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Optimal Portfolio of Low Liquid Assets with a Power Utility Function

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Abstract. When an asset is completely liquid, an investor can realize his desirable strategy. But when the asset is not sufficiently liquid, the investor cannot trade the asset continuously and his strategy is restricted. He has to consider the risk of the failure of the trade.

In this paper a risky asset is traded at the random times and an investor has a power utility function. In this situation we solve an optimal portfolio problem. We propose an asymptotic expansion of the optimal strategy. Further we discuss convergence of the value function when the asset becomes liquid.

1. Introduction

As various assets are traded in the financial market, the risk becomes diversified and the liquidity risk becomes more important. Many people study the liquidity risk from various points of view. First, Leland[6], Boyle and Vorst[1], Kusuoka[5] analyze the replication strategy of the derivatives, using the transaction cost. Secondly Subramanian and Jarrow[4] consider the liquidation strategy, using the price impact and the execution delays. In this paper we consider the liquidity risk from a different perspective. We represent the liquidity by the success rate of the trade and consider the optimal portfolio problem between the risky asset and the saving account.

We assume that an investor has a power utility function and he can trade at the random times which are exponentially distributed. When the investor trades the asset continuously, Merton[7] solves the optimal portfolio problem and gives the optimal strategy. But the investor cannot realize the Merton's strategy in our setting because he does not know when and how many times he can trade. We show how different the optimal strategy is from the Merton's strategy.

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This paper is organized as follows. The next section provides the setting of the market and the main results of this paper. Section 3 shows the value function and the optimal strategy. An asymptotic expansion of the optimal strategy is discussed in Section 4. Finally convergence of the value function is explained in Section 5.

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2. Main Results

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t; 0 \leq t \leq T\})$ be a filtered probability space satisfying the usual condition. Under P, $\{B(t); 0 \leq t \leq T, B(0) = 0\}$ is a $\{\mathcal{F}_t\}$-Brownian motion and $\{P(t); 0 \leq t \leq T, P(0) = 0\}$ is a $\{\mathcal{F}_t\}$-Poisson process with intensity λ. We denote by $\beta(t)$ the saving account and by $S(t)$ a price of a risky asset. They are assumed to be governed by $\beta(0) = 1$, $S(0) = S_0$,

$$\begin{aligned} d\beta(t) &= r\beta(t)dt, \\ dS(t) &= \mu S(t)dt + \sigma S(t)dB(t) \end{aligned}$$

where S_0, r, μ and σ are positive constants and $r < \mu$. The investor invests a part of wealth in the risky asset and the rest in the safety asset. Let the amount invested in the risky asset be $W_1(t)$ and the amount invested in the safety asset be $W_0(t)$. The investor tries to trade the risky asset worth of $V(t)$ at t but the trade succeeds only at the jump times of the Poisson process. We fix constants w_1, w_0. For any predictable locally bounded process V, we consider the following stochastic differential equations

$$W_0(t) = w_0 + \int_0^t W_0(s-)\frac{d\beta(s)}{\beta(s)} - \int_0^t V(s)dP(s), \tag{2.1}$$

$$W_1(t) = w_1 + \int_0^t W_1(s-)\frac{dS(s)}{S(s)} + \int_0^t V(s)dP(s). \tag{2.2}$$

Then these stochastic differential equations have a unique solution $W_0(t)$, $W_1(t)$ by Theorem 14.6 of Elliot[3].

We say V is an admissible strategy if V satisfies

$$-W_1(t-) \leq V(t) \leq W_0(t-)$$

for $0 \leq t \leq T$. This means that the investor cannot make a short sale of the low liquid risky asset and must not invest more risky asset than his total asset. If V is admissible, then $W_0(t)$ and $W_1(t)$ are nonnegative.

We denote by $W(t)$ the total asset and by $X(t)$ the fraction of the wealth invested in the risky asset, i.e.,

$$\begin{aligned} W(t) &= W_0(t) + W_1(t), \\ X(t) &= \frac{W_1(t)}{W_0(t) + W_1(t)}. \end{aligned}$$

Let $v(t)$ be given by

$$v(t) = \frac{V(t) + W_1(t-)}{W_0(t-) + W_1(t-)}.$$

By the Itô formula, we have

$$\begin{aligned} \text{(2.3)} \quad W(t) &= w_0 + w_1 + \int_0^t W(s-)((\mu - r)X(s-) + r)ds \\ &\quad + \int_0^t W(s-)X(s-)\sigma dB(s), \\ \text{(2.4)} \quad X(t) &= \frac{w_1}{w_0 + w_1} + \int_0^t X(s-)(1 - X(s-))(\mu - r - \sigma^2 X(s-))ds \\ &\quad + \int_0^t X(s-)(1 - X(s-))\sigma dB(s) \\ &\quad + \int_0^t (v(s) - X(s-))dP(s). \end{aligned}$$

We define a set of processes by

$$\mathcal{V}[t, T] = \{v | v \text{ is predictable and } 0 \leq v(s) \leq 1 \text{ for } t \leq s \leq T\}.$$

If V is admissible, then $v \in \mathcal{V}[0, T]$.

