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Vertical Vector Fields on Certain Complex Fibrations

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Abstract. The first part of this paper is devoted to constructing a certain special meromorphic vector field over a given complex fibration. We consider the case of ruled/elliptic fibrations in arbitrary dimensions and the case of hyperelliptic fibrations in dimension 2. The second part of the paper uses the existence of these vector fields to classify the singularities of a hyperelliptic fibration and, in particular, of a fibration whose generic fiber has genus 2. This classification will be employed in a continuation of this work to yield models for fibrations with genus 2. Our construction might also find applications to the study of elliptic threefolds.

1. Introduction

In this paper we consider certain complex Ordinary Differential Equations which are naturally associated to singular fibrations in Complex Algebraic Geometry. An introductory and simplified version of these ideas appeared in [Re3]. Here those methods will begin to be expanded. In particular, we shall consider *higher genus fibrations* as well as elliptic fibrations in *higher dimensional* complex manifold. Throughout the paper the expressions "Complex ODE" and "meromorphic vector field" are used as synonymous.

In the first part of the discussion, we shall provide statements regarding the existence of complex ODEs naturally associated to the geometry of complex manifolds as above and having special properties. These ODEs, and mainly their singularities, admit an accurate description which convey significative information on the geometry of the corresponding manifolds. Several consequences of the classification of these singularities will be exploited in subsequent papers. Thus the first half of the present paper is basically intended to set up a method of studying certain geometric questions whose applications will mostly be detailed in its continuations. The

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model we should bear in mind to understand our strategy is the study developed in [Re3] which can be summarized as follows: on elliptic (or ruled) surfaces, there is a natural complex ODE (i.e. a meromorphic vector field) which has a remarkable dynamical property called *semi-completeness*. The classification of the singularities of these vector fields leads to a generalization, in terms of differential equations, of the well-known Kodaira's picture for elliptic surfaces. This generalization makes sense whether or not the ambient manifold is compact and also applies to situations in which the orbits (i.e. "fibers") are $\mathbb{C}$ or $\mathbb{C}^*$ and hence non-compact. When applied to elliptic surfaces, our methods quickly give the description of the neighborhoods of the singular fibers which, as is well-known, has several consequences on the structure of the surface itself.

Motivated by this principle, in this paper we introduce a higher dimensional version of these vector fields which exist on elliptic (ruled) manifolds (cf. Section 3 and Theorem (3.3)). These vector fields are gentle enough to allow for a classification of their singularities when the dimension is 3. This classification will then find applications to the study of, say, elliptic threefolds. Along very similar lines, it is also observed that semi-complete vector fields exist on a certain type of $K3$-fibrations over Riemann surfaces (this happens essentially when the typical fiber has itself an elliptic fibration, cf. Theorem (4.3)).

In the second part of this paper we consider (complex) algebraic surfaces. Up to birational transformations, we think of these surfaces as carrying singular fibrations (cf. Definition (3.1)). As to algebraic surfaces, very little of an explicit nature is known for surfaces of general type. This contrasts with the case of ruled or elliptic surfaces (i.e. fibrations of genus respectively equal to 0 and 1) for which a detailed picture is available. In particular, the structure of surfaces M which are realizable as fibrations of genus 2 is still unclear as far as I know. Focusing on the point of view of [Re3], we can still construct vector fields similar to those considered in [Re3] on these surfaces. The main difference here being the fact that the resulting vector field (equivalently ODE) is not semi-complete (cf. Theorem (3.3)).

To extend our approach to the surfaces in question, we are then led to introduce the notion of k-determined vector field which generalizes semi-complete ones. The integer $k \in \mathbb{N}^*$ measures, in a certain sense, the multivalence of the integral of the vector field with respect to the time parameter

($k = 1$ means it is semi-complete, cf. Definition (4.5)). Most of the relevant properties of semi-complete vector fields admits a convenient generalization to k-determined ones. The idea is then to construct vector fields as above having k as small as possible. Apart from fibrations of genus 0 and 1, the simplest case consists of *hyperelliptic fibrations* which turn out to admit a remarkable 2-determined vector field (cf. Proposition (4.6)). Some further examples of 2-determined vector fields are also provided in connection with some $K3$-surfaces. The rest of the work, namely Sections 5 and 6, is devoted to classifying the singularities of 2-determined vector fields appearing in connection with hyperelliptic fibrations (cf. Theorem (5.1)). At least in principle our method can be pushed forward to higher values of "k" but, of course, the number of possibilities increases significatively. The case $k = 3$ might still be treatable in details.

Let us state the main application of the results in the second part of this paper. First we consider a hyperelliptic fibration $M \overset{\mathcal{P}}{\longrightarrow} S$ on a complex surface M, where S is a Riemann surface. Consider a singular fiber $\mathcal{P}^{-1}(p)$ of $\mathcal{P}$ which is a singular analytic (algebraic) curve in M. As a consequence, $\mathcal{P}^{-1}(p)$ has itself finitely many singularities as an analytic curve (or, equivalently, singularities of the foliation defined by the level sets of $\mathcal{P}$). We shall divide these singularities into two classes which will be referred to as ramified singularities and non-ramified ones. The definitions of these classes will be made precise at the end of Section 4. Roughly speaking it goes as follows. On a surface equipped with a hyperelliptic fibration we have a special meromorphic function whose restriction to the generic fiber realizes it as a hyperelliptic Riemann surface. As one moves from fiber to fiber, the ramification points of the restrictions in questions describe a possibly disconnected analytic curve $\mathcal{R}$ on the surface M. A singularity q of $\mathcal{P}^{-1}(p)$ is said to be *ramified* if it belongs to $\mathcal{R}$, otherwise we say that q is non-ramified (for more details cf. Section 4).

Given two integers $k_1, k_2 \in \mathbb{Z}$, we denote by g.c.d.$\,(k_1, k_2)$ the greatest common divisor of k_1, k_2. In the present setting, Theorem (5.1) immediately implies the following result (for more background see Section 2):

Theorem A. *Let $M \overset{\mathcal{P}}{\longrightarrow} S$ be a hyperelliptic fibration on a complex surface M, where S is a Riemann surface. Denote by $\mathcal{F}$ the singular foliation given by the level curves of $\mathcal{P}$. If p is a non-ramified singularity of $\mathcal{F}$*

then either $\mathcal{F}$ has non-vanishing eigenvalues at p or, in appropriate coordinates, $\mathcal{F}$ is given as the level curves (equivalently, the orbits) of one of the following functions (equivalently, vector fields):

1- Family **a**.

a.1 $f = xy(x-y)$ *(or* $X = x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y$*).*

a.2 $f = x^6y^4(x-y)^2$ *(or* $X = x(3y-2x)\partial/\partial x + y(4x-3y)\partial/\partial y$*).*

a.3 $f = y^4(y-x^2)^2$ *(or* $X = (3y-2x^2)\partial/\partial x + 2xy\partial/\partial y$*).*

2- Family **b**. *Here we always have* $f = x^{k_1}y^{k_2}(x-y)^{k_3}$ *where* $k = k_1+k_2+k_3 = 2k_1$ *(or* $X = x[k_3y - k_2(x-y)]\partial/\partial x + y[k_1(x-y)+k_3x]\partial/\partial y$*). Besides*

b.1 g.c.d. $(k_2, k) =$ g.c.d. $(k_3, k) = 2$.

b.2 g.c.d. $(k_2, k) = 2$ *and* g.c.d. $(k_2, k) = 1$.

b.3 g.c.d. $(k_2, k) =$ g.c.d. $(k_2, k) = 1$.

3- Family **c**. *With the same notations of item* **b**, *we have:*

c.1 $P = x^4y^3(x-y)^5$ *(or* $X = x(8y-3x)\partial/\partial x + y(9x-4y)\partial/\partial y$*).*

c.2 $P = x^8y^3(x-y)$ *(or* $X = x(4y-3x)\partial/\partial x + y(9x-8y)\partial/\partial y$*).*

c.3 $P = x^{10}y^6(x-y)^{14}$ *(or* $X = x(10y-3x)\partial/\partial x + y(12x-5y)\partial/\partial y$*).*

c.4 $P = x^{10}y^{12}(x-y)^8$ *(or* $X = x(10y-6x)\partial/\partial x + y(9x-5y)\partial/\partial y$*).*

c.5 $P = x^{10}y^{18}(x-y)^2$ *(or* $X = x(10y-9x)\partial/\partial x + y(6x-5y)\partial/\partial y$*).*

c.6 $P = x^{20}y^6(x-y)^4$ *(or* $X = x(5y-3x)\partial/\partial x + y(12x-10y)\partial/\partial y$*).*

c.7 $P = x^4y^3(x^2-y)$ *(or* $X = x(4y-3x^2)\partial/\partial x + y(6x^2-4y)\partial/\partial y$*).*

c.8 $P = x^6y^{10}(x^2-y)^2$ *(or* $X = x(12y-10x^2)\partial/\partial x + y(10x^2-6y)\partial/\partial y$*).*

c.9 $P = x^{10}y^6(x^2-y)^4$ *(or* $X = x(10y-6x^2)\partial/\partial x + y(18x^2-10y)\partial/\partial y$*).*

c.10 $P = y^3(x^3-y)$ *(or* $X = (3y-3x^3+yx^3)\partial/\partial x + 3yx^2\partial/\partial y$*).*

c.11 $P = x^6(x^2-y^3)^2$ *(or* $X = 6xy^2\partial/\partial x + (10x^2-6y^3)\partial/\partial y$*).*

c.12 $P = y^6(x^3-y)^4$ *(or* $X = (10y-6x^3)\partial/\partial x + 12yx^2\partial/\partial y$*).*

c.13 $P = x^3-y^4$ *(or* $X = 4y^3\partial/\partial x + 3x^2\partial/\partial y$*).*

c.14 $P = (x^5-y^3)^2$ *(or* $X = 3y^2\partial/\partial x + 5x^4\partial/\partial y$*).*

We also observe that all the models above are realizable by hyperelliptic fibrations as it follows from our proof. Indeed, note first that they are defined on the entire $\mathbb{C}^2$ and hence induce a singular foliation on $\mathbb{CP}(2)$.

Modulo multiplying the vector fields associated to these models by an appropriate polynomial (obtained through the proof of Proposition (5.4)), it is sufficient to perform a sequence of blow-ups in the line at infinity of $\mathbb{CP}(2)$ in order to turn the above mentioned foliation into a fibration on a suitable surface.

Our method also applies to ramified singularities of hyperelliptic fibrations of arbitrary genus. To avoid duplicate the work carried out to prove Theorem A, we just give a classification of those for genus less than or equal to 3 which is almost a by-product of the proof of Theorem A. This classification is the contents of Theorem B below.

THEOREM B. *Let $M \stackrel{\mathcal{P}}{\longrightarrow} S$ and $\mathcal{F}$ be as in Theorem A. If p is a ramified singularity of $\mathcal{F}$, then either $\mathcal{F}$ has non-vanishing eigenvalues at p or, in appropriate coordinates, $\mathcal{F}$ is given as the level curves of one of the following functions:*

- **a.1** $xy(x-y)$.
- **c.1** $x^4y^3(x-y)^5$, **c.2** $x^8y^3(x-y)$, **c.7** $x^4y^3(x^2-y)$, **c.10** $y^3(x^3-y)$ *or* **c.13** x^3-y^4.
- **b.1** $x^{k_1}y^{k_2}(x-y)^{k_3}$ *with* $k_1 = k_2+k_3$, $k_1 \leq 8$ *and* $\text{g.c.d}\,(k_2, 2k_1) = \text{g.c.d}\,(k_3, 2k_1) = 2$.
- **b.2** $x^{k_1}y^{k_2}(x-y)^{k_3}$ *with* $k_1 = k_2+k_3$, $k_1 \leq 7$, $\text{g.c.d}\,(k_2, 2k_1) = 2$ *and* $\text{g.c.d}\,(k_3, 2k_1) = 1$.
- **b.3** $x^{k_1}y^{k_2}(x-y)^{k_3}$ *with* $k_1 = k_2+k_3$, $k_1 \leq 6$ *and* $\text{g.c.d}\,(k_2, 2k_1) = \text{g.c.d}\,(k_3, 2k_1) = 1$.
- $xy(x-y)(x-\tau y)$, $\tau \in \mathbb{C}$, $\tau \notin \{0, 1\}$.
- $x^4y(x-y)$.
- $x^{k_1}y^{k_2}(x-y)^{k_3}$ *with* $k_1+k_2+k_3 = 5$ *or* 7.
- $y^2(x^2-y)^5$, $y^3(x^2-y)$, $y^4(x^2-y)$, $y^5(x^2-y)$, $y^6(x^2-y)$.
- $x^3y(x-y)$, $x^5y(x-y)$, $x^4y^2(x-y)$.
- $x^3y(x^2-y)$, $x^4y^2(x^2-y)$.
- $y(x^2-y^3)$, $y(x^2-y^3)^2$, $y^2(x^2-y^3)$, $y^3(x^2-y^3)$, $y^4(x^2-y^3)$.
- x^2-y^5, x^2-y^7.
- $x^2(y^3-x^2)^2(x^3-y^2)$, $xy(x^3-y)$.
- $y(x^2-y^5)$, $y^2(x^2-y^5)$.

- $(x^2 - y^5)(y^2 - x^3)$, $(x^3 - y^2)(x^2 - y^5)^2$.

As corollaries of the above theorems we conclude that the singular fiber of a hyperelliptic fibration $M \xrightarrow{\mathcal{P}} S$ as in the statement can have "at worst" a *non-ramified* reducible singularity consisting of a "triple" point (i.e. the mutually transverse intersection of three smooth curves). As to irreducible (unramified) singularities, the most degenerate singularities that can be obtained are the standard cusp $\{x^3 - y^2 = 0\}$ or the singularities of $\{x^3 - y^4 = 0\}$ and $\{x^5 - y^3 = 0\}$. Since compact Riemann surfaces of genus 2 are hyperelliptic, the above results immediately apply to every fibration having genus 2. Indeed, by considering the genus of the leaves of the foliations listed in Theorems A and B, we can determine which ones may appear in a fibration by Riemann surfaces of genus 2. Since one of the applications of the present work will be to the study of genus 2 fibrations (cf. [R-S]), we list all the possible singularities of these fibrations in Corollary C.

Corollary C. *Let $M \xrightarrow{\mathcal{P}} S$ be fibration on a complex surface M, where S is a Riemann surface. Assume that the genus of the typical fiber of $\mathcal{P}$ is 2. Denote by $\mathcal{F}$ the singular foliation given by the level curves of $\mathcal{P}$. If p is a singularity of $\mathcal{F}$ where the eigenvalues of $\mathcal{F}$ vanish, then $\mathcal{F}$ admits on a neighborhood of p one of the following normal forms:*

- $x(x - 2y)\partial/\partial x + y(y - 2x)\partial/\partial y$.
- $x(x - 3y)\partial/\partial x + y(y - 3x)\partial/\partial y$.
- $x(2x - 5y)\partial/\partial x + y(y - 4x)\partial/\partial y$.
- $(2y - x^2)\partial/\partial x + 2xy\partial/\partial y$.
- $(3y - x^2)\partial/\partial x + 4xy\partial/\partial y$.
- $2y\partial/\partial x - 3x^2\partial/\partial y$.
- $x(5y - 2x)\partial/\partial x + y(8x - 5y)\partial/\partial y$.
- $x(5y - 4x)\partial/\partial x + y(6x - 5y)\partial/\partial y$.
- $x(4y - 3x)\partial/\partial x + y(5x - 4y)\partial/\partial y$.
- $x(k_3 y - k_2(x - y))\partial/\partial x + y(k_3 x + k_1(x - y))\partial/\partial y$ *where* $k_1 + k_2 + k_3 = 5$.
- $(2xy - x^3)\partial/\partial x + (3x^2 y - y^2)\partial/\partial y$.

It is worth noticing that Ogg [O] has listed all the possible singular fibers of a fibration of genus 2. The main purpose of [R-S] will be to describe in details the structure of the neighborhood of these singular fibers so as to enable the reconstruction of the entire surface by putting together such neighborhoods with the remaining regular fibration. Corollary C already shows that the model 34 of [O] cannot be realized hence answering a question in that paper. Note also that the first six normal forms already exist in the case of elliptic fibrations.

According to the construction carried out in Section 4, fibrations of genus 3 necessarily carry a natural 3-determined vector field with appropriate additional properties analogous to those of the 2-determined vector fields considered in Sections 5 and 6 (cf. Section 5). The classification of the corresponding singularities can still be worked out in details with little further difficulty. The corresponding results might then be applied to study fibrations of genus 3 as well.

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2. Singular Foliations and Differential Forms

In this section, we shall recall the basic notions involved in our approach. We also recall some background material needed in other parts of the present work. The general principle guiding our investigations is the possible applications of semi-complete vector fields, as well as more general classes of vector fields, to the structure of neighborhoods of singular fibers.

A holomorphic (resp. meromorphic) vector field X on an open set $U \subseteq \mathbb{C}^n$ is given by

$$X = f_1 \frac{\partial}{\partial z_1} + \cdots + f_n \frac{\partial}{\partial z_n} \, ,$$

where the f_i's are holomorphic (resp. meromorphic) functions defined on U.

Given a complex manifold M, a singular 1-dimensional holomorphic foliation $\mathcal{F}$ defined on M consists of an atlas (V_i, ψ_i) together with non-trivial holomorphic vector fields X_i defined on $\psi(V_i) \subseteq \mathbb{C}^n$ satisfying the following: whenever $V_i \cap V_j \neq \emptyset$, the vector field $(\psi_j \circ \psi_i^{-1})_* X_i$ defined on $\psi_j(V_i \cap V_j)$ is a multiple of the restriction of X_j to the indicated set. In other words, in $\psi(V_i \cap V_j)$ one has $(\psi_j \circ \psi_i^{-1})_* X_i = \mathbf{h}_{ij} X_j$ for a non-trivial holomorphic function $\mathbf{h}_{ij}$ defined on $\psi(V_i \cap V_j)$.

From the definition above, one easily sees that the singular set of a 1-dimensional holomorphic foliation $\mathcal{F}$ can always be supposed to have codimension at least 2. It also follows that there is no interest in considering "meromorphic 1-dimensional foliations". In particular, if X is a *meromorphic* vector field defined on M, then the local orbits of X naturally defines a singular *holomorphic* foliation $\mathcal{F}$ on M. Such foliation is called the foliation *associated* to X. A (singular) holomorphic foliation of dimension 1 is also called a (singular) holomorphic foliation by *curves* (or by Riemann surfaces). All foliations considered in this work have complex dimension 1.

Consider a meromorphic vector field $X = f_1 \partial/\partial z_1 + \cdots + f_n \partial/\partial z_n$ defined on an open set U. The meromorphic functions f_i's may not be defined on the whole of U even though we consider ∞ as a value. Indeed they may also have *indeterminacy points* (thus saying that X is defined on U is a traditional abuse of language which fortunately does not lead to any misunderstood). Letting $f_i = g_i/h_i$, $i = 1, \ldots, n$, we denote by D_X the union of the sets $\{h_i = 0\}$. Of course D_X is a divisor consisting of poles and indeterminacy points of X.

Recall the following definition (see [Re3]).

Definition 2.1. The meromorphic vector field X is said to be semi-complete on U if and only if there exists a semi-global flow Φ_{sg} associated to X, i.e. a meromorphic map $\Phi_{sg} : \Omega \subseteq \mathbb{C} \times U \to U$, where Ω is an open set of $\mathbb{C} \times U$, satisfying the conditions below.

1. $$\left.\frac{d\Phi_{sg}(T,x)}{dT}\right|_{T=0} = X(x) \text{ for all } x \in U \setminus D_X;$$

2. $\Phi_{sg}(T_1 + T_2, x) = \Phi_{sg}(T_1, \Phi_{sg}(T_2, x))$ provided that both members are defined;

3. If (T_i, x) is a sequence of points in Ω converging to a point $(\hat{T}, x)$ in the boundary of Ω, then $\Phi_{sg}(T_i, x)$ converges to the boundary of $U \setminus D_X$ in the sense that the sequence leaves every compact subset of $U \setminus D_X$.

Clearly a vector field semi-complete on U is naturally semi-complete on any open set $V \subset U$.

Example 2.2. Consider the one-dimensional vector field $X = z^3 \partial / \partial z$ on $\mathbb{C}$. The orbits of X are $\{0\}$ and $\mathbb{C}^*$. If ϕ denotes the local flow of X, then we have

$$\phi(T) = \frac{x_0}{\sqrt{1 - 2Tx_0^2}} \quad \text{where} \quad x_0 = \phi(0) \; .$$

We choose $x_0 \in \mathbb{R}_+$, a "small" disc $B(\epsilon)$ around $0 \in \mathbb{C}$ of radius $\epsilon > 0$ and a "large" disc $B(r)$ centered at $0 \in \mathbb{C}$ and having radius $r > 1 + 1/2x_0^2$. Let $U \subset \mathbb{C}$ be the domain of the times T such that $1 - 2Tx_0^2$ belongs to $B(r) \setminus B(\epsilon)$. Observe that ϕ is not well defined on the whole U since the square root is not defined on a closed path going around $0 \in \mathbb{C}$.

On the other hand, let $V \subset U$ be a (maximal) domain where ϕ is well defined. By construction ϕ is uniformly bounded on V since the denominator of ϕ remains away from *zero*. Therefore if we choose a sequence $\{T_i\} \subset V$ converging to a point $\hat{T}$ in the boundary of V, $\phi(T_i)$ remains in a compact part of $\mathbb{C}$. Now the maximality of V ensures that X is not semi-complete on V and, hence, neither on U.

Example 2.3. Note that the situation discussed in Example (2.2) contrasts with the case of the vector field $X = z^2 \partial / \partial z$ which is semi-complete on $\mathbb{C}$. In fact, the local flow ϕ of X is given by

$$\phi(T) = \frac{x_0}{1 - x_o T} \quad \text{where} \quad x_0 = \phi(0) \; .$$

Clearly $\phi(T)$ is defined on $\mathbb{C} \setminus \{1/x_0\}$. Besides when $T_i \to 1/x_0$, $\phi(T_i)$ becomes unbounded and therefore leaves any compact set in $\mathbb{C}$.

Every local meromorphic vector field X can be written as $X = fY/g$ where Y is a holomorphic vector field having only isolated singularities and f, g are holomorphic functions. Furthermore f, g, Y are unique up to an invertible factor. The local orbits of Y define a singular holomorphic foliation $\mathcal{F}$ which is nothing but the foliation associated to X. We say that a holomorphic foliation $\mathcal{F}$ has eigenvalues λ_1, λ_2 at a singularity $(0,0)$ if there exists a vector field Y as above having λ_1, λ_2 as eigenvalues at $(0,0)$. Note that the eigenvalues of $\mathcal{F}$ are well-defined only up to a multiplicative constant. More generally we say that the order of $\mathcal{F}$ at $(0,0)$ is $k \in \mathbb{N}$ if the order of Y as above at $(0,0)$ is k (recall that the order of Y is the degree of the first non-vanishing jet of Y at $(0,0)$). Clearly this notion of order is well-defined.

Now we consider a meromorphic vector field X defined on a complex manifold M and denote by $\mathcal{F}$ the associated singular foliation. The foliation $\mathcal{F}$ is naturally endowed with a "foliated" meromorphic 1-form (i.e. an Abelian form) dT induced by the vector field X. Precisely, if L is a leaf of $\mathcal{F}$ that is not reduced to a singular point, then the restriction dT_L of dT to L satisfies $dT_L.X_{|L} = 1$ where $X_{|L}$ stands for the restriction of X to L. Such form dT_L is said to be the *time-form* induced by X on L. The following simple lemma is of fundamental importance for the theory of semi-complete vector fields (cf. [Re1], [Re3]).

