),$$

and

$$I_{4,r}(t) = -\int_0^t S(t-s) \sum_{j=r+1}^\infty \sigma_j(\xi(s;h))\dot{h}^j(s)ds.$$

Here $P_r \colon U_0 \to U_0$ is a orthogonal projection to the subspace spanned by $\{g_1, g_2, \dots, g_r\}$.

There exists a constant $C > 0$ independent of r such that

$$E\Big[\sup_{0\leq t\leq v} \|I_{1,r}(t)\|^{2p}\Big] \leq C\big(1 + \sup_{0\leq t\leq T} \|\dot{h}(t)\|_{U_0}\big) E\Big[\int_0^v \|\eta_r(s;h) - \xi(s;h)\|^{2p} ds\Big].$$

We immediately have

$$\sup_{0\le t\le T}\|I_{2,r}(t)\|^{2p}\le C\Big(\int_0^T\|(\rho-\rho_r)(\xi(s;h))\|ds\Big)^{2p}\to 0,\quad r\to\infty.$$

By Lemma 4.1, we have

$$E\Big[\sup_{0\le t\le T}\|I_{3,r}(t)\|^{2p}\Big]\le CE\Big[\int_0^T\Big(\sum_{j=r+1}^{\infty}\|\sigma_j(\eta_r(s;h))\|^2\Big)^p ds\Big]\to 0,\quad r\to\infty.$$

It follows readily that

$$E\Big[\sup_{0\le t\le T}\|I_{4,r}(t)\|^{2p}\Big]\le CE\Big[\Big(\int_0^T\|\sum_{j=r+1}^{\infty}\sigma_j(\xi(s;h))\dot h^j(s)\|ds\Big)^{2p}\Big]$$
$$\to 0,\quad r\to\infty.$$

Therefore, from the Gronwall inequality, we complete the proof. □

PROPOSITION 7.10. *We have*

$$\xi(\,\cdot\,;h)\in\operatorname{supp}X(\cdot).$$

PROOF. From proposition 7.8 and proposition 7.9, we have

$$1=\overline{\lim}_{r\to\infty}P\big(\eta_r(\,\cdot\,;h)\in\operatorname{supp}X(\cdot)\big)\le P\big(\xi(\,\cdot\,;h)\in\operatorname{supp}X(\cdot)\big).\ \square$$

Proposition 7.10 implies $\mathcal{L}_\infty\subset\operatorname{supp}X(\cdot)$, and therefore, from Lemma 7.1, we get the following proposition.

PROPOSITION 7.11. $\bar{\mathcal{L}}\subset\operatorname{supp}X(\cdot)$.

From Proposition 7.4 and Proposition 7.11, we obtain Theorem 1.1.

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