For any $v \in \mathcal{V}[0,T]$, (2.3) and (2.4) have a unique solution $X(t)$, $W(t)$ by Theorem 14.6 of Elliot[3]. We can show $0 \le X(t) \le 1$. Then

$$\begin{aligned} W_0(t) &= W(t)(1-X(t)), \\ W_1(t) &= W(t)X(t), \\ V(t) &= (v(t)-X(t-))W(t-) \end{aligned}$$

is a solution of (2.1), (2.2) and V is admissible. Therefore there is a one-to-one correspondence between $W_0(t)$, $W_1(t)$, $V(t)$ and $W(t)$, $X(t)$, $v(t)$. Further, V is admissible if and only if $v \in \mathcal{V}[0,T]$. Therefore we consider $W(t)$, $X(t)$, $v(t)$ and we call v a strategy instead of V. When we emphasize that a process depends on v, we denote $W(t)$, $X(t)$ by $W(t;v)$, $X(t;v)$.

We take a power-utility function. Then our problem is to find an optimal strategy v^λ which maximizes $E[W(T;v)^\alpha]$ among $v \in \mathcal{V}[0,T]$ for $0 < \alpha < 1$. And the value function of the investor is given by

$$V^\lambda(t,x,w) = \sup_{v \in \mathcal{V}[t,T]} E[W(T;v)^\alpha | \mathcal{F}_t]|_{(X(t),W(t))=(x,w)}.$$

Let x_0 be

$$x_0 = \frac{\mu - r}{(1-\alpha)\sigma^2}$$

and we assume that $0 < x_0 < 1$. When the risky asset is completely liquid, the investor can trade the risky asset continuously and the optimal portfolio problem can be solved explicitly. In fact, the optimal strategy v^∞ and the value function $V^\infty(t,x,w)$ are given by

$$\begin{aligned} v^\infty(t) &= x_0, \\ V^\infty(t,x,w) &= w^\alpha \exp\left(\alpha\left(r + \frac{(\mu-r)^2}{2(1-\alpha)\sigma^2}\right)(T-t)\right). \end{aligned}$$

For the details, see Merton[8], Duffie[2], etc. In this case, $v^\infty(t)$ is constant and $V^\infty(t,x,w)$ does not depend on x. On the other hand, the optimal strategy and the value function are more complicated when the risky asset is less liquid.

Let $A^\lambda(t,x)$ be a solution of

$$A^\lambda(t,x) = \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda \tilde{A}^\lambda(s)ds + D(t,x)e^{-\lambda t} \tag{2.5}$$

where

$$\tilde{A}^\lambda(t) = \sup_{0\le x\le 1} A^\lambda(t,x),$$
$$D(t,y) = E\left[e^{\int_0^t f(Y^y(s))ds}\right],$$
$$f(y) = \alpha(\mu-r)y + \alpha r - \frac{1}{2}\alpha(1-\alpha)\sigma^2 y^2$$

and $Y^y(t)$ is a solution of

$$\begin{aligned} dY^y(t) &= Y^y(t)(1-Y^y(t))(\mu - r - \sigma^2(1-\alpha)Y^y(t))dt \\ &\quad + Y^y(t)(1-Y^y(t))\sigma dB(t), \\ Y^y(0) &= y. \end{aligned}$$

Note that $Y^y(t)$ has a unique solution by Theorem 14.6 of Elliot[3]. If $0 \le y \le 1$, then $0 \le Y^y(t) \le 1$. Specially if $y = 0$ or 1, then $Y^y(t) = y$. In this paper we prove the following theorems.

THEOREM 2.1. *The optimal strategy exists and the value function is given by*

$$V^\lambda(t,x,w) = w^\alpha A^\lambda(T-t,x).$$

Specially if λ is sufficiently large, an optimal strategy is unique and satisfies

$$|v^\lambda(t) - x_0| \le C_0 \frac{1}{\lambda}, \quad 0 \le t \le T$$

where C_0 is some constant.

THEOREM 2.2. *For all $N \in \boldsymbol{N}$ there exists an approximation of the optimal strategy, v_N^λ such that*

$$|v^\lambda(t) - v_N^\lambda(t)| \le C_N \frac{1}{\lambda^{N+1}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N, \tag{2.6}$$

$$\left|\frac{\partial A^\lambda(t, v_N^\lambda(T-t))}{\partial x}\right| \le C_N \frac{1}{\lambda^{N+2}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N, \tag{2.7}$$

$$|\tilde{A}^\lambda(t) - A^\lambda(t, v_N^\lambda(T-t))| \le C_N \frac{1}{\lambda^{2N+3}}, \quad 0 \le t \le T,\ \lambda \ge \lambda_N \tag{2.8}$$

for some constants C_N *and* λ_N*.*

REMARK 2.1. We show how to construct v_N^λ recursively in Section 4. Further we give v_1^λ and v_2^λ concretely by Corollaries 4.1 and 4.2.

THEOREM 2.3. *For* $0 \le t \le T$*,*

$$V^\lambda(t,x,w) \to V^\infty(t,x,w)$$

and

$$\begin{aligned} &\lambda(V^\infty(t,x,w) - V^\lambda(t,x,w)) \\ &\quad \to \frac{1}{2} w^\alpha \alpha(1-\alpha)\sigma^2 e^{f(x_0)(T-t)} \\ &\quad\quad \times \left((x-x_0)^2 + x_0^2(1-x_0)^2\sigma^2(T-t)\right) \ge 0 \end{aligned} \tag{2.9}$$

as $\lambda \to \infty$ *uniformly in* $0 \le x \le 1$*.*

3. Value Function and Optimal Strategy

In this section, we show the existence and the uniqueness of the optimal strategy. After we show some lemmas, we prove Theorem 2.1.