Lemma 2.4. ([Re1]) *Assume that X is a meromorphic semi-complete vector field defined on an open set U. Denote by $\mathcal{F}$ the associated foliation and by dT the induced time-form. Let L be a regular leaf of $\mathcal{F}$ (i.e. a leaf that is not reduced to a singular point) and consider an (open) embedded curve $c : [0,1] \to L$. Then the integral of dT_L on this curve is different from zero, where dT_L stands for the restriction of dT to L.*

Example 2.5. We can now revisite the vector fields $z^3\partial/\partial z$ and $z^2\partial/\partial z$. The corresponding time-forms induced on $\mathbb{C}^*$ are dz/z^3 and dz/z^2. Letting $c : [0,1] \to \mathbb{C}^*$ be a path contained in $\mathbb{C}^*$, one has

$$\int_c \frac{dz}{z^3} = \frac{-1}{2}\left(\frac{1}{c(1)^2} - \frac{1}{c(0)^2}\right) \quad \text{and} \quad \int_c \frac{dz}{z^2} = -\left(\frac{1}{c(1)} - \frac{1}{c(0)}\right) .$$

The first integral vanishes on open curves (for example such that $c(0) = \epsilon$ and $c(1) = -\epsilon$). Thus the corresponding vector field is not semi-complete. However, if the second integral vanishes, then one has $c(0) = c(1)$ so that c is not open what is in line with the fact that $z^2\partial/\partial z$ is semi-complete.

To finish the section we shall state some basic results on (meromorphic) semi-complete vector fields. Once the notion of k-determined vector field will have been introduced, we shall see that the propositions below can easily be generalized to encompass this new class of vector fields. Proofs of the following facts are mostly elementary and can be found in [G-R] or [Re3]. The first proposition and its consequences will largely be used throughout this paper. Fix an open domain $U \subseteq \mathbb{C}^n$ and suppose that we are given a sequence of meromorphic semi-complete vector fields $\{X_i\}_{i \in \mathbb{N}}$ on U.

PROPOSITION 2.6. ([G-R]) *Assume that $\{X_i\}$ and U are as before. Suppose that the pole divisor $\mathcal{D}_i$ of X_i converges in the Haussdorff topology to some divisor $\mathcal{D}$. Suppose in addition that the order of the poles of $\{X_i\}$ is uniformly bounded and that $\{X_i\}$ converges on compact sets of $U \setminus \mathcal{D}$ towards a vector field X. Then X is a meromorphic semi-complete vector field on U.*

A rather useful consequence of Proposition (2.6) is the following statement.

COROLLARY 2.7. *Semi-complete vector fields are stable under restrictions to Zariski-open sets in the following sense: Assume that Z is a Zariski-closed (analytic) subset of a complex manifold M. A meromorphic vector field X on M is semi-complete on $M \setminus Z$ if and only if it is semi-complete on the entire M. In particular the semi-complete character of a vector field is invariant by birational transformations (i.e. the pull-back of a semi-complete vector field by a birational map is still semi-complete).*

Consider a meromorphic vector field $X = gY/h$ where Y is a holomorphic vector field whose singular set has codimension at least 2 and g, h are holomorphic functions. Denote by Y^k (resp. g^r, h^s) the first non-trivial homogeneous component of the Taylor series of Y (resp. g, h) centered at $(0,0) \in \mathbb{C}^2$ whose degree is supposed to be $k \in \mathbb{N}$ (resp. $s, r \in \mathbb{N}$). It means that k (resp. s, r) is the order of Y (resp. g, h) at $(0,0)$. The vector field $X^{\rm ho} = g^r Y^k / h^s$ will be called the *first homogeneous component of X*.

COROLLARY 2.8. *Assume that X as above is semi-complete on a neighborhood of the origin of $\mathbb{C}^n$. Then $X^{\rm ho}$ is semi-complete on the whole $\mathbb{C}^n$.*

REMARK 2.9. It is useful to point out the non-existence of a strictly meromorphic semi-complete singularity in complex dimension 1. For example the vector field $x^{-k}\partial/\partial x$ defined on $\mathbb{C}^*$ is not semi-complete around $0 \in \mathbb{C}$ for every $k \in \mathbb{N}^*$. The general case of a vector field $f\partial/\partial x$ where f has a pole at $0 \in \mathbb{C}$, can easily be derived from this fact (formally we can also apply Corollary (2.8)).

3. Fibrations and Vertical Vector Fields

In this section we are going to consider fibrations on *complex manifolds* and generalize the set up briefly considered in [Re3]. To begin with, let us consider two compact complex manifolds M^n, N^{n-1} of dimensions respectively equal to n and $n-1$ ($n \geq 2$). Whenever there is no possibility of misunderstood, we shall omit the superscripts n, $n-1$. In analogy with the two-dimensional case, we settle the following notation.

Definition 3.1. A fibration of genus g on M is a holomorphic map $\mathcal{P} : M \to N$ such that

($\imath$) $\mathcal{P}^{-1}(p)$ is a compact Riemann surface of genus g provided that p belongs to the complement of a (proper) analytic subvariety S of N.

($\imath\imath$) $\mathcal{P}$ defines a regular fibration from $M \setminus \mathcal{P}^{-1}(S)$ to $N \setminus S$.

Thus, when $n-1=1$, S is a divisor consisting of a finite number of points. The subvariety S is said to be the *singular locus* of $\mathcal{P}$. Given a point $p \in S$, the preimage $\mathcal{P}^{-1}(p)$ is called a singular fiber. Finally, if $S_i \subset S$ stands for a connected component of S, then the set $\mathcal{P}^{-1}(S_i)$ is said to be a (connected) pencil of singular fibers.

We say that M is ruled (resp. elliptic) if M carries a fibration of genus *zero* (resp. one). As mentioned, we want to construct a natural meromorphic vector field on M which is related to the triple $(M, N, \mathcal{P})$.

Let K_M (resp. K_N) be the Canonical Line Bundle of M (resp. N). This means that K_M (resp. K_N) is a line bundle whose meromorphic sections are meromorphic (non-degenerate) n-forms on M (resp. $(n-1)$-forms on N). In other words, if η is a (meromorphic) section of K_M and $(z_1, \ldots, z_n)$ are local coordinates in M, η is locally given as $f(z_1, \ldots, z_n) dz_1 \wedge \ldots \wedge dz_n$ for a meromorphic function f.

Given a fibration $\mathcal{P} : M \to N$ on M, we consider meromorphic sections θ, ϑ of K_M, K_N respectively. Let us also choose an atlas (V_i, ψ_i) for N which is compatible with ϑ. Let $W_i \subseteq \mathbb{C}^{n-1}$ be the image of V_i, that is $W_i = \psi_i(V_i)$ and denote by S the singular locus of $\mathcal{P}$. Next fix i and $\psi_i : V_i \to W_i \subset \mathbb{C}^{n-1}$. Let $U_i \subset M$ be the open set $U_i = \mathcal{P}^{-1}(V_i)$ and consider the composition map $\psi_{i,\mathcal{P}} = \psi_i \circ \mathcal{P} : U_i \to W_i \subset \mathbb{C}^{n-1}$. Using the coordinates of $\mathbb{C}^{n-1}$, we write $\psi_{i,\mathcal{P}}$ as $(\psi^1_{i,\mathcal{P}}, \dots, \psi^{n-1}_{i,\mathcal{P}})$. Finally with the preceding notations we define a vector field X_i on U_i by imposing

$$\eta_p(X_i(p),\ .\ ) = d_p\psi^1_{i,\mathcal{P}} \wedge \dots \wedge d_p\psi^{n-1}_{i,\mathcal{P}} = \vartheta \circ d_p\mathcal{P}\,, \tag{1}$$

whenever both sides are defined. In the above equation $d_p\psi^r_{i,\mathcal{P}}$ (resp. $d_p\mathcal{P}$) stands for the differential of $\psi^r_{i,\mathcal{P}}$ (resp. $\mathcal{P}$), $r = 1, \dots, n-1$, at $p \in U_i \subset M$.

Clearly the vector fields X_i agree on their intersections so that they give rise to a globally defined vector field X on M since (V_i, ψ_i) is compatible with the section ϑ of the Canonical Line Bundle of N. We call X the *vertical vector field* of $\mathcal{P}$ w.r.t η, ϑ. Nonetheless it is sometimes more confortable to work with the vector fields X_i so as to automatically have an appropriate local notion which is convenient for calculations. The following facts are immediate consequences of the above definition.

Fact 1. X_i is a meromorphic vector field on U_i (and so is X on M).

Fact 2. X_i is tangent to the fibers of $\mathcal{P}$. More precisely the singular holomorphic foliation $\mathcal{F}_i$ associated to X_i in the sense of Section 2 is nothing but (the restriction to U_i of) the fibration defined by $\mathcal{P}$.

Fact 3. Each of the $n-1$ holomorphic functions $\psi^r_{i,\mathcal{P}}$ is a first integral for X in the sense that they are constant on the orbits of X_i.

Fact 4. The divisor of zeros of X is contained in the union $D_{\text{pole}}(\eta) \cup \mathcal{P}^{-1}(V_i \cap S) \cup \mathcal{P}^{-1}(D_{\text{zero}}(\vartheta))$ where $D_{\text{pole}}(\eta)$, $D_{\text{zero}}(\vartheta)$ stands respectively for the divisor of *poles* of η and the divisor of poles of ϑ.

Fact 5. The divisor of poles of X is contained in the union $D_{\text{zero}}(\eta) \cup \mathcal{P}^{-1}(V_i \cap S) \cup \mathcal{P}^{-1}(D_{\text{pole}}(\vartheta))$ where $D_{\text{zero}}(\eta)$, $D_{\text{pole}}(\vartheta)$ stands respectively for the divisor of *poles* of η and the divisor of poles of ϑ.

Remark 3.2. In local problems we sometimes do not need the fact that X is globally defined. Indeed, it is often sufficient to work with the "local" vector field X_i defined on U_i. One advantage of doing so is that we may search for a ϑ which is regular on $V_i = \mathcal{P}(U_i)$. When such choice can effectively be made, then the divisor of zeros of X_i is contained in $D_{\text{pole}}(\eta) \cup \mathcal{P}^{-1}(V_i \cap S)$. Similarly the divisor of poles of X_i is contained in $D_{\text{zero}}(\eta) \cup \mathcal{P}^{-1}(V_i \cap S)$. In particular, the *poles* of η which are not contained in $\mathcal{P}^{-1}(S)$

correspond to *zeros* of X_i while the *zeros* of η lying in the complement of $\mathcal{P}^{-1}(S)$ correspond to *poles* of X_i.

It is natural to ask when it is possible to find η, ϑ as before such that X, defined as in (1), is semi-complete. The answer to this question is the contents of Theorem (3.3).

Theorem 3.3. *Assume that X is semi-complete. Then $\mathcal{P} : M \to N$ is a fibration of genus 0 or 1 on M. Conversely if $\mathcal{P} : M \to N$ defines a fibration of genus 0 or 1, then there are η, ϑ such that the vector field X resulting from (1) is semi-complete.*

Lemma 3.4. *Assume that the vector field X defined by $\mathcal{P}, \eta, \vartheta$ as in (1) is semi-complete. Then $\mathcal{P}$ defines a fibration of genus 0 or 1.*

Proof. Suppose for a contradiction that $\mathcal{P}$ defines a fibration of genus $\mathsf{g} \geq 2$. Consider the vector field X in question as well as a (generic) regular fiber L of $\mathcal{P}$ which is invariant by X. The fact that L is generic precisely means that the restriction of X to L is a non-trivial meromorphic vector field on L and that ϑ is regular at $q = \mathcal{P}(L) \in N$. The set of leaves L satisfying these condition is clearly open and dense in U (it is "Zariski-open").

Let dT_L denote the time-form induced on L by X. Recall that *zeros* (resp. poles) of X become *poles* (resp. zeros) of dT_L and vice-versa. Since dT_L is a meromorphic 1-form on L and the genus of L is greater than 1, it follows from Poincaré-Hopf theorem that dT_L has at least one *zero* $p \in L$. Equivalently, p is a pole of X in L. However we know that an one-dimensional meromorphic vector field is never semi-complete (cf. Remark (2.9)). It follows that the restriction of X to L is not semi-complete so that X itself is not semi-complete. The resulting contradicton proves the lemma. □

The next lemma is also simple and probably well-known to the experts.

Lemma 3.5. *Let $\mathcal{P} : M \to N$ be a fibration of genus 0 or 1. Then there is a section η of K_M which does not have zeros on the generic fibers of $\mathcal{P}$ (i.e. all the zeros of η are contained in the "singular fibers").*

Proof. We shall just sketch the proof. Let us follow the notations of [G-H] and argument by induction on the dimension of M. If D is a submanifold of M we denote by Ω^i_D the sheaf of i-forms on D. Besides $\Omega^i(M)$ denotes the sheaf of 1-forms on M and $\Omega^i_M(D)$ will denote the sheaf of i-forms with a single pole along D.

Suppose first that the dimension of M is 2. We choose $k \in \mathbb{N}$ generic fibers $L_1, \ldots, L_k$ of $\mathcal{P}$ and note that the Poincaré residue map yields the exact sequence

$$0 \to \Omega^2_M \to \Omega^2_M(L_1 + \cdots L_k) \to \oplus\, \Omega^1_{L_j} \to 0\,.$$

Passing to the exact sequence in cohomology, we obtain

$$H^0(M, \Omega^2_M(\oplus\, L_j)) \to H^0(\oplus\, L_j, \Omega^1_{\oplus\, L_j}) \to H^1(M, \Omega^2)\,.$$

If the fibers have genus 1, then $H^0(\oplus\, L_j, \Omega^1_{\oplus\, L_j}) \simeq \mathbb{C}^k$. Since the codimension of the image of $H^0(M, \Omega^2_M(\oplus\, L_j))$ in $H^0(\oplus\, L_j, \Omega^1_{\oplus\, L_j})$ is bounded by the dimension of $H^1(M, \Omega^2)$, it results that the dimension of the sections of the bundle associated to the divisor $K_M + \oplus\, L_j$ increases linearly with k. Hence, for k sufficiently large, $K_M + \oplus K_j$ is linearly equivalent to an effective divisor. Let us fix a section η of this effective divisor. On the other hand, in this context, the Adjunction Formula states that

$$-e(L) = K_M.L + L.L\ ,$$

where L is a regular fiber of $\mathcal{P}$, $e(L)$ stands for the Euler characteristic of L and $L.L$ for the self-intersection of L. Since L is a regular fiber the self-intersection $L.L$ of L necessarily vanishes. The Euler characteristic $e(L)$ also vanishes since the genus of L is 1. Thus we have $K_M.L = 0$. Because K_M is represented by the effective divisor given by zeros of η, we conclude that this divisor is contained in fibers and irreducible components of singular fibers. As an immediate consequence η does not have *zeros* when restricted to a generic regular fiber L.

When the genus of $\mathcal{P}$ is *zero* the same argument applies modulo replacing K_M by $-K_M$ and L_j by $-L_j$.

Finally for the induction step, we consider appropriate divisors D_j in N and replace the fibers L_j by the divisors $\mathcal{P}^{-1}(D_j)$. These divisors exist since they may be chosen to be linearly equivalent to the divisor defined by the singular locus of $\mathcal{P}$. Note also that all these divisors may be supposed to be smooth modulo perform a desingularization. Suppose again that the genus of $\mathcal{P}$ is 1. By the induction assumption, $\mathcal{P}^{-1}(D_j)$ admits a non-trivial vector space consisting of sections whose set of *zeros* is contained in set of (possibly singular) fibers of M having codimension at least 2 in M. Therefore for the Poincaré map the Canonical Line Bundle of these divisors may be considered as admitting non-trivial sections. Thus it is enough to repeat the argument above. Again if the genus is 0, we consider the negative of the Canonical Line Bundle. The proof of the lemma is over. □

Proof of Theorem (3.3). Suppose now that $\mathcal{P}$ defines a fibration on M with (generic) fibers of genus 0 or 1. After Lemma (3.4), we just need to find meromorphic sections η, ϑ respectively of K_M, K_N such that the corresponding vertical vector field X of (1) is semi-complete. We then fix a section ϑ of K_N and a section η of K_M with η satisfying the conclusions of Lemma (3.5). Using ϑ, η and $\mathcal{P}$ we then define a vertical vector field denoted by X. In the sequel we are going to prove that X is semi-complete.

Thanks to Facts 4 and 5, we know that X has no poles when restricted to a generic fiber L of $\mathcal{P}$. Recall that "generic" implies that the set of fibers L satisfying the condition in question is open and dense, in fact, it is Zariski-open, in M. Therefore the restriction of X to such fiber L naturally defines a holomorphic vector field $X_{|L}$ on L. Since L is compact, it follows that $X_{|L}$ is in fact complete i.e. $X_{|L}$ generates a global holomorphic flow on L. We then conclude that the restriction of X to a generic fiber L is complete (and therefore semi-complete). To prove that X itself is semi-complete (including "non-generic" fibers) we have to construct a semi-global flow associated to X on M. In order to do that, we denote by $\mathcal{D}$ the divisor of M consisting of the union of "non-generic fibers" of $\mathcal{P}$ (in the above sense). In particular $\mathcal{D}$ contains the divisor of poles of X. We then consider the mapping

$$\Phi_{sg,X} : \mathbb{C} \times (M \setminus \mathcal{D}) \longrightarrow M \setminus \mathcal{D}$$

defined by $\Phi_{sg,X}(T,p) = \Phi_{X|L_p,L_p}(T,p)$ where L_p stands for the leaf of $\mathcal{F}$ containing p and $\Phi_{X_{|L_p},L_p}$ is the flow induced on L_p by $X_{|L_p}$. It is obvious that $\Phi_{sg,X}$ fulfils all the conditions required to be a semi-global (or global) flow on $M \setminus \mathcal{D}$. Now Corollary (2.7) ensures that this flow extends to a semi-global flow on the whole of M. The theorem is proved. □

The primary interest of Theorem (3.3) results from the fact that a classification of semi-complete singularities, say in dimension 3, can be employed to study the structure of elliptic threefolds. Whereas the general classification of semi-complete singularities in dimension 3 seems to be very hard, Fact 3 ensures that the vertical vector field has 2 linearly independent first integrals. In other words, as far as applications to ruled and elliptic threefolds are concerned, we just need to classify semi-complete singularities possessing a pair of linearly independent first integrals. The last problem is very treatable and a detailed answer can effectively be given.

We want to point out that the section ϑ of K_N did not play any role in the proof of Theorem (3.3), other than the auxiliary role of defining X. The question involving the semi-complete character of X is therefore totally encoded in the choice of η. As a matter of fact the semi-complete character of X is all that is going to be used in our work and, in addition, this property remains valid if X is multiplied by a function constant along its orbits. All these arbitrary choices which does not affect the fact that X is semi-complete suggest that it might be useful to have a more intrinsic version of the above objects so as to dispense with these choices. The sheaf-theoretic language allows us to define an object which serves to our purposes and, in particular, dispenses with ϑ. The rest of the section is devoted to explaining this construction. We point out however that the material presented in the sequel will not be needed for the continuation of the paper.

To begin the construction, we drop the section ϑ and consider an atlas (V_i, ψ) for N which does not have to satisfy any extra condition. Repeating the procedure above we obtain local vector fields X_i which still satisfy the facts 1, 2, 3 and 4. However it is now necessary to examine the effect of a change of coordinates on X_i. Thus let $\psi_i : V_i \rightarrow W_i$ and $\psi_j : V_j \rightarrow W_j$, $V_i \cap V_j \neq \emptyset$, be two charts of N as before. Recall that the section η of the Canonical Line Bundle of M is fixed. First we observe that the well-known maximum principle easily yields the following lemma:

Lemma 3.6. *Let X_i, X_j be respectively the vector fields constructed by means of ψ_i and ψ_j. On $U_i \cap U_j$, we have $X_i = H_{ij} X_j$ where $H_{ij} : U_i \cap U_j \rightarrow \mathbb{C}$ is a nowhere vanishing holomorphic function. Moreover H_{ij} is constant on the fibers of $\mathcal{P}$ so that it factors as $H_{ij} = h_{ij} \circ \mathcal{P}$ where h_{ij} is defined on $V_i \cap V_j$.*

As mentioned, the preceding discussion can be interpreted in sheaf-theoretic terms as follows. Denote by $\mathcal{X}(\mathcal{M}, \mathcal{P})$ the sheaf of meromorphic vector fields on M which are tangent to the fibers of $\mathcal{P}$. Notice that N is naturally identified with the quotient $M/\stackrel{\mathcal{P}}{\sim}$ of M by the equivalence relation $\stackrel{\mathcal{P}}{\sim}$ which collapses points of M belonging to the same fiber of $\mathcal{P}$. Hence the sheaf $\mathcal{O}^*(N)$ consisting of (local) nowhere vanishing holomorphic functions on N can be identified with $\mathcal{O}^*(M/\stackrel{\mathcal{P}}{\sim})$. The space of global sections of the quotient sheaf $\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N)$ is going to be denoted by $\Gamma(\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N))$.

Given a (meromorphic) section η of the Canonical Line Bundle K_M of M, we have constructed a family of vector fields $\{X_i\}$ by means of Equation (1). After Lemma (3.6) this family of vector fields defines a section $\kappa(\eta)$ of $\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N)$. Hence, denoting by $\Gamma(\mathcal{M},K_M)$ the space of global meromorphic sections of K_M, we have proved the following:

Proposition 3.7. *There is a natural homomorphism*

$$\kappa\,:\,\Gamma(\mathcal{M},K_M)) \longrightarrow \Gamma(\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(M/\stackrel{\mathcal{P}}{\sim})) = \Gamma(\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N))\,. \tag{2}$$

Remark 3.8. The reader has certainly noticed that $\kappa(\eta_1)=\kappa(\eta_2)$ if and only if η_1,η_2 differ by a (nowhere vanishing) holomorphic function on M. Such function being necessarily constant, we conclude that κ induces an injection $\overline{\kappa}$ from $\Gamma(\mathcal{M},K_M))/\mathbb{C}^*$ to $\Gamma(\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N))$.

Given $\eta\in\Gamma(\mathcal{M},K_M))$, we say that $\kappa(\eta)$ is a *fibered field* on M (relative to $\mathcal{P}$). In practice one can work with fibered fields by choosing actual vector fields tangent to the fibers of $\mathcal{P}$ and representing the fibered field in question.

The next lemma clarifies the relations among all the objects involved in the our discussion. It shows that the choice of a section ϑ for the Canonical Line Bundle of N allows us to choose a "canonical representative" for a given fibered field which (up to a non-zero multiplicative constant) is a global meromorphic vector field X on M.

Lemma 3.9. *Assume we are given an element γ of $\Gamma(\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N))$. There exists $\eta\in\Gamma(\mathcal{M},K_M))$, unique up to multiplication by a non-zero constant, such that $\kappa(\eta)$ is equal to γ. Furthermore, choosing a* *section ϑ of the Canonical Line Bundle of N, Equation (1) provides, by means of η, ϑ, a meromorphic vector field X on M which is a representative of γ.*

PROOF. Notice that the section γ of $\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N)$ can be thought of as a collection consisting of the following data:

- A covering of N by coordinates charts $\{(V_i, \psi_i)\}$. We let $U_i = \mathcal{P}^{-1}(V_i)$.
- To each U_i as before, we have a (non-trivial) meromorphic vector field X_i tangent to the fibers of $\mathcal{P}$. Two vector fields $X_{i,1}$ and $X_{i,2}$ are equivalent if their quotient define a nowhere vanishing holomorphic function on U_i (which is necessarily constant on the fibers of $\mathcal{P}$).

