

Tyurin Parameters and Elliptic Analogue of Nonlinear Schrödinger Hierarchy

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Abstract. Two “elliptic analogues” of the nonlinear Schrödinger hierarchy are constructed, and their status in the Grassmannian perspective of soliton equations is elucidated. In addition to the usual fields u, v , these elliptic analogues have new dynamical variables called “Tyurin parameters,” which are connected with a family of vector bundles over the elliptic curve in consideration. The zero-curvature equations of these systems are formulated by a sequence of 2×2 matrices $A_n(z)$, $n = 1, 2, \dots$, of elliptic functions. In addition to a fixed pole at $z = 0$, these matrices have several extra poles. Tyurin parameters consist of the coordinates of those poles and some additional parameters that describe the structure of $A_n(z)$ ’s. Two distinct solutions of the auxiliary linear equations are constructed, and shown to form a Riemann-Hilbert pair with degeneration points. The Riemann-Hilbert pair is used to define a mapping to an infinite dimensional Grassmann variety. The elliptic analogues of the nonlinear Schrödinger hierarchy are thereby mapped to a simple dynamical system on a special subset of the Grassmann variety.

1. Introduction

Many integrable systems are expressed in the form of a Lax equation $\partial_t A(\lambda) = [B(\lambda), A(\lambda)]$ or a zero-curvature equation $[\partial_x - A(\lambda), \partial_t - B(\lambda)] = 0$, where $A(\lambda)$ and $B(\lambda)$ are matrices of rational functions of the spectral parameter λ . In other words, these Lax or zero-curvature equations are defined on the Riemann sphere. Some integrable systems, such as the elliptic Calogero-Moser system and the Landau-Lifshitz equation, have a Lax or zero-curvature representation defined on a torus, i.e., a complex elliptic curve. One will naturally expect to find a generalization to a curve of higher genus. Unfortunately, it is well known that such a naive attempt will

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be confronted with a serious difficulty that stems from the Riemann-Roch theorem [17].

Recently, Krichever presented a general scheme for constructing a Lax or zero-curvature equation on an algebraic curve Γ of arbitrary genus [8]. A central idea is to allow the matrices A, B to have extra “movable” poles $\gamma_1, \dots, \gamma_{rg} \in \Gamma$, where r is the size of the matrices. Moreover, the matrices A, B at these poles are assumed to have a special structure. A set of additional parameters are introduced to parametrize this special structure. The coordinates of poles and these parameters are called “Tyurin parameters.” This notion originates in algebraic geometry of holomorphic vector bundles over algebraic curves [16], and was applied by Krichever and Novikov in 1970’s to the study of commutative rings of differential operators [5, 6, 7]. The aforementioned difficulty can be resolved by adding Tyurin parameters as new dynamical variables.

Once applied to zero-curvature equations, Krichever’s method yielded a large class of $1 + 1$ dimensional integrable PDE’s. These equations are to be called “soliton equations” associated with an algebraic curve (though it is not known whether these equations do have a soliton or soliton-like solution). For instance, Krichever illustrates his construction for the case of a “field analogue” of the elliptic Calogero-Moser system. This raises a natural question: What is the status of these new equations in the Grassmannian perspective of soliton equations due to Sato [14] and Segal and Wilson [15]?

We address this problem in a simplified setting, namely, zero-curvature equations of 2×2 matrices defined on an elliptic curve. This system is an analogue of the usual nonlinear Schrödinger hierarchy. More precisely, we construct two distinct versions of this “elliptic analogue,” one being based on Krichever’s idea, and the other inspired by the work of Enriquez and Rubtsov [2]. Whereas Krichever’s construction requires all Tyurin parameters to be dynamical variables, Enriquez and Rubtsov keeps the position of poles constant and use the other parameters as dynamical variables. In this respect, the elliptic analogue à la Enriquez and Rubtsov’s is much closer to usual soliton equations.

Our strategy is, firstly, to derive a kind of Riemann-Hilbert problem for these systems, and secondly, to translate it to the language of an infinite dimensional Grassmann variety. This is indeed the procedure that has been used in the literature for many soliton equations and some higher dimen-

sional systems; see, e.g., the book of Mason and Woodhouse [11]. The usual Riemann-Hilbert problem, however, does not work literally in the present situation. Whereas the usual Riemann-Hilbert problem is based on *triviality* of a holomorphic vector bundle over the Riemann sphere, the systems formulated by Tyurin parameters are obviously related to a *nontrivial* holomorphic vector bundle over an algebraic curve of positive genus. An answer to this puzzle can be found in the work of Krichever and Novikov [5, 6, 7] cited above. They consider a Riemann-Hilbert problem with *degeneration points*; Tyurin parameters are nothing but the geometric data of those points. The next task is, therefore, to connect this kind of Riemann-Hilbert problems with an infinite dimensional Grassmann variety. Fortunately, a related issue has been investigated by Previato and Wilson [13]. They demonstrate therein a Grassmannian version of the “dressing method” — a classical technique in soliton theory — to solve a Riemann-Hilbert problem of the same type. Moreover, their paper shows what should be the “vacuum” (to be “dressed”) that corresponds to a holomorphic vector bundle in the Tyurin parametrization. Our goal is to develop a similar machinery for the present setting.

This paper is organized as follows. Section 2 is a brief review of the usual nonlinear Schrödinger hierarchy. This will serve as a prototype of the subsequent construction. Section 3 is devoted to the construction of the first version, à la Krichever, of the elliptic analogues. A technical clue is a generating function $U(z)$, which has been used for the usual nonlinear Schrödinger hierarchy as well. This enables one to formulate the generators of time evolutions systematically. Section 4 deals with an auxiliary linear system of the hierarchy and a pair of solutions thereof. This pair of solutions turns out to satisfy a Riemann-Hilbert problem with degeneration points on the elliptic curve. Section 5 presents main results of this paper, namely, a Grassmannian perspective of the elliptic analogue of the nonlinear Schrödinger hierarchy. An infinite dimensional Grassmann variety Gr , a special basepoint (“vacuum”) $W_0 \in \text{Gr}$ and the set $\mathcal{M} \subset \text{Gr}$ of “dressed vacua” are introduced. The Riemann-Hilbert pair determines a point of \mathcal{M} , whose motion turns out to obey a simple exponential law. The elliptic nonlinear Schrödinger hierarchy is thus mapped to a dynamical system on \mathcal{M} . In Section 6, the same story is repeated for the elliptic analogue à la Enriquez and Rubtsov. Our conclusion is shown in Section 7.

2. Nonlinear Schrödinger Hierarchy

As a prototype of the elliptic analogue, we here review a standard construction of the nonlinear Schrödinger hierarchy. Generalities and backgrounds of this kind of construction of soliton equations can be found in Frenkel's lectures [3].

2.1. A -matrix

The construction starts from the A -matrix

$$(2.1) \quad A(\lambda) = \begin{pmatrix} \lambda & u \\ v & -\lambda \end{pmatrix}$$

with a rational spectral parameter $\lambda \in \mathbf{P}^1$; u and v are fields on the x space. In view of the homogeneous grading of an underlying loop algebra, it is natural to express this matrix as

$$(2.2) \quad A(\lambda) = J\lambda + A^{(1)}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}.$$

2.2. Generating functions

A clue of the construction of the hierarchy is a Laurent series

$$U(\lambda) = \sum_{n=0}^{\infty} U_n \lambda^{-n}, \quad U_0 = J,$$

that satisfies the differential equation

$$(2.3) \quad [\partial_x - A(\lambda), U(\lambda)] = 0.$$

Although this equation itself does not determine $U(\lambda)$ uniquely, there is a *good* or *canonical* solution that takes the form

$$(2.4) \quad U(\lambda) = \phi(\lambda) J \phi(\lambda)^{-1},$$

where $\phi(\lambda)$ is a Laurent series of the form

$$\phi(\lambda) = I + \sum_{n=1}^{\infty} \phi_n \lambda^{-n}$$

and satisfies the differential equations

$$(2.5) \quad \partial_x \phi(\lambda) = A(\lambda)\phi(\lambda) - \phi(\lambda)J\lambda.$$

A solution of (2.3) of this form is indeed “good” or “canonical” in the sense that the coefficients U_n can be calculated from $A(\lambda)$ by a purely algebraic procedure (namely, without actually solving differential equations) as follows. Expanded in powers of λ , (2.3) becomes a system of differential equations

$$\partial_x U_n = JU_{n+1} - U_{n+1}J + [A^{(1)}, U_n]$$

for the coefficients U_n . On the other hand, if $U(\lambda)$ is written as (2.4), the algebraic constraint

$$(2.6) \quad U(\lambda)^2 = I$$

is automatically satisfied. This yields the algebraic relations

$$0 = JU_{n+1} + U_{n+1}J + \sum_{m=1}^n U_m U_{n+1-m}$$

of U_n 's. One can use these relations to eliminate the term $U_{n+1}J$ on the right hand sides of the foregoing differential equations. The outcome are the recurrence relations

$$(2.7) \quad 2JU_{n+1} = \partial_x U_n - [A^{(1)}, U_n] - \sum_{m=1}^n U_m U_{n+1-m}$$

that determine U_n 's successively as

$$U_1 = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -\frac{1}{2}uv & \frac{1}{2}u_x \\ -\frac{1}{2}v_x & \frac{1}{2}uv \end{pmatrix},$$

etc. Note that the matrix elements of all U_n 's thus turn out to be “local” quantities, namely, polynomials of x -derivatives of u and v .

The coefficients of $\phi(\lambda)$ are “nonlocal.” To construct $\phi(\lambda)$ from $A(\lambda)$, one expands (2.5) to the differential equations

$$(2.8) \quad \partial_x \phi_n = [J, \phi_{n+1}] + A^{(1)}\phi_n$$

for the coefficients and solves them step by step. Actually, this is not so straightforward; one has to split ϕ_n into the diagonal and off-diagonal parts,

$$\phi_n = (\phi_n)_{\text{diag}} + (\phi_n)_{\text{off-diag}},$$

and consider them separately. The differential equation for the coefficients of $\phi(\lambda)$ are thereby decomposed to the two equations

$$(2.9) \quad \partial_x (\phi_n)_{\text{diag}} = (A^{(1)} \phi_n)_{\text{diag}}$$

and

$$(2.10) \quad \partial_x (\phi_{n-1})_{\text{off-diag}} = [J, (\phi_n)_{\text{off-diag}}] + (A^{(1)} \phi_{n-1})_{\text{off-diag}}.$$

(For convenience, the index n in the second equation has been shifted.) The first equation determines $(\phi_n)_{\text{diag}}$, up to integration constants, if $\phi_1, \dots, \phi_{n-1}$ and $(\phi_n)_{\text{off-diag}}$ are given. The second equation is rather an algebraic equation that determines $(\phi_n)_{\text{off-diag}}$ from $\phi_1, \dots, \phi_{n-1}$. To construct a solution, therefore, one has to use these equations in a cyclic way:

1. Solve (2.10) for $(\phi_n)_{\text{off-diag}}$.
2. Solve (2.9) for $(\phi_n)_{\text{diag}}$.
3. Increase n by 1 and return to step 1.

