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A Characterization of Tempered Distributions with Support in a Cone by the Heat Kernel Method and its Applications

By Masanori Suwa and Kunio Yoshino

Abstract. We will characterize the space of tempered distributions with support in a proper convex cone by the heat kernel method. As applications we give a new proof of the Paley-Wiener's theorem and the Edge-of-the-Wedge theorem for tempered distributions supported by a proper convex cone.

1. Introduction

In 1987, T.Matsuzawa characterized the spaces of distributions, ultradistributions and hyperfunctions as the initial values of C^∞-solutions of the heat equation with appropriate growth rate conditions [4], [5]. In 1990, T.Matsuzawa also characterized the spaces of tempered distributions by the same method. We shall call this method " *The heat kernel method*". In this paper, we shall characterize the space of tempered distributions supported by a proper convex cone by the heat kernel method (Theorem 3.4).

As applications of the heat kernel method, T.Matsuzawa gave a new proof of the Paley-Wiener theorem for hyperfunctions supported by a ball by the heat kernel method in [4]. The proof is simpler than the former proof because we need not use the higher functional analysis and algebra techniques. Touched by this work, in 1998, S.Lee and S.-Y.Chung gave a new proof of the Paley-Wiener theorem for distributions supported by a compact convex set by the heat kernel method [2] and in 2003, M.Suwa and K.Yoshino gave a new proof of the Paley-Wiener theorem for hyperfunctions supported by a convex compact set by the heat kernel method [10]. In §4, we shall treat the Paley-Wiener theorem for tempered distributions supported by a proper convex cone by the heat kernel method. For an element of distributions or hyperfunctions supported by a convex compact set, the

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image by the Fourier-Laplace transform becomes *an entire function.* So it is trivial that we can take its boundary value. However, in our case, the image by the Fourier-Laplace transform is *not an entire function.* Hence, it is the most important point to show the existence of the boundary value in tempered distributions.

In §5, we shall give a new proof of the Edge-of-the-Wedge theorem for tempered distributions supported by a proper convex cone. Also we must notice that it is simpler than the former proof to show the existence of the boundary value in tempered distributions by the heat kernel method.

2. Preliminaries

Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$. Then we set $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$, $\langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j$, $x^2 = \langle x, x \rangle$ and $B(x_0, \delta) = \{x \in \mathbb{R}^n; |x - x_0| < \delta,\ \delta > 0\}$. If $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$, then we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and $\triangle = \sum_{j=1}^{n} \partial_{x_j}^2$. We define the heat kernel by $E(x, t) = (4\pi t)^{-n/2} \exp(-x^2/4t)$, $t > 0$. Let Ω be an open set in $\mathbb{C}^n$. We denote by $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω. $\mathcal{D}(\mathbb{R}^n)$ is the space of $\mathcal{C}^\infty$ functions with compact support. $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions and $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. We put $\mathcal{S}'_{\overline{A}} := \{T \in \mathcal{S}'(\mathbb{R}^n); \mathrm{supp}\ T \subset \overline{A}\}$. If $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, then the Fourier transform $\mathcal{F}(\varphi)(x)$ is defined by

$$\mathcal{F}(\varphi)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(\xi) e^{i\xi x} d\xi$$

and the Fourier inverse transform $\mathcal{F}^{-1}(\varphi)(\xi)$ is defined by

$$\mathcal{F}^{-1}(\varphi)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x) e^{-i\xi x} dx.$$

The convolution $(\varphi * \phi)(x)$ of functions $\varphi(x) \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(x) \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$(\varphi * \phi)(x) = \int_{\mathbb{R}^n} \varphi(x - y) \phi(y) dy.$$

If $\varphi(x)e^{i\zeta x} \in L^1(\mathbb{R}^n_x)$, then $\mathcal{LF}(\varphi)(\zeta)$ is defined by

$$\mathcal{LF}(\varphi)(\zeta) \quad = \quad \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(x)e^{i\zeta x}dx, \quad \zeta \in \mathbb{C}^n.$$

Let A be a set in $\mathbb{R}^n$. Then we denote by A° the interior of A, by $\overline{A}$ the closure of A, for $\varepsilon > 0$, $\overline{A}_\varepsilon = \{x \in \mathbb{R}^n; \mathrm{dis}(x, \overline{A}) \leq \varepsilon\}$ and by $\mathrm{ch}(A)$ the convex hull of A.

Definition 2.1 ([3],[11]). Let Γ be a cone with vertex at 0. If $\mathrm{ch}(\overline{\Gamma})$ contains no straight line, then we call Γ a proper cone. Furthermore we put

$$\Gamma' := \{\xi \in \mathbb{R}^n; \langle y, \xi\rangle \geq 0 \text{ for all } y \in \Gamma\}.$$

Then we call Γ' the dual cone of Γ.

Remark 2.2 ([11]).

1. The following conditions are equivalent:
   (a) Γ is a proper cone.
   (b) $(\Gamma')^\circ \neq \emptyset$.
2. The following equalities hold:
   (a) $(\Gamma')' = \mathrm{ch}(\overline{\Gamma})$
   (b) $(\Gamma_1 \cap \Gamma_2)' = \mathrm{ch}({\Gamma'}_1 \cup {\Gamma'}_2)$.