To construct η such that $\kappa(\eta) = \gamma$, we fix a collection (U_i, X_i) representing γ. The nature of the sheaf $\mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(N) = \mathcal{X}(\mathcal{M},\mathcal{P})/\mathcal{O}^*(M/ \overset{\mathcal{P}}{\sim})$ allows to assume in addition that $X_i = X_j$ on $U_i \cap U_j$ since they differ by an element of $\mathcal{O}^*(U_i \cap U_j)$ which is constant on the fibers of $\mathcal{P}$. Now we fix $p \in U_i \cap U_j$ and let T_pL_p be the tangent line to the fiber L_p of $\mathcal{P}$ passing through p. Next we choose vectors $v_2, \ldots, v_n$ in T_pM so that $T_pL_p, v_2, \ldots, v_n$ spans T_pM. We then define η at p by letting

$$\eta(X_i(p), v_2, \ldots, v_n) = \eta(X_j(p), v_2, \ldots, v_n) = \vartheta(d_p\mathcal{P}(v_2), \ldots, d_p\mathcal{P}(v_n)),$$

since $X_i(p) = X_j(p)$ (where ϑ is a section of the Canonical Line Bundle of N arbitrarily chosen). It promptly follows that η is globally defined and satisfies $\kappa(\eta) = \gamma$. Since κ is well-defined and one-to-one up to a multiplicative constant, it follows that η does not depend on ϑ (up to a multiplicative constant). In other words $\overline{\kappa}$ is an isomorphism. Finally, the meromorphic vector field X arising from Equation (1) combined with η, ϑ is clearly a representative of γ. □

At this point it seems very natural to say that a fibered field is semi-complete if it is possible to find a collection (U_i, V_i, X_i), $V_i = \mathcal{P}(U_i)$, such that each X_i is semi-complete on its domain.

Note that the existence of a semi-complete representative (U_i, V_i, X_i) of γ implies that any other representative is semi-complete as well since two representatives differ by a holomorphic function which is constant on the fibers. Also, γ is semi-complete if and only if the choice of a section ϑ for the Canonical Line Bundle of N allows us to construct a meromorphic vector field X on M which is semi-complete and represents γ.

4. Further Extensions and Applications

Here we shall present a simple generalization of the vector fields considered in the previous section along with some further examples. Throughout this section all manifolds are assumed to be algebraic. We also want to pay special attention to the case of (algebraic) $K3$-surfaces. Recall that a $K3$-surface is a simply connected compact complex surface with trivial Canonical Line Bundle (notation: $K_M \equiv 0$).

Fix a nowhere zero holomorphic section η of K_M and consider a non-constant meromorphic function F on M. We can define a meromorphic vector field X on M by letting

$$\eta_p(X(p),\ .\ ) = d_pF \tag{3}$$

whenever both sides are defined. Thinking of F as a potential function, the vector field X is the Hamitonian associated to F, η. Alternate X can be viewed as a vertical vector field up to birational transformations. Indeed, modulo a finite number of appropriate blow-ups, we can assume that F has no indeterminacy points and therefore defines a fibration of M over $\mathbb{CP}(1)$ (possibly having disconnected fibers). In view of Theorem (3.3), X is semi-complete if and only if the fibration mentioned above has genus 0 or 1. Since $K_M \equiv 0$, this can be rephrased as follows.

Lemma 4.1. *Assume that M satisfies $K_M \equiv 0$. Then the following is equivalent:*

1. *F does not have indeterminacy points.*
2. *F defines an elliptic fibration on M.*
3. *X is semi-complete.*

Before proving this lemma, we need some further information on the nature of indeterminacy points of a meromorphic function. Denote by $\mathcal{F}$ the foliation associated to the vector field X of (3) which coincides with the foliation defined by the level curves of F. Next let p be an indeterminacy point of F. Then p is a singular point of $\mathcal{F}$ and, besides, there are infinitely many (germs of) analytic curves passing through p and invariant by $\mathcal{F}$. A germ of an analytic curve invariant by a foliation $\mathcal{F}$ and passing through a singular point of $\mathcal{F}$ is called a *separatrix.* A singular point of a holomorphic foliation $\mathcal{F}$ admitting infinitely many separatrizes is called *dicritical*. We shall use the following easy lemma (for more details cf. Section 5 and Lemma (5.2)):

LEMMA 4.2. *Let $\mathcal{F}$ be a singular holomorphic foliation defined around $(0,0) \in \mathbb{C}^2$. Assume that the origin is a dicritical singularity of $\mathcal{F}$. Then there is a sequence of blow-up maps starting at $(0,0)$ such that the corresponding exceptional divisor possesses an irreducible component D_{i_0} which is not invariant by the corresponding proper transform of the foliation $\mathcal{F}$. Besides all the blow-ups considered in the above sequence are centered at singular points of the corresponding proper transform of $\mathcal{F}$.*

PROOF OF LEMMA (4.1). Let M and F be as in the statement. Denote by $\mathcal{D}_F$ the divisor of poles of F and consider the vector field X. Since η is holomorphic, the pole divisor of X coincides with $\mathcal{D}_F$. Notice also that indeterminacy points of F are contained in $\mathcal{D}_F$ and that they coincide with the dicritical singularities of $\mathcal{F}$.

Assume that F has no indeterminacy points. Then a generic leaf of $\mathcal{F}$ does not intersect $\mathcal{D}_F$ and, in fact, does not contain singularities of $\mathcal{F}$ on its closure. In other words, a generic leaf L is a regular leaf of a non-singular foliation. Thus the self-intersection of L, $L.L$, vanishes. On the other hand we have $K_M.L = 0$ since K_M is trivial. However the Adjunction formula says that

$$-e(L) = K_M.L + L.L \ . \tag{4}$$

In the present case, it implies that $e(L) = 0$ thus showing that F defines an elliptic fibration.

We have seen that item 1 implies item 2. The fact that item 2 implies item 3 is a consequence of Proposition (3.3). Thus it remains to check that item 3 implies item 1.

Thus we assume for a contradiciton that X is semi-complete and that F has an indeterminacy point. Modulo performing blowing-ups at these indeterminacy points we obtain a new surface $\widetilde{M}$ equipped with a semi-complete vector field $\widetilde{X}$. Besides X is the hamiltonian vector field associated to a meromorphic function $\widetilde{F}$, which does not have indeterminacy points, with respect to a section $\widetilde{\eta}$ of $K_{\widetilde{M}}$ having *zeros*. Finally, thanks to Lemma (4.2), there are generic orbits of $\widetilde{X}$ intersecting the divisor of poles of $\widetilde{\eta}$ which does not contain singularities of X on its closure. The self-intersection of a leaf L vanishes since L is a leaf of a non-singular foliation. On the other hand $K_{\widetilde{M}}.L$ is strictly positive. Thus the Adjunction formula ensures that $e(L)$ is strictly negative. Hence the restriction of X to L must have a pole. This is impossible in view of Remark (2.9). The resulting contradiction completes our proof. □

According to the above lemma, a surface with $K_M \equiv 0$ carries a semi-complete vector field as in (3) if and only if M carries an elliptic fibration. In particular, if M is $K3$, then the existence of elliptic fibrations on M can be read off the Picard lattice after [Be]. Now let us consider a family of elliptic $K3$-surfaces. A compact complex manifold M of dimension 3 (a complex threefold) will be called a $K3$-fibration (resp. elliptic $K3$-fibration) if there is a compact Riemann surface N and a holomorphic map $\mathcal{P} : M \rightarrow N$ satisfying:

1. Except for a finite set $\{p_1, \ldots, p_l\} \subset N$, $\mathcal{P}^{-1}(p)$ is a $K3$-surface (resp. elliptic $K3$-surface).
2. $\mathcal{P}$ defines a regular fibration of $M \setminus \bigcup_{i-1}^{l} \mathcal{P}^{-1}(p_i)$ over $N \setminus \{p_1, \ldots, p_l\}$.

We emphasize that is not *a priori* obvious that an elliptic $K3$-fibration is an elliptic fibration in the sense of Definition (3.1). Indeed, the point of Proposition (4.3) below is exactly the fact that the elliptic ruling of the fibers was not supposed to vary "continuously" on N.

Proposition 4.3. *Let M be an algebraic threefold which is a elliptic $K3$-fibration. Then there exists a meromorphic semi-complete vector field X on M with the following properties:*

1. *The foliation $\mathcal{F}$ associated to X leaves the fibers $\mathcal{P}^{-1}(p)$ invariant.*
2. *Restricted to a generic fiber of $\mathcal{P}$, $\mathcal{F}$ defines an elliptic fibration on this fiber.*
3. *Locally the foliation $\mathcal{F}$ possesses two linearly independent holomorphic first integral.*

Proof. We might try to construct an elliptic fibration in the sense of Section 3 on M. To avoid difficulties with non-generic fibers, we prefer to perform a direct construction which relies on the closedness of semi-complete vector fields (cf. Proposition (2.6)).

As explained in Section 3, the choice of a section ϑ for K_N allows us to reduce the problem to a local question, namely: we can suppose without loss of generality that $\mathcal{P}$ takes values in the unit disc $D \subset \mathbb{C}$. On the other hand, Poincaré's residue map gives rise to an exact sequence analogous to the exact sequence of Lemma (3.5). Since the Canonical Line Bundle of a $K3$-surface possesses a (nowhere zero) holomorphic section, the exact sequence in question, together with the Adjuntion formula, enables us to construct a section η for K_M whose divisor of poles/zeros is entirely contained in a finite number of fibers of $\mathcal{P}$ (similarly to the construction of Lemma (3.5)).

Now, if F is a meromorphic function of M, we define a meromorphic vector field X_F on M by letting

$$\eta_p(X_F(p),\ .\ ) = d_pF \wedge d_p\mathcal{P} \tag{5}$$

(where we assume that $\mathcal{P}$ takes values in $D \subset \mathbb{C}$). The proof is then reduced to find F so that the resulting X_F is semi-complete. We claim the existence of F as before such that *the restriction of F to a generic fiber $\mathcal{P}^{-1}(p)$ does not have indeterminacy points in $\mathcal{P}^{-1}(p)$*. This claim is the content of the next lemma. For the time being let us show that such F gives rise to a semi-complete vector field X_F hence proving the proposition.

Consider a generic fiber $\mathcal{P}^{-1}(p)$ of $\mathcal{P}$. Such fiber does not intersect the divisor of poles/zeros of η and, besides, $d\mathcal{P}$ is regular in the sense that $\mathcal{P}$ is a submersion on a neighborhood of $\mathcal{P}^{-1}(p)$. Thus we can define an auxiliary 2-form η_p on $\mathcal{P}^{-1}(p)$ by imposing $\eta(q) = \eta_p(q) \wedge d_q\mathcal{P}$ for $q \in \mathcal{P}^{-1}(p)$. Clearly η_p is a nowhere zero holomorphic 2-form on $\mathcal{P}^{-1}(0)$. Because $\mathcal{P}^{-1}(0)$ is also a $K3$-surface, it follows from Lemma (4.1) that the vector field $X_{F,p}$ defined by $\eta_p(X_{F,p},\ .\ ) = dF$ is semi-complete since the restriction of F to $\mathcal{P}^{-1}(0)$ does not have indeterminacy points. However the vector field $X_{F,p}$ is nothing but the restriction to $\mathcal{P}^{-1}(p)$ of X_F. Thus we conclude that the restriction of X_F to a generic fiber of $\mathcal{P}$ is semi-complete. The fact that X_F is semi-complete on the whole of M then follows from Corollary (2.7). □

To complement the proof of Proposition (4.3) we now prove Lemma (4.4).

LEMMA 4.4. *Let $M, \mathcal{P}$ be as above. Then there is a non-constant meromorphic function F on M whose restriction to a generic fiber of $\mathcal{P}$ does not* *have indeterminacy points.*

Proof. First we note that $\mathcal{P} : M \to S$ is algebraic since M, S are algebraic. Thus every fiber $\mathcal{P}^{-1}(p)$ is itself an algebraic surface. Fixed a generic fiber $\mathcal{P}^{-1}(p)$, there exists an algebraic meromorphic function (i.e. a rational function) f_p *defined on* $\mathcal{P}^{-1}(p)$ which does not have indeterminacy points. In fact, this follows from the existence of an elliptic fibration on $\mathcal{P}^{-1}(p)$ together with Lemma (4.1). Let F_p be an extension of f_p to M which always exists since all objects are algebraic.

Next observe that the set $I_{F,p}(S)$ consisting of points $q \in S$ such that F_p has an indeterminacy point on $\mathcal{P}^{-1}(q)$ is a Zariski-closed set. Indeed, letting $F_p = P_p/Q_p$ where P, Q are polynomials, it follows that $I_{F,p}(S)$ is the projection by $\mathcal{P}$ (algebraic proper) of the closed set consisting of the intersections of M, $\{P = 0\}$ and $\{Q = 0\}$. On the other hand $I_{F,p}$ is strictly contained in S, since p does not belong to it. In other words, for a generic fiber $\mathcal{P}^{-1}(q)$ (i.e. a fiber contained in a Zariski-open set), the restriction of F_p to this fiber does not have indeterminacy points. Thus it is enough to set $F = F_p$. The lemma follows. $\square$

So far we have dealt only with fibrations of genus 0 or 1. To go beyond these cases and to be able to handle, in particular, fibrations of genus 2, we introduce the following generalization of semi-complete vector fields.

Definition 4.5. Consider a meromorphic vector field X defined on a complex manifold M. Denote by $\mathcal{F}$ the foliation associated to X and, given a regular leaf L of $\mathcal{F}$, let dT_L be the restriction to L of the time-form corresponding to X. We say that X is k-determined on M if the following holds: given a regular leaf L of $\mathcal{F}$ and a point p of L, there is at most $k-1$ points $q_1, \ldots, q_{k-1} \in L$ for which there is a curve $c : [0,1] \to L$ joining p to q_i on which the integral of dT_L vanishes (for $i = 1, \ldots, k-1$).

We can now introduce the class of 2-determined vector fields with which we shall be mainly concerned from Section 5 on. This class of examples naturally arises in the context of *hyperelliptic fibrations.* To begin with consider a hyperbolic Riemann surface S together with a non-constant meromorphic function f. Considering f as a holomorphic function from S to $\mathbb{CP}(1)$, we can pull-back meromorphic forms on $\mathbb{CP}(1)$. More precisely, if $dT_{\mathbb{CP}(1)}$

is a meromorphic 1-form on $\mathbb{CP}(1)$ with a single pole which has order 2, then $dT_{S,f} = f^* dT_{\mathbb{CP}(1)}$ is a meromorphic form on S which is dual of a k-determined vector field, where k stands for the degree of f.

Suppose now that S is hyperelliptic so that we can choose f with degree 2. Denoting by g the genus of S, Hurwitz Formula tell us that f has $2\mathsf{g}+2$ ramification points, each of them with order 2. To give a certain "normalization" to the choice of a meromorphic form in $\mathbb{CP}(1)$, we let $q_1, q_2, \ldots, q_l$, $l = 2\mathsf{g}+2$ be the ramification points of f. Next we choose a meromorphic form $dT_{(1)}$ on $\mathbb{CP}(1)$ which has a pole (of order 2) at $f(q_1)$ (in particular its periods are trivial, i.e. the integral of $dT_{(1)}$ over a loop in $\mathbb{CP}(1)$ is always zero). Note that this determines $dT_{(1)}$ up to a multiplicative constant. Finally we let $dT_{S,f}$ be the pull-back of $dT_{(1)}$ by f, so that $dT_{S,f}$ is dual to a 2-determined vector field on S. We also notice that $dT_{S,f}$ has $2\mathsf{g}+1$ zeros of order 1 (i.e. locally of the form zdz) and 1 pole of order 3 (locally dz/z^3).

We want to use the above idea to study singular fibers of hyperelliptic fibration on complex surfaces. Therefore let us consider the following local setting: M is a complex (algebraic) surface together with a proper holomorphic map $\mathcal{P} : M \to D \subset \mathbb{C}$ such that $\mathcal{P}^{-1}(p)$ is a hyperelliptic Riemann surface provided that $p \neq 0$. In addition, $\mathcal{P} : M \setminus \mathcal{P}^{-1}(0) \to D \setminus \{0\}$ is a fibration. Our main result is:

Proposition 4.6. *Assume that $\mathcal{P} : M \to D \subset \mathbb{C}$ is as above. Then there is a vertical vector field X on M which is 2-determined. Furthermore X satisfies the following conditions:*

A. X admits a non-constant holomorphic first integral.

B. X has no periods, i.e. the time-form dT_L induced by X on a regular leaf $L = \mathcal{P}^{-1}(p)$ has no periods (equivalently is exact).

C. The singularities of dT_L as before on $L = \mathcal{P}^{-1}(p)$ are all simple (zdz). Besides either dT_L has one pole (which is cubic i.e. dz/z^3) or it has exactly two poles (which are quadratic i.e. dz/z^2).

To prove Proposition (4.6), let $L_{1/2} = \mathcal{P}^{-1}(1/2)$ denote the fiber sitting over $1/2 \in D \subset \mathbb{C}$. Consider also a holomorphic map $f : L_{1/2} \to \mathbb{CP}(1)$ having degree 2. Note that f is, in fact, algebraic and therefore possesses holomorphic extensions to the whole of M. We choose a holomorphic (and hence algebraic) map $F : M \to \mathbb{CP}(1)$ extending f and having finitely many indeterminacy points $\{P_1, \ldots, P_r\} \subset M$. In particular, F is continuous from $M \setminus \{P_1, \ldots, P_r\}$ to $\mathbb{CP}(1)$.

Denote by $W \subset D$ the set of points $p \in D$ such that the restriction of F to $\mathcal{P}^{-1}(p)$ has precisely $2\mathsf{g}+2$ ramification points, all of them with order 2.

Lemma 4.7. *$D \setminus W$ is a proper Zariski-closed subset of D.*

Proof. Obviously $W \neq \emptyset$ since $1/2 \in W$. Let us check that W is open for the ordinary topology. Choose a point $p \in W$ and consider a sequence of points $\{p_i\} \subset D$ such that $p_i \to p$. Denote by $Q_1^p, \ldots, Q_{2\mathsf{g}+2}^p$ the ramification points of F restricted to $\mathcal{P}^{-1}(p)$. For each $j \in \{1, \ldots, 2\mathsf{g}+2\}$ let $C_j^p \subset \mathcal{P}^{-1}(p)$ be a small circle enclosing the point Q_j^p. Without loss of generality, we can assume that the union of the C_j^p's does not contain any of the indeterminacy points $P_1, \ldots, P_r$ of F on M. Finally the small topological ball bound by C_j^p and contained in $\mathcal{P}^{-1}(p)$ is going to be denoted by B_j^p. Note that, by construction, the derivative of F restricted to $\mathcal{P}^{-1}(p)$ does not vanish on $\mathcal{P}^{-1}(p) \setminus \bigcup_{j=1}^{2\mathsf{g}+2} B_j^p$. Moreover, since each B_j^p contains exactly one ramification point Q_j^p, the index of this derivative along C_j^p is 1.

Now consider a point $p_i \in D$ sufficiently close to p. Let $C_j^{p_i}, B_j^{p_i}$ be respectively circles and balls analogous to those considered before and, this time, contained in $\mathcal{P}^{-1}(p_i)$. Naturally we also assume that $C_j^{p_i}$ (resp. $B_j^{p_i}$) converges in the natural sense to C_j^p (resp. B_j^p) when $p_i \to p$. denoting by f_{p_i} the restriction of F to $\mathcal{P}^{-1}(p_i)$, the obvious argument of continuity implies that f'_{p_i} does not vanish on $\mathcal{P}^{-1}(p_i) \setminus \bigcup_{j=1}^{2\mathsf{g}+2} B_j^{p_i}$. Similarly, the index of f'_{p_i} with respect to $C_j^{p_i}$ is again 1. It follows from the Index Formula that f'_{p_i} has exactly one zero in $B_j^{p_i}$ which, furthermore, has order 2. In other words, p_i belongs to W for i large enough so that W is open.

Finally we consider a point $\hat{p} \neq 0$ lying in the boundary of W. We also consider a sequence of points $\{p_i\} \subset W$ converging to $\hat{p}$. Again let us denote by f_{p_i} the restriction of F to $\mathcal{P}^{-1}(p_i)$. The fact that $\hat{p}$ lies in the boundary of W implies that at least two ramification points of f_{p_i} are "approximating" each other so as to collide over $\hat{p}$. In other words, $\hat{p}$ belongs to a set satisfying an algebraic condition. The algebraic condition in question is not trivial for W is open for the ordinary topology. It follows that W is Zariski-dense thus establishing the lemma. $\square$

Remark 4.8. The preceding argument allows us, in fact, to conclude that $D \setminus W$ is contained in $\{0\}$. To verify this, it is enough to apply Hurwitz Formula to $\mathcal{P}^{-1}(\hat{p})$ and repeat the argument above to show that $\mathcal{P}^{-1}(\hat{p})$ cannot be a regular fiber.

Proof of Proposition (4.6). Let us keep the preceding notations. Clearly we only need to construct X with the desired properties over $W \subset D$. For this, it will be necessary to find an appropriate *meromorphic* section η of the Canonical Line Bundle K_M of M.

First recall that the restriction of F to $\mathcal{P}^{-1}(1/2)$ is simply f which will also be denoted by $f_{1/2}$. The ramification points of $f_{1/2}$ are $q_1, \ldots, q_{2\mathrm{g}+2}$. Furthermore, if $p \in W$, the Hurwitz Formula shows that $f_p = F_{|_{\mathcal{P}^{-1}(p)}}$ realizes $\mathcal{P}^{-1}(p)$ as a ramified cover of degree 2 of $\mathbb{CP}(1)$.

To construct the desired meromorphic section η of K_M, consider the map $R : \mathcal{P}^{-1}(W \setminus \{0\}) \subset M \longrightarrow \mathbb{CP}(1) \times (W \setminus \{0\})$ given by $R(x) = (F(x), \mathcal{P}(x))$. Now we introduce a "connection" on $\mathcal{P}^{-1}(W \setminus \{0\})$ by lifting the natural horizontal connection of $\mathbb{CP}(1) \times (W \setminus \{0\})$. The corresponding monodromy is such that F is constant along the "horizontal lines", in particular, after one-tour around $0 \in D$, the monodromy induces a permutation on the sets $f^{-1}(x)$, $x \in \mathbb{CP}(1)$. It follows that the monodromy fixes the ramification points.

Let C_1 be the horizontal curve passing through q_1. This curve is closed after the above remark concerning the monodromy of our connection. We begin the construction of η by equipping $\mathbb{CP}(1) \times \{1/2\}$ with the $dT_{(1)}$ 1-form whose single pole of order 2 coincides with $F(q_1) = f_{1/2}(q_1)$. When the basis point $p = 1/2$ moves to a nearby point p^*, we choose $dT_{(1)}$ over $\mathbb{CP}(1) \times \{p^*\}$ by continuity and by declaring that the pole of $dT_{(1)}$ coincides with $F(\tilde{p})$ where $\tilde{p}$ is a point in $C_1 \cap \mathcal{P}^{-1}(p^*)$. This procedure is well-defined since we return to $q_1 \in \mathcal{P}^{-1}(1/2)$ after going around $0 \in D \subset \mathbb{C}$. Finally we equip $\mathbb{CP}(1) \times (W \setminus \{0\})$ with the natural 2-form ϑ arising from the product of $dT_{(1)}$ (constructed fiberwisely as above) with the natural form dz in $W \setminus \{0\} \subset D$. Finally it suffices to pull-back ϑ through R to obtain the desired meromorphic section η of K_M.

