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Invariants of Plane Curves and Polyak-Viro Type Formulas for Vassiliev Invariants

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Abstract. The Kontsevich integral is decomposed into two parts; one part depends on overpass or underpass of the crossing of a knot while the other depends only on the plane curve obtained by projecting the knot to the plane. In this paper, firstly, we express the latter part in terms of Arnold's invariants of plane curves J_+, J_- and St up to degree three. Secondly, we show that the Gauss diagram formulas for the Kontsevich integral agree with other types of formulas for Vassiliev invariants which are introduced by M. Polyak and O. Viro.

1. Introduction

V.A Vassiliev [18] introduced an important concept of Vassiliev invariants for knots. It is recognized that the coefficients of the perturbative series expansion of many known knot invariants, e.g., the Conway, Jones and Homfly polynomial, are Vassiliev invariants. One major step in the study of Vassiliev invariants was undertaken by M. Kontsevich, when he introduced the Kontsevich integral which expresses Vassiliev invariants of knots as iterated integrals [8].

M. Polyak and O. Viro introduced the notions of Gauss diagrams, as well as the idea of Gauss diagram formulas for Vassiliev invariants and various notations [17]. They also provided Gauss diagram formulas for invariants of degree 2, 3 and 4. Their pioneer work enables us to actually compute Vassiliev invariants in a combinatorial way up to degree four [13, 17]. M. Polyak [16] intensively studied the relation between Gauss diagrams and invariants of plane curves, pointing out the close relation between a Vassiliev invariant of degree two and invariants of plane curves.

On the physical side, E. Witten [19] established the connection between the three-dimensional Chern-Simons gauge theory and knot theory. The

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Chern-Simons gauge theory is a three dimensional model based on the functional of a connection A:

$$S_k(A) = \frac{k}{4\pi} \int_{\mathbf{R^3}} tr(A \wedge dA + i\, \frac{2}{3} A \wedge A \wedge A) \tag{1.1}$$

where a connection A is $\mathfrak{su}(N)$-valued one form on $\mathbf{R}^3$ and "tr" means the trace. The important operator in the Chern-Simons gauge theory is given by the trace over the holonomy around a knot K

$$tr(P \exp i \oint_K A), \tag{1.2}$$

which is called the Wilson loop operator. E. Witten showed that the expectation value of the Wilson loop operator in the Chern-Simons gauge theory

$$\int [DA]\ e^{iS_k(A)}\ tr(P \exp i \oint_K A) \tag{1.3}$$

gives a knot invariant, and it satisfies the HOMFLY skein relation. It is also known that the expectation value of the Wilson loop operator in the Chern-Simons gauge theory is equivalent to the Kontsevich integral coupled with a suitable weight system, i.e.,

$$W_{SU(N)} \circ \hat{Z}(L) = \int [DA]\ e^{iS_k(A)}\ tr(P \exp i \oint_K A), \tag{1.4}$$

where $W_{SU(N)}$ is the weight system defined by Lie group $SU(N)$. Therefore, the coefficients of the pertubetive series expansion of the expectation value of the Wilson loop operator in the Chern-Simons gauge theory are Vassiliev invariants.

Since then the Chern-Simons gauge theory has been studied from a variety of points of view. A.C. Hirshfeld, U. Sassenberg, T. Kloker [4, 5, 6] gave combinatorial expressions for Vassiliev invariants of links up to degree three based on the study of the configuration space integral of the Chern-Simons gauge theory. In 1998, J.M.F. Labastida and E. Perez [10] obtained combinatorial expressions for the Vassiliev invariant of knots up to degree four based on the Chern-Simons gauge theory in the temporal gauge.

Motivated by the physical side [4, 5, 6, 9, 10], we gave the Gauss diagram formulas for the Kontsevich integral (GDK formulas) of links up to degree four in [13]. In this paper, we discuss the relation between our GDK formulas and two other results, namely, invariants of plane curves and the Polyak-Viro type formulas.

First, we discuss the relation between the GDK formulas and invariants of plane curves introduced by V. I. Arnold [1]. The GDK formulas consist of two parts; one part depends on overpass or underpass of the crossings of knot diagrams while the other does not. Therefore, it is natural to ask whether the latter part can be expressed by invariants of plane curves introduced by V. I. Arnold [1]. In this paper, we answer this question affirmatively up to degree three and indeed express the latter part in terms of invariants of plane curves.(Theorem 1)

Secondly, we discuss the relation between the GDK formulas and the Polyak-Viro type formulas [17]. The difference between the GDK formulas and the Polyak-Viro type formulas is as follows. First, the GDK formulas use *unoriented* Gauss diagrams, while the Polyak-Viro type formulas use *oriented* Gauss diagrams. Secondly, the GDK formulas use $\alpha(K)$ and the concept of splitting the crossing, while the Polyak-Viro type formulas do not. It is necessary to clarify the relationship between the GDK formulas and the Polyak-Viro type formulas. So our goal is to derive the Polyak-Viro type formulas of knots up to degree three from the GDK formulas. (Theorems 2, 3)

We also give some supplementary explanations for the GDK formulas which we did not present in [13].

Acknowledgements. The author would like to express his hearty thanks to Professor T. Kohno for suggesting the problem and for his advice and constant encouragement.