By the definition we have $D(0,x) = 1$ and

$$\frac{\partial D(0,x)}{\partial t} = f(x).$$

Since

$$f(x) = -\frac{1}{2}\alpha(1-\alpha)\sigma^2\left(x - \frac{\mu - r}{(1-\alpha)\sigma^2}\right)^2 + \frac{\alpha}{2(1-\alpha)}\frac{(\mu-r)^2}{\sigma^2} + \alpha r \le f(x_0),$$

we have

$$\begin{aligned} 0 < e^{\min(f(0),f(1))t} \le D(t,x) \le e^{f(x_0)t}, \\ \min(f(0),f(1)) \le \frac{\partial D(0,x)}{\partial t} \le f(x_0). \end{aligned} \tag{3.1}$$

LEMMA 3.1. *There exists a unique solution* $\tilde{A}^\lambda(t)$ *of*

$$\tilde{A}^\lambda(t) = \sup_{0 \le x \le 1} \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda \tilde{A}^\lambda(s)ds + D(t,x)e^{-\lambda t}. \tag{3.2}$$

Further $\tilde{A}^\lambda(t)$ *satisfies*

$$0 \le \tilde{A}^\lambda(t) \le e^{f(x_0)t}. \tag{3.3}$$

PROOF. Let

$$\begin{aligned}
\tilde{A}_0^\lambda(t) &= e^{f(x_0)t}, \\
\tilde{A}_n^\lambda(t) &= \sup_{0 \le x \le 1} \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda \tilde{A}_{n-1}^\lambda(s)ds + D(t,x)e^{-\lambda t}, \quad n \ge 1.
\end{aligned}$$

Note that $\tilde{A}_n^\lambda(t)$ is non-negative. By the definition and (3.1),

$$\begin{aligned}
&|\tilde{A}_n^\lambda(t) - \tilde{A}_{n-1}^\lambda(t)| \\
&\le \sup_{0 \le x \le 1} \left| \int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda(\tilde{A}_{n-1}^\lambda(s) - \tilde{A}_{n-2}^\lambda(s))ds \right| \\
&\le \int_0^t e^{f(x_0)(t-s)}e^{-\lambda(t-s)}\lambda \left| \tilde{A}_{n-1}^\lambda(s) - \tilde{A}_{n-2}^\lambda(s) \right| ds.
\end{aligned}$$

Substituting

$$a_n^\lambda(t) = |\tilde{A}_n^\lambda(t) - \tilde{A}_{n-1}^\lambda(t)|e^{(\lambda - f(x_0))t}$$

in the above inequality, we get

$$a_n^\lambda(t) \le \lambda \int_0^t a_{n-1}^\lambda(s)ds \le \lambda^{n-1} \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_2} a_1^\lambda(t_1)dt_1 \cdots dt_{n-2}dt_{n-1}.$$

By the definition and the integration by parts,

$$\begin{aligned}
a_1^\lambda(t) &= \sup_{0 \le x \le 1} E[\int_0^t (f(x_0) - f(Y^x(t-s)))e^{-\int_0^{t-s} f(x_0) - f(Y^x(u))du}e^{\lambda s}ds] \\
&\le K \int_0^t e^{\lambda s}ds
\end{aligned}$$

where $K = \sup_{0 \le x \le 1} f(x_0) - f(x)$. Therefore we obtain

$$\begin{aligned}
a_n^\lambda(t) &\le \lambda^{n-1}K \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} e^{\lambda t_0}dt_0 \cdots dt_{n-2}dt_{n-1} \\
&= \frac{K}{\lambda}\left(e^{\lambda t} - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right).
\end{aligned}$$

Then we have

$$|\tilde{A}_n^\lambda(t) - \tilde{A}_{n-1}^\lambda(t)| \le \frac{K}{\lambda} e^{f(x_0)t} \left(1 - e^{-\lambda t} \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!}\right).$$

Because the right side converges to 0, $\tilde{A}_n^\lambda(t)$ has the limit $\tilde{A}^\lambda(t) \ge 0$ satisfying (3.2).

Suppose that $\hat{A}^\lambda(t)$ is a second solution of (3.2). Since

$$|\tilde{A}^\lambda(t) - \hat{A}^\lambda(t)| \quad \le \quad \int_0^t e^{f(x_0)(t-s)} e^{-\lambda(t-s)} \lambda \left|\tilde{A}^\lambda(s) - \hat{A}^\lambda(s)\right| ds,$$

we get

$$|\tilde{A}^\lambda(t) - \hat{A}^\lambda(t)| e^{(\lambda - f(x_0))t} \quad \le \quad \lambda \int_0^t e^{(\lambda - f(x_0))s} \left|\tilde{A}^\lambda(s) - \hat{A}^\lambda(s)\right| ds.$$

Then we obtain

$$|\tilde{A}^\lambda(t) - \hat{A}^\lambda(t)| e^{(\lambda - f(x_0))t} = 0$$

by the Gronwall inequality. Therefore $\tilde{A}^\lambda(t)$ is a unique solution.