The first step of this cycle is to construct $(\phi_1)_{\text{off-diag}}$ as a solution of (2.10) ($n = 1$); note that the only data necessary here is $\phi_0 = I$. Starting with this step, one can proceed as $(\phi_1)_{\text{off-diag}} \rightarrow (\phi_1)_{\text{diag}} \rightarrow (\phi_2)_{\text{off-diag}} \rightarrow (\phi_2)_{\text{diag}} \rightarrow \dots$. Changing integration constants in the solution of (2.9) amounts to the right action $\phi(\lambda) \rightarrow \phi(\lambda)C(\lambda)$ by a diagonal matrix $C(\lambda) = \text{diag}(c_1(\lambda), c_2(\lambda))$ of Laurent series with constant coefficients.

2.3. Construction of hierarchy

Having constructed the generating function $U(\lambda)$, one can formulate the hierarchy as the system of the Lax equations

$$(2.11) \quad [\partial_{t_n} - A_n(\lambda), U(\lambda)] = 0,$$

where $A_n(\lambda)$ denotes the ‘‘polynomial part’’ of $U(\lambda)\lambda^n$:

$$(2.12) \quad A_n(\lambda) = U_0\lambda^n + U_1\lambda^{n-1} + \dots + U_n.$$

Since $U_1 = A^{(1)}$, $A_1(\lambda)$ coincides with $A(\lambda)$, so that x can be identified with the first time variable t_1 . As we shall show later in a more complicated situation, one can derive the zero-curvature equations

$$(2.13) \quad [\partial_{t_m} - A_m(\lambda), \partial_{t_n} - A_n(\lambda)] = 0$$

from these Lax equations of $U(\lambda)$. Actually, another set of zero-curvature equations, i.e.,

$$(2.14) \quad [\partial_{t_m} - A_m^-(\lambda), \partial_{t_n} - A_n^-(\lambda)] = 0,$$

can be derived for the Laurent “tail”

$$(2.15) \quad A_n^-(\lambda) = A_n(\lambda) - U(\lambda)\lambda^n = -U_{n+1}\lambda^{-1} - U_{n+2}\lambda^{-2} - \dots$$

as well. These “dual” zero-curvature equations are the Frobenius integrability condition of the linear system

$$(2.16) \quad \partial_{t_n} \phi(\lambda) = A_n^-(\lambda)\phi(\lambda),$$

which thereby determine the time evolutions of $\phi(\lambda)$. This linear system turns out to be equivalent to the usual auxiliary linear system

$$(2.17) \quad \partial_{t_n} \psi(\lambda) = A_n(\lambda)\psi(\lambda)$$

upon identifying

$$\psi(\lambda) = \phi(\lambda) \exp\left(\sum_{n=1}^{\infty} t_n J \lambda^n\right) \quad (t_1 = x).$$

3. Construction of Elliptic Analogue à la Krichever

Let Γ be a nonsingular elliptic curve realized as the torus $\mathbf{C}/(2\omega_1\mathbf{Z} + 2\omega_3\mathbf{Z})$, and z the complex coordinate of \mathbf{C} , which is also understood as a local coordinate of Γ . The polynomial matrices $A(\lambda), A_n(\lambda)$ in the nonlinear Schrödinger hierarchy are replaced by matrices $A(z), A_n(z)$ of meromorphic functions on Γ . They have a fixed pole at $z = 0$ (which amounts to $\lambda = \infty$ in the nonlinear Schrödinger hierarchy) and two “movable” poles at $z = \gamma_1, \gamma_2$, $\gamma_1 \neq \gamma_2$.

3.1. A -Matrix on elliptic curve

The role of the A -matrix in the usual nonlinear Schrödinger hierarchy is now played by a 2×2 matrix $A(z)$ ($z \in \Gamma$) of meromorphic functions on Γ with the following properties:

1. $A(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. As $z \rightarrow 0$,

$$(3.1) \quad A(z) = \begin{pmatrix} z^{-1} & u \\ v & -z^{-1} \end{pmatrix} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$(3.2) \quad A(z) = \frac{\beta_s {}^t \alpha_s}{z - \gamma_s} + O(1),$$

where α_s and β_s are two-dimensional column vectors that do not depend on z . α_s is normalized as $\alpha_s = {}^t(\alpha_s, 1)$.

γ_s and α_s in this definition are the Tyurin parameters in the present setting. u and v are counterparts of those in the nonlinear Schrödinger hierarchy. All these parameters are understood to be dynamical, i.e., a function of x (and the time variables t_n to be introduced later). We have thus altogether six dynamical variables $\gamma_1, \gamma_2, \alpha_1, \alpha_2, u, v$.

LEMMA 1. *If $\alpha_1 \neq \alpha_2$, a matrix $A(z)$ of meromorphic functions on Γ with these properties does exist. It is unique and can be written explicitly in terms of the Weierstrass zeta function $\zeta(z)$ as*

$$(3.3) \quad A(z) = \sum_{s=1,2} \beta_s {}^t \alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)) + \begin{pmatrix} \zeta(z) & u \\ v & -\zeta(z) \end{pmatrix},$$

where

$$(3.4) \quad \beta_1 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -1 \\ -\alpha_2 \end{pmatrix}, \quad \beta_2 = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 1 \\ \alpha_1 \end{pmatrix}.$$

PROOF. The defining properties of $A(z)$ imply that $A(z)$ can be written as

$$A(z) = \sum_{s=1,2} \beta_s {}^t\alpha_s \zeta(z - \gamma_s) + J\zeta(z) + C,$$

where C is a constant matrix. By the residue theorem, the coefficients have to satisfy the linear relation

$$\sum_{s=1,2} \beta_s {}^t\alpha_s + J = 0$$

that ensures that $A(z)$ is single valued on Γ . Solving these equations for β_s leads to the formula stated in the lemma. On the other hand, matching with the Laurent expansion of $A(z)$ at $z = 0$ yields to the relation

$$A^{(1)} = \sum_{s=1,2} \beta_s {}^t\alpha_s \zeta(-\gamma_s) + C,$$

which determines C . \square

The Tyurin parameters γ_s and α_s are required to satisfy the equations

$$(3.5) \quad \partial_x \gamma_s + \text{Tr} \beta_s {}^t\alpha_s = 0,$$

$$(3.6) \quad \partial_x {}^t\alpha_s + {}^t\alpha_s A^{(s,1)} = \kappa_s {}^t\alpha_s,$$

where $A^{(s,1)}$ stands for the constant term of the Laurent expansion of $A(z)$ at $z = \gamma_s$,

$$A^{(s,1)} = \lim_{z \rightarrow \gamma_s} \left(A(z) - \frac{\beta_s {}^t\alpha_s}{z - \gamma_s} \right),$$

and κ_s is a constant to be determined by the equation itself. More explicitly, we have

$$(3.7) \quad \partial_x \gamma_1 = \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2}, \quad \partial_x \gamma_2 = -\frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2},$$

$$(3.8) \quad \partial_x \alpha_1 = -2\alpha_1 \zeta_{12} - v + \alpha_1^2 u, \quad \partial_x \alpha_2 = 2\alpha_2 \zeta_{12} - v + \alpha_2^2 u,$$

where

$$\zeta_{12} = \zeta(\gamma_1) - \zeta(\gamma_2) - \zeta(\gamma_1 - \gamma_2),$$

and the constants κ_s take the form

$$(3.9) \quad \kappa_s = -\frac{2\alpha_s}{\alpha_1 - \alpha_2} \zeta_{12} + \alpha_s u.$$

As Krichever's lemma [8, Lemma 5.2] shows, these equations ensure that the auxiliary linear system $\partial_x \psi(z) = A(z)\psi(z)$ has a 2×2 matrix solution that is holomorphic at $z = \gamma_s$ and invertible except at these points. One will notice from (3.7) and (3.8) that not all of the six dynamical variables $\gamma_1, \gamma_2, \alpha_1, \alpha_2, u, v$ are independent; for instance, one can solve (3.8) for u and v to eliminate u and v as auxiliary dynamical variables. In the following, however, we shall treat these six variables on a equal footing.

3.2. Generating functions

We now proceed to the construction of two generating functions

$$\phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^n, \quad U(z) = J + \sum_{n=1}^{\infty} U_n z^n.$$

The first generating function $\phi(z)$ is a Laurent series that satisfies the differential equation

$$(3.10) \quad \partial_x \phi(z) = A(z)\phi(z) - \phi(z)Jz^{-1}.$$

Here $A(z)$ is understood to be its Laurent expansion

$$(3.11) \quad A(z) = Jz^{-1} + \sum_{n=1}^{\infty} A^{(n)} z^{n-1}$$

at $z = 0$; the first few coefficients of this expansion read

$$A^{(1)} = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

$$A^{(2)} = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} \alpha_1 \wp(\gamma_1) - \alpha_2 \wp(\gamma_2) & \wp(\gamma_1) - \wp(\gamma_2) \\ \alpha_1 \alpha_2 (\wp(\gamma_1) - \wp(\gamma_2)) & \alpha_2 \wp(\gamma_1) - \alpha_1 \wp(\gamma_2) \end{pmatrix},$$

etc.

LEMMA 2. *A Laurent series solution $\phi(z)$ of (3.10) does exist.*

PROOF. Expanded in powers of z , (3.10) yields the differential equations

$$\partial_x \phi_n = [J, \phi_{n+1}] + \sum_{m=1}^{n+1} A^{(m)} \phi_{n+1-m}$$

for the coefficients ϕ_n . One can decompose these equations into the diagonal and off-diagonal parts. The diagonal part becomes the equation

$$\partial_x (\phi_n)_{\text{diag}} = \sum_{m=1}^{n+1} (A^{(m)} \phi_{n+1-m})_{\text{diag}},$$

which determines the diagonal part $(\phi_n)_{\text{diag}}$ of ϕ_n up to integration constants. The off-diagonal part gives the algebraic relation

$$\partial_x (\phi_n)_{\text{off-diag}} = [J, (\phi_{n+1})_{\text{off-diag}}] + \sum_{m=1}^{n+1} (A^{(m)} \phi_{n+1-m})_{\text{off-diag}}.$$

The off-diagonal part $(\phi_{n+1})_{\text{off-diag}}$ of ϕ_{n+1} is thus determined from ϕ_1, \dots, ϕ_n . \square

The second generating function $U(z)$ can be obtained from $\phi(z)$ as

$$(3.12) \quad U(z) = \phi(z) J \phi(z)^{-1},$$

which satisfies the differential equation

$$(3.13) \quad [\partial_x - A(z), U(z)] = 0,$$

and the algebraic constraint

$$(3.14) \quad U(z)^2 = I.$$

As we have seen in the case of the nonlinear Schrödinger hierarchy, this algebraic constraint singles out a unique Laurent series solution of (3.13), and the Laurent coefficients can be calculated by a set of recurrence relations.