2. The Gauss Diagram Formulas for the Kontsevich Integral

In this section, we review the Gauss diagram formulas for the Kontsevich integral (GDK formulas) as in [13, 17]. We limit our consideration up to degree three.

2.1. The Kontsevich integral

A chord diagram is n-oriented circles $\cup_{i=1}^{n} S_i^1$ together with finite chords whose endpoints are marked on $\cup_{i=1}^{n} S_i^1$. By convention, we set the orientation of $\cup_{i=1}^n S_i^1$ counterclockwise.

Let L be an embedding of n-oriented circles into $\mathbb{R}^3 = \mathbb{C}\times\mathbb{R}$, where $\mathbb{C}$ is parameterized by z and $\mathbb{R}$ is parameterized by t. Let $\hat{\mathcal{A}}$ be the quotient of the linear span of chord diagrams by the four term relation and the framing independence relation. We define the Kontsevich integral on $\hat{\mathcal{A}}$ as in [3]

$$
\begin{aligned}
(2.1) \qquad Z(L) \;=\; & \sum_{m=0}^{\infty} \frac{1}{(i\pi)^m} \int_{t_{\max}>t_1>\cdots>t_m>t_{\min}} \\
& \times \sum_{\substack{\text{applicable pairings} \\ P=\{(z_i,z_i')\}}} (-1)^{\sharp P\downarrow} D_p \bigwedge_{i=1}^{m} \frac{dz_i - dz_i'}{z_i - z_i'} \in \hat{\mathcal{A}}
\end{aligned}
$$

We have used a slightly different normalization from [3, 8].

It is known that the Kontsevich integral is invariant under only horizontal deformation of L. Therefore we define the modified Kontsevich integral by

$$
(2.2) \qquad \hat{Z}(L) = Z(L)Z(U_0)^{-m(L)},
$$

where $m(L)$ denotes the number of maximal points of link L and U_0 is a knot given in Fig. 1. It is known that $\hat{Z}(L)$ is invariant under arbitrary deformations of the link L.

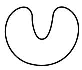
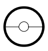
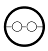
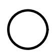
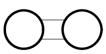
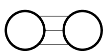


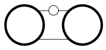
Fig. 1. U_0

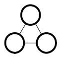
Let $L = \{K_1, \cdots, K_n\}$ be an arbitrary n-component link where K_i is its component. The Kontsevich integral is factorized as follows:

(2.3) $\hat{Z}(L) = \exp\Big\{(-\frac{1}{2})$  $\sum_{i=1}^{n} v_2(K_i) + (-\frac{1}{2})^2$  $\sum_{i=1}^{n} v_{3.1}(K_i) + \cdots\Big\}$

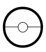
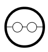
$\times\Big\{$  $+$  $\sum_{1\le i<j\le n} \frac{1}{2}(v_1(\{K_i,K_j\}))^2$

$+$  $\sum_{1\le i<j\le n} \frac{1}{3!}(v_1(\{K_i,K_j\}))^3$

$$+ $$  $$(-\frac{1}{2}) \sum_{1\le i<j\le n} v_{3.2}(\{K_i, K_j\})$$

$$+ $$  $$\sum_{1\le i<j<k\le n} v_1(\{K_i, K_j\})v_1(\{K_j, K_k\})v_1(\{K_k, K_i\}) + \cdots\Big\}.$$

If we consider one-component knot K instead of link L, the above Kontsevich integral is reduced to the following simple form:

(2.4) $$\hat{Z}(K) = \exp\Big\{(-\frac{1}{2})$$  $$v_2(K) + (-\frac{1}{2})^2$$  $$v_{3.1}(K) + \cdots\Big\}.$$

We give some comments on v_1, v_2, $v_{3.1}$ and $v_{3.2}$. v_1 is twice the linking number, which is a Vassiliev invariant of degree one for two-component links. v_2 is a Vassiliev invariant of degree two for one-component knots. $v_{3.1}$ and $v_{3.2}$ are Vassiliev invariants of degree three for one-component knots and for two-component links respectively. We shall recall the GDK formulas for v_1, v_2, $v_{3.1}$ and $v_{3.2}$ in the following subsections.

2.2. The Gauss diagram formulas for the Kontsevich integral

In this subsection, we introduce a concept of Gauss diagram of links and define a pairing of Gauss diagrams, which has nice properties for acutual computations. The GDK formulas for v_1, v_2, $v_{3.1}$ and $v_{3.2}$ are expressed by this pairing.

Let $X = \cup_{i=1}^n S_i^1$ be n-oriented circles and $\vec{y} : X \to \mathbb{R}^2$ an immersion. An n-component *oriented link diagram* L is its image $L = \{K_1, \cdots, K_n\}$ ($K_i = \vec{y}(S_i^1)$) together with the information of overpass or underpass at each crossing. We assign a sign $\pm$ to each crossing as the information of overpass or underpass.

Let D be a chord diagram and $C(D)$ the set of all chords of D. By an integer-labelling of D, we mean a map $\kappa : C(D) \to \mathbf{Z}$. An *Integer-Labeled Chord Diagram* (IL Diagram) is a pair $\{D, \kappa\}$ of a chord diagram D together with an integer-labelling κ.