Definition 2.3 ([9],[12]). Let $A \subset \mathbb{R}^n$ be a closed set and satisfy the following condition:

$$\begin{aligned}&\exists d > 0\ \exists \omega \geq 0\ \exists q \geq 1,\ \forall x_1 \in A,\ \forall x_2 \in A,\ |x_1 - x_2| \leq d,\\ &\ \exists \gamma,\ \gamma \subset A,\ \text{a curve binding } x_1 \text{ and } x_2,\\ &\text{such that } l \leq \omega |x_1 - x_2|^{1/q},\ \text{where } l \text{ is the length of a curve } \gamma.\end{aligned}$$

Then we call A a regular closed set.

Remark 2.4. We note that a closed convex set is a regular closed set.

3. Tempered Distributions as the Boundary Values of Smooth Solutions of Heat Equations

First, we recall the results about tempered distributions as the initial values of smooth solutions of heat equations.

THEOREM 3.1 (Matsuzawa [6],[7]). *Let* $T \in \mathcal{S}'(\mathbb{R}^n)$ *and* $U(x,t) = \langle T_y, E(x-y,t)\rangle$. *Then* $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfies the following conditions*:

(1) $$(\partial_t - \triangle)\, U(x,t) = 0.$$

(2) $$U(x,t) \longrightarrow T \quad in \quad \mathcal{S}'(\mathbb{R}^n) \quad (t \to 0_+).$$

(3) *There exist some positive constants* N, M *and* C *such that*

$$|U(x,t)| \le Ct^{-N}(1+|x|)^M, \quad (0 < t < 1).$$

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfying* (1) *and* (3), *there exists a unique* $T \in \mathcal{S}'(\mathbb{R}^n)$ *such that* $\langle T_\xi, E(x-\xi,t)\rangle = U(x,t)$.

LEMMA 3.2 ([9], [12]). *Let* $A \subset \mathbb{R}^n$ *be a regular closed set. If* $T \in \mathcal{S}'_A$, *then there exists a tempered measure* μ_α $(|\alpha| \le m)$, *supp* $\mu_\alpha \subset A$, *such that*

$$T = \sum_{|\alpha| \le m} D^\alpha \mu_\alpha.$$

LEMMA 3.3 (Bros-Epstein-Glaser [1],[8]). *Let* Γ *be a proper open convex cone in* $\mathbb{R}^n$ *and let* $T \in \mathcal{S}'_{\overline{\Gamma}}$. *Then there exists a polynomially bounded continuous function* G *with support in* $\overline{\Gamma}$ *and a partial differential operator* $P(D)$ *so that* $T = P(D)G$.

We have the following theorem:

THEOREM 3.4. *Suppose that* Γ *is a proper convex cone in* $\mathbb{R}^n$. *Let* $T \in \mathcal{S}'_{\overline{\Gamma}}$ *and* $U(x,t) = \langle T_y, E(x-y,t)\rangle$. *Then* $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfies the following conditions*:

(4) $$(\partial_t - \triangle)\, U(x,t) = 0.$$

(5) $$U(x,t) \longrightarrow T \quad in \quad \mathcal{S}'(\mathbb{R}^n) \quad (t \to 0_+).$$

(6) *There exist some positive constants* N, M *and* C *such that*

$$|U(x,t)| \le Ct^{-N}(1+|x|)^M \exp\left(-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}\right), \quad (0 < t < 1, x \in \mathbb{R}^n).$$

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *satisfying* (4) *and* (6), *there exists a unique* $T \in \mathcal{S}'_{\overline{\Gamma}}$ *such that* $\langle T_y, E(x-y,t)\rangle = U(x,t)$.

Proof. Let $T \in \mathcal{S}'_{\overline{\Gamma}}$. By Theorem 3.1, we have (4) and (5). Let $0 < t < 1$. By Lemma 3.3, we have

$$\begin{aligned} T(x) &= P(D)G(x), \quad \text{supp } G(x) \subset \overline{\Gamma}, \\ &\exists M \geq 0,\ \exists C \geq 0, \quad |G(x)| \leq C(1+|x|)^M, \quad (x \in \mathbb{R}^n). \end{aligned}$$

Since

$$\begin{aligned} U(x,t) &= \langle T_y, E(x-y,t)\rangle \\ &= \langle G(y), P(-D_y)E(x-y,t)\rangle \\ &= \int_{\overline{\Gamma}} G(y)P(-D_y)E(x-y,t)dy, \end{aligned}$$

we have

$$\begin{aligned} |U(x,t)| &\leq \int_{\overline{\Gamma}} |G(y)P(-D_y)E(x-y,t)|dy \\ &\leq C\int_{\overline{\Gamma}} (1+|y|)^{M+2n}|P(-D_y)E(x-y,t)|\frac{1}{(1+|y|)^{2n}}dy. \end{aligned}$$

For the heat kernel we have the following estimate [7]:

$$|D^\alpha E(x,t)| \leq \frac{\alpha!}{(4\pi t)^{n/2}}\left(\frac{en}{2t|\alpha|}\right)^{|\alpha|/2}\exp\left(-\frac{x^2}{8t}\right).$$