To complete the proof we denote by X the vertical vector field relative to $\mathcal{P}, \eta$. Observe that the restriction of X to the fiber L_p, $p \in W \setminus \{0\}$, is nothing but the lift by f_p of the holomorphic vector field on $\mathbb{CP}(1)$ having a zero of order 2 at $F(\tilde{p})$ where $\tilde{p} = C_1 \cap \mathcal{P}^{-1}(p)$. In particular X is 2-

determined. Also we observe that X possesses a holomorphic first integral given by $\mathcal{P}$ itself (as it happens for every vertical vector field). Besides, since a holomorphic vector field on $\mathbb{CP}(1)$ which has a zero of order 2 has no periods, it results that X has no periods either. As to Condition C, we just need to notice that C_1 may or may not be constituted by ramification points of f_p. When C_1 is constituted by these ramification points, one has the first case mentioned in Condition C. The second case happens otherwise. □

Now we are able to provide a precise definition of ramified and non-ramified singularities for fibrations. Recall that these notions were briefly mentioned in the Introduction. Let us consider a fibration $\mathcal{P} : M \rightarrow D \subset \mathbb{C}$ and the 2-determined vector field X constructed in Proposition (4.6). The singular fiber $\mathcal{P}^{-1}(0)$ may have singularities which coincide with the singularities of the foliation $\mathcal{F}$ associated to X (equivalently defined by the level sets of $\mathcal{P}$). A singularity $p \in \mathcal{P}^{-1}(0)$ of $\mathcal{F}$ is said to be *non-ramified* if, on a neighborhood of p, the divisor of zeros and poles of X is invariant by $\mathcal{F}$ which means that this divisor is contained in the components of $\mathcal{P}^{-1}(0)$ passing through p. Otherwise we say that p is *ramified.*

Example 4.9. Consider the vector field $X = (2y - x^2)\partial/\partial x + 2xy\partial/\partial y$. Viewed as a meromorphic vector field on $\mathbb{CP}(2)$, the foliation $\mathcal{F}$ associated to X possesses 3 singularities (each of them with infinitely many separatrizes) in the "line at infinity" Δ. After performing an appropriate number of blow-ups at these singularities, we obtain an surface M equipped with a elliptic (and therefore hyperelliptic) fibration whose fibers are the leaves of the proper transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$. If $\widetilde{X}$ stands for the proper transform of X on M, the mentioned fibration has exactly two singular fibers which are respectively given as the zero-divisor and the pole-divisor of $\widetilde{X}$. The zero-divisor consists of two rational curves (of self-intersection -2 each) with a quadratic tangency (the local model of the fibration is obviously given by the local orbits of X around $(0, 0) \in \mathbb{C}^2$). The unique singularity of this singular fiber is the tangency point between the two rational curves (which is the proper transform of the origin $(0, 0)$). Since the vector field X does not appear in the list of Theorem A, we conclude that such singularity is *always ramified.*

Let us finish this section with two other examples of 2-determined vector fields which arises in slightly different situations. The first example concerns certain fibrations of genus 3. Consider a complex surface M along with a proper holomorphic map $\mathcal{P} : M \to D \subset \mathbb{C}$ such that $\mathcal{P}^{-1}(p)$ is a Riemann surface of genus 3 provided that $p \neq 0$. Furthermore we assume that $\mathcal{P} : M \setminus \mathcal{P}^{-1}(0) \to D \setminus \{0\}$ is a fibration. It is well known that a Riemann surface of genus 3 admits a holomorphic map of degree 3 over $\mathbb{CP}(1)$. This allows us to construct a 3-determined vertical vector field on M having properties analogous to those of the vector field described in Proposition (4.6). However there are families of genus 3 curves that are not hyperelliptic but admit a degree 2 holomorphic map onto an elliptic curve (these curves are sometimes called bi-elliptic, cf. [H-M]). If $\mathcal{P}$ is a fibration consisting of this type of curves, then it is possible to obtain a 2-determined vertical vector field on M. Precisely, one has:

Proposition 4.10. *Let $\mathcal{P} : M \to D \subset \mathbb{C}$ be as above. Then M carries a vertical vector field X which is 2-determined. Besides the restriction of the corresponding time-form dT_L to a generic fiber $L = \mathcal{P}^{-1}(p)$ is a holomorphic Abelian form with 4 simple singularities (zdz).*

Proof. We choose and fix an elliptic curve S and consider a degree 2 holomorphic map $f : \mathcal{P}^{-1}(1/2) \to S$ whose existence is part of our assumptions. Next we equip S with the (unique up to constant) holomorphic Abelian form dT_S which is everywhere regular. Clearly dT_S is also semi-complete.

The pull-back of dT_S by f gives a 2-determined vector field $X_{|L}$ on $L = \mathcal{P}^{-1}(1/2)$. Since dT_S is everywhere regular, $X_{|L}$ has no *zeros* and its poles correspond to the ramification points of $f : L = \mathcal{P}^{-1}(1/2) \to S$. By Hurwitz Formula this number is 4. Now using the extension F of f to M resulting from the algebraic nature of our problem, we can repeat the arguments used in the proof of Proposition (4.6) to globalize $X_{|L}$ into a vertical vector field X with the required properties. □

For our last example, consider an algebraic $K3$-surface M embedded in some projective space. The examples that we have in mind are indeed (families of) complete intersection $K3$-surfaces. As is well known (cf. for example [G-H]), there are three such families, each depending on 19 parameters, namely: quartics in $\mathbb{CP}(3)$, the complete intersection of a quadric and a cubic in $\mathbb{CP}(4)$ and the complete intersection of three quadrics in $\mathbb{CP}(5)$.

Fixed one such surface, its intersection with a pencil of hyperplanes defines a pencil on M whose generic element is a canonical curve (cf. [G-H]). This pencil is going to be referred to as the *canonical pencil* (defined up to linear equivalence). For a generic algebraic $K3$-surface the canonical pencil represents, in the natural sense, the generator of the Picard group of the surface. In the sequel we are going to provide examples of situations where the canonical pencil carries a 2-determined vertical vector field.

First let us consider homogeneous coordinates $[x_0, x_1, \ldots, x_n]$ for $\mathbb{CP}(n)$, $n \geq 3$. On $\mathbb{CP}(n)$, let F be the meromorphic function defined by $F(x_0, \ldots, x_n) = x_1/x_2$. Also denote by σ the involution of $\mathbb{CP}(n)$ induced by $\sigma(x_0, x_1, x_2, x_3, \ldots, x_n) = (x_0, -x_1, -x_2, x_3, \ldots, x_n)$. Clearly one has $F \circ \sigma = F$.

Proposition 4.11. *Let M be a complete intersection $K3$-surface on $\mathbb{CP}(n)$, $n = 3, 4, 5$. If σ preserves M, then the canonical pencil of M carries a 2-determined vertical vector field.*

Remark 4.12. It is not hard to characterize those surfaces M which are preserved by σ. For example on $\mathbb{CP}(3)$, M is given as the zero-set of a homogeneous polynomial

$$P = \sum_{i+j+k+l=4} c_{ijkl} x_0^i x_1^j x_2^k x_3^l \,.$$

The set of these polynomials is acted upon by $\mathrm{PGL}\,(4, \mathbb{C})$. Thus M is invariant by σ if and only if it is possible to find a representative (with respect to the action of $\mathrm{PGL}\,(4, \mathbb{C})$) satisfying $c_{ijkl} = 0$ whenever $j + k = 1$ or 3.

Proof of Proposition (4.11). The restriction of F to M clearly induces the canonical pencil on M. We then define a vertical vector field X on M by means of the formula

$$\eta_p(X(p),\ .\ ) = d_pF$$

whenever both sides are defined and where η stands for a nowhere zero holomorphic section of K_M.

In order to prove that X as above is 2-determined, the first step is to find out the topology of the generic element of the canonical pencil on M. For this let us observe that the indeterminacy points of F on M are given by the intersection of M with the projective set characterized by $\{x_1 = x_2 = 0\}$. If this intersection is generic, then it consists of $2n-2$, $q_1, \ldots, q_{2n-2}$, points which are dicritical singularities for $\mathcal{F}$ (where $\mathcal{F}$ is the foliation associated to X or, equivalently, given by the pencil). It is easy to check that both eigenvalues of $\mathcal{F}$ at any q_i are equal to 1. In other words, there are local coordinates u, v around q_i in which $\mathcal{F}$ is given by the 1-form

$$\omega = u\,dv \; - \; v\,du \; .$$

In particular each generic leaf L of $\mathcal{F}$ passes exactly once through each q_i. Therefore such L is smooth and of self-intersection equal to $2n-2$. The Adjunction formula then says that the Euler characteristic is $2-2n$ and hence the genus is n.

Next notice that σ takes L to L since it preserves F. Besides σ also preserves η since it is an automorphism of M. It results that σ preserves X. Finally let M be the singular quotient M/σ and denote by $\widetilde{\mathsf{M}}$ the corresponding desingularization. Clearly $\widetilde{\mathsf{M}}$ is equipped with a vector field $\widetilde{Y}$ induced by X. To prove the statement, it suffices to check that $\widetilde{Y}$ is semi-complete. Indeed, we shall see that its orbits are elliptic curves.

To prove the last claim, we note that the mentioned orbit is the quotient of L by σ. Now applying Hurwitz formula to this ramified covering and taking into account that the ramification points are que q_i's ($i = 1, \ldots, 2n-2$), we conclude that the quotient is an elliptic curve. It remains to check that the vector field Y is holomorphic on L/σ. For this we need to consider the points q_i at which X has simple poles. These poles however are cancelled by the ramification of σ at q_i. This shows that Y is holomorphic. The proposition is proved. □

5. 2-Determined Singularities

The remainder of this paper is concerned with the classification of the singularities of 2-determined vector fields appearing in connection with hyperelliptic surfaces in the sense of Section 4. Recall that a first integral for a vector field X (foliation $\mathcal{F}$) is a function which is constant along the orbits of X (resp. leaves of $\mathcal{F}$). Throughout this section only *non-constant* first integrals will be considered. Clearly the singularities of the 2-determined vector fields in question possess a holomorphic first integral given by the natural projection.

Consider a (meromorphic) vector field X whose underlying foliation is $\mathcal{F}$. Recall that X is said to have a *period* if there is a regular leaf L of $\mathcal{F}$ and closed curve $c \subset L$ such that

$$\int_c dT_L \neq 0 \,.$$

The 2-determined vertical vector field X constructed over hyperelliptic fibrations in Section 4 does not have periods. Hence, in the course of this section, we shall deal with 2-determined meromorphic vector fields X defined on a neighborhood of $(0,0) \in \mathbb{C}^2$, satisfying the conditions A, B and C of Proposition (4.6) and also the two conditions below:

D. If the pole divisor of the (foliated) time-form dT is not empty, then it consists of either a single connected component with order 3 or of one or two components with order 2 each.

E. If $\mathcal{D}_X$ is the divisor of zeros and poles of X, then for a sufficiently small neighborhood U of every singular point of $\mathcal{F}$ the intersection $U \cap \mathcal{D}_X$ is invariant by $\mathcal{F}$.

Theorem 5.1. *Let X be a 2-determined vector field defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ and satisfying Conditions A-E. Denote by $\mathcal{F}$ the foliation associated to X and suppose that both eigenvalues of $\mathcal{F}$ at $(0,0) \in \mathbb{C}^2$ are equal to zero. Then $\mathcal{F}$ is given by one of normal forms of the families* **a**, **b** *or* **c** *of Theorem A.*

Obviously Theorem (5.1) implies Theorem A in the Introduction. Note also that, due to condition E, only non-ramified singularities are under consideration.

We set $X = fY/g$ where f, g are holomorphic functions and Y is a *holomorphic* vector field for which the origin is either a regular point or an isolated singularity. Again we denote by $\mathcal{F}$ the local singular holomorphic foliation associated to X, Y. In what follows, we always assume that $\mathcal{F}$ admits a (non-constant) holomorphic first integral. In particular, the leaves of $\mathcal{F}$ are locally closed. Recall that a leaf L of $\mathcal{F}$ is said to be a *separatrix* if its closure $\overline{L}$ defines an analytic curve passing through $(0,0) \in \mathbb{C}^2$ (note that $\overline{L}$ may be singular at $(0,0)$). Since leaves are locally closed, every leaf of $\mathcal{F}$ accumulating on the origin of $\mathbb{C}^2$ must be a separatrix of $\mathcal{F}$.

Next consider a complex surface M a point $p \in M$. Recall that the blow-up of M at p is a new surface $\widetilde{M}$ together with a proper holomorphic map $\pi : \widetilde{M} \to M$ such that

($\imath$) $\pi^{-1}(p) = D_\pi$ is a rational curve of self-intersection -1.

($\imath\imath$) π is a holomorphic diffeomorphism from $\widetilde{M} \setminus D_\pi$ to $M \setminus \{p\}$.

In particular π defines a birational equivalence between $\widetilde{M}$ and M. Also the process of blowing-up points can be iterated so that we may consider sequences of surfaces and (proper) maps

$$M \stackrel{\pi=\pi_{(1)}}{\longleftarrow} \widetilde{M} = \widetilde{M}_{(1)} \stackrel{\pi_{(2)}}{\longleftarrow} \cdots \stackrel{\pi_{(n)}}{\longleftarrow} \widetilde{M}_{(n)}$$

where each $\widetilde{M}_{(i)}$ is obtained by blowing-up a point of $\widetilde{M}_{(i-1)}$. Let us denote by $D_{(i)}$ (resp. $E_{(i)}$) the exceptional divisor of $\widetilde{M}_{(i)}$ introduced by $\pi_{(i)}$ (resp. the total exceptional divisor of $\pi_{(i)}^{-1} \circ \cdots \circ \pi_{(1)}^{-1}(p)$). With this notation we can state a rather useful lemma which is a well-known version of Seidenberg's theorem [Se] adapted to our context.

Lemma 5.2. *Suppose that $\mathcal{F}$ is a singular holomorphic foliation defined on a neighborhood U of $(0,0) \in \mathbb{C}^2$ and possessing a meromorphic first integral. There exists a finite sequence of blow-up maps*

$$(\mathcal{F}, U) = (\mathcal{F}_{(0)}, U_{(0)}) \stackrel{\pi=\pi_{(1)}}{\longleftarrow} (\widetilde{\mathcal{F}}_{(1)}, \widetilde{U}_{(1)}) \stackrel{\pi_{(2)}}{\longleftarrow} \cdots \stackrel{\pi_{(n)}}{\longleftarrow} (\widetilde{\mathcal{F}}_{(n)}, \widetilde{U}_{(n)})$$

verifying the following conditions:

1. $\pi_{(i)}$ is a blow-up map centered at a point of $E_{(i-1)} \subset \widetilde{\mathbb{C}}^2_{(i-1)}$ and $\widetilde{U}_{(i)} = \pi_{(i)}^{-1}(\widetilde{U}_{(i-1)})$.

2. $\widetilde{\mathcal{F}}_{(i)}$ is induced from $\widetilde{\mathcal{F}}_{(i-1)}$ by means of $\pi_{(i)}$.

3. All the singularities of $\widetilde{\mathcal{F}}_{(n)}$ belong to $E_{(n)}$. Furthermore each singularity has two non-vanishing eigenvalues λ_1, λ_2 whose quotient is rational and negative (i.e. $\lambda_1/\lambda_2 \in \mathbb{Q}_-$).

The reader has certainly observed that the exceptional divisor $E_{(n)}$ is constituted by rational curves in a tree-like arrangement. The irreducible components of $E_{(n)}$ are precisely these n rational curves denoted by $D_{(1)}, \ldots, D_{(n)}$. Furthermore the components $D_{(i)}$ are invariant by $\widetilde{\mathcal{F}}_{(i)}$. Otherwise, there would be infinitely many leaves of $\widetilde{\mathcal{F}}_{(i)}$ crossing $D_{(i)}$. The blow-down of these leaves would therefore produce infinitely many separatrizes for $\mathcal{F}$ since the blow-down map is proper. Nonetheless, a foliation having a holomorphic first integral can have only finitely many separatrizes.

We say that a rational function $P = P_\alpha/P_\beta$ is *homogeneous* if both P_α, P_β are homogeneous polynomials (possibly with different degrees). We have:

Lemma 5.3. *Consider the linear vector field $Z = x\partial/\partial x + \lambda y\partial/\partial y$ where $\lambda \in \mathbb{R}_-$, where $\lambda = -n/m$ ($n, m \in \mathbb{N}$ without non-trivial common factors). Let $P = P_\alpha/P_\beta$ be a non-constant homogeneous rational function. Suppose that PZ is 2-determined and that at least one between m, n is greater than 1. Then $P = x^a y^b$ with $bn - am \in \{-2, -1, 0, 1, 2\}$ provided that $m + n \geq 5$. Furthermore, for small m, n the following cases can also occur:*

1. $x^a y^b (x-y)^{-1}(mx\partial/\partial x - ny\partial/\partial y)$ where $m+n = 4$ and $bn-am-n = -2$.

2. $x^a y^b (x-y)^{-1}(mx\partial/\partial x - ny\partial/\partial y)$ where $m+n = 3$ and $bn-am-n$ equals -1 or -2.

Proof. Observe that the orbit L of Z passing through the point (x_0, y_0) possesses a parametrization given by $\mathsf{A} : T \mapsto (x_0 e^T, y_0 e^{-nT/m})$. Therefore the restriction to L of the vector field PZ is given in the coordinate T by $P(x_0 e^T, y_0 e^{-n/mT})\partial/\partial T$. Note that the parametrization A is one-to-one when restricted to the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$ which is identified to $\mathbb{CP}(1)$ minus two points by means of the map $T \mapsto \exp(T/m) = z$. In the coordinate $z = \exp(T/m)$, the restriction of PZ to L becomes $zP(x_0 z^m, y_0 z^{-n})\partial/\partial z$, up to a multiplicative constant. Since $\{x = 0\}$ and $\{y = 0\}$ are all the separatrizes of Z, we set $P = x^a y^b Q_\alpha/Q_\beta$ with $a, b \in \mathbb{Z}$ and Q_α, Q_β homogeneous polynomials. The time-form dT_L in the coordinate $z \in \mathbb{C}^*$ is given by $z^{bn-am-1}Q_\beta(x_0 z^m, y_0 z^{-n})/Q_\alpha(x_0 z^m, y_0 z^{-n})$. Next we decompose Q_α (resp. Q_β) into a product $Q_\alpha = (a_1 x - b_1 y)\cdots(a_s x - b_s y)$ (resp. $Q_\beta = (c_1 x - d_1 y)\cdots(c_r x - d_r y)$). Up to a multiplicative constant the time-form dT_L becomes

$$dT_L = z^{bn-am-1}\frac{(c_1 x_0 z^m - d_1 y_0 z^{-n})\cdots(c_r x_0 z^m - d_r y_0 z^{-n})}{(a_1 x_0 z^m - b_1 y_0 z^{-n})\cdots(a_s x_0 z^m - b_s y_0 z^{-n})}dz$$

$$= \quad z^{bn-am-1} z^{(s-r)n} \frac{(c_1 x_0 z^{m+n} - d_1 y_0) \cdots (c_r x_0 z^{m+n} - d_r y_0)}{(a_1 x_0 z^{m+n} - b_1 y_0) \cdots (a_s x_0 z^{m+n} - b_s y_0)} dz \,.$$

Since $m + n \geq 3$, it follows that Q_α is constant, otherwise dT_L would have more than two poles. Hence we have $dT_L = z^{(bn-am-1-rn)}(c_1 x_0 z^{m+n} - d_1 y_0) \ldots (c_r x_0 z^{m+n} - d_r y_0)$. Whereas dT_L is not defined at $z = 0$, since $\exp(T/m)$ is never *zero*, the fact that dT_L is induced by a 2-determined vector field implies that dT_L has a meromorphic extension to $z = 0$. Besides, if $z = 0$ is a pole of dT_L, then it must have order smaller than or equal to 3 since a differential form with a pole of order greater than 3 cannot be induced by a 2-determined vector field. Similarly $z = 0$ cannot be a *zero* of dT of order greater than 1. Summarizing we have $-3 \leq bn - am - 1 - rn \leq 1$.

On the other hand, the factors $(a_i x_0 z^{m+n} - b_i y_0)$ are pairwise distinct since the zeros of dT_L on L have order 1 by assumption (note that each factor $a_i x_0 z^{m+n} - b_i y_0$ corresponds to $m + n$ *zeros* in the coordinate T). If Q_β is constant, the preceding shows that $-3 \leq bn - am - 1 \leq 1$ i.e. $bn - am \in \{-2, -1, 0, 1, 2\}$.

Next we suppose that Q_β is not constant. Fixed a point z_0, there may exist at most another point z_1 such that the integral of dT_L over a path joining z_0, z_1 vanishes. In particular, the degree of the polynomial $z^{(bn-am-1-rn)}(c_1 x_0 z^{m+n} - d_1 y_0) \cdots (c_r x_0 z^{m+n} - d_r y_0)$ is not greater than 1. Since $m + n \geq 3$ and $bn - am - 1 - rn \geq -3$, we have $r \geq 1$ and $m + n \leq 4$. We consider each possibility:

• If $m + n = 4$, then $r = 1$ and $bn - am - 1 - n = -3$ thus producing the form 1.

• If $m + n = 3$, then $r = 1$ and $bn - am - 1 - n = -3$ or -2. This gives us form 2 and completes the proof of the lemma. □

We do not need to deal with the special case $m = n = 1$ since it will not appear in our discussion. Now we consider a meromorphic 2-determined vector field X defined on a neighborhood of $(0, 0) \in \mathbb{C}^2$. We let $X = fY/g$ as in the beginning of the section and denote by $\widetilde{X}$ the blow-up of X. Similarly $\mathcal{F}$ will stand for the foliation associated to X and $\widetilde{\mathcal{F}}$ will be its blow-up.

Before continuing let us introduce two basic definitions. Assume that $\mathcal{F}$ is a singular holomorphic foliation defined on a neighborhood of a singular point p. Let $\mathcal{S}$ be a smooth separatrix of $\mathcal{F}$ at p. We want to define the *order* $\mathcal{F}$ with respect to $\mathcal{S}$ at p, $\mathrm{ord}_{\mathcal{S}}(\mathcal{F}, p)$, and the *index* of $\mathcal{S}$ w.r.t.

$\mathcal{F}, p$, $\mathrm{Ind}_p(\mathcal{F}, \mathcal{S})$ (cf. [C-S]). To introduce these definitions, let us consider coordinates (x, y) where $\mathcal{S}$ is given by $\{y = 0\}$ and a holomorphic 1-form $\omega = F(x, y)\, dy - G(x, y)\, dx$ defining $\mathcal{F}$ and having an isolated singularity at p. Then, we let

$$\mathrm{ord}_{\mathcal{S}}(\mathcal{F}, p) = \mathrm{ord}\,(F(x, 0)) \quad \text{at} \quad 0 \in \mathbb{C} \quad \text{and} \tag{6}$$

$$\mathrm{Ind}_p(\mathcal{F}, \mathcal{S}) = \mathrm{Res}\, \frac{\partial}{\partial y}\left(\frac{G}{F}\right)(x, 0)\, dx\,. \tag{7}$$

In the above formulas $\mathrm{ord}\,(F(x, 0))$ stands for the order of the function $x \mapsto F(x, 0)$ at $0 \in \mathbb{C}$ and Res for the residue of the 1-form in question.