We shall define a *Gauss Diagram* and *ML Diagram* as special cases of IL Diagrams.

An IL diagram $\{G, \epsilon\}$ is called a *Gauss Diagram* if $\epsilon(c) = \pm 1$ ($c \in C(G)$). An integer-labelling ϵ of the Gauss diagram is called a *sign-labelling*.

Let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. Define a Gauss diagram $P(\{L :$

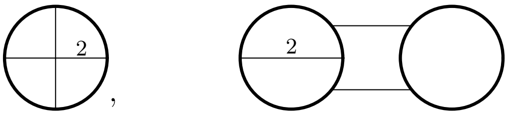
$a_1, \cdots, a_m\}$) as follows. For each a_i, set $\vec{y}^{-1}(a_i) = \{s(a_i), s'(a_i)\}$ as the inverse image of a_i. For each crossing a_i, we join $s(a_i), s'(a_i)$ by a chord on X and label this chord by the sign of a_i ($i = 1, \cdots, m$). We define a Gauss diagram $P(\{L : a_1, \cdots, a_m\})$ to be the result.

Specially, if $\{a_1, \cdots, a_m\}$ are all the crossings of L (this means we select all the crossings of L) , we write $G(L) = P(\{L : a_1, \cdots, a_m\})$ and call it the Gauss diagram of L.

An IL diagram $\{D, m\}$ is called a *Multiplicity-Labeled Diagram* (ML Diagram) if $m(c) = 1, 2$ ($c \in C(D)$). In figures, we draw a chord c with $m(c) = 1$ by a thin line and a chord c with $m(c) = 2$ by a thin line with a letter "2" as follows:

——— m(c)=1, $\overset{2}{———}$ m(c)=2.

We give two examples of ML diagrams:



We will define a pairing of Gauss diagrams and ML diagrams, by which the Vassiliev invariants v_1, v_2, $v_{3.1}$ and $v_{3.2}$ are expressed.

Let $\hat{G} = \{G, \epsilon\}$ be a Gauss diagram and $\hat{D} = \{D, m\}$ a ML diagram. Let $\psi : D \to G$ be an embedding of D into G which maps the circles of D to those of G preserving the orientations and each chord of D to a chord of G. Let $C(G)$ be the set of all chords of G. For ψ, define a map $\kappa_\psi : C(G) \to \{0, 1, 2\}$ by

$$\kappa_\psi(c) = \begin{cases} m(\psi^{-1}(c)) & \text{if } c \in \psi(D) \\ 0 & \text{if } c \notin \psi(D) \end{cases}$$

Two embedding ψ, φ are said to be equal if $\kappa_\psi = \kappa_\varphi$. The equivalence class of an embedding ψ is denoted by $[\psi]$.

Let $C(D)$ be the set of all chords of D. Define $\mathcal{E}([\psi])$ by

$$\mathcal{E}([\psi]) = \prod_{c \in C(D)} \{\epsilon(\psi(c))\}^{m(c)},$$

where the product is taken over all chords of D. Notice that this definition is well defined.

Define a pairing of Gauss diagrams and ML diagrams $\langle \hat{G}, \hat{D} \rangle_\chi$ by

$$\langle \hat{G}, \hat{D} \rangle_\chi = \sum_{[\psi]} \mathcal{E}([\psi]),$$

where the sum is taken over all the distinct equivalence classes $[\psi]$.

We will introduce some concepts for the GDK formulas.

First, we will introduce a concept of splitting the crossings of a link diagram. Let $\{L : a_1, \cdots, a_m\}$ be a link diagram L where we select some distinct crossings $a_1, \cdots, a_m$ out of all crossings of L. By a splitting information, we mean a finite sequence $[s_1, \cdots, s_m]$ ($s_i = \alpha, \beta, \gamma$), where α, β, γ are formal letters. For example, $[\alpha, \beta, \alpha, \gamma]$ ($m = 4$).

We shall define a link diagram $Q(\{L : a_1, \cdots, a_m\}, [s_1, \cdots, s_m])$ as follows. For $1 \leq i \leq m$, we replace each crossing a_i by Fig. 2 or Fig. 3, and give any orientation to the resulting diagram. Define $Q(\{L : a_1, \cdots, a_m\}, [s_1, \cdots, s_m])$ to be the resulting oriented link diagram.

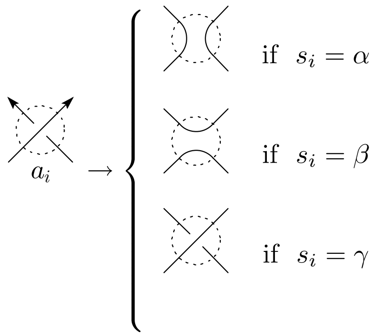


Fig. 2. Replacing a crossing of overpass

Remark 1. The calculation in the sequel does not depend on the orientation we give to the resulting diagram.