So

$$\begin{aligned} &|U(x,t)| \\ &\leq C_1 t^{-N}(1+|x|)^{M+2n}\int_{\overline{\Gamma}}(1+|x-y|)^{M+2n}e^{-\frac{|x-y|^2}{16t}}e^{-\frac{|x-y|^2}{16t}}\frac{1}{(1+|y|)^{2n}}dy \\ &\leq C_2 t^{-N}(1+|x|)^{M+2n}e^{-\frac{\text{dis}(x,\overline{\Gamma})^2}{16t}}\int_{\overline{\Gamma}}\frac{1}{(1+|y|)^{2n}}dy \\ &\leq C_3 t^{-N}(1+|x|)^{M+2n}e^{-\frac{\text{dis}(x,\overline{\Gamma})^2}{16t}}. \end{aligned}$$

Hence we have

$$|U(x,t)| \leq Ct^{-N}(1+|x|)^{M+2n}\exp\left(-\frac{\text{dis}(x,\overline{\Gamma})^2}{16t}\right).$$

Now we will prove the converse. Assume that $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfies (4) and (6). Then

$$\begin{aligned} |U(x,t)| &\leq Ct^{-N}(1+|x|)^M e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}} \\ &\leq Ct^{-N}(1+|x|)^M, \quad (0<t<1, x\in\mathbb{R}^n). \end{aligned}$$

By Theorem 3.1, there exists $T \in \mathcal{S}'$ such that

$$U(x,t) = \langle T_y, E(x-y,t)\rangle.$$

Let $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ and $K = \mathrm{supp}(\varphi) \subset \mathbb{R}^n \backslash \overline{\Gamma}$. Then

$$\begin{aligned} \left|\int_{\mathbb{R}^n} U(x,t)\varphi(x)dx\right| &= \left|\int_{\mathrm{supp}\ \varphi} U(x,t)\varphi(x)dx\right| \\ &\leq Ct^{-N}\int_K (1+|x|)^M|\varphi(x)|e^{-\frac{\mathrm{dis}(x,\overline{\Gamma})^2}{16t}}dx. \end{aligned}$$

Set $\delta := \mathrm{dis}(K,\overline{\Gamma}) > 0$. Then for some constant $C_1 > 0$ we have

$$\begin{aligned} \left|\int_{\mathbb{R}^n} U(x,t)\varphi(x)dx\right| &\leq Ct^{-N}\int_K (1+|x|)^M|\varphi(x)|e^{-\frac{\mathrm{dis}(K,\overline{\Gamma})^2}{16t}}dx \\ &\leq C_1 t^{-N} e^{-\frac{\delta^2}{16t}}. \end{aligned}$$

So we have

$$\lim_{t\to 0_+}\int_{\mathbb{R}^n} U(x,t)\varphi(x)dx = 0.$$

On the other hand, by (5), $U(x,t) \to T$, $(t \to 0_+)$ in $\mathcal{S}'$. Hence we have $\mathrm{supp}\ T \subset \overline{\Gamma}$. □

For (6) in Theorem 3.4, we have the following lemma:

LEMMA 3.5. *Let* $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ *and satisfies* $(\partial_t - \triangle)U(x,t) = 0$. *Then condition* (6) *in Theorem* 3.4 *is equivalent to the following conditions*:

(7) *There exist some positive constants* N, M *and* C *such that*

$$|U(x,t)| \leq Ct^{-N}(1+|x|)^M, \quad (0<t<1, x\in\mathbb{R}^n),$$

and $U(x,t) \to 0, (t \to 0_+)$, *uniformly on all compact sets in* $\mathbb{R}^n\backslash\overline{\Gamma}$.

PROOF. (6) $\Rightarrow$ (7) is obvious. Now we suppose (7). By the estimate in (7) and Theorem 3.1, there exists $T \in \mathcal{S}'$ such that $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Let $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$, supp $(\varphi) \subset \mathbb{R}^n \backslash \overline{\Gamma}$. Then by (2) in Theorem 3.1 and the assumption in (7), we have

$$\langle T, \varphi \rangle \;=\; \lim_{t \to 0_+} \int_{\mathbb{R}^n} U(x,t)\varphi(x)dx = 0.$$

It means that $T \in \mathcal{S}'_{\overline{\Gamma}}$. By Theorem 3.4, we have (6). □

By Lemma 3.5, we have the following corollary:

COROLLARY 3.6. *Let $T \in \mathcal{S}'_{\overline{\Gamma}}$ and $U(x,t) = \langle T_y, E(x-y,t)\rangle$. Then $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfies the following conditions*:

(8) $$(\partial_t - \triangle)\, U(x,t) = 0.$$

(9) $$U(x,t) \longrightarrow T \quad in \quad \mathcal{S}'(\mathbb{R}^n) \quad (t \to 0_+).$$

(10) *There exist some positive constants N, M and C such that*

$$|U(x,t)| \le Ct^{-N}(1+|x|)^M, \;(0 < t < 1, x \in \mathbb{R}^n)$$

and $U(x,t) \to 0, (t \to 0_+)$, uniformly on all compact sets in $\mathbb{R}^n \backslash \overline{\Gamma}$.