LEMMA 3. *The coefficients U_n of $U(z)$ satisfy the recurrence relations*

$$(3.15) \quad 2JU_{n+1} = \partial_x U_n - \sum_{m=1}^{n+1} [A^{(m)}, U_{n+1-m}] - \sum_{m=1}^n U_m U_{n+1-m}.$$

PROOF. (3.13) yields the differential equations

$$\partial_x U_n = JU_{n+1} - U_{n+1}J + \sum_{m=1}^{n+1} [A^{(m)}, U_{n+1-m}]$$

for the coefficients U_n . The algebraic constraint $U(z)^2 = 1$ gives the algebraic relations

$$0 = JU_{n+1} + UJ_{n+1} + \sum_{m=1}^n U_m U_{n+1-m}.$$

Combining them, one obtains the recurrence relation. \square

One can thus calculate U_n 's successively from the Laurent coefficients $A^{(n)}$ of $A(z)$ as

$$\begin{aligned} U_1 &= \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \\ U_2 &= \begin{pmatrix} -\frac{1}{2}uv & \frac{1}{2}u_x \\ -\frac{1}{2}v_x & \frac{1}{2}uv \end{pmatrix} \\ &\quad + \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} 0 & \wp(\gamma_1) - \wp(\gamma_2) \\ \alpha_1 \alpha_2 (\wp(\gamma_1) - \wp(\gamma_2)) & 0 \end{pmatrix}, \end{aligned}$$

etc. In particular, the matrix elements of all U_n 's turn out to be a polynomial of x -derivatives of u, v, γ_s, α_s .

3.3. Construction of hierarchy

Generators of time evolution are 2×2 matrices $A_n(z)$, $n = 1, 2, \dots$, of meromorphic functions on Γ with the following properties:

1. $A_n(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. As $z \rightarrow 0$,

$$(3.16) \quad A_n(z) = U(z)z^{-n} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$(3.17) \quad A_n(z) = \frac{\beta_{n,s} {}^t \alpha_s}{z - \gamma_s} + O(1),$$

where $\beta_{n,s}$ is a two-dimensional column vector that does not depend on z .

LEMMA 4. *If $\alpha_1 \neq \alpha_2$, a matrix $A_n(z)$ of meromorphic functions on Γ with these properties does exist. It is unique and can be written explicitly as*

$$(3.18) \quad A_n(z) = \sum_{s=1,2} \beta_{n,s} {}^t \alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)) + \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \partial_z^m \zeta(z) U_{n-1-m} + U_n.$$

The vectors $\beta_{n,s}$ are determined by the linear equation

$$(3.19) \quad \sum_{s=1,2} \beta_{n,s} {}^t \alpha_s + U_{n-1} = 0$$

that ensures the single-valuedness of $A_n(z)$ on Γ .

PROOF. Repeat the same reasoning as the case of $A(z)$. \square

Solving the last linear equation, one can eventually find an explicit form of $A_n(z)$. For instance, $A_1(z)$ coincides with $A(z)$, and $A_2(z)$ takes the form

$$(3.20) \quad A_2(z) = \sum_{s=1,2} \beta_{2,s} {}^t \alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)) + J\wp(z) + U_1 \zeta(z) + U_2,$$

where

$$(3.21) \quad \beta_{2,1} = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} u\alpha_2 \\ -v \end{pmatrix}, \quad \beta_{2,2} = \frac{1}{\alpha_1 - \alpha_2} \begin{pmatrix} -u\alpha_1 \\ v \end{pmatrix}.$$

Let us assume the genericity condition

$$(3.22) \quad \alpha_1 \neq \alpha_2$$

throughout the following consideration. We now formulate an elliptic analogue of the nonlinear Schrödinger hierarchy as the system of the Lax equations

$$(3.23) \quad [\partial_{t_n} - A_n(z), U(z)] = 0$$

for the generating function $U(z)$ and the differential equations

$$(3.24) \quad \partial_{t_n} \gamma_s + \text{Tr} \beta_{n,s} {}^t \alpha_s = 0,$$

$$(3.25) \quad \partial_{t_n} {}^t \alpha_n + {}^t \alpha_s A_n^{(s,1)} = \kappa_{n,s} \alpha_s$$

for the Tyurin parameters. Here $A_n^{(s,1)}$ denotes the constant term of the Laurent expansion of $A_n(z)$ at $z = \gamma_s$, i.e.,

$$A_n^{(s,1)} = \lim_{z \rightarrow \gamma_s} \left(A_n(z) - \frac{\beta_{n,s} {}^t \alpha_s}{z - \gamma_s} \right),$$

and $\kappa_{n,s}$ is a constant determined by the differential equation itself. As in the case of (3.5) and (3.6), the two equations (3.24) and (3.25) for the Tyurin parameters are the necessary and sufficient conditions for the auxiliary linear system $\partial_{t_n} \psi(z) = A_n(z) \psi(z)$ to have a 2×2 matrix solution that is holomorphic at $z = \gamma_s$ and invertible except at these points.

3.4. Zero-curvature equations

Commutativity of the time evolutions in t_n 's is by no means obvious from the construction. If one can derive the zero-curvature equations for $A_n(z)$'s, commutativity of the time evolutions is an immediate consequence. As it turns out below, however, the zero-curvature equations in the present setting possess richer contents.

Let us first derive a “dual” expression of the curvature components

$$(3.26) \quad F_{mn}(z) = [\partial_{t_m} - A_m(z), \partial_{t_n} - A_n(z)].$$

Let $A_n^+(z)$ denote the “tail” part in the Laurent expansion of (3.16). Namely,

$$(3.27) \quad A_n^+(z) = A_n(z) - U(z)z^{-n},$$

which has a Laurent expansion of the form

$$A_n^+(z) = (A_n^{(n+1)} - U_{n+1})z + (A_n^{(n+2)} - U_{n+2})z^2 + \cdots,$$

where $A_n^{(m)}$'s denote the coefficients of the Laurent expansion

$$A_n(z) = \sum_{m=0}^{\infty} A_n^{(m)} z^{m-n}$$

of $A_n(z)$ at $z = 0$. The Lax equations (3.23) of $U(z)$ can be rewritten in the “dual” form

$$(3.28) \quad \partial_{t_n} U(z) = [A_n^+(z) - U(z)z^{-n}, U(z)] = [A_n^+(z), U(z)].$$

The curvature components turn out to have a similar dual expression as follows.

LEMMA 5. *If the Lax equations (3.23) are satisfied, the curvature components $F_{mn}(z)$ can be written in the dual form*

$$(3.29) \quad F_{mn}(z) = [\partial_{t_m} - A_m^+(z), \partial_{t_n} - A_n^+(z)].$$

PROOF. Differentiating $A_m(z) = U(z)z^{-m} + A_m^+(z)$ by t_n and using the Lax equation (3.23), one has

$$\begin{aligned} \partial_{t_n} A_m(z) &= [A_n(z), U(z)]z^{-m} + \partial_{t_n} A_m^+(z) \\ &= [A_n^+(z), U(z)]z^{-m} + \partial_{t_n} A_m^+(z), \end{aligned}$$

and exchanging m and n ,

$$\partial_{t_m} A_n(z) = [A_m^+(z), U(z)]z^{-n} + \partial_{t_m} A_n^+(z).$$

As for the commutator $[A_m(z), A_n(z)]$,

$$\begin{aligned} [A_m(z), A_n(z)] &= [U(z)z^{-m} + A_m^+(z), U(z)z^{-n} + A_n^+(z)] \\ &= [A_m^+(z), U(z)]z^{-m} - [A_n^+(z), U(z)]z^{-n} \\ &\quad + [A_m^+(z), A_n^+(z)]. \end{aligned}$$

Collecting these pieces yields the dual expression of the curvature. \square

LEMMA 6. *If (3.24) and (3.25) are satisfied, the zero-curvature equations*

$$(3.30) \quad [\partial_{t_m} - A_m(z), \partial_{t_n} - A_n(z)] = 0$$

can be derived from the Lax equations (3.23).

PROOF. The following method of proof originates in the early work of Krichever and Novikov [7]. The curvature component $F_{mn}(z)$ is a matrix of meromorphic functions on Γ . Suppose that $F_{mn}(z)$ turns out to satisfy the following conditions:

1. $F_{mn}(z)$ is holomorphic at all points of Γ other than possible poles at γ_1, γ_2 .
2. As $z \rightarrow 0$, $F_{mn}(z) = O(z)$.
3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$F_{mn}(z) = \frac{\beta_{mn,s} {}^t\alpha_s}{z - \gamma_s} + O(1),$$

where $\beta_{mn,s}$ is a two-dimensional column vector.

Such a matrix of function can be expressed as

$$F_{mn}(z) = \sum_{s=1,2} \beta_{mn,s} {}^t\alpha_s (\zeta(z - \gamma_s) + \zeta(\gamma_s)).$$

By the residue theorem, the coefficients satisfy the relation

$$\sum_{s=1,2} \beta_{mn,s} {}^t\alpha_s = 0,$$

which, under the the genericity condition (3.22), imply that $\beta_{mn,s} = 0$, hence $F_{mn}(z) = 0$. Thus the proof is reduced to confirming that $F_{mn}(z)$ does have the three properties. The first and second properties are now obvious; in particular, the dual expression of $F_{mn}(z)$ and the fact that

$A_n^+(z) = O(z)$ imply that $F_{mn}(z) = O(z)$ as $z \rightarrow 0$. What is left is to check the third property. To this end, note that

$$F_{mn}(z) = \left[\begin{array}{l} \partial_{t_m} - \frac{\beta_{m,s} \mathop{\text{t}}\alpha_s}{z - \gamma_s} - A_m^{(s,1)} + O(z - \gamma_s), \\ \partial_{t_n} - \frac{\beta_{n,s} \mathop{\text{t}}\alpha_s}{z - \gamma_s} - A_n^{(s,1)} + O(z - \gamma_s) \end{array} \right]$$

as $z \rightarrow \gamma_s$. Expanded to powers of $z - \gamma_s$, one can readily see, by (3.24), that the coefficient of $(z - \gamma_s)^{-2}$ vanishes. It is also easy to see, by (3.25), that the coefficient of $(z - \gamma_s)^{-1}$ is a rank-one matrix of the factorized form $\beta_{mn,s} \mathop{\text{t}}\alpha_s$. \square

One can conversely derive the Lax equations (3.23) from the zero-curvature equations.