Remark 2. Notice that we do not split a_i, if $s_i = \gamma$. But we introduce it for convenience. We give a trivial example:

$$Q(\{K : a\}, [\gamma]) = K.$$

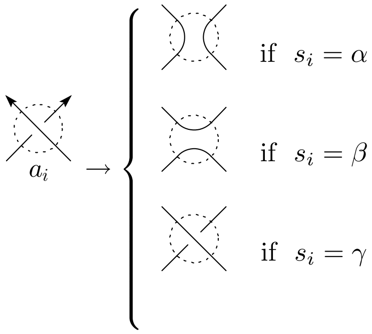


Fig. 3. Replacing a crossing of underpass

In this case, we do nothing at all at a_i.

REMARK 3. In this paper, we do not consider the case $s_i = \beta$, although it is needed for the GDK formulas at degree four [13].

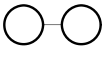
Let $L = \{K_1, K_2, \cdots, K_n\}$ be an n-component link diagram. Define $S(L)$ to be the formal sum of each component K_i

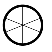
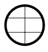
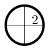
$$S(L) = \sum_{i=1}^{n} K_i.$$

Define $\alpha(L)$ to be a trivial link diagram of n-separated trivial knots which is obtained by switching the sign of the crossings of L properly. There are several ways to obtain $\alpha(L)$ from L. So $\alpha(L)$ cannot be uniquely determined from L. But the calculation in the sequel does not depend on the way we choose.

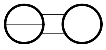
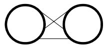
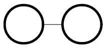
We give the Gauss diagram formulas for $v_1, v_2, v_{3.1}, v_{3.2}$, which have a nice property for acutual computations.

FORMULA 1. (GDK formulas) Let K be an one-component knot diagram and $\{K_1, K_2\}$ a two-component link diagram. The Vassiliev invariants $v_1, v_2, v_{3.1}, v_{3.2}$ have the explicit combinatorial expressions as follows:

(2.5) • $v_1(\{K_1, K_2\}) = \Big\langle G(\{K_1, K_2\}),$  $\Big\rangle_{\chi}$, (2.6) • $v_2(K) = -\frac{1}{6} + \Big\langle G(K),$  $\Big\rangle_\chi - \Big\langle G(\alpha(K)),$  $\Big\rangle_\chi,$

(2.7) • $v_{3.1}(K) = \Big\langle G(K),\ 2$  $+$  $+\frac{1}{2}$  $\Big\rangle_\chi - I_{3.1}(K),$

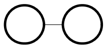
(2.8) • $v_{3.2}(\{K_1, K_2\}) = \Big\langle G(\{K_1, K_2\}),$

 $+$  $+\frac{1}{3}$  $\Big\rangle_\chi$

$-I_{3.2}(\{K_1, K_2\}).$

Set $R = G \circ \alpha \circ S \circ Q$. Here $I_{3.1}, I_{3.2}$ are given as follows:

(2.9) • $I_{3.1}(K) = \sum_a \Big\langle P(\{K : a\}),$  $\Big\rangle_\chi \Big\langle R(\{K : a\}, [\gamma] - [\alpha]),$  $\Big\rangle_\chi,$

(2.10) • $I_{3.2}(\{K_1, K_2\}) = \sum_a \Big\langle (P(\{K_1, K_2 : a\}),$  $\Big\rangle_\chi$

$\times \Big\langle R(\{K_1, K_2 : a\}, [\alpha] - [\gamma]),$  $\Big\rangle_\chi,$

where the sum $\sum_a$ is taken over all the crossings. □

Remark. The GDK formulas (2.6), (2.7) and (2.8) are expressed by $\alpha(L)$. Since the concept of $\alpha(L)$ depends only on its shadow of L, we expect that functions of $\alpha(L)$ may be expressed by Arnold's invariants of plane curves. This expectation is true for the GDK formulas up to degree three. In the next section, we will rewrite the GDK formulas (2.6), (2.7) and (2.8) in terms of J_+, J_- and St without using $\alpha(L)$.

Remark. These formulas were also obtained by using quantum field theoretical methods. A.C. Hirshfeld, U. Sassenberg, T. Kloker [4, 5, 6] gave the combinatorial expression for Vassiliev invariants of links up to degree three based on the study of the configuration space integral of the Chern-Simons gauge theory. In 1998, J.M.F. Labastida and E. Perez [10] obtained the combinatorial expression for the Vassiliev invariant of knots up to degree four based on the Chern-Simons gauge theory in the temporal gauge. The GDK formulas (2.6), (2.7) and (2.8) coincide with those obtained by these methods in [4, 10], [5, 10] and [6] respectively.

2.3. Another expression for the Gauss diagram formulas

Let us transform the GDK formulas for $v_{3.1}, v_{3.2}$ into more convenient forms to compute. Consider $R = G \circ \alpha \circ S \circ Q$ in (2.9) and (2.10) more carefully.

DEFINITION 2.1. Define two new knot diagrams $K_+^{[1]}, K_-^{[1]}$ by

$$\{K_+^{[1]}, K_-^{[1]}\} = Q(\{K : a\}, [\alpha]). \tag{2.11}$$

If a crossing a belongs to both K_1 and K_2, define $K^{[2]}$ by

$$K^{[2]} = Q(\{K_1, K_2 : a\}, [\alpha]). \tag{2.12}$$

See Fig. 4 and Fig. 5.