Conversely, for a function $U(x,t) \in \mathcal{C}^\infty(\mathbb{R}^n \times (0,\infty))$ satisfying (8) *and* (10), *there exists a unique $T \in \mathcal{S}'_{\overline{\Gamma}}$ such that $\langle T_y, E(x-y,t)\rangle = U(x,t)$.*

4. Paley-Wiener's Theorem for $\mathcal{S}'_{\overline{\Gamma}}$

In [2], S.Lee and S.-Y.Chung proved Paley-Wiener's theorem for distributions with compact support by the heat kernel method. In this section, we give a new proof of Paley-Wiener's theorem for $\mathcal{S}'_{\overline{\Gamma}}$ by the similar way. Γ denotes a proper open convex cone in this section.

DEFINITION 4.1 ([3], [11]). For $T \in \mathcal{S}'_{\overline{\Gamma}}$, putting $\zeta = \xi + i\eta \in \mathbb{R}^n + i(\overline{\Gamma}')^\circ$, we define the Laplace transform of T by

$$\mathcal{LF}(T)(\xi + i\eta) \;:=\; (2\pi)^{-n/2}\langle T_x, e^{i\zeta x}\rangle.$$

The right hand side means $\langle T_x, e^{i\zeta x}\rangle = \langle T_x, \chi(x)e^{i\zeta x}\rangle$ where $\chi(x) \in \mathcal{C}^\infty(\mathbb{R}^n)$ which satisfies

$$\chi(x) = \begin{cases} 1 & , x \in \overline{\Gamma}_\varepsilon \\ 0 & , x \notin \overline{\Gamma}_{2\varepsilon}, \quad \varepsilon > 0. \end{cases}$$

First we need the following lemma:

LEMMA 4.2. *For every $\varphi \in \mathcal{D}(\mathbb{R}^n)$, let*

$$\begin{aligned} V_\eta(x,t) &= e^{-t\eta^2 - \eta x}(E(x,t) * \varphi(-x)) \\ &= e^{-t\eta^2 - \eta x}\int_{\mathbb{R}^n} E(x-y,t)\varphi(-y)dy, \quad 0 < t < 1. \end{aligned}$$

Then for every $N \geq 0$ and $\beta \in \mathbb{N}_0^n$, there exists a constant $C \geq 0$ such that

$$\sup_{x \in \mathbb{R}^n} |(1+|x|^2)^N \partial_x^\beta (V_\eta(x,t) - \varphi(-x)e^{-\eta x})| \leq C\sqrt{t}, \quad 0 < t < 1.$$

PROOF.

$$\begin{aligned} &V_\eta(x,t) - \varphi(-x)e^{-\eta x} \\ &= e^{-t\eta^2}\int_{\mathbb{R}^n} E(x-y,t)e^{-\eta(x-y)}\varphi(-y)e^{-\eta y}dy - \varphi(-x)e^{-\eta x} \\ &= e^{-t\eta^2}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}\varphi(u-x)e^{-\eta(x-u)}du \\ &\quad - e^{-t\eta^2}\varphi(-x)e^{-\eta x}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}du \\ &= e^{-t\eta^2}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}(\varphi(-(x-u))e^{-\eta(x-u)} - \varphi(-x)e^{-\eta x})du \\ &= e^{-t\eta^2}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}(\phi(x-u) - \phi(x))du, \end{aligned}$$

where $\phi(x) = \varphi(-x)e^{-\eta x}$. Applying the mean value theorem and Peetre's inequality [2]

$$(1+|\xi|^2)^s(1+|\eta|^2)^{-s} \leq 2^{|s|}(1+|\xi-\eta|^2)^{|s|}$$

for $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
&(1+|x|^2)^N|\partial^\beta\phi(x-u)-\partial^\beta\phi(x)| \\
&\le \sup_{0<\theta<1}(1+|x|^2)^N|\nabla\partial^\beta\phi(x-\theta u)||u| \\
&\le \sup_{0<\theta<1} C(1+|x|^2)^N(1+|x-\theta u|^2)^{-N}|u| \\
&\le C2^N(1+|u|^2)^N|u|.
\end{aligned}$$

Then for every $N>0$ and $\beta \in \mathbb{N}_0^n$ and $0<t<1$, we have

$$\begin{aligned}
&(1+|x|^2)^N|\partial^\beta V_\eta(x,t)-\partial^\beta\phi(x)| \\
&= (1+|x|^2)^N e^{-t\eta^2}\left|\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}(\partial^\beta\phi(x-u)-\partial^\beta\phi(x))du\right| \\
&\le e^{-t\eta^2}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}(1+|x|^2)^N|\partial^\beta\phi(x-u)-\partial^\beta\phi(x)|du \\
&\le C2^N e^{-t\eta^2}\int_{\mathbb{R}^n} E(u,t)e^{-\eta u}(1+|u|^2)^N|u|du \\
&= C\frac{2^N}{\pi^{n/2}}e^{-t\eta^2}\int_{\mathbb{R}^n} e^{-v^2}e^{-\sqrt{4t}\eta v}(1+4tv^2)^N\sqrt{4t}|v|dv \\
&\le C_1\sqrt{4t}\int_{\mathbb{R}^n} e^{-(v+\sqrt{t}\eta)^2}(1+4v^2)^N|v|dv \\
&\le C_1\sqrt{4t}e^{t\eta^2}\int_{\mathbb{R}^n} e^{-\frac{1}{2}v^2}(1+4v^2)^N|v|dv \\
&\le C_2\sqrt{t}.
\end{aligned}$$