By (3.2), we have

$$e^{(\lambda - f(x_0))t} \tilde{A}^\lambda(t) \quad \le \quad 1 + \lambda \int_0^t e^{(\lambda - f(x_0))s} \tilde{A}^\lambda(s) ds.$$

By the Gronwall inequality, (3.3) is proved. □

REMARK 3.1. By Lemma 3.1 (2.5) has a unique solution. Similarly we can show that when $x_0 \le 0$ or $1 \le x_0$, (2.5) has a unique solution.

LEMMA 3.2. *The optimal strategy exists and the value function is given by*

$$V^\lambda(t, x, w) \quad = \quad w^\alpha A^\lambda(T - t, x).$$

PROOF. We denote the generator of $Y^x(t)$ by $\mathcal{L}$. Since $D(t, x) \in C^2([0, T] \times [0, 1])$,

$$\begin{aligned} \mathcal{L} D(t, x) \quad &= \quad x(1 - x)(\mu - r - \sigma^2(1 - \alpha)x) \frac{\partial D(t, x)}{\partial x} \\ &\quad + \frac{1}{2} x^2 (1 - x)^2 \sigma^2 \frac{\partial^2 D(t, x)}{\partial x^2}. \end{aligned}$$

Let

$$\mu_A(t,x,v) = -\frac{\partial A^\lambda(t,x)}{\partial t} + \mathcal{L}A^\lambda(t,x) + f(x)A^\lambda(t,x) + \lambda(A^\lambda(t,v) - A^\lambda(t,x)).$$

Since $D(t,x)$ is a solution of

$$\frac{\partial D(t,x)}{\partial t} = \mathcal{L}D(t,x) + f(x)D(t,x)$$

and $D(0,x) = 1$, we get

$$\frac{\partial A^\lambda(t,x)}{\partial t} = \mathcal{L}A^\lambda(t,x) + f(x)A^\lambda(t,x) + \lambda(\tilde{A}^\lambda(t) - A^\lambda(t,x)).$$

Therefore we obtain

$$\sup_{0\le v\le 1} \mu_A(t,x,v) = 0$$

for $0 \le x \le 1$.

Using the Itô formula, for $v \in \mathcal{V}[0,T]$,

$$\begin{aligned}
&d(W(t;v)^\alpha A^\lambda(T-t, X(t;v))) \\
&= W(t-)^\alpha \mu_A(T-t, X(t-), v(t))dt \\
&\quad + W(t-)^\alpha \alpha X(t-) A^\lambda(T-t, X(t-))\sigma dB(t) \\
&\quad + W(t-)^\alpha X(t-)(1-X(t-))\frac{\partial A^\lambda(T-t, X(t-))}{\partial x}\sigma dB(t) \\
&\quad + W(t-)^\alpha (A^\lambda(T-t, v(t)) - A^\lambda(T-t, X(t-)))(dP(t) - \lambda dt).
\end{aligned}$$

Since $\mu_A(T-t, X(t-), v(t)) \le 0$, we obtain

$$E[W(T;v)^\alpha | \mathcal{F}_t]|_{(X(t),W(t))=(x,w)} \le w^\alpha A^\lambda(T-t,x).$$

Because $A^\lambda(T-t,x)$ is continuous with respect to x, there exists $\hat{v}(t)$ which satisfies

$$A^\lambda(T-t,\hat{v}(t)) = \sup_{0\le x\le 1} A^\lambda(T-t,x).$$

Because $\hat{v}$ is a deterministic process and satisfies

$$E[W(T;\hat{v})^\alpha | \mathcal{F}_t]|_{(X(t),W(t))=(x,w)} = w^\alpha A^\lambda(T-t,x),$$

v^λ is an optimal strategy. The result follows. □

By Lemmas 3.1 and 3.2,

$$\begin{aligned} w^\alpha \tilde{A}^\lambda(t) &\geq V(T-t, 0, w) \\ &\geq E[W(T; v \equiv 0)^\alpha | \mathcal{F}_{T-t}]|_{X(T-t)=0, W(T-t)=w} = w^\alpha e^{\alpha r t} \end{aligned}$$

and then

$$\tilde{A}^\lambda(t) \geq e^{\alpha r t} \geq 1. \tag{3.4}$$

Let

$$\begin{aligned} B_n &= \sup\left\{ \left| \frac{\partial^n D(t,x)}{\partial t^i \partial x^{n-i}} \right| \middle| 0 \leq i \leq n,\, 0 \leq t \leq T,\, 0 \leq x \leq 1 \right\}, \\ L_n(t) &= \int_0^t s^n e^{-s} ds, \\ M_n(\lambda, t) &= \int_0^{\lambda t} s^n e^{-s} (\tilde{A}^\lambda(t - \frac{s}{\lambda}) - 1) ds \end{aligned}$$

for $i, j, n \geq 0$. Also $g^\lambda(t,x)$ is defined by

$$g^\lambda(t,x) = \frac{\lambda(A^\lambda(t,x) - M_0(\lambda,t) - 1)}{M_1(\lambda,t) + L_0(\lambda t)}. \tag{3.5}$$

Note that $g^\lambda(t,x)$ has an absolute maximum at the same point as $A^\lambda(t,x)$. By (3.4) and Lemma 3.1,

$$0 \leq M_n(\lambda,t) \leq (e^{f(x_0)t} - 1) L_n(\lambda t). \tag{3.6}$$

Also we can show

$$0 \leq L_n(\lambda t) = n! \left(1 - e^{-\lambda t} \sum_{i=0}^n \frac{(\lambda t)^i}{i!} \right) \leq n!$$

and

$$\frac{L_n(\lambda t)}{L_m(\lambda t)} \leq \frac{n!}{m!} \tag{3.7}$$

for $0 \leq m \leq n$.