LEMMA 7. *The Lax equations (3.23) can be derived from the zero-curvature equations (3.30).*

PROOF. Substituting $A_m(z) = A_m^+(z) + U(z)z^{-m}$ in the zero-curvature equation yields

$$[\partial_{t_m} - A_m^+(z) - U(z)z^{-m}, \partial_{t_n} - A_n(z)] = 0,$$

which one can further rewrite as

$$[\partial_{t_n} - A_n(z), U(z)] = [\partial_{t_n} - A_n(z), \partial_{t_m} - A_m^+(z)]z^m.$$

Since $A_m^+(z) = O(1)$ and $A_n(z) = O(z^{-n})$ as $z \rightarrow 0$, the right hand side of the last equation is $O(z^{m-n})$, so that

$$[\partial_{t_n} - A_n(z), U(z)] = O(z^{m-n}).$$

Letting $m \rightarrow \infty$, one obtains the Lax equation (3.23) as expected. \square

We thus eventually arrive at the following conclusion.

THEOREM 1. *As far as (3.24) and (3.25) are satisfied, the Lax equations (3.23) and the zero-curvature equations (3.30) are equivalent.*

Let us conclude the present consideration with a comment on (3.24) and (3.25). Although these equations look somewhat distinct from the other equations, these equations themselves are directly related to the zero-curvature equations

$$(3.31) \quad [\partial_{t_n} - A_n(z), \partial_x - A(z)] = 0$$

between $A_n(z)$ and $A(z)$. Namely, if (3.5) and (3.6) are satisfied (this should be understood as part of the definition of $A(z)$), (3.24) and (3.25) follow from these zero-curvature equations. One can indeed derive these equations from the Laurent expansion of the left hand side of (3.31) at $z = \gamma_s$. In this respect, one may consider the zero-curvature equations (3.31) as the defining equation of a hierarchy. This is indeed the way Krichever formulates a hierarchy.

4. Riemann-Hilbert Problem

In this and next sections, we encounter various initial value problems with regard to the time variables $t = (t_1, t_2, \dots)$, in which t_1 is identified with x . For this reason, let us make the notations slightly more strict. Namely, we write a t -dependent quantity always indicating its t -dependence explicitly as $A_n(t, z)$, $\gamma_s(t)$, $\alpha_s(t)$, etc. Otherwise, a quantity is understood to be independent of t .

4.1. Laurent series solution of auxiliary linear system

As a consequence of (3.29), we have the “dual” zero-curvature equations

$$(4.1) \quad [\partial_{t_m} - A_m^+(t, z), \partial_{t_n} - A_n^+(t, z)] = 0.$$

These equations are the Frobenius integrability condition of the linear system

$$(4.2) \quad \partial_{t_n} \phi(t, z) = A_n^+(t, z) \phi(t, z).$$

One can redefine the generating function $\phi(t, z)$ to satisfy these equations as well.

THEOREM 2. *Upon being suitably modified, the generating function $\phi(t, z)$ satisfies the foregoing linear system or, equivalently,*

$$(4.3) \quad \partial_{t_n} \phi(z) = A_n(t, z) \phi(t, z) - \phi(t, z) J z^{-n}.$$

In particular,

$$(4.4) \quad \psi(t, z) = \phi(t, z) \exp\left(\sum_{n=1}^{\infty} t_n J z^{-n}\right) \quad (t_1 = x)$$

gives a Laurent series solution of the auxiliary linear system

$$(4.5) \quad \partial_{t_n} \psi(t, z) = A_n(t, z) \psi(t, z).$$

PROOF. One can construct a Laurent series $\tilde{\phi}(t, z) = I + \tilde{\phi}_1 z + \dots$ as a solution of the initial value problem

$$\partial_{t_n} \tilde{\phi}(t, z) = A_n^+(t, z) \tilde{\phi}(t, z), \quad \tilde{\phi}(t, z)|_{t_2=t_3=\dots=0} = \phi(t, z)|_{t_2=t_3=\dots=0}.$$

The Frobenius integrability condition of this system is ensured by the zero-curvature equation of $A_n^+(t, z)$'s. Moreover, since $A_n^+(t, z) = O(z)$, the solution persists to be of the form $I + O(z)$. Now consider the new Laurent series

$$\tilde{U}(t, z) = \tilde{\phi}(t, z) J \tilde{\phi}(t, z)^{-1},$$

which satisfies the differential equations

$$\partial_{t_n} \tilde{U}(t, z) = [\partial_{t_n} \tilde{\phi}(t, z) \cdot \tilde{\phi}(t, z)^{-1}, \tilde{U}(t, z)] = [A_n^+(t, z), \tilde{U}(t, z)].$$

On the other hand, one knows that $U(t, z)$, too, satisfies differential equations of the same form, i.e., (3.28). Since $\tilde{U}(t, z)$ and $U(t, z)$ have the same initial data at $t_2 = t_3 = \dots = 0$, uniqueness of solution of the initial value problem implies that $\tilde{U}(t, z) = U(t, z)$, i.e.,

$$U(t, z) = \tilde{\phi}(t, z) J \tilde{\phi}(t, z)^{-1},$$

so that one can rewrite the foregoing differential equation for $\tilde{\phi}(t, z)$ as

$$\partial_{t_n} \tilde{\phi}(t, z) = (A_n(t, z) - U(t, z) z^{-n}) \tilde{\phi}(t, z) = A_n(t, z) \tilde{\phi}(t, z) - \tilde{\phi}(t, z) J z^{-n}.$$

Thus $\tilde{\phi}(t, z)$ turns out to fulfill all requirements. \square

4.2. Global solution of auxiliary linear system

The Laurent series solution $\psi(t, z)$ of the auxiliary linear system, by its nature, carries no information on the global structure of $A_n(t, z)$'s on Γ . To fill this gap, we now introduce another solution $\chi(t, z)$ that is globally defined on Γ with several singular points. As it turns out, these two distinct solutions of the same auxiliary linear system play the role of the Riemann-Hilbert (or factorization) pair in the usual nonlinear Schrödinger hierarchy.

To avoid delicate problems, we assume in the following that the solutions of the hierarchy under consideration are (real or complex) analytic in a neighborhood of the initial point $t = 0$.

$\chi(t, z)$, by definition, is a solution of the auxiliary linear system

$$(4.6) \quad \partial_{t_n} \chi(t, z) = A_n(t, z) \chi(t, z)$$

that satisfies the initial condition

$$(4.7) \quad \chi(0, z) = I.$$

Since the auxiliary linear system is a collection of ordinary differential equations, any solution remains nonsingular as far as the coefficients of the equations are nonsingular. Consequently, if z is in a subset of Γ where $A_n(0, z)$'s are holomorphic, such a solution $\chi(t, z)$ does exist in a (possibly small) neighborhood of $t = 0$ in the t -space. Since all singularities of $A_n(0, z)$ on Γ are located at the three points $0, \gamma_1(0), \gamma_2(0)$, we can conclude that the singularities of $\chi(t, z)$ on Γ are confined to a neighborhood of these three points as far as t is sufficiently close to 0.

To elucidate the nature of singularities on Γ more precisely, we expand $\chi(t, z)$ into a Taylor series at $t = 0$ and examine the Taylor coefficients as a function of z . Note that this is reasonable, because this Taylor series has a nonzero radius of convergence as far as $z \neq 0, \gamma_1(0), \gamma_2(0)$.

The Taylor coefficients of $\chi(t, z)$ at $t = 0$ can be evaluated by successively differentiating the differential equations as

$$\begin{aligned} \partial_{t_n} \chi(t, z) &= A_n(t, z) \chi(t, z), \\ \partial_{t_m} \partial_{t_n} \chi(t, z) &= (\partial_{t_m} A_n(t, z) + A_n(t, z) A_m(t, z)) \chi(t, z), \\ \partial_{t_k} \partial_{t_m} \partial_{t_n} \chi(t, z) &= \left(\partial_{t_k} \partial_{t_m} A_n(t, z) + \partial_{t_k} (A_n(t, z) A_m(t, z)) \right. \\ &\quad \left. + (\partial_{t_m} A_n(t, z)) A_k(t, z) \right. \\ &\quad \left. + A_n(t, z) A_m(t, z) A_k(t, z) \right) \chi(t, z), \end{aligned}$$

etc. Letting $t = 0$, we are left with a noncommutative polynomial of derivatives of A_n 's. We can deduce from these calculations the following precise information.

LEMMA 8. *The derivatives $\partial_{t_{n_1}} \cdots \partial_{t_{n_p}} \chi(t, z)|_{t=0}$ of all orders of $\chi(t, z)$ at $t = 0$ are a matrix of meromorphic functions of z on Γ with poles at $z = 0, \gamma_1(0), \gamma_2(0)$ and holomorphic at other points. As $z \rightarrow \gamma_s(0)$, $s = 1, 2$,*

$$(4.8) \quad \partial_{t_{n_1}} \cdots \partial_{t_{n_p}} \chi(t, z)|_{t=0} = \frac{\beta_{n_1, \dots, n_1, s}(0) {}^t \alpha_s(0)}{z - \gamma_s(0)} + O(1),$$

where $\beta_{n_1, \dots, n_1, s}(0)$ is a two-dimensional constant column vector.

PROOF. As illustrated above, the derivatives of $\chi(t, z)$ of all order can be written as

$$(4.9) \quad \partial_{t_{n_1}} \cdots \partial_{t_{n_p}} \chi(t, z) = A_{n_1, \dots, n_p}(t, z) \chi(t, z).$$