COROLLARY 1. *We denote the sign of the crossing a by $sign(a)$. Then, the GDK formulas in Formula 1 are transformed into the following forms:*

$$I_{3.1}(K) = \sum_a sign(a) \Big\langle G(\alpha(K)) - \sum_{s=\pm} G(\alpha(K_s^{[1]})), \bigoplus \Big\rangle_\chi, \tag{2.13}$$

$$I_{3.2}(\{K_1, K_2\}) = \sum_a sign(a) \Big\langle G(\alpha(K^{[2]})) - \sum_{i=1,2} G(\alpha(K_i)), \bigoplus \Big\rangle_\chi, \tag{2.14}$$

where the first sum is taken over all crossings of knot diagram K and the second sum is taken over all crossings where one branch comes from K_1 and the other branch comes from K_2.

PROOF. We insert the following obvious relations

$$R(\{K : a\}, [\gamma] - [\alpha]) = G(\alpha(K)) - \sum_{s=\pm} G(\alpha(K_s^{[1]})), \tag{2.15}$$

$$R(\{K_1, K_2 : a\}, [\alpha] - [\gamma]) = G(\alpha(K^{[2]})) - \sum_{i=1,2} G(\alpha(K_i)) \tag{2.16}$$

into $I_{3.1}$ and $I_{3.2}$ respectively. $\Big\langle P(\{K : a_1\}), \ominus \Big\rangle_\chi$ and $\Big\langle (P(\{K_1, K_2 : a_2\}), \bigcirc\!\!-\!\!\bigcirc \Big\rangle_\chi$ in $I_{3.1}$ and $I_{3.2}$ are equal to the sign of the crossing a_1 and a_2 respectively, if a_2 belongs to both K_1 and K_2. This proves Corollary 1. □

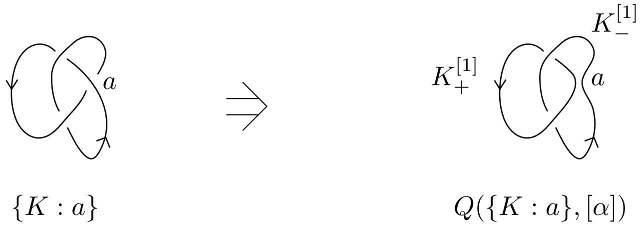


Fig. 4. Define $K_+^{[1]}$ and $K_-^{[1]}$ by $\{K_+^{[1]}, K_-^{[1]}\} \equiv Q(\{K : a\}, [\alpha])$.

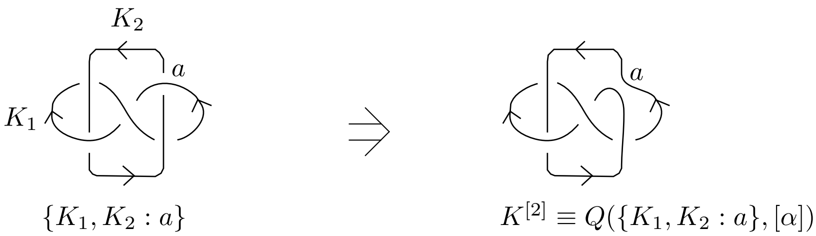


Fig. 5. Define a knot diagram $K^{[2]}$ by $K^{[2]} \equiv Q(\{K_1, K_2 : a\}, [\alpha])$.

3. The Gauss Diagram Formulas for the Kontsevich Integral and Invariants of Plane Curves

In this section, we discuss the relation between the GDK formulas and invariants of plane curves introduced by V. I. Arnold [1].

3.1. Invariants of plane curves

In this subsection, we review the invariants of plane curves as in [1, 2, 16].

We call an oriented immersed curve a *plane curve*. We change plane curves as shown in Fig.6, which are called *perestroikas*. There are three types of perestroika, *direct self-tangency perestroika*, *inverse self-tangency perestroika* and *triple point perestroika*.

We will give some definitions for the triple point perestroika. A *vanishing triangle* is the triangle formed by the three branches of a curve just before

1. direct self-tangency
2. inverse self-tangency
3. triple point

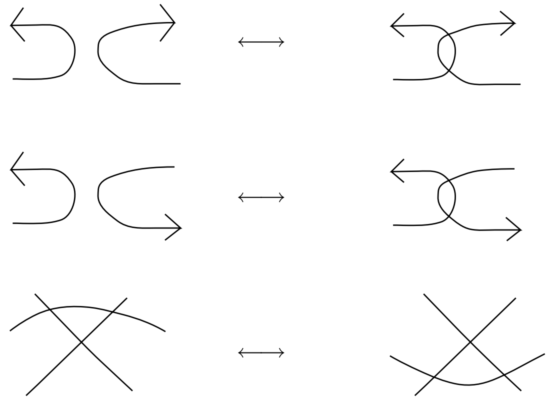


Fig. 6. Perestroikas

and just after the triple point perestroika in Fig. 6. A *newborn* (resp. *dying*) vanishing triangle is a vanishing triangle exsisting just after (resp. just before) the triple point perestroika.

We will define the *sign of a vanishing triangle* as follows. The orientation of the plane curve defines a cyclic ordering of the edges of the vanishing triangle. Hence the edges of the triangle acquire orientations induced by this ordering. But each edge also has its own direction, which may or may not coincide with the orientation defined by the ordering. For each vanishing triangle we define q by the number of the edge for which the orientation defined by the ordering coincides with its direction (see Fig. 7). The sign of vanishing triangle is defined to be $(-1)^q$.