So we obtain the lemma. □

By Lemma 4.2, we have the following corollary:

Corollary 4.3. *$V_\eta(x,t) \to \varphi(-x)e^{-\eta x}$ in $\mathcal{S}(\mathbb{R}^n)$, as $t \to 0_+$.*

Theorem 4.4 (Paley-Wiener [8], [12]). *Suppose that Γ is a proper open convex cone in $\mathbb{R}^n$. If $F(\zeta) = \mathcal{F}(e^{-\eta x}T)(\xi)$ with some $T \in \mathcal{S}'_{\overline{\Gamma}}$ and $\zeta = \xi + i\eta \in \mathbb{R}^n + i(\overline{\Gamma}')^\circ$, then $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + i(\overline{\Gamma}')^\circ)$. Furthermore there exists an $m>0$ and for every cone C satisfying $\{\eta \in \overline{C}; \eta \ne 0\} \subset (\overline{\Gamma}')^\circ$, there exists a constant $M_C \ge 0$ such that*

$$|F(\zeta)| \le M_C(1+|\zeta|)^m(1+|\eta|^{-m}), \quad \zeta \in \mathbb{R}^n + iC. \tag{11}$$

Conversely, for $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + i(\overline{\Gamma}')^\circ)$ *satisfying the conditions* (11), *there exists* $T \in \mathcal{S}'_{\overline{\Gamma}}$ *such that*

$$F(\zeta) \;=\; \mathcal{F}(e^{-\eta x}T)(\xi).$$

PROOF. For the necessity, we refer the reader to [11].

Now we will prove the sufficiency. Let $\zeta = \xi + i\eta \in \mathbb{R}^n + i(\overline{\Gamma}')^\circ$. We set

$$U(x,t) \;=\; (2\pi)^{-n/2}\int_{\mathbb{R}^n} F(\zeta)e^{-t\zeta^2}e^{-i\zeta x}d\xi, \quad t>0. \tag{12}$$

It is obvious that

$$(\partial_t - \triangle)\,U(x,t) \;=\; 0.$$

We notice that $U(x,t)$ is independent of $\eta \in (\overline{\Gamma}')^\circ$. Then we have

$$\begin{aligned}
&|U(x,t)|\\
&\le (2\pi)^{-n/2}\int_{\mathbb{R}^n} |F(\zeta)|e^{-t\xi^2+t\eta^2+\eta x}d\xi\\
&\le Me^{t\eta^2+\eta x}(1+|\eta|^{-m})\int_{\mathbb{R}^n}(1+|\xi+i\eta|)^m e^{-t\xi^2}d\xi\\
&\le Me^{t\eta^2+\eta x}(1+|\eta|^{-m})\int_{\mathbb{R}^n}(1+|\xi_1|+|\eta_1|+\cdots+|\xi_n|+|\eta_n|)^m e^{-t\xi^2}d\xi\\
&\le Me^{t\eta^2+\eta x}(1+|\eta|^{-m})\\
&\qquad\times\int_{\mathbb{R}^n}\{(1+|\xi_1|)(1+|\eta_1|)\cdots(1+|\xi_n|)(1+|\eta_n|)\}^m e^{-t\xi^2}d\xi\\
&\le 2^{2mn}Me^{t\eta^2+\eta x}(1+|\eta|^{-m})\\
&\qquad\times\int_{\mathbb{R}^n}\{(1+|\xi_1|^2)(1+|\eta_1|^2)\cdots(1+|\xi_n|^2)(1+|\eta_n|^2)\}^m e^{-t\xi^2}d\xi\\
&= M_1e^{t\eta^2+\eta x}(1+|\eta|^{-m})\{(1+|\eta_1|^2)\cdots(1+|\eta_n|^2)\}^m\\
&\qquad\times\int_{\mathbb{R}^n}\{(1+|\xi_1|^2)\cdots(1+|\xi_n|^2)\}^m e^{-t\xi^2}d\xi\\
&= M_2t^{-(m+1/2)n}e^{t\eta^2+\eta x}(1+|\eta|^{-m})\{(1+|\eta_1|^2)\cdots(1+|\eta_n|^2)\}^m.
\end{aligned}$$

Let $|x| \geq 1$. For η of (12), we choose $\eta'' \in (\overline{\Gamma}')^{\circ}$ such that $|\eta''| = \dfrac{1}{|x|}$. For $0 < t < 1$, we have

$$\begin{aligned}
&|U(x,t)| \\
&\leq \; M_2 t^{-(m+1/2)n} e^{t|\eta''|^2 + |\eta''||x|}(1 + |\eta''|^{-m})\{(1 + |\eta''{}_1|^2)\cdots(1 + |\eta''{}_n|^2)\}^m \\
&\leq \; M_3 t^{-(m+1/2)n}(1 + |x|)^m.
\end{aligned}$$

Let $|x| \leq 1$. For η of (12), we choose $\eta'' \in (\overline{\Gamma}')^{\circ}$ such that $|\eta''| = 1$. For $0 < t < 1$, we have

$$|U(x,t)| \;\; \leq \;\; M'' t^{-(m+1/2)n}.$$

Therefore, we have

$$|U(x,t)| \;\; \leq \;\; M'' t^{-(m+\frac{1}{2})n}(1 + |x|)^m, \quad (0 < t < 1, x \in \mathbb{R}^n).$$

By Theorem 3.1, there exists $T \in \mathcal{S}'$ such that $U(x,t) = \langle T_y, E(x-y,t)\rangle$.