Differentiating this equation by t_m yields the recurrence relations

$$A_{m, n_1, \dots, n_p}(t, z) = \partial_{t_m} A_{n_1, \dots, n_p}(t, z) + A_{n_1, \dots, n_p}(t, z) A_m(t, z).$$

for the coefficients $A_{n_1, \dots, n_p}(t, z)$. One can prove, by induction on p , that $A_{n_1, \dots, n_p}(t, z)$ is a matrix of meromorphic functions of z on Γ with poles at $z = 0, \gamma_1(t), \gamma_2(t)$, and

$$(4.10) \quad A_{n_1, \dots, n_p}(t, z) = \frac{\beta_{n_1, \dots, n_p, s}(t) {}^t \alpha_s(t)}{z - \gamma_s(t)} + O(1)$$

as $z \rightarrow \gamma_s(t)$, where $\beta_{n_1, \dots, n_p, s}(t)$ is a two-dimensional column vector. Assume that the Laurent expansion (4.10) holds for $A_{n_1, \dots, n_p}(t, z)$. The Laurent expansion of $A_{m, n_1, \dots, n_p}(t, z)$ can be read off from the recurrence relation as

$$\begin{aligned} & A_{m, n_1, \dots, n_p}(t, z) \\ &= \beta_{n_1, \dots, n_p, s}(t) {}^t \alpha_s(t) (\partial_{t_m} \gamma_s(t) + {}^t \alpha_s(t) \beta_s(t)) (z - \gamma_s(t))^{-2} \\ & \quad + \left(\partial_{t_m} \beta_{n_1, \dots, n_p, s}(t) \cdot {}^t \alpha_s(t) + \beta_{n_1, \dots, n_p, s}(t) \cdot \partial_{t_m} {}^t \alpha_s(t) + \right. \\ & \quad \quad \left. + A_{n_1, \dots, n_p, s}^{(s,1)} \beta_s(t) {}^t \alpha_s(t) + \right. \\ & \quad \quad \left. + \beta_{n_1, \dots, n_p, s}(t) {}^t \alpha_s(t) A_m^{(s,1)}(t) \right) (z - \gamma_s(t))^{-1} \\ & \quad + O(1), \end{aligned}$$

where $A_{n_1, \dots, n_p}^{(s,1)}(t)$ denotes the constant term in the Laurent expansion (4.10). By (3.24), the coefficient of $(z - \gamma_s(t))^{-2}$ vanishes; by (3.25), the terms containing $\partial_{t_m} {}^t\alpha_s(t)$ and ${}^t\alpha_s(t)A_m^{(s,1)}(t)$ in the coefficient of $(z - \gamma_s(t))^{-1}$ cancel out. Thus $A_{m, n_1, \dots, n_p}(t, z)$, too, turns out to have a Laurent expansion of the expected form. This completes the proof of (4.10). Lastly, letting $t = 0$ in (4.9), one eventually arrives at the statement of the lemma. \square

All Taylor coefficients of $\chi(t, z)$ at $t = 0$ thus turn out to have poles at the same position, namely, the three points $0, \gamma_1(0), \gamma_2(0)$. Moreover, whereas the order of pole at $z = 0$ is unbounded, the poles at $z = \gamma_s(0)$, $s = 1, 2$, are of the first order. Accordingly, $\chi(t, z)$ has an essential singularity at $z = 0$ and simple poles at the other two points. The leading part of the Laurent expansion at $z = \gamma_s(0)$ takes the familiar form

$$(4.11) \quad \chi(t, z) = \frac{\beta_{\chi, s}(t) {}^t\alpha_s(0)}{z - \gamma_s(0)} + O(1),$$

where $\beta_{\chi, s}(t)$ is a two-dimensional column vector that depends on t . Note that the pole of $\chi(t, z)$ at $z = \gamma_s(0)$ disappears when $t = 0$ (because $\chi(0, z) = I$).

Lastly, let us mention another important property of $\chi(t, z)$.

LEMMA 9. *$\det \chi(t, z)$ is a meromorphic function on Γ with simple poles at $z = \gamma_s(0)$, $s = 1, 2$, and simple zeroes at $z = \gamma_s(t)$, $s = 1, 2$. ${}^t\alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$.*

PROOF. The auxiliary linear system $\partial_{t_n}\chi(t, z) = A_n(t, z)\chi(t, z)$ induces the linear system

$$\partial_{t_n} \det \chi(t, z) = \text{Tr } A_n(t, z) \det \chi(t, z)$$

for $n = 1, 2, \dots$. Since $A_n(t, z) = U(t, z)z^{-n} + O(z)$ as $z \rightarrow 0$ and $\text{Tr } U(t, z) = 0$, one finds that the coefficients of this linear system for $\det \chi(t, z)$ has no singularity at $z = 0$, but rather a zero, namely,

$$\text{Tr } A_n(t, z) = O(z) \quad (z \rightarrow 0).$$

This implies that $\chi(t, z)$ has no singularity at $z = 0$. In view of the initial condition $\chi(0, z) = I$, one can conclude that $\det \chi(t, z)|_{z=0} = 1$. One can

thus confirm that $\det \chi(t, z)$ is a meromorphic function on Γ with poles at $z = \gamma_s(t)$, $s = 1, 2$, and holomorphic at other points. Since the residue matrix of $\chi(t, z)$ at $z = \gamma_s(t)$ is a rank-one matrix, $\det \chi(t, z)$ has a simple pole there. The position of zeroes of $\det \chi(t, z)$ can be deduced from the linear equation

$$\partial_x \chi(t, z) = A(t, z) \chi(t, z)$$

(or from any any member of the auxiliary linear system). Extracting the residue at $z = \gamma_s(t)$ yields the relation

$$0 = \beta_s(t) {}^t \alpha_s(t) \chi(t, \gamma_s(t))$$

which, because $\beta_s(t) \neq \mathbf{0}$, reduces to the relation

$${}^t \alpha_s(t) \chi(t, \gamma_s(t)) = \mathbf{0}.$$

Thus ${}^t \alpha_s(t)$ turns out to be a left null vector of $\chi(t, \gamma_s(t))$. On the other hand, rewriting the linear system as

$$A = \partial_x \chi(t, z) \cdot \chi(t, z)^{-1},$$

one can see that the zeroes $\gamma_s(t)$ of $\det \chi(t, z)$ are simple. If they are a multiple zero, the matrix A will have a multiple pole; this contradicts the construction of the matrix A . \square

These results show that $\chi(t, z)$ is exactly the solution mentioned in Krichever's lemma [8, Lemma 5.2], namely a matrix solution holomorphic at the movable poles of $A(t, z)$.

In summary, $\chi(t, z)$ has the following properties.

THEOREM 3. *$\chi(t, z)$ has an essential singularity at $z = 0$ and simple poles at $z = \gamma_s(0)$, $s = 1, 2$, and is holomorphic at other points of Γ . As $z \rightarrow \gamma_s(0)$, $\chi(t, z)$ behaves as (4.11) shows. Moreover, $\det \chi(t, z)$ is a meromorphic function on Γ with simple poles at $z = \gamma_s(0)$, $s = 1, 2$, and simple zeros at $z = \gamma_s(t)$, $s = 1, 2$. ${}^t \alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$.*

4.3. Riemann-Hilbert problem with degeneration points

We now have two distinct solutions of the same linear system, namely, the Laurent series solution $\psi(t, z)$ and the solution $\chi(t, z)$ carrying global information on Γ . The “matrix ratio” of these two solutions is a constant matrix, i.e.,

$$\partial_{t_n} \left(\chi(t, z)^{-1} \phi(t, z) \exp \left(\sum_{n=1}^{\infty} t_n J z^{-n} \right) \right) = 0.$$

Equating this matrix ratio with its value at $t = 0$, we are led to the relation

$$(4.12) \quad \chi(t, z)^{-1} \phi(t, z) \exp \left(\sum_{n=1}^{\infty} t_n J z^{-n} \right) = \phi(0, z)$$

or, equivalently,

$$(4.13) \quad \phi(0, z) \exp \left(- \sum_{n=1}^{\infty} t_n J z^{-n} \right) = \chi(t, z)^{-1} \phi(t, z).$$

The last relation may be thought of as a kind of Riemann-Hilbert problem concerning a small circle $|z| = a$ on the torus Γ . The input of this problem are the initial values $\gamma_s(0), \alpha_s(0)$ and $\phi(0, z)$. The left hand side of (4.13) is a $\mathrm{GL}(2, \mathbf{C})$ -valued function on the circle, in other words, a $\mathrm{GL}(2, \mathbf{C})$ loop group element. The problem is to factorize it to two factors. The second factor $\phi(t, z)$ is a loop group element that can be extended to a matrix of holomorphic functions on the inside of the circle. The first factor $\chi(t, z)$ is a loop group element that can be similarly extended to the outside of the circle, but *not holomorphic everywhere*; $\chi(t, z)$ is required to have poles at $z = \gamma_s(0)$, $s = 1, 2$, with the structure described in (4.11). Moreover, in addition to these poles, $\chi(t, z)$ have *degeneration points*, i.e., zeros of the determinant at $z = \gamma_s(t)$, $s = 1, 2$. These zeroes are nothing but the poles of $A_n(t, z)$'s.

Thus the Riemann-Hilbert problem relevant to the present setting is a Riemann-Hilbert problem with movable degeneration points (and extra fixed poles) on a torus. A similar Riemann-Hilbert problem appears in Krichever's work [5] on commutative rings of differential operators. In that case, the Riemann-Hilbert problem is formulated on the “spectral curve”

of the commutative ring under consideration, and the genus of the spectral curve can be an arbitrary positive integer.

Krichever converts the Riemann-Hilbert problem to an integral equation and solves it by a standard procedure. The same method can be applied to the present setting as well, though we shall not seek this approach here. An alternative approach, as demonstrated by Previato and Wilson [13], is to translate the Riemann-Hilbert problem to the language of an infinite dimensional Grassmann variety. We shall present this method in the next section.

4.4. Back to hierarchy

It will be instructive to show how to derive a solution of (3.23), (3.24) and (3.25) from the Riemann-Hilbert problem. This is more or less parallel to the procedure that Krichever and Novikov employ in their work [6, 7].

Notice, first of all, that $\chi(t, z)$ is a matrix version of the “vector Baker-Akhiezer function” in their terminology. This is an immediate consequence of the Riemann-Hilbert problem: $\chi(t, z)$ has an essential singularity of the exponential type at $z = 0$, and fixed simple poles at $z = \gamma_s(0)$, $s = 1, 2$. Accordingly, the determinant $\det \chi(t, z)$ has zeros at $\gamma_s(t)$, $s = 1, 2$, that depend on t . Let us consider the generic situation where $\gamma_s(t)$ ’s are simple zeros of $\det \chi(t, z)$. The matrices $A_n(t, z)$, now defined by

$$A_n(t, z) = \partial_{t_n} \chi(t, z) \cdot \chi(t, z)^{-1},$$

thereby has simple poles at $\gamma_s(t)$. As simple linear algebraic calculations show, the residue of $\chi(t, z)^{-1}$ at the degeneration point $\gamma_s(t)$ is a rank-one matrix. Consequently, the residue of $A_n(t, z)$, too, is a rank-one matrix and takes the factorized form $\beta_{n,s}(t) {}^t \alpha_s(t)$ with a common vector $\alpha_s(t)$ independent of n . The dynamical Tyurin parameters $\gamma_s(t), \alpha_s(t)$, $s = 1, 2$, are thus obtained. According to a general theorem of Krichever and Novikov (restated in Krichever’s recent paper [8]), these parameters satisfy the differential equations (3.24) and (3.25).

One can now derive the Lax equation (3.23) as follows. Differentiating the Riemann-Hilbert relation (4.13) yields another expression of $A_n(t, z)$,

$$A_n(t, z) = \partial_{t_n} \phi(t, z) \cdot \phi(t, z)^{-1} + U(t, z)z^{-n},$$

where $U(t, z)$ is defined as

$$U(t, z) = \phi(t, z)J\phi(t, z)^{-1}.$$

The Lax equations (3.23) are thereby satisfied automatically. Moreover, the second expression of $A_n(t, z)$ also shows the singular behavior of $A_n(t, z)$ as $z \rightarrow 0$:

$$A_n(t, z) = U(t, z)z^{-n} + O(z).$$

Thus $A_n(t, z)$'s turn out to have all properties that we have assumed in the construction of the hierarchy.