A self-tangency perestroika is called *positive* (resp. negative) if it increases the number of double points by 2 (resp. -2). A triple point perestroika is called *positive* (resp. negative) if the sign of the newborn vanishing triangle is positive (resp. negative).

It is known that invariants J_+, J_- and St of plane curves are characterized by the following properties.

(1) J^+ does not change under inverse self-tangency or triple-point perestoikas but increases by 2 (resp. -2) under a positive (resp. negative) direct self-tangency perestroika.

(2) J^- does not change under direct self-tangency or triple-point pere-

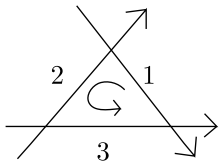


Fig. 7. Sign of vanishing triangle ($q = 1$)

stroikas but decreases by 2 (resp. -2) under a positive (resp. negative) inverse self-tangency perestroika.

(3) St does not change under self-tangency perestroikas but increases by 1 (resp. -1) under a positive (resp. negative) triple point perestroika.

(4) Set the initial values for the curves K_i ($i = 0, 1, 2, \cdots$) shown in Fig.8 as follows:

$$J^+(K_0) = 0, \quad J^+(K_{i+1}) = -2i, \quad (i = 0, 1, 2, \cdots) \tag{3.1}$$

$$J^-(K_0) = -1, \quad J^-(K_{i+1}) = -3i, \quad (i = 0, 1, 2, \cdots) \tag{3.2}$$

$$St(K_0) = 0, \quad St(K_{i+1}) = i, \quad (i = 0, 1, 2, \cdots) \tag{3.3}$$

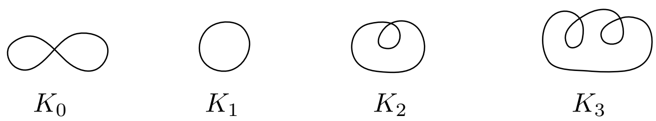


Fig. 8. K_i, ($i = 0, 1, 2, \cdots$)

3.2. The relation to invariants of plane curves

Let K be a knot diagram. Define $\beta(K)$ to be a plane curve obtained from K by ignoring its signs. For convenience, set

$$F_1(\alpha(K)) = \left\langle G(\alpha(K)), \bigoplus \right\rangle_\chi \tag{3.4}$$

$$F_2(\beta(K)) = -(St + \frac{1}{2}J_+)(\beta(K)). \tag{3.5}$$

Let P_{2A}, P_{2B}, P_3 be the operation of the direct self-tangency perestroika, inverse self-tangency perestroika and triple point perestroika respectively. Let R_{2A}, R_{2B}, R_3 be the operation of the Reidemeister move acting on knot diagrams naturally corresponding to P_{2A}, P_{2B}, P_3 respectively. Therefore, it is clear that

$$\beta(R_j(K)) = P_j(\beta(K)) \quad (j = 2A, 2B, 3). \tag{3.6}$$

LEMMA 3.1. *If* $F_1(K) = F_2(\beta(K))$, *then* $F_1(R_j(K)) = F_2(P_j(\beta(K)))$ $(j = 2A, 2B, 3)$.

PROOF. It is enough to show that F_1 and F_2 are changed by the same amount under the Reidemeister moves R_j and the operation of perestroikas P_j

There are three cases $(j = 2A, 2B, 3)$. We consider each case from our definitions of the pairing of Gauss diagrams and perestroikas of plane curves.
(1) $(j = 2A)$ F_1 and F_2 decrease by 1 under the Reidemeister move R_{2A} and direct self-tangency perestroika P_{2A} respectively.
(2) $(j = 2B)$ F_1 and F_2 do not change under the Reidemeister move R_{2B} and inverse self-tangency perestroika P_{2B} respectively.
(3) $(j = 3)$ There are two cases shown in Fig. 9. We shall prove Case A. We can prove Case B in the same way. Let $(\epsilon_1, \epsilon_2, \epsilon_3)$ be the three signs of the crossing which consist of the triangle of Reidemeister move R_3. Therefore there are 8 cases for the three signs $(\epsilon_1, \epsilon_2, \epsilon_3)$. Two of them $(\epsilon_1, \epsilon_2, \epsilon_3) = (+,+,+), (-,-,-)$ are forbidden since the Reidemeister move R_3 is impossible. For the other six cases, F_1 and F_2 decrease by 1 under the Reidemeister move R_3 and triple point perestroika P_3 respectively. □

LEMMA 3.2. *If* $F_1(K) = F_2(\beta(K))$, *then* $F_1(R_j(K)) = F_2(\beta(R_j(K)))$ $(j = 2A, 2B, 3)$.

PROOF. It is clear from Lemma 3.1 and (3.6). □

LEMMA 3.3.

$$F_1(\alpha(K)) = F_2(\beta(K)). \tag{3.7}$$

Fig. 9. Case A and B

REMARK. M. Polyak [16] obtained an identity similar to (3.7) in a different situation.