Lemma 3.3. *Suppose that*

$$\lambda \geq \max\left(2H_2, \frac{2H_1}{H_3}\right)$$

where

$$\begin{aligned} H_1 &= \frac{1}{\alpha(1-\alpha)\sigma^2} B_3 e^{f(x_0)T}, \\ H_2 &= \frac{1}{\alpha(1-\alpha)\sigma^2} B_4 e^{f(x_0)T}, \\ H_3 &= \min\left(1-x_0, x_0\right). \end{aligned}$$

Then there uniquely exists $h^\lambda(t)$ which satisfies

$$\begin{aligned} \frac{\partial g^\lambda(t, x_0 + h^\lambda(t))}{\partial x} &= 0, \\ h^\lambda(t) &\leq H_3. \end{aligned}$$

Further $h^\lambda(t)$ satisfies

$$h^\lambda(t) \leq 2H_1 \frac{1}{\lambda}.$$

Proof. By (2.5)

$$\begin{aligned} A^\lambda(t,x) &= \lambda \int_0^t D(t-s,x) e^{-\lambda(t-s)} \tilde{A}^\lambda(s) ds + D(0,x) \\ &\quad + \int_0^t e^{-\lambda s} \frac{\partial D(s,x)}{\partial t} - \lambda e^{-\lambda s} D(s,x) ds \\ &= \int_0^{\lambda t} D(\frac{u}{\lambda}, x) e^{-u} \left(\tilde{A}^\lambda(t - \frac{u}{\lambda}) - 1\right) du \\ &\quad + \frac{1}{\lambda} \int_0^{\lambda t} \frac{\partial}{\partial t} D(\frac{u}{\lambda}, x) e^{-u} du + 1. \end{aligned} \tag{3.8}$$

By Taylor's theorem,

$$\begin{aligned} D(\frac{u}{\lambda}, x) &= 1 + f(x)\frac{u}{\lambda} + \left(\frac{u}{\lambda}\right)^2 \int_0^1 (1-s) \frac{\partial^2}{\partial t^2} D(s\frac{u}{\lambda}, x) ds, \\ \frac{\partial}{\partial t} D(\frac{u}{\lambda}, x) &= f(x) + \frac{u}{\lambda} \int_0^1 \frac{\partial^2}{\partial t^2} D(s\frac{u}{\lambda}, x) ds. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} A^\lambda(t,x) &= M_0(\lambda,t) + \frac{1}{\lambda}(M_1(\lambda,t) + L_0(\lambda t))f(x) \\ &+ \frac{1}{\lambda^2}\int_0^{\lambda t} u^2 e^{-u}\left(\tilde{A}^\lambda(t-\frac{u}{\lambda}) - 1\right)\int_0^1 (1-s)\frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \\ &+ \frac{1}{\lambda^2}\int_0^{\lambda t} ue^{-u}\int_0^1 \frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu + 1. \end{aligned}$$

By (3.5) we have

$$g^\lambda(t,x) = f(x) + \frac{1}{\lambda}(g_1^\lambda(t,x) + g_2^\lambda(t,x)) \tag{3.9}$$

where

$$\begin{aligned} g_1^\lambda(t,x) &= \int_0^{\lambda t} u^2 e^{-u}\left(\tilde{A}^\lambda(t-\frac{u}{\lambda}) - 1\right) \\ &\times \int_0^1 (1-s)\frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \Big/ (M_1(\lambda,t) + L_0(\lambda t)), \\ g_2^\lambda(t,x) &= \int_0^{\lambda t} ue^{-u}\int_0^1 \frac{\partial^2}{\partial t^2}D(s\frac{u}{\lambda},x)dsdu \Big/ (M_1(\lambda,t) + L_0(\lambda t)). \end{aligned}$$

Differentiating $g^\lambda(t,x)$ with respect to x and substituting $x = x_0 + h$,

$$\frac{\partial g^\lambda(t,x_0+h)}{\partial x} = -h\alpha(1-\alpha)\sigma^2 + \frac{1}{\lambda}\left(\frac{\partial g_1^\lambda(t,x_0+h)}{\partial x} + \frac{\partial g_2^\lambda(t,x_0+h)}{\partial x}\right).$$