5. Grassmannian Perspective

We here translate the Riemann-Hilbert problem to the language of an infinite dimensional Grassmann variety. This leads to a mapping of the elliptic nonlinear Schrödinger hierarchy to a multi-time dynamical system on a subset (the set of dressed vacua) of the infinite dimensional Grassmann variety.

5.1. Formulation of Grassmann variety

Two different models of infinite dimensional Grassmann varieties have been used in the literature of integrable systems. One is Sato's algebraic or complex analytic model based on a vector space of (formal or convergent) Laurent series [14]. The other is Segal and Wilson's functional analytic model based on the Hilbert space of square-integrable functions on a circle [15]. Which to choose is rather a problem of taste; both of them work well in the present context. Let us use Sato's model in the following. Actually, Sato's formulation contains a continuous family of different models. Among them, we choose one of the presumably simplest models.

Let V denote the vector space of all 2×2 matrices of Laurent series

$$X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in \mathfrak{gl}(2, \mathbf{C}),$$

that converges in a neighborhood of $z = 0$ except at $z = 0$; $\mathfrak{gl}(2, \mathbf{C})$ denotes the vector space of 2×2 complex matrices without any algebraic constraints. This vector space is a matrix analogue of $V^{\text{ana}(\infty)}$ in Sato's list of models [14]; as noted therein, one can introduce a natural linear topology in this vector space.

We construct an infinite dimensional Grassmann variety Gr from this vector space V and the vector subspace

$$(5.1) \quad V_+ = \{X(z) \in V \mid X_n = 0 \text{ for } n \leq 0\}$$

of all $X(z) \in V$ that are holomorphic and vanish at $z = 0$. The Grassmann variety Gr consists of all closed vector subspaces $W \subset V$ for which the composition of the inclusion map $W \hookrightarrow V$ and the canonical projection $V \rightarrow V/V_+$ is a Fredholm map of index 0:

$$(5.2) \quad \begin{aligned} \text{Gr} &= \{W \subset V \mid \dim \text{Ker}(W \rightarrow V/V_+) \\ &= \dim \text{Coker}(W \rightarrow V/V_+) < \infty\}. \end{aligned}$$

The so called “big cell” $\text{Gr}^\circ \subset \text{Gr}$ is an open subset that consists of subspaces for which the map $W \rightarrow V/V_+$ is an isomorphism:

$$(5.3) \quad \text{Gr}^\circ = \{W \in \text{Gr} \mid W \simeq V/V_+\}.$$

5.2. Vacuum and dressing

Following the idea of Previato and Wilson [13], we now introduce a special element $W_0(\gamma, \alpha)$ of the big cell determined by constant Tyurin parameters $\gamma = (\gamma_1, \gamma_2)$ and $\alpha = (\alpha_1, \alpha_2)$. This is a matrix version of the “vacuum” that Previato and Wilson suggest to use for a holomorphic vector bundle in the Tyurin parametrization.

LEMMA 10. *Let $\gamma = (\gamma_1, \gamma_2)$ be a pair of distinct points of Γ , $\gamma_1 \neq \gamma_2$, and $\alpha = (\alpha_1, \alpha_2)$ a pair of constants satisfying the genericity condition $\alpha_1 \neq \alpha_2$. Then, for any integer $n \geq 0$ and the matrix indices $i, j = 1, 2$, there is a unique 2×2 matrix $w_{n,ij}(z)$ of meromorphic functions on Γ with the following properties:*

1. $w_{n,ij}(z)$ has poles at $z = 0, \gamma_1, \gamma_2$ and is holomorphic at other points.
2. As $z \rightarrow 0$, $w_{n,ij}(z) = E_{ij}z^{-n} + O(z)$, where E_{ij} , $i, j = 1, 2$, are the standard basis of $\text{gl}(2, \mathbf{C})$.
3. As $z \rightarrow \gamma_s$, $s = 1, 2$,

$$w_{n,ij}(z) = \frac{\beta_{n,ij,s} {}^t \alpha_s}{z - \gamma_s} + O(1),$$

where $\alpha_s = {}^t(\alpha_s, 1)$, and $\beta_{n,ij,s}$ is another two-dimensional constant column vector.

The subspace

$$(5.4) \quad W_0(\gamma, \alpha) = \langle w_{n,ij}(z) \mid n \geq 0, i, j = 1, 2 \rangle$$

spanned by (the Laurent series of) $w_{n,ij}(z)$'s is an element of the big cell.

PROOF. One can confirm the existence and uniqueness of $w_{n,ij}(z)$ in the same way as the case of $A(z)$ and $A_n(z)$. Since the leading terms $E_{ij}z^{-n}$ of the Laurent expansion at $z = 0$ are in one-to-one correspondence with the elements of the standard basis $\{E_{ij}z^{-n} \mid n \geq 0, i, j = 1, 2\}$ of V/V_+ , the linear map $W_0(\gamma, \alpha) \rightarrow V/V_+$ is obviously surjective. To prove the injectivity, note that any element $X(z)$ of $W_0(\gamma, \alpha) \cap V_+$ is a matrix of functions holomorphic at all points of Γ other than possible poles at $z = \gamma_1, \gamma_2$, behaves as

$$X(z) = \frac{\beta_{X,s} {}^t\alpha_s}{z - \gamma_s} + O(1)$$

at these points (where $\beta_{X,s}$ is a two-dimensional column vector), and has a zero at $z = 0$. Such a matrix of function is equal to 0 as one can see by the same reasoning as the proof of the zero-curvature equation (3.30). Therefore $W_0(\gamma, \alpha) \cap V_+ = \{0\}$, hence the injectivity of the linear map $W_0(\gamma, \alpha) \rightarrow V/V_+$ follows. \square

This special base point $W_0(\gamma, \alpha)$ of the big cell plays the role of vacuum in the ‘‘dressing method.’’ This is a complicated vacuum with nontrivial structure that stems from an underlying holomorphic vector bundle over Γ . We ‘‘dress’’ this vacuum to obtain an element W of the big cell that represents a general solution of our hierarchy. Dressing is achieved by multiplying a Laurent series $\phi(z)$ from the right side as

$$(5.5) \quad W = W_0(\gamma, \alpha)\phi(z), \quad \phi(z) = I + \sum_{n=1}^{\infty} \phi_n z^n, \quad \phi_n \in \mathfrak{gl}(2, \mathbf{C}).$$

Our goal in the following is to show that our hierarchy can be mapped to a multi-time dynamical system on the set

$$(5.6) \quad \mathcal{M} = \{W \in \text{Gr}^\circ \mid W = W_0(\gamma, \alpha)\phi(z), \\ \gamma = (\gamma_1, \gamma_2) \in \Gamma^2, \alpha = (\alpha_1, \alpha_2) \in \mathbf{C}^2, \\ \gamma_1 \neq \gamma_2, \alpha_1 \neq \alpha_2, \phi_n \in \mathfrak{gl}(2, \mathbf{C})\}$$

of these dressed vacua.

5.3. Interpretation of Riemann-Hilbert problem

We now translate the Riemann-Hilbert problem (4.13) to the language of dressed vacua. Because of several reasons, the following consideration is limited to a small neighborhood of $t = 0$. Firstly, this is to ensure that the conditions $\gamma_1(t) \neq \gamma_2(t)$ and $\alpha_1(t) \neq \alpha_2(t)$ are satisfied; this issue is related to boundaries of the Tyurin parametrization of holomorphic vector bundles. Secondly, if t gets large, the dressed vacuum $W(t) \in \mathcal{M}$ representing a solution of (4.13) can hit the boundary of the big cell, so that more careful analysis is required.

The first step is the following.

LEMMA 11. $W_0(\gamma(t), \alpha(t))\chi(t, z) \subseteq W_0(\alpha(0), \gamma(0))$.

PROOF. Let $w_{n,ij}(t, z)$, $n \geq 0$, $i, j = 1, 2$, denote the elements of the basis of $W_0(\gamma(t), \alpha(t))$ defined in Lemma 10. $w_{n,ij}(t, z)$ has poles at $z = 0, \gamma_1(t), \gamma_2(t)$, and behaves as

$$w_{n,ij}(t, z) = \frac{\beta_{n,ij,s}(t) {}^t\alpha_s(t)}{z - \gamma_s(t)} + O(1)$$

as $z \rightarrow \gamma_s(t)$. Upon multiplication with $\chi(t, z)$, the poles at $z = \gamma_s(t)$ are cancelled out because ${}^t\alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$ (see Theorem 3). Thus one finds that $w_{n,ij}(t, z)\chi(t, z)$ has an essential singularity at $z = 0$, simple poles at $z = \gamma_s(0)$, $s = 1, 2$, and is holomorphic at other points of Γ . The leading part of the Laurent expansion at $z = \gamma_s(0)$ takes the form

$$w_{n,ij}(t, z)\chi(t, z) = \frac{w_{n,ij}(t, \gamma_s(0))\beta_{\chi,s}(t) {}^t\alpha_s(0)}{z - \gamma_s(0)} + O(1),$$

so that the residue matrix has such a factorized form as (column vector) \cdot ${}^t\alpha_s(0)$. One can thus confirm that $w_{n,ij}(t, z)\chi(t, z)$ fulfills all conditions to be an element of $W_0(\gamma(0), \alpha(0))$. \square

The next step is to show that the inclusion relation in this lemma is actually an equality. To this end, we prove the following lemmas.

LEMMA 12. $\chi(t, z)^{-1}$ has an essential singularity at $z = 0$, simple poles at $\gamma_s(t)$, $s = 1, 2$, and is holomorphic at other points. As $z \rightarrow \gamma_s(t)$,

$$(5.7) \quad \chi(t, z)^{-1} = \frac{\beta_{\chi^{-1}, s}(t) {}^t\alpha_s(t)}{z - \gamma_s(t)} + O(1),$$

where $\beta_{\chi^{-1}, s}(t)$ is a two-dimensional column vector.