Let $p_1, \ldots, p_r$ denote the singularities of $\widetilde{\mathcal{F}}$ belonging to $\pi^{-1}(0)$. Since the exceptional divisor $\pi^{-1}(0)$ constitutes a separatrix for every p_i, we can consider both $\mathrm{ord}_{\pi^1(0)}(\widetilde{\mathcal{F}}, p_i)$ and $\mathrm{Ind}_{p_i}(\widetilde{\mathcal{F}}, \pi^{-1}(0))$. With these notations, one has

$$\mathrm{ord}_{(0,0)}(\mathcal{F}) + 1 = \sum_{i=1}^{r} \mathrm{ord}_{\pi^{-1}(0)}(\widetilde{\mathcal{F}}, p_i)\,, \tag{8}$$

$$\sum_{i=1}^{r} \mathrm{Ind}_{p_i}(\widetilde{\mathcal{F}}, \pi^{-1}(0)) = -1\,. \tag{9}$$

In fact, Formulas (8) and (9) can be found respectively in [M-M] and [C-S]. Their proofs depend on reasonably straightforward calculations and on the Residue Theorem.

On the other hand, the order of $\pi^{-1}(0)$ as a divisor of zeros or poles of $\widetilde{X}$ is

$$\mathrm{ord}_{\pi^{-1}(0)}\widetilde{X} = \mathrm{ord}_{(0,0)}(\mathcal{F}) + \mathrm{ord}_{(0,0)}(f) - \mathrm{ord}_{(0,0)}(g) - 1\,. \tag{10}$$

In particular, when this order is *zero*, $\widetilde{X}$ is regular on $\pi^{-1}(0)$. The next propostion is the core of the proof of Theorem (5.1) (the notation $(\epsilon_1, \epsilon_2, \epsilon_3)$ will be explained later on).

Proposition 5.4. *Let $X = fY/g$ and $\mathcal{F}$ be as in the beginning of the section and assume that both eigenvalues of $\mathcal{F}$ at $(0, 0)$ are zero. Denote by $\widetilde{\mathcal{F}}$ the blow-up of $\mathcal{F}$ which has singularities $p_1, \ldots, p_r$ in $\pi^{-1}(0)$. Suppose that $\widetilde{\mathcal{F}}$ has non-vanishing eigenvalues at each p_i, $i = 1, \ldots, r$. Then Y (and* *therefore X up to a multiplicative factor) has one of the following normal forms:*

a.

a.1 $x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(-1,2,2)$.

a.2 $x(3y-2x)\partial/\partial x + y(4x-3y)\partial/\partial y$ *(with first integral* $x^6y^4(x-y)^2$*) and* $(\epsilon_1,\epsilon_2,\epsilon_3)=(1,2,-1)$.

b. *Here the first integral* P *of* $\mathcal{F}$ *is given by* $P = x^{k_1}y^{k_2}(x-y)^{k_3}$ *with* $k=k_1+k_2+k_3=2k_1$ $(k_1,k_2,k_3\neq 0)$ *and*

b.1 $\text{g.c.d.}\,(k_2,k)=\text{g.c.d.}\,(k_3,k)=2$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,1,-1)$.

b.2 $\text{g.c.d.}\,(k_2,k)=2$, $\text{g.c.d.}\,(k_3,k)=1$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,1,-2)$.

b.3 $\text{g.c.d.}\,(k_2,k)=\text{g.c.d.}\,(k_3,k)=1$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-2)$.

In the cases **b**, $Y=x[k_3y-k_2(x-y)]\partial/\partial x + y[k_1(x-y)+k_3x]\partial/\partial y$.

c. *With the same notations of item* **b**, *we have:*

c.1 $P=x^4y^3(x-y)^5$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-2)$ *and* $Y=x(8y-3x)\partial/\partial x + y(9x-4y)\partial/\partial y$.

c.2 $P=x^8y^3(x-y)$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-2)$ *and* $Y=x(4y-3x)\partial/\partial x + y(9x-8y)\partial/\partial y$.

c.3 $P=x^{10}y^6(x-y)^{14}$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-1)$ *and* $Y=x(10y-3x)\partial/\partial x + y(12x-5y)\partial/\partial y$.

c.4 $P=x^{10}y^{12}(x-y)^8$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-1)$ *and* $Y=x(10y-6x)\partial/\partial x + y(9x-5y)\partial/\partial y$.

c.5 $P=x^{10}y^{18}(x-y)^2$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-1)$ *and* $Y=x(10y-9x)\partial/\partial x + y(6x-5y)\partial/\partial y$.

c.6 $P=x^{20}y^6(x-y)^4$, $(\epsilon_1,\epsilon_2,\epsilon_3)=(2,2,-1)$ *and* $Y=x(5y-3x)\partial/\partial x + y(12x-10y)\partial/\partial y$.

Remark 5.5. Modulo a linear change of coordinates, the vector field **a.2** becomes $x(2x-5y)\partial/\partial x + y(y-4x)\partial/\partial y$ which is a form more familiar to our previous works, cf. for example [Re3].

Fix a singularity p_i $(i=1,\dots,r)$ of $\widetilde{\mathcal{F}}$. Recall that $\pi^{-1}(0)$ is invariant by $\widetilde{\mathcal{F}}$ and denote by $m_i, -n_i$ the eigenvalues of $\widetilde{\mathcal{F}}$ at p_i. There exist coordinates

(x_i, t_i) $(\{x_i = 0\} \subset \pi^{-1}(0))$ around p_i in which $\widetilde{X}$ has the form

$$\begin{aligned}(11) \qquad \widetilde{X} \; = \; & x_i^{(\mathrm{ord}_{\pi^{-1}(0)}(\widetilde{X}))} t_i^{d_i} \frac{F_{\alpha,i}(x_i, t_i)}{F_{\beta,i}(x_i, t_i)} \\ & \times [m_i x_i (1 + \text{h.o.t.}) \partial/\partial x_i - n_i t_i (1 + \text{h.o.t.}) \partial/\partial t_i] \,,\end{aligned}$$

where $d_i \in \mathbb{Z}$ and $F_{\alpha,i}, F_{\beta,i}$ are holomorphic functions which are not divisible by x_i or t_i. However, Condition E ensures that both $F_{\alpha,i}, F_{\beta,i}$ are different from *zero* at p_i. In the present case, Formulas (8) and (9) provide

$$(12) \qquad r = \mathrm{ord}_{(0,0)}(\mathcal{F}) + 1 \quad \text{and} \quad \sum_{i=1}^{r} m_i/n_i = 1 \,.$$

Since m_i/n_i is strictly positive, we see that n_i is strictly greater than 1.

On the other hand, let X^{ho} be the first homogeneous component of $X = fY/g$ at $(0,0) \in \mathbb{C}^2$ (cf. Section 2). Since X is 2-determined, it follows that X^{ho} is at most 2-determined (this is the analogue of Corollary (2.8)). Denote by $\mathcal{F}^{\mathrm{ho}}$ the singular foliation associated to X^{ho} and notice that $\mathcal{F}^{\mathrm{ho}}$ can naturally be considered as a foliation in $\mathbb{CP}(2)$ which, in turn, is viewed as the natural compactification of $\mathbb{C}^2$. From this point of view, $\mathcal{F}^{\mathrm{ho}}$ leaves the "line at infinity" Δ of $\mathbb{CP}(2)$ invariant.

Lemma 5.6. *$\mathcal{F}^{\mathrm{ho}}$ has exactly r singularities $p'_1, \ldots, p'_r$ in Δ. Furthermore the eigenvalues of p'_i are m_i, n_i.*

Proof. Let $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ be the blow-up of $\mathcal{F}^{\mathrm{ho}}$ at $(0,0) \in \mathbb{C}^2$. Clearly the singularities of $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ on $\pi^{-1}(0)$ coincide with the singularities $p_1, \ldots, p_r$ of $\widetilde{\mathcal{F}}$ on $\pi^{-1}(0)$. The corresponding eigenvalues are the same as well.

Now notice that $\mathcal{F}^{\mathrm{ho}}$ is invariant by homotheties so that the singularities of $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ on $\pi^{-1}(0)$ correspond to the radial lines of $\mathbb{C}^2$ which are invariant by $\mathcal{F}^{\mathrm{ho}}$. Clearly the intersection of these invariant lines with the line at infinity Δ produces singularities for $\mathcal{F}^{\mathrm{ho}}$ in Δ. The converse also holds and this implies that the singularities of $\mathcal{F}^{\mathrm{ho}}$ in Δ are in one-to-one correspondence with those of $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ in $\pi^{-1}(0)$. A direct inspection also shows that the eigenvalues are the same up to the sign so that the statement follows. □

Lemma 5.7. *The foliation $\mathcal{F}^{\mathrm{ho}}$ admits a non-constant polynomial first integral P.*

PROOF. Let F be a non-constant holomorphic first integral for $\mathcal{F}$ whose existence is guaranteed by Condition A. Consider also a holomorphic vector field Y (with isolated singularities to simplify) whose associated foliation is $\mathcal{F}$. Denote by F^d (resp. Y^k) the first non-trivial homogeneous component of the Taylor series of F (resp. Y) centered at the origin. Clearly the foliation associated to Y^k is nothing but $\mathcal{F}^{\text{ho}}$. On the other hand, the condition that F is constant along the orbits of Y can be written as $\langle \nabla \mathsf{F}\, , Y\rangle = 0$ where $\nabla \mathsf{F}$ stands for the gradient of F and $\langle\ ,\ \rangle$ for the usual inner product. Replacing F and Y be their respective Taylor series and comparing degrees, it follows that $\langle \nabla \mathsf{F}^d\, , Y^k\rangle = 0$. Therefore F^d is a polynomial first integral for $\mathcal{F}^{\text{ho}}$. The lemma is proved. □

LEMMA 5.8. *The foliations $\mathcal{F}$, $\mathcal{F}^{\text{ho}}$ are holomorphically conjugate on a neighborhood of $(0,0) \in \mathbb{C}^2$. Furthermore the vector field X^{ho} does not have periods.*

PROOF. We first observe that $\mathcal{F}$ and $\mathcal{F}^{\text{ho}}$ are holomorphically conjugate on a neighborhood of $(0,0) \in \mathbb{C}^2$. The argument that follows is standard and essentially a special case of the argument developed in [M-M]. In order to construct the desired conjugacy between $\mathcal{F}$ and $\mathcal{F}^{\text{ho}}$, let us consider the respective blow-ups $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}^{\text{ho}}$. We fix a local transverse section Σ at a point $q \in \pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$. By using Σ, we can identify the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ w.r.t. $\widetilde{\mathcal{F}}$ (resp. $\widetilde{\mathcal{F}}^{\text{ho}}$) with a pseudogroup Γ (resp. Γ^{ho}) of holomorphic transformations of Σ fixing $0 \simeq \Sigma \cap \pi^{-1}(0)$. Since $\mathcal{F}^{\text{ho}}$ is associated to a homogeneous vector field, it follows that Γ^{ho} is a finite (cyclic) group of rotations whose order is the least common multiple of the n_i's. We claim that Γ is holomorphically conjugate to Γ^{ho}. Indeed, Γ must have finite order since $\mathcal{F}$ possesses a holomorphic first integral. Thus every element $f \in \Gamma$ such that $f'(0) = 1$ must coincide with the identity and thus Γ is conjugate to its linear part. However this linear part is clearly cyclic with order equal to the least common multiple of the n_i's. Next let $h : \Sigma \to \Sigma$ be a local conjugacy between Γ, Γ^{ho}. We note that $\widetilde{\mathcal{F}}^{\text{ho}}$ is transverse to the natural Hopf fibration of $\widetilde{\mathbb{C}}^2$, namely the fibration of $\widetilde{\mathbb{C}}^2$ over $\pi^{-1}(0)$ whose fibers are the proper transforms of the radial lines of $\mathbb{C}^2$. On the other hand $\widetilde{\mathcal{F}}$ also possesses a transverse fibration obtained through its first integral. By using the classical method of "lifting paths" we can extend the conjugacy h to a compact part of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$. Finally the fact that both foliations are linearizable at each p_i immediately implies that h can, indeed, be extended to a neighborhood of each p_i completing the proof of the first part of the statement.

Now suppose for a contradiction the existence of a closed path $c : [0,1] \to L^{\rm ho}$ on which the integral of the time-form $dT_{L^{\rm ho}}$ induced by $X^{\rm ho}$ does not vanish. Let X_n be a sequence of vector fields converging to $X^{\rm ho}$ and obtained from renormalizations of X. Precisely this means the following. Let $\Lambda : \mathbb{C}^2 \to \mathbb{C}^2$ be the homothety $(x,y) \mapsto (\lambda x, \lambda y)$ for some $\lambda \in \mathbb{C}^*$. We choose a sequence λ_n converging to $0 \in \mathbb{C}$ and set $X_n = \lambda_n^{1-d}.(\Lambda^* X)$, where $\Lambda^* X$ stands for the pull-back of X by Λ and d is the degree of the first non-trivial homogeneous component of the Taylor series of X at the origin. For n large enough, the path c can be lifted into a path c_{L_n} contained in a leaf L_n of $\mathcal{F}_n$ (the foliation associated to X_n). Clearly the integral of the time-form induced by X_n on L_n, dT_{L_n}, over c_{L_n} is different from *zero* provided that n is large enough. On the other hand, since all $\mathcal{F}_n$ have the same first integral up to renormalization, it follows that L_n covers $L^{\rm ho}$ with constant degree on a neighborhood of c. In other words, a multiple $l.c_n$ of c_n lifts into a closed curve on L_n. The integral of dT_{L_n} over $l.c_n$ is necessarily different from *zero* which contradicts the assumption that X does not have periods. The lemma is proved. □

Next let us fix a regular leaf $L^{\rm ho}$ of $\mathcal{F}^{\rm ho}$ and consider the time-form $dT_{L^{\rm ho}}$ induced by $X^{\rm ho}$. Recall that Lemma (5.7) provides a polynomial first integral P for $\mathcal{F}^{\rm ho}$. Therefore the closure $\overline{L}^{\rm ho}$ of $L^{\rm ho}$ in $\mathbb{CP}(2)$ is algebraic. In the sequel, we identify $\overline{L}^{\rm ho}$ with its normalization. Since $dT_{L^{\rm ho}}$ does not have periods, it results that $dT_{L^{\rm ho}}$ is the differential of a meromorphic function F_{dT} defined on $\overline{L}^{\rm ho}$. Naturally F_{dT} is nothing but a holomorphic function from $\overline{L}^{\rm ho}$ to $\mathbb{CP}(1)$. Thus we can consider its topological degree. We claim that this degree is 2. The claim does not follow *a priori* from the construction of X in Section 4. Indeed, whereas the time-form induced by X is the lift by a degree 2 map of the exact (semi-complete) form on $\mathbb{CP}(1)$, $X^{\rm ho}$ is obtained as the limit of a sequence of vector fields conjugate to X. Thus is not obvious that the degree of F_{dT} is necessarily 2. To verify the claim, we first assume that this degree is 3 or greater. Then there are pairwise distinct points p_0, p_1, p_2 in $\overline{L}^{\rm ho}$ such that $F_{dT}(p_0) = F_{dT}(p_1) = F_{dT}(p_2)$.

Since $dT_{L^{\rm ho}}$ is the differential of F_{dT}, we conclude that

$$0 \;=\; \int_{c_1} dT_{L^{\rm ho}} \;=\; \int_{c_2} dT_{L^{\rm ho}}\,.$$

The last equation contradicts the fact that $X^{\rm ho}$ is 2-determined. However if the degree of F_{dT} is 1, then $\overline{L}^{\rm ho}$ is a rational curve and the corresponding vector field is semi-complete. From the classification of meromorphic semi-complete vector fields in [Re3], we conclude that $\mathcal{F}^{\rm ho}$ has non-vanishing eigenvalues at $(0,0)$. This is obviously impossible in the present case. Summarizing one has:

Lemma 5.9. *F_{dT} realizes $\overline{L}^{\rm ho}$ as a degree 2 ramified covering over $\mathbb{CP}(1)$. The poles of $dT_{L^{\rm ho}}$ are contained in the ramification points of F_{dT}. Furthermore, if $dT_{L^{\rm ho}}$ has a single zero, then it has order 3 and coincides with a ramification point of F_{dT} as well. Otherwise F_{dT} has exactly 2 zeros of order 2 each.*

In view of Lemmas (5.8) and (5.9), we can work with the vector field $X^{\rm ho}$ rather than with the original vector field X. In particular, the divisor of zeros (resp. poles) of $X^{\rm ho}$ is given as the zero-set of a homogeneous polynomial and therefore consists of a finite number of lines through $(0,0) \in \mathbb{C}^2$. Let us denote by $\widetilde{X}^{\rm ho}$ the blow-up of $X^{\rm ho}$ and fix a singularity p_i of $\widetilde{\mathcal{F}}^{\rm ho}$. Using Formula (11) and the fact that $F_{\alpha,i}$, $F_{\beta,i}$ are different from *zero* at p_i, we immediately obtain

$$\begin{aligned}(13)\qquad \widetilde{X}^{\rm ho} \;&=\; x_i^{({\rm ord}_{\pi^{-1}(0)}\,(\widetilde{X}))} t_i^{d_i} \\ &\quad \times \left[m_i x_i(1+{\rm h.o.t.})\partial/\partial x_i - n_i t_i (1+{\rm h.o.t.})\partial/\partial t_i\right],\end{aligned}$$

up to a constant multiplicative factor.

For each $i = 1, \ldots, r$, let us set $\epsilon_i = ({\rm ord}_{\pi^{-1}(0)}\,(\widetilde{X}))m_i - n_i d_i$ (note that ${\rm ord}_{\pi^{-1}(0)}\,(\widetilde{X}) = {\rm ord}_{\pi^{-1}(0)}\,(\widetilde{X}^{\rm ho})$). Lemma (5.3) then says that $\epsilon_i \in \{-2,-1,0,1,2\}$. By considering the proper transforms of f, g, we see that $\sum_{i=1}^r d_i = {\rm ord}_{(0,0)}(f) - {\rm ord}_{(0,0)}(g)$ since $F_{\alpha,i}(0)/F_{\beta,i}(0) \neq 0$. However, by definition of ϵ_i, we have $d_i = ({\rm ord}_{\pi^{-1}(0)}\,(\widetilde{X}))m_i/n_i - \epsilon_i/n_i$. Using Formulas (10) and (12), it follows that

$$\begin{aligned}
&\mathrm{ord}_{(0,0)}(f) - \mathrm{ord}_{(0,0)}(g) \\
&\quad = \sum_{i=1}^{r} d_i = \sum_{i=1}^{r} (\mathrm{ord}_{\pi^{-1}(0)}\,(\widetilde{X})) \sum_{i=1}^{r} m_i/n_i - \sum_{i=1}^{r} \epsilon_i/n_i \\
&\quad = -\sum_{i=1}^{r} \epsilon_i/n_i + \mathrm{ord}_{(0,0)}(\mathcal{F}) - 1 + \mathrm{ord}_{(0,0)}(f) - \mathrm{ord}_{(0,0)}(g)\ .
\end{aligned}$$

However we know from (12) that $\mathrm{ord}_{(0,0)}(\mathcal{F}) = r-1$ so that $\sum_{i=1}^{r} \epsilon_i/n_i = r-2$. Equivalently

$$\sum_{i=1}^{r} (1 - \epsilon_i/n_i) = 2\,. \tag{14}$$

Lemma 5.10. *$\widetilde{\mathcal{F}}$ has exactly 3 singularities on $\pi^{-1}(0)$ (i.e. $r = 3$).*

Proof. Recall that $\mathcal{F}^{\mathrm{ho}}$ has a polynomial first integral P (cf. Lemma (5.7)). Let $\mathsf{P} = (x - c_1 y)^{k_1} \cdots (x - c_l y)^{k_l}$, $k = k_1 + \cdots + k_l$, $c_i \in \mathbb{C}$. Recall also that $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ has r singularities $p_1, \ldots, p_r$ in $\pi^{-1}(0)$. Clearly the singularities $p_1, \ldots, p_r$ correspond to r straight lines passing through $(0,0) \in \mathbb{C}^2$ and invariant by $\mathcal{F}^{\mathrm{ho}}$.

A generic leaf L^{ho} of $\mathcal{F}^{\mathrm{ho}}$ is a covering of $\mathbb{CP}(1) \setminus \{p_1, \ldots, p_r\}$ whose degree is k. This allows us to find the Euler characteristic $e(L^{\mathrm{ho}})$ of the normalization $\overline{L}^{\mathrm{ho}}$ of the closure of L^{ho} by means of the formula

$$e(L^{\mathrm{ho}}) \;=\; (2-r)k \;+\; \sum_{i=1}^{r} \mathrm{g.c.d.}\,(k_i, k)\,, \tag{15}$$

where $\mathrm{g.c.d.}\,(k_i, k)$ stands for the greatest common divisor between k_i, k. On the other hand, note that all the ramification points of F_{dT} have ramification index equal to 2 since the degree of F_{dT} is precisely 2. Denoting by s this number of ramification points, Hurwitz Formula yields

$$e(L^{\mathrm{ho}}) = 2(2-s) + s = 4 - s\,. \tag{16}$$

Since $F_{\alpha,i}(0)/F_{\beta,i}(0) \neq 0$, a ramification point of F_{dT} is necessarily contained in the intersections of (a branch of) $\overline{L}^{\mathrm{ho}}$ with the line at infinity Δ. In particular $s \leq k$.

Now we notice that Formula (8) guarantees that $r \geq 3$. Suppose for a contradiction that $r \geq 4$. Note that $\sum_{i=1}^{r} \text{g.c.d.}\,(k_i, k) \leq \sum_{i=1}^{r} k_i = k$, so that $e(L^{\text{ho}}) \leq (3-r)k$ (cf. (15)). However Formula (16) shows that $e(L^{\text{ho}}) \geq 4 - k$. Hence $4 - k \leq (3-r)k$ which is impossible since $r \geq 4$. □

Proof of Proposition (5.4). Let us keep the preceding notations. After Lemma (5.10), we know that the polynomial first integral of $\mathcal{F}^{\text{ho}}$, P, can be written as $\mathsf{P} = x^{k_1} y^{k_2} (x-y)^{k_3}$. We also point out that the possibility of having k_1, k_2, k_3 with a non-trivial common factor is not excluded and might be thought of as representing a "multiple leaf". Such degeneracies does not really have a geometric meaning in terms of foliations since $\mathcal{F}$ and $\mathcal{F}^{\text{ho}}$ are conjugate (cf. Lemma (5.8)). However they correspond to the correct order of the vector fields X, X^{ho} tangent to these foliations. Next set $k = k_1 + k_2 + k_3$ and fix a singularity p_i' of $\mathcal{F}^{\text{ho}}$ in Δ $(i = 1, 2, 3)$ whose eigenvalues are m_i, n_i. The singularity p_i' corresponds to the intersection of Δ with the same line involved with $p_i \in \pi^{-1}(0)$ (i.e. p_i, p_i' are "dual" in this sense). Let us introduce coordinates u_i, v_i around $p_i' \in \Delta$ such that $\{u_i = 0\} \subset \Delta$ and $\{v_i = 0\}$ is contained in the line mentioned above. Finally notice that the order of the vector field X^{ho} on Δ is the negative of the order of its blow-up, $\widetilde{X}^{\text{ho}}$, on $\pi^{-1}(0)$ which, in turn, coincides with $\text{ord}_{\pi^{-1}(0)}\,(\widetilde{X})$. Summarizing, in the coordinates u_i, v_i above, one has

$$X^{\text{ho}} = u_i^{-\text{ord}_{\pi^{-1}(0)}\,(\tilde{X})} v_i^{d_i} (m_i u_i \partial/\partial u_i + n_i v_i \partial/\partial v_i)\,.$$

Therefore a branch of $\overline{L}^{\text{ho}}$ passing through p_i' admits the parametrization $t \mapsto (t^{m_i}, \text{const}\, t^{n_i})$ for an appropriate constant const. The restriction to $\overline{L}^{\text{ho}}$ of the vector field X^{ho} is therefore given in the coordinate t by $t^{(-m_i \text{ord}_{\pi^{-1}(0)}\,(\tilde{X}) + n_i d_i)} t\partial/\partial t = t^{1-\epsilon_i} \partial/\partial t$. Equivalently, the time-form dT induced by X^{ho} on $\overline{L}^{\text{ho}}$ is given by

$$dT = t^{\epsilon_i - 1} dt\,. \tag{17}$$

In particular, we see that $\epsilon_i \neq 0$, otherwise the form dT would not be exact. If $\epsilon_i = 1$, then $\{t = 0\}$ is a regular point of F_{dT}. However F_{dT} has a ramification point at $t = 0$ provided that $\epsilon_i = \pm 2$. More precisely, if $\epsilon_i = -2$, then X^{ho} has a divisor of zeros consisting of a single component which has order 3. If $\epsilon_i = -1$ then the divisor of zeros of X^{ho} must consist of one or two components each of them having order 2, these points are regular for F_{dT}.