By (3.6) and (3.7), we have

$$\begin{aligned} \left|\frac{\partial g_1^\lambda(t,x)}{\partial x}\right| &\le \frac{B_3}{2}\frac{M_2(\lambda,t)}{M_1(\lambda,t) + L_0(\lambda t)} \\ &\le \frac{B_3}{2}\frac{(e^{f(x_0)t} - 1)L_2(\lambda t)}{L_0(\lambda t)} \le B_3(e^{f(x_0)t} - 1), \\ \left|\frac{\partial g_2^\lambda(t,x)}{\partial x}\right| &\le B_3\frac{L_1(\lambda t)}{M_1(\lambda,t) + L_0(\lambda t)} \le B_3. \end{aligned}$$

Similarly we get

$$\begin{aligned} \left|g_1^\lambda(t,x)\right| &\le B_2(e^{f(x_0)t} - 1), \\ \left|g_2^\lambda(t,x)\right| &\le B_2 \end{aligned} \tag{3.10}$$

and

$$\left|\frac{\partial^2 g_1^\lambda(t,x)}{\partial x^2}\right| \le B_4(e^{f(x_0)t}-1),$$

$$\left|\frac{\partial^2 g_2^\lambda(t,x)}{\partial x^2}\right| \le B_4. \tag{3.11}$$

We solve

$$\frac{\partial g^\lambda(t,x_0+h)}{\partial x}=0$$

by the successive approximation. Let

$$\begin{aligned} h_1(t) &= 0,\\ h_n(t) &= \frac{1}{\alpha(1-\alpha)\sigma^2}\\ &\quad\times\left(\frac{\partial g_1^\lambda(t,x_0+h_{n-1}(t))}{\partial x}+\frac{\partial g_2^\lambda(t,x_0+h_{n-1}(t))}{\partial x}\right)\frac{1}{\lambda}, \quad n\ge 2. \end{aligned}$$

Then

$$\begin{aligned} |h_2(t)-h_1(t)| &= \left|\frac{1}{\alpha(1-\alpha)\sigma^2}\left(\frac{\partial g_1^\lambda(t,x_0)}{\partial x}+\frac{\partial g_2^\lambda(t,x_0)}{\partial x}\right)\frac{1}{\lambda}\right|\\ &\le \frac{B_3e^{f(x_0)t}}{\alpha(1-\alpha)\sigma^2}\frac{1}{\lambda}\le H_1\frac{1}{\lambda}. \end{aligned}$$

We assume that for $2\le k\le n$

$$|h_k(t)-h_{k-1}(t)|\le \frac{1}{2^{k-2}}|h_2(t)|.$$

From the assumption and (3.11), we get

$$\begin{aligned} |h_{n+1}(t)-h_n(t)| &\le \frac{1}{\alpha(1-\alpha)\sigma^2}\sup_{0\le x\le 1}\left|\frac{\partial^2 g_1^\lambda(t,x)}{\partial x^2}+\frac{\partial^2 g_2^\lambda(t,x)}{\partial x^2}\right|\\ &\quad\times\frac{1}{\lambda}|h_n(t)-h_{n-1}(t)|\\ &\le \frac{1}{\alpha(1-\alpha)\sigma^2}B_4e^{f(x_0)t}\frac{1}{\lambda}|h_n(t)-h_{n-1}(t)| \end{aligned}$$

$$\leq H_2 \frac{1}{\lambda} |h_n(t) - h_{n-1}(t)| \leq \frac{1}{2} |h_n(t) - h_{n-1}(t)|$$
$$\leq \frac{1}{2^{n-1}} |h_2(t)|.$$

Therefore we have for all $n \geq 2$,

$$|h_n(t) - h_{n-1}(t)| \leq \frac{1}{2^{n-2}} |h_2(t)|.$$

Since

$$\begin{aligned} |h_n(t)| &\leq \sum_{k=2}^{n} |h_k(t) - h_{k-1}(t)| + |h_1(t)| \\ &\leq \sum_{k=2}^{n} \frac{1}{2^{k-2}} |h_2(t)| \leq 2|h_2(t)| \leq 2H_1 \frac{1}{\lambda}, \end{aligned}$$

we obtain for all $n \geq 2$,

$$|h_n(t)| \leq 2H_1 \frac{1}{\lambda} \leq H_3.$$

Then $h_n(t)$ has the limit $h(t)$ satisfying

$$\frac{\partial g^{\lambda}(t, x_0 + h(t))}{\partial x} = 0$$

and

$$|h(t)| \leq 2H_1 \frac{1}{\lambda} \leq H_3.$$

Suppose that $\tilde{h}(t)$ is a second solution which satisfies

$$\frac{\partial g^{\lambda}(t, x_0 + \tilde{h}(t))}{\partial x} = 0$$

and

$$|\tilde{h}(t)| \leq H_3.$$

By (3.11) we obtain

$$\begin{aligned} |h(t) - \tilde{h}(t)| &\leq \frac{B_4 e^{f(x_0)t}}{\alpha(1-\alpha)\sigma^2} \frac{1}{\lambda} |h(t) - \tilde{h}(t)| \leq \frac{H_2}{\lambda} |h(t) - \tilde{h}(t)| \\ &\leq \frac{1}{2} |h(t) - \tilde{h}(t)|. \end{aligned}$$

Therefore $h(t)$ is a unique solution and the results follow. □

LEMMA 3.4. *For all* $0 \le t \le T$ *and* $0 \le x \le 1$

$$\left| g^\lambda(t,x) - f(x) \right| \le B_2 e^{f(x_0)T} \frac{1}{\lambda},$$

that is, $g^\lambda(t,x)$ *converges to* $f(x)$ *uniformly when* λ *tends to* ∞.