PROOF. It is shown in Theorem 3 that ${}^t\alpha_s(t)$ is a left null vector of $\chi(t, \gamma_s(t))$. A clue to the proof of the lemma is the fact that the left null space (i.e., the left zero-eigenspace) of $\chi(t, \gamma_s(t))$ is, actually, one-dimensional and spanned by ${}^t\alpha_s(t)$. If the left null space is two-dimensional, $\chi(t, \gamma_s(t))$ itself is a zero matrix, so that $\det \chi(t, z)$ has a double zero at $z = \gamma_s(t)$; this contradicts the present setting. [Remark: The same reasoning holds for an $r \times r$ analogue of the present case as well. Namely, if the left null space of $\chi(t, \gamma_s(t))$ has k dimensions, then $\det \chi(t, z)$ has a zero of the k -th order at $z = \gamma_s(t)$.] Bearing this fact in mind, one can prove the statement of the lemma as follows. Theorem 3 implies that $\gamma_s(t)$ is a simple zero of $\chi(t, z)^{-1}$. Extracting the residue from the obvious identity $\chi(t, z)^{-1}\chi(t, z) = I$ yields the relation

$$\operatorname{Res}_{z=\gamma_s(t)} \chi(t, z)^{-1} dz \cdot \chi(t, \gamma_s(t)) = 0.$$

This implies that the residue matrix of $\chi(t, z)^{-1}$ at $z = \gamma_s(t)$ is a rank-one matrix of the factorized form (column vector) \cdot (row vector). The row vector on the right side is accordingly a left null vector of $\chi(t, \gamma_s(t))$. By the aforementioned fact, one can choose this row vector to be equal to ${}^t\alpha_s(t)$. Thus the residue matrix turns out to have a factorized form as shown in the statement of the lemma. The other properties of $\chi(t, z)^{-1}$, too, can be readily derived from Theorem 3. \square

LEMMA 13. ${}^t\alpha_s(0)$ is a left null vector of $\chi(t, z)^{-1}|_{z=\gamma_s(0)}$.

PROOF. The identity $\chi(t, z)\chi(t, z)^{-1} = I$ yields the relation

$$\operatorname{Res}_{z=\gamma_s(0)} \chi(t, z) dz \cdot \chi(t, z)^{-1}|_{z=\gamma_s(0)} = 0,$$

which, by (4.11), takes the form

$$\beta_{\chi, s}(t) {}^t\alpha_s(0)\chi(t, z)^{-1}|_{z=\gamma_s(0)} = 0.$$

Since $\beta_{\chi, s}(t) \neq \mathbf{0}$, this implies that

$${}^t\alpha_s(0)\chi(t, z)^{-1}|_{z=\gamma_s(0)} = 0. \quad \square$$

These lemmas show that the inverse matrix $\chi(t, z)^{-1}$ has essentially the same properties as $\chi(t, z)$ except that the position of poles and degeneration points are exchanged. Consequently, one can repeat the proof of Lemma 11, replacing the role of $\chi(t, z)$, $W_0(\gamma(t), \alpha(t))$ and $W_0(\gamma(0), \alpha(0))$ with those of $\chi(t, z)^{-1}$, $W_0(\gamma(0), \alpha(0))$ and $W_0(\gamma(t), \alpha(t))$, to derive the inclusion relation

$$W_0(\gamma(0), \alpha(0))\chi(t, z)^{-1} \subseteq W_0(\gamma(t), \alpha(t)).$$

Thus the equality

$$(5.8) \quad W_0(\gamma(t), \alpha(t))\chi(t, z) = W_0(\alpha(0), \gamma(0)).$$

follows as expected.

Having this equality, one can readily convert the Riemann-Hilbert problem to the language of dressed vacua as follows. The Riemann-Hilbert relation (4.13) yields the relation

$$W_0(\gamma(t), \alpha(t))\phi(t, z) = W_0(\gamma(t), \alpha(t))\chi(t, z)\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right).$$

By (5.8), $W_0(\gamma(t), \alpha(t))$ absorbs $\chi(t, z)$ to become $W_0(\gamma(0), \alpha(0))$. The outcome is the relation

$$W_0(\gamma(t), \alpha(t))\phi(t, z) = W_0(\gamma(0), \alpha(0))\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right),$$

which means that the dressed vacuum $W(t) = W_0(\gamma(t), \alpha(t))\phi(t, z) \in \mathcal{M}$ obeys the exponential law

$$(5.9) \quad W(t) = W(0) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right).$$

Conversely, one can obtain a solution of the Riemann-Hilbert problem from the exponential flows (5.9) as follows. (This is a variation of the dressing method of Previato and Wilson [13].) Given a set of initial values $\gamma(0), \alpha(0)$ and $\phi(0, z)$, let us consider the exponential flows (5.9) sending $W(0) = W_0(\gamma(0), \alpha(0))\phi(0, z)$ to $W(t)$. A clue is, again, the fact that $W(t)$ remains in the big cell as far as t is sufficiently small. In that case, the linear map $W(t) \rightarrow V/V_+$ is an isomorphism. Let $\phi(t, z)$ denote the inverse image of $I \in V/V_+$ by this isomorphism. Being equal to I modulo V_+ , $\phi(t, z)$ is a Laurent series of the form

$$\phi(t, z) = I + \sum_{n=1}^{\infty} \phi_n(t) z^n.$$

On the other hand, as an element of

$$W(t) = W_0(\gamma(0), \alpha(0))\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right),$$

$\phi(t, z)$ can also be written as

$$\phi(t, z) = \chi(t, z)\phi(0, z) \exp\left(-\sum_{n=1}^{\infty} t_n J z^{-n}\right)$$

with an element $\chi(t, z)$ of $W_0(\gamma(0), \alpha(0))$. Recalling the definition of $W_0(\gamma(0), \alpha(0))$, one finds that $\chi(t, z)$ is a matrix of functions with all properties in the statement of Theorem 3. The associated Tyurin parameters $(\gamma_s(t), \alpha_s(t))$ are determined as the position of zeros of $\chi(t, z)$ and the normalized left null vector of $\chi(t, z)$ at those degeneration points.

Thus we have been able to show the following fundamental picture of our hierarchy as a dynamical system embedded in the Grassmann variety.

THEOREM 4. *The elliptic analogue of the nonlinear Schrödinger hierarchy can be mapped, by the correspondence $W(t) = W_0(\gamma(t), \alpha(t))\phi(t, z)$,*

to a dynamical system on the set \mathcal{M} of dressed vacua in the Grassmann variety Gr . The motion of $W(t)$ obeys the exponential law (5.9). Conversely, the exponential flows on \mathcal{M} yield a solution of the Riemann-Hilbert problem. (4.13).

Let us conclude this section with a few remarks.

1. Our approach owes much to the work of Previato and Wilson [13]. They use a similar Grassmannian version of the dressing method as a tool to reformulate the work of Krichever and Novikov [5, 6, 7] on commutative rings of differential operators. Accordingly, the detail of the dressing procedure is quite different from ours. In particular, they take Krichever's "algebraic spectral data" [5] as the input; dressing is achieved by a matrix solution of linear differential equations determined by these data. In our case, the dressing matrix is the product of $\phi(0, z)$ and the exponential matrix generating the exponential flows (5.9).
2. In every aspect, the construction of the mapping to the Grassmann variety is related to the geometry of holomorphic vector bundles over Γ . First of all, the Tyurin parameters $(\gamma_s(t), \alpha_s(t))$ themselves correspond to a holomorphic vector bundle that deforms as t varies. The subspace $W_0(\gamma, \alpha) \subset V$ can be identified with the space of holomorphic sections of the associated $\mathfrak{sl}(2, \mathbf{C})$ bundle over the punctured torus $\Gamma \setminus \{z = 0\}$. $\phi(t, z)$ is related to changing local trivialization of this bundle at $z = 0$. Note, in particular, that the data of local trivialization plays the role of dynamical variables. This is to be contrasted with the work of Previato and Wilson; in their case, a set of functional parameters in the algebraic spectral data play a similar role in place of the data of local trivialization. In this respect, our approach is more close to Li and Mulase's approach [12, 9] to the classification of commutative rings of differential operators, in which the choice of local trivialization is treated as an independent data.

6. Construction à la Enriquez and Rubtsov

Enriquez and Rubtsov [2] parametrize the $\mathfrak{sl}(2, \mathbf{C})$ Hitchin system on an algebraic curve of genus $g \geq 2$ [4] by $3g$ (rather than $2g$) pairs (γ_s, α_s) ,

$s = 1, \dots, 3g$, of Tyurin parameters. The roles of parameters are also different from Krichever's formulation. Namely, whereas the directional vectors $\alpha_s = {}^t(\alpha_s, 1)$ remain dynamical, the poles γ_s are fixed.

We borrow their idea to construct another elliptic analogue of the nonlinear Schrödinger hierarchy. This hierarchy has three pairs (γ_s, α_s) , $s = 1, 2, 3$, as Tyurin parameters; γ_s are constant and α_s are variables. In addition to these Tyurin parameters, the hierarchy contains the nonlinear Schrödinger fields u, v . The foregoing consideration on the elliptic analogue of the Krichever type can be extended to this case with minimal modifications.

6.1. Construction of A -matrix

The A -matrix $A(z)$ is a 2×2 matrix of meromorphic functions on Γ characterized by the following properties:

1. $A(z)$ has poles at $z = 0, \gamma_1, \gamma_2, \gamma_3$ and holomorphic at other points.
2. As $z \rightarrow 0$,

$$A(z) = \begin{pmatrix} z^{-1} & u \\ v & -z^{-1} \end{pmatrix} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2, 3$,

$$(6.1) \quad A(z) = \lambda_s \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} (z - \gamma_s)^{-1} + O(1),$$

where λ_s is a constant to be determined below.

One can write $A(z)$ itself more explicitly as

$$(6.2) \quad A(z) = \sum_{s=1,2,3} \lambda_s \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} (\zeta(z - \gamma_s) + \zeta(\gamma_s)) \\ + \begin{pmatrix} \zeta(z) & u \\ v & -\zeta(z) \end{pmatrix}.$$

By the residue theorem, the coefficients have to satisfy the linear equations

$$(6.3) \quad \sum_{s=1,2,3} \lambda_s \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This yields the three linear equations

$$(6.4) \quad \sum_{s=1,2,3} \lambda_s = 0, \quad \sum_{s=1,2,3} \alpha_s \lambda_s = -1, \quad \sum_{s=1,2,3} \alpha_s^2 \lambda_s = 0,$$

which can be solve for λ_s 's as

$$(6.5) \quad \lambda_1 = \frac{\alpha_3^2 - \alpha_2^2}{\Delta}, \quad \lambda_2 = \frac{\alpha_1^2 - \alpha_3^2}{\Delta}, \quad \lambda_3 = \frac{\alpha_2^2 - \alpha_1^2}{\Delta}$$

as far as the Vandermonde determinant

$$\Delta = \Delta(\alpha_1, \alpha_2, \alpha_3) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{vmatrix}$$

does not vanish. This condition

$$(6.6) \quad \Delta \neq 0$$

is the “genericity condition” in the present setting. We assume this condition throughout the consideration in the following.