Denote by s_1 (resp. s_2, s_3) the number of ramification points of F_{dT} associated to the branches of $\overline{L}^{\text{ho}}$ passing through p_1 (resp. p_2, p_3). Clearly $s_1 + s_2 + s_3 = s$. Furthermore $\epsilon_i = \pm 1$ implies that $s_i = 0$ and $\epsilon_i = \pm 2$ implies that $s_i = \text{g.c.d.}\,(k_i, k)$ (recall that $k = k_1 + k_2 + k_3$).

First let us discuss the case in which some n_i ($i = 1, 2, 3$) is equal to 2. There may exist at most one such n_i (cf. Formula (12)) and the corresponding m_i must be 1. Without loss of generality, we let $n_1 = 2$, $m_1 = 1$ (more generally we shall always assume that $n_1 = \min\{n_1, n_2, n_3\}$).

• Suppose that $\epsilon_1 = 2$. In this case, Equation (14) becomes

$$\frac{\epsilon_2}{n_2} + \frac{\epsilon_3}{n_3} = 0\,.$$

Moreover $s_1 = \text{g.c.d.}\,(k_1, k) = k/2$ since $2 = n_1 = k/\text{g.c.d.}\,(k_1, k)$ (similarly one also has $m_1 = k_1/\text{g.c.d.}\,(k_1, k)$). Thus we conclude that

$$4 - s_2 - s_3 = \text{g.c.d.}\,(k_2, k) + \text{g.c.d.}\,(k_3, k)\,.$$

1. $\epsilon_2 = 1$, $\epsilon_3 = -1$. In this case $n_2 = n_3$ and therefore $k/\text{g.c.d.}\,(k_2, k) = k/\text{g.c.d.}\,(k_3, k)$. On the other hand $s_2 = s_3 = 0$, thus $\text{g.c.d.}\,(k_2, k) = \text{g.c.d.}\,(k_3, k) = 2$. Thus we obtain form **b.1**.

2. $\epsilon_2 = 1$, $\epsilon_3 = -2$. Here $n_3 = 2n_2$ or equivalently $\text{g.c.d.}\,(k_2, k) = 2\text{g.c.d.}\,(k_3, k)$. Since $s_2 = 0$ and $s_3 = \text{g.c.d.}\,(k_3, k)$, we have case **b.2**.

3. $\epsilon_2 = 2$, $\epsilon_3 = -2$. Again $n_2 = n_3$ so that $\text{g.c.d.}\,(k_2, k) = \text{g.c.d.}\,(k_3, k)$. Since $s_2 = \text{g.c.d.}\,(k_2, k)$ (resp. $s_3 = \text{g.c.d.}\,(k_3, k)$), we must have case **b.3**.

It is very easy to check that the possibilities $\epsilon_1 = -1$ and $\epsilon_1 = -2$ do not lead to any solution of the above equations. Thus we just need to consider the case $\epsilon_1 = 1$.

• $\epsilon_1 = 1$. Note that $s_1 = 0$ so that $4 - s_2 - s_3 = -k/2 + \text{g.c.d.}\,(k_2, k) + \text{g.c.d.}\,(k_3, k)$. On the other hand, one has $\epsilon_2/n_2 + \epsilon_3/n_3 = 1/2$. If both ϵ_2, ϵ_3 are positive, then we immediately obtain a contradiction ($4 = 0$). So let us assume that $\epsilon_2 < 0$. In this case $\epsilon_3/n_3 > 1/2$ so that $n_3 < 4$ and therefore $n_3 = 3$ and $\epsilon_3 = 2$. Hence $\epsilon_2/n_2 = -1/6$.

Suppose first $\epsilon_2 = -1$ so that $n_2 = 6$. This leads to $k/6 + \text{g.c.d}\,(k_2, k) = 4$ and, on the other hand, $k = 6\text{g.c.d}\,(k_2, k)$. Thus $\text{g.c.d}\,(k_2, k) = 2$ and $k = 12$.

The resulting P would be $x^6y^2(x-y)^4$. Note that this vector field is not included in the family **b.1** since $\epsilon_1 = 1$ (and the family **b.1** has $\epsilon_1 = 2$). Nonetheless, up to permutation of the lines $\{y = 0\}$ and $\{x = y\}$ it is the member **a.2** of family **a**. Finally the case $\epsilon_2 = -2$ implies that $n_2 = 12$. In turn, this leads to $3\text{g.c.d.}\,(k_2, k) = 4$ which is impossible.

To finish the discussion of the case $n_1 = 2$, we just need to check that ϵ_1 cannot be -1 or -2. Indeed, Formula (14) gives us

$$\frac{1-\epsilon_2}{n_2} + \frac{1-\epsilon_3}{n_3} = 1 + \frac{\epsilon_1}{2}\,.$$

We know that n_2 and n_3 are at least 3 so that the left hand side is greater than or equal to $2/3$. However the right hand side is not larger than $1/2$ provided that $\epsilon_1 = -1$ or -2. This shows that the mentioned cases cannot occur.

The next step is to assume that $n_1 = 3$. Clearly we must have $\epsilon_1 \neq -2$ since $\epsilon_2/n_2 + \epsilon_3/n_3 \leq 4/3$. Besides $\epsilon_1 = -1$ implies that $\epsilon_2 = \epsilon_3 = 2$ and $n_2 = n_3 = 3$. On the other hand

$$4 - s_2 - s_3 = -k + \text{g.c.d.}(k_1, k) + \text{g.c.d.}(k_2, k) + \text{g.c.d.}(k_3, k) = 0$$

since $\text{g.c.d.}(k_i, k) = k/3$. Thus $4 = 2k/3$, i.e. $k = 6$. Hence $\text{g.c.d.}(k_i, k) = 2$ which is form **a.1**.

It remains to analyse the cases $\epsilon_1 = 1$, 2. Let us start with the easy case $\epsilon_1 = 1$ so that $\epsilon_2/n_2 + \epsilon_3/n_3 = 2/3$. We also have

$$4 - s_2 - s_3 = -2k/3 + \text{g.c.d.}(k_2, k) + \text{g.c.d.}(k_3, k)\,.$$

It is clear that one between ϵ_2, ϵ_3 has to be negative so that we can assume that $\epsilon_2 > 0$ and $\epsilon_3 < 0$. In particular, $\epsilon_2/n_2 > 2/3$ which implies that $n_2 < 3$. The resulting contradiction shows that the case $\epsilon_1 = 1$ cannot be produced.

Now we fix $\epsilon_1 = 2$. Then we have $\epsilon_2/n_2 + \epsilon_3/n_3 = 1/3$ and

$$4 - s_2 - s_3 = -k/3 + \text{g.c.d.}(k_2, k) + \text{g.c.d.}(k_1, k)\,.$$

Again one between ϵ_2, ϵ_3 has to be negative. Without loss of generality we assume that $\epsilon_2 > 0$ and $\epsilon_3 < 0$. Therefore $\epsilon_2/n_2 > 1/3$ so that $\epsilon_2 = 2$ as a consequence of the fact that $n_2 \geq 3$.

• Suppose that $n_2 = 3$ so that $4 - s_3 = k/3 + \text{g.c.d.}(k_3, k)$ and $\epsilon_3/n_3 = -1/3$. If $\epsilon_3 = -1$ then $n_1 = n_2 = n_3 = 3$ and $k = 6$ so that we are in the same situation considered above. Similarly $\epsilon_3 = -2$ yields $n_3 = 6$ and $k = 6$. Thus $k_3 = 1$, 5 whereas $k_2 = 2$, 4 and $k_3 = 2$, 4. This is not consistent with $k_1 + k_2 + k_3 = k = 6$.

• The case $n_2 = 4$ implies $\epsilon_3/n_3 = -1/6$ and $4 - s_3 = k/6 + \text{g.c.d.}(k_3, k)$. If $\epsilon_3 = -1$ then $n_3 = 6$ and $4 = k/6 + k/6$ i.e. $k = 12$. Thus $\text{g.c.d.}(k_1, k) = 4$, $\text{g.c.d.}(k_2, k) = 3$ and $\text{g.c.d.}(k_3, k) = 2$. In other words $k_1 = 4$, 8; $k_2 = 3$, 9 and $k_3 = 2$, 10 so that all the resulting possibilities are incompatible with $k_1 + k_2 + k_3 = 12$. Next we consider the case $\epsilon_3 = -2$ so that $n_3 = 12$ and $4 = k/6 + k/12 = k/12$ i.e. $k = 12$. Now one has $\text{g.c.d.}(k_1, k) = 4$, $\text{g.c.d.}(k_2, k) = 3$ and $\text{g.c.d.}(k_3, k) = 1$ which is equivalent to $k_1 = 4$, 8; $k_2 = 3$, 9 and $k_3 = 1$, 5, 7, 11. The only possibilities are **c.1** and **c.2**.

Finally we have to consider the case $n_2 = 5$. Clearly $\epsilon_3/n_3 = 1/15$ and $4 - s_3 = k/15 + \text{g.c.d.}(k_3, k)$. If $\epsilon_3 = -1$ then $n_3 = 15$ and $4 = 2k/15$ so that $k = 30$. Thus $\text{g.c.d.}(k_1, k) = 10$, $\text{g.c.d.}(k_2, k) = 6$ and $\text{g.c.d.}(k_3, k) = 2$. Hence we have the forms **c.3**, **c.4**, **c.5** and **c.6**. Next we consider $\epsilon_3 = -2$ so that $n_3 = 30$ and $k = 30$. The possibilities are now $k_1 = 10$, 20; $k_2 = 6$, 12, 18, 24 but k_3 has to be odd. This is clearly incompatible with $k_1 + k_2 + k_3 = 30$.

The next step is to consider $n_1 = 4$. Clearly $\epsilon_2/n_2 + \epsilon_3/n_3 \leq 1$ so that ϵ_1 has to be 1 or 2. Let us first suppose that $\epsilon_1 = 1$ so that $\epsilon_2/n_2 + \epsilon_3/n_3 = 3/4$ and $4 - s_2 - s_3 = -3k/4 + \text{g.c.d.}\,(k_2, k) + \text{g.c.d.}\,(k_3, k)$. As usual ϵ_2, ϵ_3 cannot be simultaneously positive so that we can suppose $\epsilon_2 > 0$ and $\epsilon_3 < 0$. In particular $\epsilon_2/n_2 > 3/4$ and therefore $n_2 < 8/3 < 3$. This is impossible since $n_2 \geq 4$. Thus we conclude that $\epsilon_1 = 2$. However in this case $\epsilon_2/n_2 + \epsilon_3/n_3 = 1/2$ and, in addition,

$$4 - s_2 - s_3 = -k/2 + \text{g.c.d.}\,(k_2, k) + \text{g.c.d.}\,(k_3, k)\,.$$

Once again we can suppose that $\epsilon_2 > 0$ and $\epsilon_3 < 0$. Therefore $\epsilon_2/n_2 > 1/2$ which implies $n_2 < 4$ yielding a contradiction.

The last case to be considered is $n_1 = 5$. As above $\epsilon_2/n_2 + \epsilon_3/n_3 < 1$ so that $\epsilon_1 > 0$. If $\epsilon_1 = 1$, then we have $\epsilon_2/n_2 + \epsilon_3/n_3 = 4/5$ and the only possibility is $\epsilon_2 = \epsilon_3 = 2$ and $n_2 = n_3 = 5$. Therefore $4 = -k + k = 0$ which is a contradiction. Thus we must have $\epsilon_1 = 2$. However $\epsilon_1 = 2$ still implies (just as before) that either ϵ_2 or ϵ_3 is negative. Letting $\epsilon_2 > 0$ and

$\epsilon_3 < 0$. we conclude that $\epsilon_1/n_2 + \epsilon_2/n_2 > 1$ which is again impossible. It follows that $n_1 \neq 5$. A similar argument works for $n_1 = 6$ and, obviously, we cannot have $n_1 = \min\{n_1, n_2, n_3\} \geq 7$ by virtue of (14). The proposition is proved. □

6. Proofs of Theorem (5.1) and of Theorem B

This section is primarily devoted to the proof of Theorem (5.1). The arguments involved in this proof will also be adapted to provide a proof for Theorem B. Let $X = fY/g$ and $\mathcal{F}$ be as in the statement of this theorem. Consider a resolution tree for $\mathcal{F}$

$$(18)\quad (\mathcal{F}, U) = (\mathcal{F}_{(0)}, U_{(0)}) \stackrel{\pi=\pi_{(1)}}{\longleftarrow} (\widetilde{\mathcal{F}}_{(1)}, \widetilde{U}_{(1)}) \stackrel{\pi_{(2)}}{\longleftarrow} \cdots \stackrel{\pi_{(n)}}{\longleftarrow} (\widetilde{\mathcal{F}}_{(n)}, \widetilde{U}_{(n)}) \, ,$$

where U is a neighborhood of $(0,0) \in \mathbb{C}^2$ like that of Lemma (5.2). Recall that all singularities p_i of $\widetilde{\mathcal{F}}_{(n)}$ have two eigenvalues λ_1/λ_2 different from *zero* and such that $\lambda_1/\lambda_2 \in \mathbb{Q}_-$. Naturally to each foliation $\widetilde{\mathcal{F}}_{(i)}$ there corresponds a meromorphic vector field $\widetilde{X}_{(i)}$ obtained through the original vector field X. Recall also that E_n stands for the total exceptional divisor whose irreducible components are $D_1, \ldots, D_n$. The proposition below is fundamental to the proof of Theorem (5.1).

PROPOSITION 6.1. *Let X be a vector field as in the statement of Theorem (5.1). Denote by $\mathcal{F}$ the foliation associated to X and consider a smooth separatrix $\mathcal{S}$ of $\mathcal{F}$. Then $\mathrm{Ind}_{(0,0)}(\mathcal{F}, \mathcal{S})$ is strictly negative.*

To begin with, let us make two general and elementary remarks which will be used throughout the proof of Proposition (6.1). First let $\mathcal{F}$ be a foliation defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ and consider a separatrix $\mathcal{S}$ of $\mathcal{F}$. Denote by $\widetilde{\mathcal{F}}$ the blow-up of $\mathcal{F}$ and by $\widetilde{\mathcal{S}}$ the proper transform of $\mathcal{S}$. Naturally $\widetilde{\mathcal{S}}$ constitutes a separatrix for some singularity p of $\widetilde{\mathcal{F}}$. The following relation will implicitly be used several times in the sequel

$$(19)\qquad \mathrm{Ind}_p(\widetilde{\mathcal{F}}, \widetilde{\mathcal{S}}) = \mathrm{Ind}_{(0,0)}(\mathcal{F}, \mathcal{S}) - 1 \, ,$$

provided that $\mathcal{S}$ is smooth at $(0,0) \in \mathbb{C}^2$.

For the second remark, we keep the same notations and consider a singularity $p \in \pi^{-1}(0)$ of $\widetilde{\mathcal{F}}$. Here we suppose that the proper transform $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ is smooth at p. As to the original separatrix $\mathcal{S} = \pi(\widetilde{\mathcal{S}})$ of $\mathcal{F}$, one has:

Fact. *$\mathcal{S} = \pi(\widetilde{\mathcal{S}})$ is smooth at $(0,0) \in \mathbb{C}^2$ if and only if $\widetilde{\mathcal{S}}$ is transverse to $\pi^{-1}(0)$ at p.*

This fact will be useful for the proof of Proposition (6.1) since we only need to keep record of the index of smooth separatrizes. Clearly, a singular separatrix cannot be part of an irreducible component D_i of E_n. As mentioned, these remarks will be assumed without further comments in the proof below.

Proof of Proposition (6.1). The statement is obvious if $\mathcal{F}$ has non-vanishing eigenvalues at $(0,0)$. In general, we consider a resolution tree as in (18) for $\mathcal{F}$. If $n = 1$, then the statement follows from a direct inspection in the normal forms provided by Proposition (5.4). In the sequel, we are going to proceed by recurrence from the last level of the resolution tree (18). First we blow-down (collapse) the rational curve $D_{(n)}$ which is an irreducible component of the total exceptional divisor $E_{(n)}$ having self-intersection -1. If $p^{(n-1)}$ stands for the singularity of $\widetilde{\mathcal{F}}_{(n-1)}$ obtained in this way, we notice that $\widetilde{\mathcal{F}}_{(n-1)}$ admits one of the normal forms of Proposition (5.4) on a neighborhood of $p^{(n-1)}$. The next step is to search for an irreducible component, say $D_{(i_0)}$, of the corresponding exceptional divisor $E_{(n-1)}$, satisfying the following:

• $D_{(i_0)}$ has self-intersection -1.

• $D_{(i)}$ contains only singularities of $\widetilde{\mathcal{F}}_{(n-1)}$ where $\widetilde{\mathcal{F}}_{(n-1)}$ has non-vanishing eigenvalues.

If there exists such component $D_{(i)}$ we can suppose without loss of generality that $D_{(i)} = D_{(n-1)}$ and we collapse $D_{(i)} = D_{(n-1)}$. In particular, this collapsing produces a new singularity having again one of the normal forms indicated in Proposition (5.4). Continuing inductively, after a finite number of steps, we find $n_0 < n$ such that, if p is a singularity of the corresponding foliation $\widetilde{\mathcal{F}}_{(n_0)}$, then either $\widetilde{\mathcal{F}}_{(n_0)}$ has two non-vanishing eigenvalues at p or $\widetilde{\mathcal{F}}_{(n_0)}$ admits one of the normal forms of Proposition (5.4) on a neighborhood of p.

Let $D_{(n_0)}$ be an irreducible component of $E_{(n_0)}$ with self-intersection -1. Denoting by $p_1^{(n_0)}, \ldots, p_l^{(n_0)}$ the singularities of $\widetilde{\mathcal{F}}_{(n_0)}$ in $D_{(n_0)}$, we can suppose without loss of generality that the eigenvalues of $\widetilde{\mathcal{F}}_{(n_0)}$ at $p_1^{(n_0)}$ vanish.

Therefore $\widetilde{\mathcal{F}}_{(n_0)}$ admits one of the normal forms of Proposition (5.4) around $p_1^{(n_0)}$. Note also that $D_{(n_0)}$ defines a separatrix for $\widetilde{\mathcal{F}}_{(n_0)}$ at $p_1^{(n_0)}$. Because the index of $D_{(n_0)}$ with respect to $\widetilde{\mathcal{F}}_{(n_0)}$ at each $p_i^{(n_0)}$, $i = 1, \ldots, l$, is strictly negative we have that $\mathrm{Ind}_{p_1^{(n_0)}}(\widetilde{\mathcal{F}}_{(n_0)}, D_{(n_0)}) \geq -1$.

The only models possessing a separatrix whose index is not strictly less than -1 are **a.2**, **b.1**, **b.2**, **b.3** and **c.2**, **c.5**, **c.6**. The model **c.2** (resp. **c.5**, **c.6**) has a separatrix of index $-1/2$ (resp. $-2/3$, $-1/2$) whereas each of the models **a.2**, **b.1**, **b.2**, **b.3** has a separatrix of index -1.

Suppose first that the above model is **b.1**, **b.2** or **b.3**. Then it results from Formula (9) that $l = 1$ i.e. $p_1^{(n_0)}$ is the unique singularity of $\widetilde{\mathcal{F}}_{(n_0)}$ on $D_{(n_0)}$. We claim that this situation is impossible. Indeed, Condition E ensures that all the components of the divisor of zeros/poles of the corresponding vector field $\widetilde{X}_{(n_0)}$ are invariant by $\widetilde{\mathcal{F}}_{(n_0)}$. Hence, on a neighborhood of $D_{(n_0)}$, the divisor of zeros/poles of $\widetilde{X}_{(n_0)}$ is contained in the separatrizes of $\widetilde{\mathcal{F}}_{(n_0)}$ at $p_1^{(n_0)}$ (one of them being $D_{(n_0)}$ itself). Nonetheless the order of $\widetilde{X}_{(n_0)}$ on its separatrizes can be determined from the possible values of $(\epsilon_1, \epsilon_2, \epsilon_3)$ for the normal forms **b.1**, **b.2**, **b.3** in Proposition (5.4). It turns out that these orders are not compatible with Formula (10) which gives the desired contradiction. Alternatively, by using the notion of asymptotic order to be introduced below, we can say that the asymptotic order of the vector field $\widetilde{X}_{(n_0)}$ along its separatrix of self-intersection -1 is 3 and this provides a contradiction with Proposition (6.3).

On the other hand, if the model in question is **a.2**, then it is possible to collapse $D_{(n_0)}$ to obtain singularity **a.3** (again we can notice that the asymptotic order involved here equals 2 and is therefore compatible with Proposition (6.3)).

Finally suppose that the normal form of $\widetilde{\mathcal{F}}_{(n_0)}$ around $p_1^{(n_0)}$ is **c.1** (the cases **c.5**, **c.6** are analogous). Hence $\mathrm{Ind}_{p_1^{(n_0)}}(\widetilde{\mathcal{F}}_{(n_0)}, D_{(n_0)}) = -1/2$ (resp. $-2/3$, $-1/2$).

• Suppose first that there is another singularity among $p_2^{(n_0)}, \ldots, p_l^{(n_0)}$ where the eigenvalues of $\widetilde{\mathcal{F}}_{(n_0)}$ vanish. Denoting by $p_2^{(n_0)}$ this singularity, it follows that $p_2^{(n_0)}$ has the normal form **c.1** or **c.6** (this possibility would immediately be excluded if the normal form of $\widetilde{\mathcal{F}}_{(n_0)}$ around $p_1^{(n_0)}$ were **c.5**). In particular, Formula (9) implies that $l = 2$ in this case.

• Suppose now that all the remaining singularities $p_2^{(n_0)}, \dots, p_l^{(n_0)}$ are such that the eigenvalues of $\widetilde{\mathcal{F}}_{(n_0)}$ do not vanish at them. If $l = 2$, then $\widetilde{\mathcal{F}}^{(n_0)}$ has a separatrix of index -2 (resp. -3, -2 when the normal form around $p_1^{(n_0)}$ is **c.5**, **c.6**). In this case, the collapsing of $D_{(n_0)}$ leads to the singularity **c.7** (resp. **c.8**, **c.9**). On the other hand, if $l \geq 3$, then each p_i, $i = 2, \dots, l$, has a separatrix transverse to $D_{(n_0)}$ whose index is strictly smaller than -2 (resp. -3, -2). In particular, the collapsing of $D_{(n_0)}$ gives rise to a singularity with $l+1$ separatrizes, $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_{l+1}$, such that $\mathcal{S}_1, \mathcal{S}_2$ are tangent to each other. The remaining separatrizes $\mathcal{S}_3, \dots, \mathcal{S}_{l+1}$ also have a common tangent and they are transverse to $\mathcal{S}_1, \mathcal{S}_2$. Let us denote by $R_1^{\mathrm{c}.5}$ the class of singularities arising from the collapsing of $D_{(n_0)}$ when the normal form of $\widetilde{\mathcal{F}}_{(n_0)}$ around $p_1^{(n_0)}$ is **c.5**.