Let $x_0 \notin \overline{\Gamma}$. By Proposition 2.2, there exists a $\eta_0 \in (\overline{\Gamma}')^{\circ}$ with $|\eta_0| = 1$ such that $\eta_0 x_0 = -2\delta < 0$. Then we have

$$\sup_{x \in B(x_0,\delta)} \eta_0 x \;\; = \;\; -2\delta + \delta = -\delta.$$

Let $\eta' = \eta_0/\sqrt{t}$ with $0 < t < 1$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\mathrm{supp}(\varphi) \subset B(x_0,\delta)$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |U(x,t)||\varphi(x)|dx \\
&\leq \; \int_{B(x_0,\delta)} \left\{ \int_{\mathbb{R}^n} |F(\xi + i\eta')| e^{-t\xi^2 + t\eta'^2} e^{\eta' x} d\xi \right\} |\varphi(x)| dx \\
&\leq \; M_1 \int_{B(x_0,\delta)} e^{\frac{\eta_0 x}{\sqrt{t}}} |\varphi(x)| \int_{\mathbb{R}^n} (1 + |\xi + i\eta'|)^m e^{-t\xi^2} d\xi dx \\
&\leq \; M_2 2^{2mn} e^{-\frac{\delta}{\sqrt{t}}} \\
&\qquad \times \int_{\mathbb{R}^n} \{(1 + |\xi_1|^2)(1 + |\eta_1|^2)\cdots(1 + |\xi_n|^2)(1 + |\eta_n|^2)\}^m e^{-t\xi^2} d\xi \\
&\leq \; M_3 t^{-mn} e^{-\frac{\delta}{\sqrt{t}}} \int_{\mathbb{R}^n} (1 + |\xi_1|^2)^m (1 + |\xi_2|^2)^m \cdots (1 + |\xi_n|)^m e^{-t\xi^2} d\xi \\
&\leq \; M t^{-(2mn + n/2)} e^{-\frac{\delta}{\sqrt{t}}}.
\end{aligned}$$

So we have

$$\langle T, \varphi\rangle = \lim_{t\to 0_+}\int_{\mathbb{R}^n} U(x,t)\varphi(x)dx = 0.$$

Since $x_0 \notin \overline{\Gamma}$ is arbitrary, we have $T \in \mathcal{S}'_{\overline{\Gamma}}$.

Let $\zeta = \xi + i\eta \in \mathbb{R}^n + i(\overline{\Gamma}')^\circ$. For $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle U(x,t), \varphi(x)\rangle &= \left\langle \int_{\mathbb{R}^n} F(\zeta)e^{-t\zeta^2}e^{-i\zeta x}d\xi, \varphi(x)\right\rangle \\
&= \langle F(\zeta), e^{-t\zeta^2}\mathcal{LF}(\varphi(x))(-\zeta)\rangle \\
&= \langle F(\zeta), \mathcal{LF}(E(x,t))(\zeta) \times \mathcal{LF}(\varphi(x))(-\zeta)\rangle \\
&= \langle F(\zeta), \mathcal{LF}(E(x,t) * \varphi(-x))(\zeta)\rangle \\
&= e^{t\eta^2}\langle F(\zeta), \mathcal{F}(e^{-t\eta^2}(E(x,t) * \varphi(-x))e^{-\eta x})(\xi)\rangle \\
&= e^{t\eta^2}\langle \mathcal{F}(F(\zeta)), e^{-t\eta^2-\eta x}(E(x,t) * \varphi(-x))\rangle.
\end{aligned}$$

By Corollary 3.6,

$$\begin{aligned}
\langle T, \varphi\rangle &= \lim_{t\to 0_+}\langle U(x,t), \varphi(x)\rangle \\
&= \lim_{t\to 0_+} e^{t\eta^2}\langle \mathcal{F}(F(\zeta)), e^{-t\eta^2-\eta x}(E(x,t) * \varphi(-x))\rangle \\
&= \lim_{t\to 0_+}\langle \mathcal{F}(F(\zeta)), e^{-t\eta^2-\eta x}(E(x,t) * \varphi(-x))\rangle.
\end{aligned}$$