The Tyurin parameters are required to satisfy the differential equations (3.5) and (3.6). Note that the residue matrices of $A(z)$ at $z = \gamma_s$ can be factorized as

$$\lambda_s \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} = \lambda_s \begin{pmatrix} 1 \\ -\alpha_s \end{pmatrix} \begin{pmatrix} \alpha_s & 1 \end{pmatrix},$$

so that the role of β_s is now played by $\lambda_s^t(1, -\alpha_s)$. (3.5) reduces to

$$(6.7) \quad \partial_x \gamma_s = -\text{Tr} \lambda_s \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} = 0,$$

thus being consistent with the assumption that γ_s are understood to be constant. On the other hand, (3.6) takes the form

$$(6.8) \quad \begin{aligned} \partial_x \alpha_s &= \sum_{r \neq s} \lambda_r (\alpha_s - \alpha_r)^2 (\zeta(\gamma_s - \gamma_r) + \zeta(\gamma_r)) \\ &\quad + \alpha_s^2 u - 2\alpha_s \zeta(\gamma_s) + v \end{aligned}$$

with the constant κ_s uniquely determined as

$$(6.9) \quad \kappa_s = \sum_{r \neq s} \lambda_r (\alpha_s - \alpha_r) (\zeta(\gamma_s - \gamma_r) + \zeta(\gamma_r)) + \alpha_s u - \zeta(\gamma_s).$$

6.2. Construction of hierarchy

The construction of time evolutions is fully parallel to the previous case. Firstly, we construct a 2×2 matrix of generating functions

$$U(z) = \sum_{n=1}^{\infty} U_n z^n, \quad U_0 = J,$$

as a solution of the equations

$$[\partial_x - A(z), U(z)] = 0, \quad U(z)^2 = I.$$

The coefficients U_n are uniquely determined by a set of recurrence relations; the matrix elements thus turn out to be a differential polynomial of α_s ($s = 1, 2, 3$), u and v .

Having this generating function as local data at $z = 0$, we now proceed to the construction of the generators $A_n(z)$ of time evolutions. $A_n(z)$ is a 2×2 matrix of meromorphic functions on Γ with the following properties:

1. $A_n(z)$ has poles at $z = 0, \gamma_1, \gamma_2, \gamma_3$ and holomorphic at other points.
2. As $z \rightarrow 0$,

$$A_n(z) = U(z)z^{-n} + O(z).$$

3. As $z \rightarrow \gamma_s$, $s = 1, 2, 3$,

$$(6.10) \quad A_n(z) = \lambda_{n,s} \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} (z - \gamma_s)^{-1} + O(1),$$

where $\lambda_{n,s}$ is a constant to be determined below.

$A_n(z)$ is uniquely determined by these conditions, and can be written as

$$(6.11) \quad A_n(z) = \sum_{s=1,2,3} \lambda_{n,s} \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} (\zeta(z - \gamma_s) + \zeta(\gamma_s)) \\ + \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \partial_z^m \zeta(z) U_{n-1-m} + U_n.$$

The coefficients $\lambda_{n,s}$ are determined by the linear equations

$$(6.12) \quad \sum_{s=1,2,3} \lambda_{n,s} \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} + U_{n-1} = 0$$

or, equivalently,

$$(6.13) \quad \begin{aligned} \sum_{s=1,2,3} \lambda_{n,s} &= -(U_{n-1})_{12}, \\ \sum_{s=1,2,3} \alpha_s \lambda_{n,s} &= -(U_{n-1})_{11} = (U_{n-1})_{22}, \\ \sum_{s=1,2,3} \alpha_s^2 \lambda_{n,s} &= -(U_{n-1})_{21}. \end{aligned}$$

Of course, these linear equations are uniquely solvable as far as the genericity condition $\Delta \neq 0$ is satisfied.

Lastly, the hierarchy is defined by the system of Lax equations

$$(6.14) \quad [\partial_{t_n} - A_n(z), U(z)] = 0$$

for $U(z)$ and the differential equations

$$(6.15) \quad \partial_{t_n} {}^t \alpha_s + {}^t \alpha_s A_n^{(s,1)} = \kappa_{n,s} {}^t \alpha_s,$$

for ${}^t \alpha_s = (\alpha_s, 1)$. The differential equations for γ_s reduce to

$$(6.16) \quad \partial_{t_n} \gamma_s = -\text{Tr} \lambda_{n,s} \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} = 0$$

as expected. One can derive the zero-curvature equations $[\partial_{t_m} - A_m(z), \partial_{t_n} - A_n(z)] = 0$ by the same procedure as in the previous case.

6.3. Riemann-Hilbert problem and Grassmann variety

The Riemann-Hilbert problem and the mapping to an infinite dimensional Grassmann variety can be derived in almost the same form as the previous case. The present case is conceptually rather simpler, because the poles γ_s do not move. To avoid confusion, we again move to the convention that the t -dependence is always explicitly indicated as $A(t, z)$, $A_n(t, z)$, $\alpha_s(t)$, etc. Note that γ_s 's are constant throughout the present setting.

The Riemann-Hilbert pair consists of a Laurent series solution $\psi(t, z)$ and a global solution $\chi(t, z)$ of the same auxiliary linear system. The former takes the form

$$\psi(t, z) = \phi(t, z) \exp\left(\sum_{n=1}^{\infty} t_n J z^{-n}\right), \quad \phi(t, z) = I + \sum_{n=1}^{\infty} \phi_n(t) z^n.$$

The prefactor $\phi(t, z)$ is connected with the generating function $U(t, z)$ (the t -dependence is now shown explicitly) as $U(t, z) = \phi(t, z) J \phi(t, z)^{-1}$. The second solution $\chi(t, z)$ of the auxiliary linear system is characterized by the initial condition $\chi(0, z) = I$. One can prove, by the same technique as the previous case, that $\chi(t, z)$ has essential singularity at $z = 0$ and poles at $z = \gamma_1, \gamma_2, \gamma_3$, and behave as

$$(6.17) \quad \chi(t, z) = \lambda_{\chi, s} \begin{pmatrix} \alpha_s(0) & 1 \\ -\alpha_s(0)^2 & -\alpha_s(0) \end{pmatrix} (z - \gamma_s)^{-1} + O(1)$$

as $z \rightarrow \gamma_s$. These two solutions $\chi(t, z), \psi(t, z)$ of the auxiliary linear system obeys a relation of the same form as (4.13). On the other hand, since $A_n(z)$'s are trace-free, both $\chi(t, z)$ and $\phi(t, z)$ are now unimodular, i.e.,

$$(6.18) \quad \det \chi(t, z) = \det \phi(t, z) = 1.$$

Consequently, unlike the previous case, $\chi(t, z)$ has no degeneration point.

We use the same Grassmann variety Gr to embed the hierarchy. The definition of the base point $W_0(\gamma, \alpha)$ for the present case, too, is essentially the same, except that we now use the three pairs (γ_s, α_s) , $s = 1, 2, 3$, as the input. The basis $\{w_{n,ij}(z) \mid n \geq 0, i, j = 1, 2\}$ of $W_0(\gamma, \alpha)$ consists of the matrices $w_{n,ij}(z)$ of meromorphic functions on Γ uniquely determined by these parameters as in the statement of Lemma 10; the third condition therein has to be modified as

$$(6.19) \quad w_{n,ij}(z) = \lambda_{n,ij,s} \begin{pmatrix} \alpha_s & 1 \\ -\alpha_s^2 & -\alpha_s \end{pmatrix} (z - \gamma_s)^{-1} + O(1) \quad (z \rightarrow \gamma_s).$$

By the correspondence $\phi(t, z) \mapsto W(t) = W_0(\gamma, \alpha(t))\phi(t, z)$, the hierarchy is converted to a dynamical system on the set \mathcal{M} of dressed vacua. The motion of $W(t)$ again turns out to obey the same exponential law as (5.9).

7. Conclusion

We have elucidated the status of the two elliptic analogues of the nonlinear Schrödinger hierarchy in the Grassmannian perspective of Sato [14] and Segal and Wilson [15]. Each of these systems are mapped to a dynamical system in the Grassmann variety Gr . The phase space of the dynamical system is the set \mathcal{M} of dressed vacua $W = W_0(\gamma, \alpha)\phi(z)$. The motion of the dressed vacuum $W(t) = W_0(\gamma(t), \alpha(t))\phi(t, z)$ under time evolutions of the hierarchy obeys a simple exponential law. This is just the restriction of universal exponential flows on the Grassmann variety itself. Thus the situation is fully parallel to many classical soliton equations that have been understood in the Grassmannian perspective.

It is straightforward to generalize the 2×2 system of the Krichever type to an $r \times r$ system [8]. In that case, one has to use several U -matrices rather than a single one. The Tyurin parameters consist of r pairs $(\gamma_s, \alpha_s) \in \Gamma \times \mathbf{P}^{r-1}$, $s = 1, \dots, r$, of a point γ_s of Γ and an r dimensional directional vector α_s . As a special case, one can obtain an elliptic analogue of the so called N wave system, etc.

If one does not insist on an explicit description of the system, one can generalize the results of this paper to an algebraic curve Γ of genus g with a marked point P_0 . The Tyurin parameters for the construction of the Krichever type consist of $2g$ pairs $(\gamma_s, \alpha_s) \in \Gamma \times \mathbf{P}^1$ of a point of Γ and a two dimensional directional vector [8]. The construction à la Enriquez and Rubtsov requires $3g$, rather than $2g$, pairs of Tyurin parameters [2]. Upon choosing a local parameter z in a neighborhood of P_0 , one can start the construction of the fundamental matrix $A(P)$ ($P \in \Gamma$) of meromorphic functions on Γ and the matrix $U(z)$ of Laurent series. A convenient choice of z is to define it as the (multivalued) primitive function $z(P) = \int_{P_0}^P \omega$ of a holomorphic differential ω on Γ without zero at P_0 . The matrices $A_n(P)$, like $A(P)$, are characterized by a set of conditions on the poles. Namely, they are matrices of meromorphic functions on Γ with poles at P_0 and γ_s 's, and behave as

$$A_n(P) = U(z(P))z(P)^{-n} + O(z(P)) \quad (P \rightarrow P_0),$$

$$A_n(P) = \frac{\beta_{n,s} {}^t \alpha_s}{z(P) - z(\gamma_s)} + O(1) \quad (P \rightarrow \gamma_s).$$

The existence and the uniqueness of these matrices are ensured by the

Riemann-Roch theorem. Formulating these systems in a more explicit form is a problem left for future research.

Lastly, let us mention some other approaches to soliton equations associated with algebraic curves. Ben-Zvi and Frenkel [1] and Levin, Olshanetsky and Zotov [10] propose to construct those equations as a $1 + 1$ dimensional analogue of the Hitchin systems [4]. The framework of Ben-Zvi and Frenkel is conceptually similar to ours, though they use a Grassmann variety in a different way. The work of Li and Mulase [12, 9] is also closely related to the present issue. Our construction of dressed vacua has obviously a counterpart in their description of commutative rings of differential operators in the language of infinite dimensional Grassmann variety.

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