PROOF. By (3.9) and (3.10),

$$|g^\lambda(t,x) - f(x)| \le |g_1^\lambda(t,x) + g_2^\lambda(t,x)| \frac{1}{\lambda} \le B_2 e^{f(x_0)T} \frac{1}{\lambda}. \ \square$$

PROOF OF THEOREM 2.1. By Lemma 3.2 we have shown the first half of Theorem 2.1. Let

$$\begin{aligned} \epsilon_1 &= \frac{1}{2}\alpha(1-\alpha)\sigma^2 H_3^2 = \sup_{|x-x_0| \le H_3} f(x_0) - f(x), \\ H_4 &= 2B_2 e^{f(x_0)T}/\epsilon_1. \end{aligned}$$

If $\lambda \ge H_4$, by Lemma 3.4 we obtain

$$\left| g^\lambda(t,x) - f(x) \right| \le \frac{\epsilon_1}{2}.$$

Suppose that

$$\lambda \ge \max\left(2H_2, \frac{2H_1}{H_3}, H_4\right).$$

If $|x - x_0| > H_3$, then $f(x_0) - f(x) > \epsilon_1$ and then

$$\begin{aligned} g^\lambda(t,x_0) - g^\lambda(t,x) &> \left(f(x_0) - \frac{1}{2}\epsilon_1\right) - \left(f(x) + \frac{1}{2}\epsilon_1\right) \\ &> (f(x_0) - f(x)) - \epsilon_1 > 0. \end{aligned}$$

Therefore $g^\lambda(t,x)$ has a maximum in $|x - x_0| \le H_3$. From Lemma 3.3 $x_0 + h^\lambda(t)$ is a unique extreme point of $g^\lambda(t,x)$ in $|x - x_0| \le H_3$. The result follows. □

REMARK 3.2. By the above proof, the optimal strategy can be represented by

$$v^\lambda(T-t) = x_0 + h^\lambda(t).$$

4. Asymptotic Expansion of the Optimal Strategy

In this section we propose a procedure of an asymptotic expansion of h^λ and then we give the poof of Theorem 2.2.

Let

$$D_{i,j} = \frac{\partial^{i+j} D(0, x_0)}{\partial t^i \partial x^j}.$$

Then we have

$$\begin{aligned}
D_{0,0} &= 1, \\
D_{0,k} &= 0, \quad k \geq 1, \\
D_{1,0} &= f(x_0) = \frac{\alpha}{2(1-\alpha)} \frac{(\mu - r)^2}{\sigma^2} + \alpha r, \\
D_{1,1} &= f'(x_0) = 0, \\
D_{1,2} &= f''(x_0) = -\alpha(1-\alpha)\sigma^2, \\
D_{1,k} &= 0, \quad k \geq 3, \\
D_{2,0} &= \left(\frac{\alpha}{2(1-\alpha)} \frac{(\mu - r)^2}{\sigma^2} + \alpha r \right)^2 + \frac{1}{2}\alpha(\alpha - 1)x_0^2(1 - x_0)^2\sigma^4, \\
D_{2,1} &= \alpha(\alpha - 1)x_0(1 - x_0)(1 - 2x_0)\sigma^4
\end{aligned}$$

since

$$\begin{aligned}
\frac{\partial D(t,x)}{\partial t} &= E[f(Y^x(t)) e^{\int_0^t f(Y^x(s))ds}], \\
\frac{\partial^2 D(t,x)}{\partial t^2} &= E[f(Y^x(t))^2 e^{\int_0^t f(Y^x(s))ds} \\
&\quad + f'(Y^x(t)) e^{\int_0^t f(Y^x(s))ds} Y^x(t)(1 - Y^x(t)) \\
&\quad \times (\mu - r - \sigma^2(1-\alpha)Y^x(t)) \\
&\quad + \frac{1}{2} f''(Y^x(t)) e^{\int_0^t f(Y^x(s))ds} Y^x(t)^2 (1 - Y^x(t))^2 \sigma^2].
\end{aligned}$$

For $l, m, N \in \mathbf{N}$ and $0 \leq x_0 + h \leq 1$, by Taylor's theorem,

$$\frac{\partial^{m+l} D(t, x_0 + h)}{\partial t^m \partial x^l} = \sum_{n=0}^{N-1} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k h^{n-k} D_{m+k,l+n-k} + R_N(t, h, m, l)$$

where

$$\begin{aligned} R_N(t,h,m,l) &= \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} t^k h^{N-k} \\ &\quad \times \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!} \frac{\partial^{m+l+N} D(st, x_0+sh)}{\partial t^{m+k} \partial x^{l+N-k}} ds. \end{aligned}$$