More generally let us denote by R_1 the class of singularities constituted by the singularities of foliations that can be obtained by collapsing $D_{(n_0)}$ in the above cases (in particular $R_1^{\mathrm{c}.5} \subset R_1$). Also the singularities **a.3**, **c.7**, **c.8** and **c.9** belong to R_1. Besides it is convenient to denote by R_0 the singularities listed in Proposition (5.4). The preceding discussion shows that the separatrizes of a singularity in R_1 have strictly negative index. Hence, if the recurrence procedure ends with a R_1 singularity, the proposition is proved.

Conversely if the recurrence procedure is not ended yet, we then continue as before. To go beyond R_1 singularities, we need to arrive to an irreducible component $D_{(n_1)}$ of self-intersection -1 together with a foliation $\widetilde{\mathcal{F}}_{(n_1)}$ satisfying the following conditions:

1. $\widetilde{\mathcal{F}}_{(n_1)}$ has singularities $p_1^{(n_1)}, \dots, p_l^{(n_1)}$ on $D_{(n_1)}$.

2. $p_1^{(n_1)}$ is a singularity of the class R_1 (in particular the eigenvalues of $\widetilde{\mathcal{F}}_{(n_1)}$ at $p_1^{(n_1)}$ vanish and $p_1^{(n_1)}$ does not belong to the class R_0).

3. A singularity $p_i^{(n_1)} \in \{p_2^{(n_1)}, \dots, p_l^{(n_1)}\}$ either belong to $R_0 \cup R_1$ or $\widetilde{\mathcal{F}}_{(n_1)}$ has non-vanishing eigenvalues at $p_i^{(n_1)}$.

As before $D_{(n_1)}$ defines a smooth separatrix for $\widetilde{\mathcal{F}}_{(n_1)}$ at $p_1^{(n_0)}$ whose index is not smaller than -1. However it follows from the above discussion that the singularities in the class R_1 are have only (smooth) separatrizes whose index is smaller than or equal to -1. More precisely the only singularities in R_1 having a smooth separatrix of index -1 are those belonging to $R_1^{\mathrm{c.5}}$ or to the list **a.3**, **c.7**, **c.8** and **c.9**. According to Formula (9) and to the fact number *3* above, $p_1^{(n_1)}$ must be the unique singularity of $\widetilde{\mathcal{F}}_{(n_1)}$ in $D_{(n_1)}$ in these cases. However the same argument employed for the models **b.1**, **b.2**, **b.3** now exclude the case **a.3**. This can directly be seen from the fact that the asymptotic order of this model with respect to its separatrix of self-intersection -1 is 3). Next we collapse $D_{(n_1)}$ and denote by R_2 the class of singularities that can be obtained in this way. We also denote by $R_2^{\mathrm{c.5}}$ the subclass of R_2 corresponding to the cases in which the normal form of $\widetilde{\mathcal{F}}_{(n_1)}$ around $p_1^{(n_1)}$ belongs to $R_1^{\mathrm{c.5}}$. Again a direct inspection shows that every singularity in R_2 verifies the conclusion of our statement.

Let us continue inductively our procedure. It is clear that the only way to go beyond singularities in the class R_2 is by finding an irreducible component $D_{(n_2)}$ of self-intersection -1 together with a foliation $\widetilde{\mathcal{F}}_{(n_2)}$ which satisfy the conditions below:

4. $\widetilde{\mathcal{F}}_{(n_2)}$ has singularities $p_1^{(n_2)}, \ldots, p_l^{(n_2)}$ on $D_{(n_2)}$.

5. $p_1^{(n_2)}$ is a singularity of the class R_2.

6. A singularity $p_i^{(n_2)} \in \{p_2^{(n_2)}, \ldots, p_l^{(n_2)}\}$ either belong to $R_0 \cup R_1 \cup R_2$ or $\widetilde{\mathcal{F}}_{(n_2)}$ has non-vanishing eigenvalues at $p_i^{(n_2)}$.

However, it turns out that the only singularities in R_2 which have a smooth separatrix whose index is greater than or equal to -1 are **c.7**, **c.8** and **c.9**. Indeed, the smooth separatrizes of singularities in $R_2^{\mathrm{c.5}}$ have indices strictly smaller than -1. In particular, it follows that $\widetilde{\mathcal{F}}_{(n_2)}$ has a unique singularity $p_1^{(n_2)}$ on $D_{(n_2)}$ and this singularity belongs to the list **c.7**, **c.8** and **c.9**. The rest of the proof of the proposition is now clear. □

To deduce Theorem (5.1) we just need to identify which singularities, among those obtained in the preceding proof, actually verify the Conditions A, B, C, D and E. The discussion will be organized into two parts. In the course of these parts, the proof of Proposition (6.1) will be reviewed.

Going back to the resolution tree (18), let $p^{(n-1)} \in E_{n-1}$ be the center of $\pi_{(n)}$ and set $D_{(n)} = \pi_{(n)}^{-1}(p^{(n-1)})$. By assumption, $\widetilde{\mathcal{F}}_{(n-1)}$ has trivial eigenvalues at $p^{(n-1)}$. Hence, on a neighborhood of $p^{(n-1)}$, $\widetilde{\mathcal{F}}_{(n-1)}$ admits one of the normal forms indicated in Proposition (5.4). Similar conclusion holds for the corresponding vector field $\widetilde{X}_{(i)}$. The proof of Theorem (5.1) consists of a closer analysis of the structure of the resolution tree (18).

Case 1: Suppose that the normal form mentioned above is **a.1**, **a.2**, **c.1**, **c.3** or **c.4**.

Proof of Theorem (5.1) in Case 1. If $n = 1$, there is nothing to be proved so that we can suppose $n \geq 2$. With this assumption $p^{(n-1)}$ lies in some rational curve $D_{(n-1)}$ whose self-intersection is -1. Clearly $D_{(n-1)}$ defines a separatrix for $\widetilde{\mathcal{F}}_{(n-1)}$ at the point $p^{(n-1)}$.

Let us denote by $p^{(n-1)} = p_1^{(n-1)}, \ldots, p_l^{(n-1)}$ the singularities of $\widetilde{\mathcal{F}}_{(n-1)}$ lying in $D_{(n-1)}$. According to Proposition (6.1), the remaining singularities $p_2^{(n-1)}, \ldots, p_l^{(n-1)}$ of $\widetilde{\mathcal{F}}_{(n-1)}$ on $D_{(n-1)}$ satisfy $\mathrm{Ind}_{p_i^{(n-1)}}(\widetilde{\mathcal{F}}_{(n-1)}, D_{(n-1)}) < 0$ (strictly, for $i \in \{2, \ldots, l\}$). Thanks to Formula (9), it follows that $\mathrm{Ind}_{p^{(n-1)}}(\widetilde{\mathcal{F}}_{(n-1)}, D_{(n-1)}) \leq 1$.

On the other hand a direct inspection of the foliations associated with the normal forms **a.1**, **a.2**, **c.1** **c.3** or **c.4** shows that their indices with respect their separatrizes is always less than or equal to -1. In fact, only the model **a.2** possesses one separatrix whose index is -1. Given that $n \geq 2$, it results that the model in question is **a.2** and that $p^{(n-1)} = p_1^{(n-1)}$ is the unique singularity of $\widetilde{\mathcal{F}}_{(n-1)}$ in $D_{(n-1)}$. By collapsing $D_{(n-1)}$ we obtain the normal forms **a.3**.

Next we assume for a contradiction that $n \geq 3$. With similar notations, it follows the existence of a singularity $p^{(n-2)}$ of $\widetilde{\mathcal{F}}_{(n-2)}$ which belongs to a rational curve $D_{(n-2)}$ whose self-intersection is -1. Furthermore $\widetilde{\mathcal{F}}_{(n-2)}$ admits the normal form **a.3** on a neighborhood of $p^{(n-2)}$. Again a direct inspection shows that this singularity cannot belong to a rational curve of self-intersection -1 (note also that the asymptotic order of the model **a.3** along its separatrix of self-intersection -1 is 3). The resulting contradiction shows that $n \leq 2$ and completes the proof. □

Case 2: Suppose that the normal form mentioned above is **b.1**, **b.2**, **b.3**, **c.2**, **c.5** or **c.6**.

As already explained, the models **b.1**, **b.2**, **b.3** cannot exist on a rational curve of self-intersection -1. Thus, in these cases, we have $n = 1$ and the corresponding normal form (**b.1**, **b.2**, or **b.3**) must be the normal form of the original foliation $\mathcal{F}$ around $(0, 0)$.

Thus we just have to analyse the cases **c.2**, **c.5** and **c.6** in order to complete the proof of Theorem (5.1). Here a further idea will be needed, namely the previously mentioned notion of asymptotic order. Let $X = fY/g$ be a meromorphic vector field defined on a neighborhood of $(0,0) \in \mathbb{C}^2$ and denote by $\mathcal{F}$ the foliation associated to X (or Y). Assume that $\mathcal{S}$ is a smooth separatrix of $\mathcal{F}$. We want to define a notion of order for X along $\mathcal{S}$ (called the asymptotic order of X on $\mathcal{S}$) even in the case where $\mathcal{S}$ is a component of the divisor of zeros or poles of X. In order to do that, we consider local coordinates (x, y) around $(0,0) \in \mathbb{C}^2$ in which $\mathcal{S}$ is given by $\{y = 0\}$. In these coordinates, we have

$$X = y^d [f_1 \partial/\partial x + y^m h_2 \partial/\partial y]$$

where $d \in \mathbb{Z}$, $m \in \mathbb{N}^*$ and h_1, h_2 are meromorphic functions for which $\{y = 0\}$ is not a component of the divisor of zeros or poles. We then define the *asymptotic order of* X *along* $\mathcal{S}$ $\operatorname{ord\,asy}_{(0,0)}(X, \mathcal{S})$, by the formula

$$\operatorname{ord\,asy}_{(0,0)}(X, \mathcal{S}) = k + d.\mathrm{Ind}_{(0,0)}(\mathcal{F}, \mathcal{S})\,.$$

The next proposition is a direct by-product of the "division" techniques introduced in [Re2].

Proposition 6.2. *Let* X, $\mathcal{F}$ *and* $\mathcal{S}$ *be as above. Assume that* X *is* 2-*determined. Then* $-1 \leq \operatorname{ord\,asy}_{(0,0)}(X, \mathcal{S}) \leq 3$.

The second needed result has a more global nature, its proof is elementary in essence and can be found in [Re4]. Again X is a meromorphic vector field whose associated foliation is $\mathcal{F}$. Assume that C is a rational curve invariant by $\mathcal{F}$ and denote by $p_1, \ldots, p_l$ the singularities of $\mathcal{F}$ on C.

Proposition 6.3. *Let* X, $\mathcal{F}$ *and* C *be as above. Then one has*

$$\sum_{i=1}^{l} \operatorname{ord\,asy}_{p_i}(X, \mathsf{C}) = 2.$$

Now we go back to the vector field $X = fY/g$ defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. In practice, X and its associated foliation $\mathcal{F}$ will be as in one of the models **c.2**, **c.5**, **c.6**. When X, $\mathcal{F}$ are as in the model **c.2** (resp. **c.5**,

c.6), we denote by $\mathcal{S}_1^c$ the separatrix of $\mathcal{F}$ whose index is $-1/2$ (resp. $-2/3$, $-1/2$). Direct calculations based on the data provided by Proposition (5.4) yield:

1. $\operatorname{ord}\operatorname{asy}_{(0,0)}(X, \mathcal{S}_1^c) = 2$ for the models **c.2** and **c.6**.
2. $\operatorname{ord}\operatorname{asy}_{(0,0)}(X, \mathcal{S}_1^c) = 1 + 2/3$ for the model **c.5**.

Let us return to the proof of Theorem (5.1). Our recurrent procedure of collapsing appropriate irreducible components of the exceptional divisors in question can be summarized, in the present case, by the following lemma.

Lemma 6.4. *There exists n_1, $0 < n_1 < n$, in the resolution tree (18) of $\mathcal{F}$ such that the following holds:*
1. The corresponding foliation $\widetilde{\mathcal{F}}_{(n_1)}$ has a singularity $p_1^{(n_1)}$ around which $\widetilde{\mathcal{F}}_{(n_1)}$ has a normal form belonging to the list **c.2**, **c.5**, **c.6**.
2. $p_1^{(n_1)}$ lies in a rational curve $D_{(n_1)}$ of self-intersection -1 which is contained in the (total) exceptional divisor associated to $\widetilde{\mathcal{F}}_{(n_1)}$.
3. Let $p_2^{(n_1)}, \dots, p_l^{(n_1)}$ be the remaining singularities of $\widetilde{\mathcal{F}}_{(n_1)}$ in $D_{(n_1)}$. Given $p_i^{(n_1)}$, $i = 2, \dots, l$, either $\widetilde{\mathcal{F}}_{(n_1)}$ has non-vanishing eigenvalues at $p_i^{(n_1)}$ or $\widetilde{\mathcal{F}}_{(n_1)}$ admits one of the normal forms **c.2**, **c.5**, **c.6** *around $p_i^{(n_1)}$.*

Remark 6.5. Consider again a vector field X satisfying our standard assumptions and defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. Suppose that X has, in fact, the form

$$X = t^a x^b h(t,x)[nt(1 + \text{h.o.t})\partial/\partial t - mx(1 + \text{h.o.t.})\partial/\partial x] \tag{20}$$

where $h(0,0) \neq 0$. The asymptotic order of X on $\{x = 0\}$ is given by $1 + \epsilon/n$ with $\epsilon \in \{-2, -1, 0, 1, 2\}$. In particular, $\operatorname{ord}\operatorname{asy}_{(0,0)}(X, \{x = 0\})$ is strictly positive provided that $n \geq 3$. This remark will be used in the sequel.

Proof of Theorem (5.1) in Case 2. We suppose that the normal form of $\widetilde{\mathcal{F}}_{(n_1)}$ around $p_1^{(n_1)}$ is **c.2** (resp. **c.5**, **c.6**). The rational curve $D_{(n_1)}$ clearly must coincide with $\mathcal{S}_1^c$. Thus we have $\operatorname{ord}\operatorname{asy}_{(0,0)}(\widetilde{X}_{(n_1)}, \mathcal{S}_1^c) = 2$ (resp. 5/3, 2) and $\operatorname{Ind}_{p_1^{(n_1)}}(\widetilde{\mathcal{F}}_{(n_1)}, D_{(n_1)}) = -1/2$ (resp. $-2/3$, $-1/2$), where $\widetilde{X}_{(n_1)}$ stands for the corresponding blow-up of X.

The combination of Formula (9), Proposition (6.1) and Proposition (6.3) promptly implies that $\widetilde{\mathcal{F}}_{(n_1)}$ has two eigenvalues different from *zero* at all of its remaining singularities $p_2^{(n_1)}, \ldots, p_l^{(n-1)}$ in $D_{(n_1)}$. Next we observe that $\widetilde{X}_{(n_1)}$ has the form (20) on a neighborhood of each $p_i^{n_1}$, $i = 2, \ldots, l$. Therefore Remark (6.5) and Proposition (6.3) show that all the "n's" involved must be equal to 2 (resp. 2 or 3, 2). Now Formula (9) immediately implies that $l = 2$ and, with the notations of Equation (20), $n = 2$, $m = 1$ (resp. $n = 3$, $m = 1$, $n = 2$, $m = 1$). These cases give rise to the models **c.7**, **c.8**, **c.9**.

To complete the proof of Theorem (5.1) in Case 2. All we need to do is to follow the indices associated to the separatrizes of these vector fields (relying of course on Proposition (6.1)). In fact, these models have only one separatrix whose index is exactly -1. The indices of the other separatrizes being strictly smaller than -1. The rest of the proof results at once from this fact. □

The remainder of this article is devoted to the proof of Theorem B. As we are going to see, this proof is a simple adaptation of the methods used to prove Theorem A. Again we consider a vector field $X = fY/g$ defined on a neighborhood of $(0,0) \in \mathbb{C}^2$. We are going to suppose that X satisfies conditions A, B, C and D of Section 5. However, Condition E is going to be replaced by the following condition:

F. The (local) vector field X is realized by a global vertical vector field, as in the statement of Proposition (4.6), which is defined on a hyperelliptic fibration of genus less than or equal to 3.

The main step of our proof consists of obtaining a suitable analogue of Proposition (5.4). Denote by $\mathcal{F}$ the foliation associated to X and let $\widetilde{\mathcal{F}}$ be its blow-up. Let $p_1, \ldots, p_r$ be the singularities of $\widetilde{\mathcal{F}}$ in $\pi^{-1}(0)$. With these notations, the desired analogue of Proposition (5.4) can be stated as

PROPOSITION 6.6. *Assume that $\widetilde{\mathcal{F}}$ has non-zero eigenvalues at each p_i, $i = 1, \ldots, r$. Then, up to an invertible factor, X has one of the normal forms indicated below:*

1. Non-ramified cases: the forms are **a.1**, **a.2**, **c.1**, **c.2** *or*

b.1 *with* $k_1 \leq 8$*;*

b.2 *with* $k_1 \leq 7$*;*

b.3 *with* $k_1 \leq 6$.

2. Ramified cases:

r.a.0 $H(x,y)X$ *where* H *is a holomorphic function admitting the normal form* $y^2 = x^2 h(x)$, $h(0) = 0$ *(or* $(\alpha_1 x - \beta_1 y)(\alpha_2 x g(x) - \beta_2 y)$*) and* X *has one of the following forms:*

- $[x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y]$.
- $[x(x-2y)\partial/\partial x + y(2y-3x)\partial/\partial y]$.
- $[(2(\tau_1+1)x^2y - x^3 - 3\tau_1 xy^2)\partial/\partial x + (3x^2y - 2(\tau_1+1)xy^2 + \tau_1 y^3)\partial/\partial y]$.

r.a.1 $(x-\tau_1 y)^{-1}(x-\tau_2 y)^{-1}[x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3) = (1,-2,-2)$ *(8 ramification points).*

r.a.2 $(x-\tau_1 y)^{-1}(x-\tau_2 y)^{-1}[x(x-2y)\partial/\partial x + y(y-2x)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3) = (2,-1,-1)$ *or* $(-2,1,1)$ *(7 ramification points).*

r.a.3 $(x-\tau y)^{-1}[x(x-2y)\partial/\partial x + y(4y-5x)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3) = (-1,1,1)$ *or* $(1,-1,-1)$ *(6 ramification points) or* $\epsilon_1 = \pm 1$ *and* $\epsilon_2 + \epsilon_3 = 2\epsilon_1$ *(7 ramification points).*

r.a.4 $(x-\tau y)^{-1}[x(x-2y)\partial/\partial x + y(2y-3x)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3) = (-1,1,1)$ *or* $(1,-1,-1)$ *(4 ramification points, all of them being poles).*

r.a.5 $(x-\tau y)^{-1}[x(3x-4y)\partial/\partial x + y(4y-5x)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3) = (-1,1,1)$ *or* $(1,-1,-1)$ *(8 ramification points).*

r.d.1 $(x-\tau_2 y)^{-1}[(2(\tau_1+1)x^2y - x^3 - 3\tau_1 xy^2)\partial/\partial x + (3x^2y - 2(\tau_1+1)xy^2 + \tau_1 y^3)\partial/\partial y]$ *with* $(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4) = (-2,2,2,2)$ *(8 ramification points).*

r.e.1 $(x-\tau_1 y)^{-1}[x(k_3 y - k_2(x-y))\partial/\partial x + y(k_3 x + k_1(x-y))\partial/\partial y$ *where* $k_1+k_2+k_3 = 5$, $(\epsilon_1,\epsilon_2,\epsilon_3) = (2,-1,-1)$ *or* $(-2,1,1)$ *(6 ramification points).*

r.f.1 $(x-\tau_1 y)^{-1}[x(k_3 y - k_2(x-y))\partial/\partial x + y(k_3 x + k_1(x-y))\partial/\partial y$ *where* $k_1+k_2+k_3 = 7$, $(\epsilon_1,\epsilon_2,\epsilon_3) = (2,-1,-1)$ *or* $(-2,1,1)$ *(8 ramification points).*

In the sequel, we resume the notations of Section 5 and suppose that X, $\mathcal{F}$ are as in the statement of Proposition (6.6). Again we denote by $X^{\rm ho}$ the first homogeneous component of X and by $\mathcal{F}^{\rm ho}$ the foliation associated to $X^{\rm ho}$. It is immediate to check that Lemmas (5.6), (5.8) and (5.9) still hold in the present context. In fact, their proofs do not require Condition E of Section 5.

Next we consider a regular leaf L^{ho} of $\mathcal{F}^{\mathrm{ho}}$ as well as its closure $\overline{L}^{\mathrm{ho}}$ in $\mathbb{CP}(2)$. Recall that the genus of L^{ho} (or of $\overline{L}^{\mathrm{ho}}$) is by definition the genus of the (smooth) Riemann surface obtained by normalization ("desingularization") of $\overline{L}^{\mathrm{ho}}$. Note also that we make no distinction between $\overline{L}^{\mathrm{ho}}$ and its normalization (or "desingularization" or non-singular model) which is also denoted by $\overline{L}^{\mathrm{ho}}$. Now one has:

LEMMA 6.7. *The genus of* L^{ho} *(i.e. of* $\overline{L}^{\mathrm{ho}}$*) is not greater than* 3.

PROOF. The proof relies on condition F. Precisely, there is a global vertical vector field Z as in the statement of Proposition (4.6) which is defined on a surface M and satisfies the following:

1. The foliation $\mathcal{F}_Z$ associated to Z defines a hyperelliptic fibration on M whose fibers have genus ≤ 3.
2. There is a singularity $p \in M$ of $\mathcal{F}_Z$ around which $\mathcal{F}_Z$ is conjugate to the restriction of $\mathcal{F}^{\mathrm{ho}}$ to a neighborhood of $(0,0) \in \mathbb{C}^2$.

Consider a ball $B(p) \subset M$ with center at p and a holomorphic diffeomorphism $\mathsf{h} : B(p) \to U$ that conjugates the restriction of $\mathcal{F}_Z$ to $B(p)$ and that of $\mathcal{F}^{\mathrm{ho}}$ to U, where U stands for a neighborhood of $(0,0) \in \mathbb{C}^2$. Let L_Z be a regular leaf of $\mathcal{F}_Z$. Without loss of generality, we can suppose that L_Z intersects transversely the boundary $\partial B(p)$ of $B(p)$. Thus $L_Z \cap \partial B(p)$ consists of a finite number of simple closed curves ("circles") $c_1, \dots, c_l$. Let L_Z^B be the part of L_Z contained in $B(p)$, i.e. $L_Z^B = L_Z \cap B(p)$. The leaf L_Z is therefore constructed by attaching surfaces (of real dimension 2) with boundaries $S_1, \dots, S_r$ to L_Z^B along the curves $c_1, \dots, c_l$. Note that on such surface S_i may be glued along several curves c_j (i.e. its boundary may have more than one connected component).