By Corollary 4.3,

$$\begin{aligned}
&\lim_{t\to 0_+}\langle \mathcal{F}(F(\zeta))(x), e^{-t\eta^2-\eta x}(E(x,t) * \varphi(-x))\rangle \\
&\quad = \langle \mathcal{F}(F(\zeta))(x), \varphi(-x)e^{-\eta x}\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle T, \varphi\rangle &= \langle \mathcal{F}(F(\zeta))(x), \varphi(-x)e^{-\eta x}\rangle \\
&= \langle F(\zeta), \mathcal{F}^{-1}(\varphi(x)e^{\eta x})(\xi)\rangle \\
\Leftrightarrow \quad \langle T, \varphi(x)e^{-\eta x}\rangle &= \langle F(\zeta), \mathcal{F}^{-1}(\varphi(x))(\xi)\rangle \\
\Leftrightarrow \quad \langle e^{-\eta x}T, \varphi(x)\rangle &= \langle F(\zeta), \mathcal{F}^{-1}(\varphi(x))(\xi)\rangle \\
\Leftrightarrow \quad \langle \mathcal{F}(e^{-\eta x}T), \varphi\rangle &= \langle F(\zeta), \varphi\rangle. \ \Box
\end{aligned}$$

5. Edge-of-the-Wedge Theorem

In this section, we give a new proof of Edge-of-the-Wedge theorem.

First we see the following lemma:

LEMMA 5.1. *Let* $\eta \in (\overline{\Gamma}')^\circ$, $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$ *and* $\chi(x)$ *be a function in Definition* 4.1 *for* $\overline{\Gamma}$. *For every* $N \geq 0$ *and* $\beta \in \mathbb{N}_0^n$, *there exists a constant* C *such that*

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}&(1+|x|^2)^N|\partial^\beta(\chi(x)e^{-\eta x}\mathcal{F}(\varphi)(x)-\chi(x)\mathcal{F}(\varphi)(x))|\\ &\leq\ C|\eta|,\quad (|\eta|<1).\end{aligned}$$

PROOF. For every $N \geq 0$ and $\beta \in \mathbb{N}_0^n$,

$$\begin{aligned}&(1+|x|^2)^N|\partial^\beta(\chi(x)e^{-\eta x}\mathcal{F}(\varphi)(x)-\chi(x)\mathcal{F}(\varphi)(x))|\\ &= (1+|x|^2)^N|\partial^\beta\{\chi(x)\mathcal{F}(\varphi)(x)(e^{-\eta x}-1)\}|\\ &\leq \sum_{0\leq k\leq\beta}\binom{\beta}{k}(1+|x|^2)^N|\partial^{\beta-k}\mathcal{F}(\varphi)(x)||\partial^k(\chi(x)(e^{-\eta x}-1))|\\ &= \sum_{0\leq k\leq\beta}\binom{\beta}{k}(1+|x|^2)^{N+1}|\partial^{\beta-k}\mathcal{F}(\varphi)(x)|\\ &\quad\times\{|\partial^k\chi(x)(e^{-\eta x}-1)+\sum_{1\leq l\leq k}\binom{k}{l}\partial^{k-l}\chi(x)\partial^l(e^{-\eta x}-1)|\}\frac{1}{1+|x|^2}\\ &\leq \sum_{0\leq k\leq\beta}\binom{\beta}{k}(1+|x|^2)^{N+1}|\partial^{\beta-k}\mathcal{F}(\varphi)(x)|\\ &\quad\times\{|\partial^k\chi(x)e^{-\theta\eta x}\eta x|+\sum_{1\leq l\leq k}\binom{k}{l}|\partial^{k-l}\chi(x)||\eta|^l e^{-\eta x}\}\frac{1}{1+|x|^2},\end{aligned}$$

where some $\theta \in (0,1)$. Since $\mathcal{F}(\varphi)(x) \in \mathcal{S}$, we have

$$(1+|x|^2)^N|\partial^\beta(\chi(x)e^{-\eta x}\mathcal{F}(\varphi)(x)-\chi(x)\mathcal{F}(\varphi)(x))|\ \leq\ C|\eta|.\ \square$$

By the above lemma, we have the following corollary:

COROLLARY 5.2.

$$\chi(x)e^{-\eta x}\mathcal{F}(\varphi)(x) \quad \rightarrow \quad \chi(x)\mathcal{F}(\varphi)(x), \quad \eta \rightarrow 0,\ \eta \in C,\ \textit{in}\ \mathcal{S},$$

where C *is a cone in Theorem* 4.4.

PROPOSITION 5.3. *Let* Γ *be a proper open convex cone with vertex at* 0 *and let* $F(\zeta)$ *be a holomorphic function in* $\mathbb{R}^n + i(\overline{\Gamma}')^\circ$*. Suppose that there exists a constant* $m > 0$ *and that for every cone* C *satisfying* $\overline{C} \cap \{\eta \neq 0\} \subset (\overline{\Gamma}')^\circ$*, there exists a constant* $M_C \geq 0$ *such that*

$$|F(\zeta)| \quad \leq \quad M_C(1+|\zeta|)^m(1+|\eta|^{-m}), \quad \zeta \in \mathbb{R}^n + iC.$$

Then $\lim_{\substack{\eta\to 0\\ \eta\in C}} F(\xi + i\eta)$ *converges as* $\eta \rightarrow 0$*,* $\eta \in C$ *in* $\mathcal{S}'$*.*

PROOF. By Theorem 4.4, there exists $T \in \mathcal{S}'_{\overline{\Gamma}}$ such that

$$F(\zeta) = \mathcal{F}(e^{-\eta x}T)(\xi). \tag{13}$$

Let $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$. By (13),

$$\langle F(\xi + i\eta), \varphi(\xi)\rangle \quad = \quad \langle T, \chi(x)e^{-\eta x}\mathcal{F}(\varphi)(x)\rangle.$$