PROOF. From our definition of $\alpha(K)$, it can be expressed as

$$\alpha(K) = R_{j_n} R_{j_{n-1}} \cdots R_{j_1}(K_i), \tag{3.8}$$

where K_i is given by Fig. 8. From our definition of F_1 and F_2, it is clear that

$$F_1(K_i) = F_2(\beta(K_i)). \tag{3.9}$$

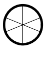
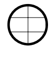
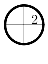
From (3.9) and Lemma 3.2, we can prove

$$F_1(R_{j_n} R_{j_{n-1}} \cdots R_{j_1}(K_i)) = F_2(\beta(R_{j_n} R_{j_{n-1}} \cdots R_{j_1}(K_i))) \tag{3.10}$$

by induction on n. This shows (3.7). □

THEOREM 1. *Each of the GDK formulas (2.6), (2.7) and (2.8) has another expression as follows.*

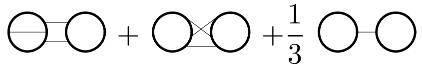
(3.11) • $v_2(K) = -\frac{1}{6} + \Big\langle G(K),$  $\Big\rangle_\chi + St(\beta(K)) + \frac{1}{2} J_+(\beta(K)),$

(3.12) • $v_{3.1}(K) = \Big\langle G(K), \ 2$  $+$  $+ \frac{1}{2}$  $\Big\rangle_\chi$

$$+ \sum_a sign(a) \Big\{ St(\beta(K)) - St(\beta(K_+^{[1]})) - St(\beta(K_-^{[2]}))$$

$$+\frac{1}{2}J_+(\beta(K)) - \frac{1}{2}J_+(\beta(K_+^{[1]})) - \frac{1}{2}J_+(\beta(K_-^{[1]}))\Big\}$$

(3.13) $$\bullet \quad v_{3.2}(\{K_1, K_2\}) = \Big\langle G(\{K_1, K_2\}),$$



$$\Big\rangle_\chi$$

$$+\sum_a sign(a)\Big\{St(\beta(K^{[2]})) - St(\beta(K_1)) - St(\beta(K_2)) + \frac{1}{2}J_+(\beta(K^{[2]})) - \frac{1}{2}J_+(\beta(K_1)) - \frac{1}{2}J_+(\beta(K_2))\Big\},$$

where the first sum is taken over all crossings of knot diagram K and the second sum is taken over all crossings where one branch comes from K_1 and the other branch comes from K_2.

Proof. Inserting (3.7) into (2.6), (2.13) and (2.14), we obtain (3.11), (3.12) and (3.13) respectively. □

4. The Gauss Diagram Formulas for the Kontsevich Integral and Polyak-Viro Type Formulas

In this section, we discuss the relation between the GDK formulas and the Polyak-Viro type formulas [17].

4.1. The preparation for Polyak-Viro type formulas

In this subsection, we shall fix notations as in M. Polyak and O. Viro [17]. This subsection is devoted to the oriented version of the definition of Subsection 2.2. We use arrows (oriented chords) instead of unoriented chords.

An *oriented Gauss Diagram* and *oriented ML Diagram* are defined as in Subsection 2.2, but we use arrows (oriented chords) instead of unoriented chords. See Fig. 10 for examples of oriented ML diagrams.

Let K be a knot diagram. For each crossing a of K, we set $\vec{y}^{-1}(a) = \{s(a), s'(a)\}$ as the inverse image of a, and join $s(a), s'(a)$ by an arrow oriented from the lower branch to the upper branch, and label this arrow by the sign of a. We define an oriented Gauss diagram $G^{\mathcal{A}}(K)$ ($\mathcal{A}$ denotes arrows) to be the result. See Fig. 11.

Fig. 10. Two examples for oriented ML diagrams

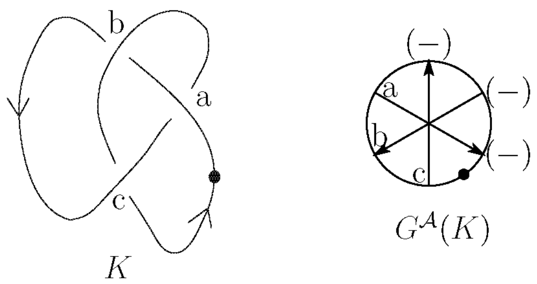


Fig. 11. an oriented Gauss diagram $G^{\mathcal{A}}(K)$

Let $\hat{G}^{\mathcal{A}} = \{G^{\mathcal{A}}, \epsilon\}$ and $\hat{D}^{\mathcal{A}} = \{D^{\mathcal{A}}, m\}$ ($\mathcal{A}$ denotes arrows) be an oriented Gauss diagram and oriented ML diagram respectively. Define a pairing $\langle \hat{G}^{\mathcal{A}}, \hat{D}^{\mathcal{A}} \rangle_{\chi}$ as in Subsection 2.2, but $\psi : D^{\mathcal{A}} \to G^{\mathcal{A}}$ preserves all the orientations of arrows.