Setting $L^{\mathrm{ho},U} = L^{\mathrm{ho}} \cap U$, it follows that $L^{\mathrm{ho},U}$ and L_Z^B are diffeomorphic. Furthermore L^{ho} is obtained from $L^{\mathrm{ho},U}$ by gluing real surfaces (of dimension 2) along the circles $\mathsf{h}(c_1), \dots, \mathsf{h}(c_l)$ contained in ∂U. To prove the statement, it suffices to verify that L^{ho} is obtained from $L^{\mathrm{ho},U}$ by attaching topological disks to the curves c_i. In fact, this way of performing the gluing of surfaces automatically minimizes the genus of the resulting (compact, boundariless, real 2-dimensional) surface. Hence the statement will follow.

Finally to check that L^{ho} is obtained by attaching topological disks to $L^{\mathrm{ho},U}$ as indicated above, we proceed as follows. Recall that $\mathcal{F}^{\mathrm{ho}}$ is invariant by homotheties so that $L^{\mathrm{ho},U}$ is diffeomorphic to the restriction of L^{ho} to any arbitrarily large ball of $\mathbb{C}^2$. In other words, to understand what are the surfaces glued along the $\mathsf{h}(c_i)$'s to obtain L^{ho}, we just need to understand the compactification of L^{ho} viewed as the compactification of its affine part (contained in $\mathbb{C}^2$) into the entire leaf L^{ho} (contained in $\mathbb{CP}(2)$). Since $\mathcal{F}^{\mathrm{ho}}$ possesses a polynomial first integral (cf. Section 5), it promptly results that such compactification consists of adding a point (a dicritical singularity of $\mathcal{F}^{\mathrm{ho}}$ at the line at infinity) to each branch of L^{ho} passing through the singularity in question. The claim follows at once. The proof of the lemma is over. □

Now we go back to the proof of Proposition (6.6). If the origin $(0,0) \in \mathbb{C}^2$ is a non-ramified singularity of X, then X must have one of the normal forms indicated in Proposition (5.4). However, since the genus of the closure of a regular leaf in the corresponding foliation is not greater than 3, the only possible normal forms for X are those listed in the statement of Proposition (6.6) (cf. the "non-ramified" case).

Therefore we suppose from now on that $(0,0)$ is a ramified singularity of X and therefore of X^{ho} as well. Consider the vertical vector field Z mentioned above which realizes X is a hyperelliptic fibration. Recall that the number of poles of Z on a generic fiber is at most 8 (all of them being simple, in fact, the number of poles is bounded by $2\mathsf{g}+2$). Similarly, on each generic fiber, either Z has a single zero whose order is exactly 3 (in which case it has at most 7 poles) or it has exactly two zeros each of them with order 2.

The lemma below allows us to work with X^{ho} in most of our forthcoming discussion.

Lemma 6.8. *The foliations $\mathcal{F}$ and $\mathcal{F}^{\mathrm{ho}}$ are holomorphically conjugate on a neighborhood of the origin. Furthermore the vector field X^{ho} satisfies conditions A, B, C, D and F.*

Proof. The fact that $\mathcal{F}$ and $\mathcal{F}^{\mathrm{ho}}$ are holomorphically conjugate is nothing but Lemma (5.8). According to Lemma (5.9), F_{dT} realizes L^{ho} as a degree 2 ramified cover of $\mathbb{CP}(1)$. Since the time-form induced by X^{ho} on L^{ho} agrees with the differential of F_{dT}, it follows that it is 2-determined and exact (in particular without periods). Finally Condition F results from Lemma (6.7). □

Lemma (6.8) does not allow us to completely substitute X by X^{ho} since the divisor of zeros/poles of the latter consists of finitely many radial lines whereas the divisor of zeros/poles of the former may contain singular components. However, since the associated foliations $\mathcal{F}$ and $\mathcal{F}^{\mathrm{ho}}$ are conjugate, we can set $X = fX^{\mathrm{ho}}$ where f is a meromorphic function.

Again we consider the blow-up $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ of $\mathcal{F}^{\mathrm{ho}}$. Recall that $\widetilde{\mathcal{F}}^{\mathrm{ho}}$, $\widetilde{\mathcal{F}}$ have the same singularities $p_i \in \pi^{-1}(0)$, $i = 1, \dots, r$. Thus there are local coordinates (x_i, t_i) $(\{x_i = 0\} \subset \pi^{-1}(0))$ around p_i in which $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ is given by a vector field having the form (13).

Lemma 6.9. *The divisor of zeros of X is invariant by $\mathcal{F}^{\mathrm{ho}}$. The components of the divisor of poles of X which is not invariant by $\mathcal{F}^{\mathrm{ho}}$ either consists of an irreducible singular curve of the form $y^2 = x^2 h(x)$ or of $s \in \{1, 2\}$ smooth irreducible components each of them with order* 1.

Proof. The proof is almost immediate. Consider the blow-up $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ of $\mathcal{F}^{\mathrm{ho}}$ and the regular leaf $\pi^{-1}(0) \setminus \{p_1, \dots, p_r\}$ of $\widetilde{\mathcal{F}}^{\mathrm{ho}}$. Formula (12) and Formula (9) clearly imply that the holonomy of $\pi^{-1}(0) \setminus \{p_1, \dots, p_r\}$ has order at least 3. Hence the divisor of zeros of X must be invariant by $\mathcal{F}^{\mathrm{ho}}$ otherwise there would be leaves intersecting this divisor three or more times. This contradicts the fact that the intersection of this divisor with a generic leaf consists of a single point.

As to the divisor of poles D_{poles} of X, first suppose that it contains a singular irreducible component. Let $\widetilde{D}_{\mathrm{poles}}$ be the proper transform of D_{poles} and notice that each singular irreducible component of D_{poles} gives rise to a component of $\widetilde{D}_{\mathrm{poles}}$ which is either singular or tangent to $\pi^{-1}(0)$. One such component locally intersects a leaf of $\widetilde{\mathcal{F}}^{\mathrm{ho}}$ at least twice. Since the holonomy has order at least 3 and a generic leaf intersects D_{poles} no more than eight times, one can have exactly one irreducible singular component which intersects $\pi^{-1}(0)$ at a single point p. Around p one can introduce coordinates (x, t), $\{x = 0\} \subset \pi^{-1}(0)$, where $\widetilde{D}_{\mathrm{poles}}$ is given by an equation of the form $t^2 = h(x)$. The statement results at once.

Now assume that all the irreducible components of D_{poles} are smooth. By construction the order of each irreducible component is one. Therefore all we need to do is to check there cannot be more than two such components. Again the order of the holonomy associated to the leaf $\pi^{-1}(0) \setminus \{p_1, \dots, p_r\}$

is 3 or larger. Thus each leaf intersects each irreducible component of D_{poles} in no less than three points. Thus $s \leq 2$. The lemma is proved. □

Again we define $\epsilon_i = (\text{ord}_{\pi^{-1}(0)}(\widetilde{X}))m_i - n_i d_i$ and note that $\text{ord}_{\pi^{-1}(0)}(\widetilde{X}) = \text{ord}_{\pi^{-1}(0)}(\widetilde{X}^{\text{ho}})$. According to Lemma (5.3) one has $\epsilon_i \in \{-2, -1, 0, 1, 2\}$.

Let us first deal with the case in which the divisor of zeros/poles of X contains singular components.

Lemma 6.10. *Suppose that the divisor of poles of X has a singular component $y^2 = x^2 h$. Then X has one of the forms indicated in item r.a.0.*

Proof. As already seen the order of the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ is at least 3 and at most 4. If this order is 3 the statement is immediate. In the second case, the least common multiple of the n_i's is 4. The Index Formula (9) then implies the statement. □

We can now suppose that the irreducible components of the pole divisor of X are smooth. Given the normal form (13) of $\widetilde{X}$, Equation (14) becomes

$$\sum_{i=1}^{r} (1 - \epsilon_i / n_i) = 2 + s \tag{21}$$

as follows, for instance, from employing Proposition (6.3).

The case $s = 0$ was already discussed and is a particular case of Theorem (5.1). The case $s = 2$ implies that the holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ with respect to $\widetilde{\mathcal{F}}^{\text{ho}}$ must have order 3 or 4. This again leads to the forms indicated in r.a.0.

Proof of Proposition (6.6). Clearly we just need to deal with $s = 1$. At this point we can suppose without loss of generality that the divisor of zeros of X is a radial line through $(0, 0) \in \mathbb{C}^2$ and, in fact, we now substitute X by X^{ho}.

The holonomy of $\pi^{-1}(0) \setminus \{p_1, \ldots, p_r\}$ can have order at most equal to 8 which means that the minimum common multiple of the n_i's is not larger than 8.

Other than Formula (21), we also have the following relations:

$$e(L^{\rm ho}) = (2-r)k + \sum_{i=1}^{r} {\rm g.c.d.}(k_i, k) \ \text{ and } \ e(L^{\rm ho}) = 4 - s\,. \tag{22}$$

First let us suppose that $r \geq 4$. From (22), it follows that $e(L^{\rm ho}) \leq (3-r)k$. Since $s \leq 8$, we have

$$-4 \leq 4 - s \leq (3-r)k$$

so that $r = 4$ and $k \leq 4$. It is easy to check that the only possibility is $k = 4$; $m_1 = m_2 = m_3 = m_4 = 1$ and $n_1 = n_2 = n_3 = n_4 = 4$. In addition, we must have $\epsilon_1 = \epsilon_2 = \epsilon_3 = 2$ and $\epsilon_4 = -2$ and this leads to form **r.d.1**.

Now we can assume that $r = 3$ so that Equation (21) becomes $\sum_{i=1}^{3} \epsilon_i/n_i = 0$. In the present setting we can also write $s = s_1 + s_2 + s_3 + k$ (cf. Formulas (22)). Again we let $n_1 = \min\{n_1, n_2, n_3\}$.

• Suppose that $n_1 = 2$ and $\epsilon_1 = -2$. The leaf $L^{\rm ho}$ has a ramification point at infinity corresponding to the direction of $\{x = 0\}$. At this point the corresponding time-form has a pole of order 3 so that it has to be unique (i.e. there must exist a single brach of $L^{\rm ho}$ passing through this point). In other words, the greatest common divisor between k_1 and $k_1 + k_2 + k_3 = 2k_1$ must be 1 what is obviously impossible. Now let us suppose that $\epsilon_1 = 2$. Formula (21) provides

$$\frac{\epsilon_2}{n_2} + \frac{\epsilon_3}{n_3} = -1\,.$$

The solutions are $\epsilon_2 = \epsilon_3 = -2$ and $n_2 = n_3 = 4$ or $\epsilon_2 = \epsilon_3 = -2$ and $n_2 = 3$, $n_3 = 6$ (cf. also the relations in (12)). These solutions nonetheless cannot be realized since each direction with $\epsilon_i = -2$ produces a point at which the time-forms induced by $X^{\rm ho}$ has a cubic pole.

Suppose now that $\epsilon_1 = -1$ (again $\epsilon_1 = 1$ is analogous). In this case Formulas (22) provide

$$4 - s_2 - s_3 = \sum_{i=1}^{3} {\rm g.c.d.}\,(k_i, k) = k/2 + {\rm g.c.d.}\,(k_2, k) + {\rm g.c.d.}\,(k_3, k)\,.$$

It results that $k \leq 4$. Actually, one easily checks that the only possibility is $k = 4$ with $n_1 = 2$, $n_2 = n_3 = 4$. In addition, we must have $\epsilon_2 = \epsilon_3 = 1$ so that the model in question is given by form **r.a.4**.

• Next we suppose that $n_1 = 3$. Since the minimum common multiple among the n_i's is at most 8, it follows that $n_2, n_3 \in \{3, 6\}$. Hence the equation $\sum_{i=1}^{3} m_i/n_i = 1$ (cf. 12) admits only two solutions. The first being $n_1 = n_2 = n_3 = 3$ so that, in addition, we must have $\sum_{i=1}^{3} \epsilon_i = 0$ and $s_1 + s_2 + s_3 = 1$. Indeed, the equation $s_1 + s_2 + s_3 = 1$ is a consequence of (22). This means that the normal form in question is **r.a.2**. The second solution is $m_1 = 2, m_2 = m_3 = 1$ and $n_2 = n_3 = 6$. The reader will easily check that this case corresponds to form **r.a.3**.

• Now assume $n_1 = 4$. Just as before $n_2, n_3 \in \{4, 8\}$. This time the equation $\sum_{i=1}^{3} m_i/n_i = 1$ admits a single solution namely $m_1 = m_3 = 1, m_2 = 3$ and $n_2 = n_3 = 8$. The corresponding form is **r.a.5**.

• Now consider that $n_1 \geq 5$. Again the fact that the minimum common multiple of the n_i's is at most 8 ensures that $n_1 = n_2 = n_3$. When $n_1 = 6$ or 8, the condition $\sum_{i=1}^{3} m_i/n_i = 1$ (equivalently $\sum_{i=1}^{3} m_i = 4$ or 6) immediately provides a contradiction excluding these possibilities (recall that m_i, n_i do not have non-trivial common factors).

If $n_1 = 5$ (resp. 7), we have $\sum_{i=1}^{3} m_i/n_i = 1$, $\sum_{i=1}^{3} \epsilon_i = 0$ and

$$s_1 + s_2 + s_3 = 1 \, .$$

The solutions are those indicated in the normal forms **r.e.1** and **r.f.1**. This completes the proof of the proposition. □

Finally we are going to provide the proof of Theorem B. As mentioned this proof is now very similar to the proof of Theorem A so that we only summarize its main steps.

Let $M \xrightarrow{\mathcal{P}} S$ and $\mathcal{F}$ be as in the statement of this theorem. We consider the 2-determined vector field X provided by Proposition (4.6) which is associated to $M, \mathcal{P}$ and $\mathcal{F}$. Recall that X satisfies conditions A, B, C, D and F. Next we consider a singularity p of $\mathcal{F}$ in which the eigenvalues of $\mathcal{F}$ are *zero*. To prove that $\mathcal{F}$ (or X) has one of the indicated normal forms on a neighborhood of p, we need to employ the same recurrence procedure used in the proof of Theorem A. First we consider a resolution tree (18) for $\mathcal{F}$ at p. Next we observe that a straightforward adaptation of the proof of Proposition (6.1) yields the following:

Lemma 6.11. *Assume that X is as above and consider its associated* *foliation $\mathcal{F}$. Given a singularity q of $\mathcal{F}$ and a separatrix $\mathcal{S}$ of $\mathcal{F}$ at p, the index* $\mathrm{Ind}_q(\mathcal{F},\mathcal{S})$ *is strictly negative.*

Proof of Theorem B. Let X and $\mathcal{F}$ be as above. Using the same notations of the proof of Theorem A, we consider an appropriate blow-up $\widetilde{\mathcal{F}}_{(n-1)}$ of $\mathcal{F}$. The foliation $\widetilde{\mathcal{F}}_{(n-1)}$ has a singularity $p^{(n-1)}$ such that, on a neighborhood of $p^{(n-1)}$, $\widetilde{\mathcal{F}}_{(n-1)}$ possesses one of the normal forms listed in Proposition (6.6). If $\widetilde{\mathcal{F}}_{(n-1)}$ is different from $\mathcal{F}$, then the singularity $p^{(n-1)}$ must belong to a rational curve $D_{(n-1)}$ of self-intersection -1 and invariant by $\widetilde{\mathcal{F}}_{(n-1)}$.

Before continuing we observe that the leaves of our foliations cannot have more than 8 ramification points. Some of these points however, being at infinity, may disappear after implosion (i.e. once we take the homogeneous model of a singularity obtained by implosion, this new model may have less ramification points at infinity than the original model before implosion). This leads us to talk about *crossings*, these are ramification points obtained by intersection of leaves with non-invariant components of the divisor of poles of the corresponding vector field (which are in the affine part of $\mathbb{C}^2$ and therefore are, in fact, present in any neighborhood of the origin). Clearly crossings points are "stable" under blow-down so that their number cannot be reduced in the above mentioned procedure.

We now consider the normal form of $\widetilde{\mathcal{F}}_{(n-1)}$ around $p^{(n-1)}$. If all the separatrizes of $\widetilde{\mathcal{F}}_{(n-1)}$ at $p^{(n-1)}$ have indices strictly smaller than -1, then such singularity cannot belong to a rational curve of self-intersection -1 in view of Lemma (6.11). It follows that the singularity ($p^{(n-1)}$) in question is *terminal* in the sense that $\widetilde{\mathcal{F}}_{(n-1)} = \mathcal{F}$. Besides, if $\widetilde{\mathcal{F}}_{(n-1)}$ has a separatrix of index -1 (and no separatrix with index larger than -1), then $p^{(n-1)}$ must be the unique singularity of $\widetilde{\mathcal{F}}_{(n-1)}$ over a -1-curve. Thus the next step in our recurrence procedure must be the collapsing of the separatrix in question producing a new model of foliation ($\widetilde{\mathcal{F}}_{(n-2)}$). Proceeding in the way described above, our "first generation" of singularities having at least one separatrix of index larger than -1 consists of the following foliations:

c.2 $x^8y^3(x-y)$ where $\{x=0\}$ has index $-1/2$ and $\mathrm{ord\,asy}_{(0,0)}(X,\{x=0\})=2$.

b.1 $x^4y^3(x-y) \to y^3(x^2-y)$ where $\{y=0\}$ has index $-2/3$ and $\mathrm{ord\,asy}_{(0,0)}(X,\{y=0\})=4/3$.

b.2

a- $x^7y^6(x-y) \to y^6(x^2-y)$ where $\{y=0\}$ has index $-1/3$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{y=0\}) = 4/3$.

b- $x^7y^2(x-y)^5 \to y^2(x^2-y)^5$ where $\{x^2-y=0\}$ has index $-4/5$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{x^2-y=0\}) = 3/5$.

c- $x^5y^4(x-y) \to y^4(x^2-y)$ where $\{y=0\}$ has index $-1/2$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{y=0\}) = 3/2$.

b.3 $x^6y^5(x-y) \to y^5(x^2-y)$ where $\{y=0\}$ has index $-2/5$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{y=0\}) = 7/5$.

r.e.1 ($k_1=3$, $k_2=k_3=1$) $x^3y(x-y)$ where $\{x=0\}$ has index $-2/3$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{x=0\})1+\epsilon/5$, $\epsilon \in \{-2,-1,1,2\}$.

r.f.1

a- $x^5y(x-y)$ where $\{x=0\}$ has index $-2/5$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{x=0\}) = 1+\epsilon/5$, $\epsilon \in \{-2,-1,1,2\}$.

b- $x^4y^2(x-y)$ where $\{x=0\}$ has index $-3/4$ and $\operatorname{ord}\operatorname{asy}_{(0,0)}(X,\{x=0\}) = 1+\epsilon/5$, $\epsilon \in \{-2,-1,1,2\}$.

Note that the case **c.2** has *zero* crossings, whereas the cases **b.1**, **b.2** and **b.3** have exactly one crossing. Finally the case **r.e.1** has five crossings and the cases **r.f.1** have seven crossings.

Besides the Index Formula (9), the sum of the asymptotic orders must be an integer. If this integer is different from 2, then this difference is compensated by the intersection of the exceptional curve under consideration with non-invariant components of the divisor of poles of X. Such possibility produces more crossings for the singularity obtained by collapsing the exceptional curve.

A singularity in **r.f.1** can only be combined with singularities having at most 1 crossing (since the number of crossings cannot exceeds 8). From the list above and given the formulas for the indices and asymptotic orders, it follows that the only possibility is to combine a **r.f.1** singularity with a linear one. In fact, note that this claim essentially amounts to eliminating **c.2**, **c.3**, **b.3** and **b.2**a which can easily be done by using the mentioned formulas. In the case of $x^5y(x-y)$, the eigenvalues of this linear singularity are -3, 5. This yields the singularity $x^3y(x^2-y)$ which has a separatrix of index $-2/3$ and asymptotic order equal to $1+\epsilon/3$. Similarly, in the case $x^4y^2(x-y)$, the Index Formula (9) immediately guarantees that the only possibility is to combine it with a linear singularity of eigenvalues -1, 4. The resulting singularity is $x^4y^2(x^2-y)$ which is terminal.

Continuing the formation of our "second generation" of singularities, let us consider the case **r.e.1** $x^3y(x-y)$. According to the Index Formula, this might combine with **b**.**2**a. However the formula for the asymptotic order rules this combination out. Hence $x^3y(x-y)$ can only be combined with a linear singularity of eigenvalues -1, 3. However the asymptotic orders are

$$1+\frac{\epsilon_1}{5} \quad \text{and} \quad 1+\frac{\epsilon_2}{3} \quad \text{with} \quad \epsilon_1,\,\epsilon_2 \in \{-2,-1,1,2\}$$

and therefore do not add up to an integer. In other words, this case cannot be produced so that the singularity $x^3y(x-y)$ is, in fact, terminal.

Consider now the case **c.2**. The same analysis shows that **c.2** can only be combined with a linear singularity of eigenvalues -1, 2. This gives form **c.7**.

As to the family **b**, the case **b.1** can be combined with a linear singularity of eigenvalues -1, 3 to produce the terminal singularity $y(x^2-y^3)$. Analogously the case **b.2**b yields the terminal singularity $y(x^2-y^3)^2$. The form **b.2**c yields $y^2(x^2-y^3)$ which has a smooth separatrix of index -1. The collapsing of this separatrix in turn leads to the terminal x^2-y^5. Finally the cases **b.2**a ($y^6(x^2-y)$) and **b3** ($y^5(x^2-y)$) lead respectively to the models $y^4(x^2-y^3)$ and $y^3(x^2-y^3)$.

Summarizing the second generation of non-terminal singularities is constituted by the models:

- $x^3y(x^2-y)$ (from **r.f.1**).
- **c**.**7** (from **c.2**).
- $y^4(x^2-y^3)$ (from **b.2**a).
- $y^3(x^2-y^3)$ (from **b.3**).

The next step is to repeat the analysis by considering singularities from both first and second generation. The argument is as before. The Index Formula allows the case $x^3y(x^2-y)$ to be combined with **b.2**a. Note that these two singularities brought together have already 8 crossings. On the other hand, the sum of the corresponding asymptotic orders forces $\epsilon=-1$

(since no new crossing can be produced) and we arrive to the singularity $x^2(y^3-x)^2(x^3-y^2)$ which is terminal. Furthermore $x^3y(x^2-y)$ can also be combined with a linear singularity of eigenvalues -1, 3. The resulting singularity is $xy(x^3-y)$ which is also terminal.

The case **c.7** is easy since separatrizes have indices -1. This leads successively to **c.10** and to the terminal **c.13**.

Let us now consider $y^4(x^2-y^3)$ that has a smooth separatrix of index $-1/2$. First we notice that it can be combined with a linear singularity to produce $y^2(x^2-y^5)$ (having a smooth separatrix of index -1) and thus leading to the terminal singularity x^2-y^7. Other than a linear singularity, the Index Formula allows us to combine $y^4(x^2-y^3)$ and **b.2**c. This produces the singularity $(x^2-y^5)(y^2-x^3)$ which is obviously terminal since all separatrizes are singular.

Finally we consider the case $y^3(x^2-y^3)$ whose smooth separatrix has index $-2/3$. Combined with a linear singularity of eigenvalues -1, 3, this leads to the terminal singularity $y(x^2-y^5)$. According to the Index Formula the only other combination possible is with **b.2**a. This produces $(x^3-y^2)(x^2-y^5)^2$ which is obviously terminal for all separatrizes are singular. To finish the proof of the theorem it is now sufficient to eliminate the redundant cases. □

Proof of Corollary C. It suffices to repeat the preceding argument restricting the discussion to foliations associated to genus 2 fibrations. The leaves of these foliations can have at most 6 ramification points. One then obtains the models indicated in the statement. □

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