By Corollary 5.2,

$$\begin{aligned}\lim_{\substack{\eta\to 0\\ \eta\in C}} \langle F(\xi + i\eta), \varphi(\xi)\rangle \quad &= \quad \langle T, \chi(x)\mathcal{F}(\varphi)(x)\rangle \\ &= \quad \langle \mathcal{F}(\chi(x)T)(\xi), \varphi(\xi)\rangle.\end{aligned}$$

Since $T \in \mathcal{S}'$, $\chi(x)T \in \mathcal{S}'$ and $\mathcal{F}(\chi(x)T)(\xi) \in \mathcal{S}'$. Therefore,

$$\lim_{\substack{\eta\to 0\\ \eta\in C}} F(\xi + i\eta) \quad = \quad \mathcal{F}(\chi(x)T)(\xi), \quad \textit{in}\ \mathcal{S}'. \ \square$$

THEOREM 5.4 (Edge-of-the-Wedge Theorem). *Let* Γ_1*,* Γ_2 *be proper open convex cones with vertex at* 0 *and let* $F_j(\zeta)$ *be holomorphic functions in* $\mathbb{R}^n + i(\overline{\Gamma_j}')^\circ$*,* $j = 1, 2$*. Suppose that there exist constants* $m_j > 0$ *and* *for every cones* C_j *satisfying* $\overline{C_j} \cap \{\eta \neq 0\} \subset (\overline{\Gamma}'_j)^\circ$, *there exist constants* $M_{C_j} \geq 0$ *such that*

$$|F_j(\zeta)| \quad \leq \quad M_{C_j}(1+|\zeta|)^{m_j}(1+|\eta|^{-m_j}), \quad \zeta \in \mathbb{R}^n + iC_j, \quad j = 1,2.$$

If

$$\lim_{\substack{\eta\to 0\\ \eta\in C_1}} F_1(\xi+i\eta) \quad = \quad \lim_{\substack{\eta\to 0\\ \eta\in C_2}} F_2(\xi+i\eta), \quad in\ \mathcal{S}', \tag{14}$$

then there exists $F(\zeta) \in \mathcal{H}(\mathbb{R}^n + i(\mathrm{ch}(\overline{\Gamma_1}' \cup \overline{\Gamma_2}'))^\circ)$ *such that*

$$\begin{aligned} F(\zeta)|_{\mathbb{R}^n+i(\overline{\Gamma_1}')^\circ} &= F_1(\zeta), \\ F(\zeta)|_{\mathbb{R}^n+i(\overline{\Gamma_2}')^\circ} &= F_2(\zeta). \end{aligned}$$

Proof. Let $\varphi(x) \in \mathcal{D}(\mathbb{R}^n)$. By Proposition 5.3 and assumption (14),

$$\begin{aligned} \langle \mathcal{F}(\chi_1(x)T_1)(\xi), \varphi(\xi)\rangle &= \lim_{\substack{\eta\to 0\\ \eta\in C_1}} \langle F_1(\xi+i\eta), \varphi(\xi)\rangle \\ &= \lim_{\substack{\eta\to 0\\ \eta\in C_2}} \langle F_2(\xi+i\eta), \varphi(\xi)\rangle \\ &= \langle \mathcal{F}(\chi_2(x)T_2)(\xi), \varphi(\xi)\rangle. \end{aligned}$$

Here $T_1 \in \mathcal{S}'_{\overline{\Gamma_1}}$, $T_2 \in \mathcal{S}'_{\overline{\Gamma_2}}$ are tempered distributions for $F_1(\zeta)$, $F_2(\zeta)$ in Theorem 4.4, and $\chi_1(x)$, $\chi_2(x)$ be functions for $\overline{\Gamma_1}$, $\overline{\Gamma_2}$ in Definition 4.1 respectively.

Therefore, we have $\chi_1(x)T_1(x) = \chi_2(x)T_2(x) =: T(x)$ and $\mathrm{supp}T \subset (\overline{\Gamma_1} \cap \overline{\Gamma_2})$. By Remark 2.2 and Theorem 4.4,

$$\begin{aligned} F(\zeta) &\in \mathcal{H}(\mathbb{R}^n + i(\mathrm{ch}(\overline{\Gamma_1}' \cup \overline{\Gamma_2}'))^\circ), \\ F(\zeta)|_{\mathbb{R}^n+i(\overline{\Gamma_1}')^\circ} &= F_1(\zeta), \\ F(\zeta)|_{\mathbb{R}^n+i(\overline{\Gamma_2}')^\circ} &= F_2(\zeta). \ \square \end{aligned}$$

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Masanori Suwa
Department of Mathematics
Sophia University
7-1 Kioichô, Chiyoda-ku
Tokyo, Japan
E-mail: m-suwa@mm.sophia.ac.jp

Kunio Yoshino
Department of Mathematics
Sophia University
7-1 Kioichô, Chiyoda-ku
Tokyo, Japan
E-mail: k_yosino@mm.sophia.ac.jp