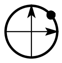
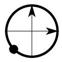
We shall use also a version which uses base points. *Based-oriented Gauss diagram* and *Based-oriented ML diagram* are obtained from oriented Gauss diagram and oriented ML diagram respectively by marking a point on the circles of the diagram. The base point is distinct from the endpoints of arrows. Define a pairing of an based-oriented Gauss diagram and based-oriented ML diagram $\langle \hat{G}^{\mathcal{A}}, \hat{D}^{\mathcal{A}} \rangle_{\chi}$ as in Subsection 2.2, but in this case, $\psi : D^{\mathcal{A}} \to G^{\mathcal{A}}$ maps the base point of $D^{\mathcal{A}}$ to the base point of $G^{\mathcal{A}}$.

4.2. The relation to the Polyak-Viro type formulas

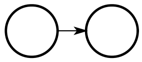
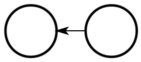
In this subsection, we derive the Polyak-Viro type formulas [17] from the GDK formulas (2.6), (2.7).

The next lamma is proved in [17], but for the matter of convenience for the reader we prove it below.

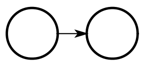
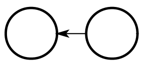
LEMMA 4.1.

(4.1) $\left\langle G^{\mathcal{A}}(K),\right.$  $\left.\right\rangle_{\chi} = \left\langle G^{\mathcal{A}}(K),\right.$  $\left.\right\rangle_{\chi}$

PROOF. Let L be a two-component link diagram. We use the obvious identity

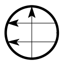
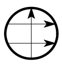
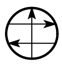
(4.2) $\left\langle G^{\mathcal{A}}(L),\right.$  $\left.\right\rangle_{\chi} = \left\langle G^{\mathcal{A}}(L),\right.$  $\left.\right\rangle_{\chi}$,

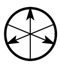
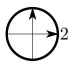
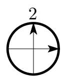
where we naturally extend the definition of pairng to two-component link diagrams. Lemma 4.1 is obtained by expanding the following identity:

(4.3) $0 = \sum_{c} sign(c) \left\{ \left\langle G(\{K_c^+, K_c^-\}),\right.\right.$  $-$  $\left.\left.\right\rangle_{\chi} \right\}$

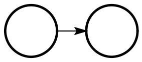
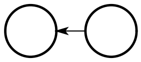
where the sum is taken over all crossings c, and $sign(c)$ denotes the sign of c. □

LEMMA 4.2.

(4.4) $\left\langle G^{\mathcal{A}}(K),\right.$  $+$  $-$  $-$ 

$+$  -3  $+\frac{1}{2}$  $+\frac{1}{2}$  $\left.\right\rangle_{\chi} = 0$

PROOF. The proof is similar to that of Lemma 4.1. Lemma 4.2 is obtained by expanding the following identity:

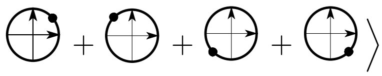
(4.5) $0 = \sum_{c} sign(c) \left\{ \left\langle G(\{K_c^+, K_c^-\}),\right.\right.$  $-$  $\left.\left.\right\rangle_{\chi} \right\}^2$

where the sum is taken over all crossings c. □

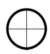
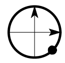
THEOREM 2. *The GDK formula (2.6) has another expression as follows:*

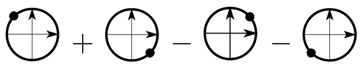
(4.6) $v_2 = -\frac{1}{6} + 4 \left\langle G^{\mathcal{A}}(K),\right.$  $\left.\right\rangle_{\chi}$, *which is the same as the Polyak-Viro type formula [17].*

Proof. We expand unoriented ML diagram into based-oriented ML diagram as follows:

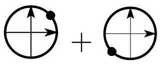
(4.7) $\Big\langle G(K),$  $\Big\rangle_\chi = \Big\langle G^{\mathcal{A}}(K),$  $\Big\rangle_\chi.$

We set $\alpha(K)$ to be a descending diagram which is obtained by switching all the undercrossings to overcrossings when going along the knot from the base point. Then we obtain

(4.8) $\Big\langle G(\alpha(K)),$  $\Big\rangle_\chi = \Big\langle G^{\mathcal{A}}(\alpha(K)),$  $\Big\rangle_\chi$

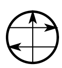
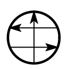
$= \Big\langle G^{\mathcal{A}}(K),$  $\Big\rangle_\chi.$

Notice that if the sign of the crossing is changed, the direction of the corresponding arrow is reversed and the sign of coefficient of the based oriented ML diagram is changed. Inserting (4.7) and (4.8) into (2.6), we obtain

(4.9) $v_2(K) = -\frac{1}{6} + 2\Big\langle G^{\mathcal{A}}(K),$  $\Big\rangle_\chi.$

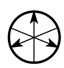
Using Lemma 4.1, we obtain (4.6). □

Theorem 3. *The GDK formula (2.7) has another expression as follows:*

(4.10) $v_{3.1} = \Big\langle G^{\mathcal{A}}(K),\ 8$  $+2$  $+2$  $\Big\rangle,$

which is the same as the Polyak-Viro type formula [17].

Proof. Using the similar argument to (4.8), (2.9) is transformed into:

(4.11) $I_{3.1}(K) = \Big\langle G^{\mathcal{A}}(K),$  -3  $\Big\rangle_\chi.$

Inserting Lemma 4.2 and (4.11) into (2.7), we obtain (4.10). □

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