Differentiating (3.8) with respect to x and substituting the above equations,

$$\begin{aligned} (4.1) \qquad & \frac{\partial A^\lambda(t,x_0+h)}{\partial x} \\ &= \int_0^{\lambda t} \left(\sum_{n=0}^{N} \frac{1}{n!} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{u}{\lambda}\right)^k h^{n-k} D_{k,n-k+1} \right. \\ &\quad \left. + R_{N+1}(\frac{u}{\lambda}, h, 0, 1) \right) e^{-u} \left(\tilde{A}^\lambda(t-\frac{u}{\lambda}) - 1 \right) du \\ &\quad + \frac{1}{\lambda} \int_0^{\lambda t} \left(\sum_{n=0}^{N-1} \frac{1}{n!} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(\frac{u}{\lambda}\right)^k h^{n-k} D_{k+1,n-k+1} \right. \\ &\quad \left. + R_N(\frac{u}{\lambda}, h, 1, 1) \right) e^{-u} du \\ &= \sum_{n=2}^{N} \sum_{k=1}^{n} \frac{h^{n-k}}{\lambda^k} D_{k,n-k+1} \\ &\quad \times \left(\begin{pmatrix} n \\ k \end{pmatrix} \frac{M_k(\lambda,t)}{n!} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \frac{L_{k-1}(\lambda t)}{(n-1)!} \right) \\ &\quad + S_N(\lambda,t,h,1) \end{aligned}$$

where

$$\begin{aligned} S_N(\lambda,t,h,l) &= \int_0^{\lambda t} R_{N+1}(\frac{u}{\lambda}, h, 0, l) e^{-u} \left(\tilde{A}^\lambda(t-\frac{u}{\lambda}) - 1 \right) du \\ &\quad + \frac{1}{\lambda} \int_0^{\lambda t} R_N(\frac{u}{\lambda}, h, 1, l) e^{-u} du. \end{aligned}$$

Since

$$|R_N(t,h,m,l)| \leq \sum_{k=0}^{N} \begin{pmatrix} N \\ k \end{pmatrix} t^k |h|^{N-k} B_{N+m+l} \frac{1}{N!},$$

we get by (3.6) and (3.7)

$$\begin{aligned}(4.2)\qquad & \left|\frac{S_N(\lambda,t,h,l)}{L_0(\lambda t)}\right| \\ & \le \frac{B_{N+l}}{(N+1)!}\sum_{k=0}^{N+1}\binom{N+1}{k}\frac{|h|^{N+1-k}}{\lambda^k}\frac{M_k(\lambda,t)}{L_0(\lambda t)} \\ & \quad +\frac{B_{N+l+1}}{N!}\sum_{k=0}^{N}\binom{N}{k}\frac{|h|^{N-k}}{\lambda^{k+1}}\frac{L_k(\lambda t)}{L_0(\lambda t)} \\ & \le \frac{B_{N+l}}{(N+1)!}\sum_{k=0}^{N+1}\binom{N+1}{k}\frac{|h|^{N+1-k}(e^{f(x_0)t}-1)k!}{\lambda^k} \\ & \quad +\frac{B_{N+l+1}}{N!}\sum_{k=0}^{N}\binom{N}{k}\frac{|h|^{N-k}k!}{\lambda^{k+1}}.\end{aligned}$$

Suppose that $T_1, T_2, \ldots$ are independently uniformly distributed in $[0,1]$ under P. Let

$$\begin{aligned} J^{\lambda}(t;h) &= e^{-\lambda t}\sum_{n=0}^{+\infty}\frac{(\lambda t)^n}{n!}F(n+1,t;h), \\ F(n,t;h) &= E_n[\prod_{i=1}^{n} D(tT_i, x_0+h(t\hat{T}_i))] \end{aligned}$$

where $x_0+h\in\mathcal{V}[0,T]$, $\hat{T}_i = 1-\sum_{j=1}^{i-1}T_j$ and $E_n[\cdot] = E[\cdot|\sum_{i=1}^{n}T_i=1]$. By (3.1),

$$0 < e^{\min(f(0),f(1))t} \le F(n,t;h) \le e^{f(x_0)t}$$

and then

$$(4.3)\qquad 0 < e^{\min(f(0),f(1))t} \le J^{\lambda}(t;h) \le e^{f(x_0)t}.$$

LEMMA 4.1. *$\tilde{A}^{\lambda}(t)$ satisfies*

$$(4.4)\qquad \tilde{A}^{\lambda}(t) = J^{\lambda}(t;h^{\lambda}).$$

PROOF. By the definition

$$\tilde{A}^\lambda(t) = \int_0^t D(t-s, x_0 + h^\lambda(t))e^{-\lambda(t-s)}\lambda \tilde{A}^\lambda(s)ds + D(t, x_0 + h^\lambda(t))e^{-\lambda t}.$$

Therefore we get for all $N \in \boldsymbol{N}$,

$$\begin{aligned} e^{\lambda t_0}\tilde{A}^\lambda(t_0) &= D(t_0, x_0 + h^\lambda(t_0)) \\ &+ \sum_{n=1}^{N} \lambda^n \int_0^{t_0} \cdots \int_0^{t_{n-1}} \left(\prod_{i=0}^{n-1} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \\ &\times D(t_n, x_0 + h^\lambda(t_n))dt_n \cdots dt_1 \\ &+ \lambda^{N+1} \int_0^{t_0} \cdots \int_0^{t_N} \left(\prod_{i=0}^{N} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \\ &\times e^{\lambda t_{N+1}} \tilde{A}^\lambda(t_{N+1})dt_{N+1} \cdots dt_1. \end{aligned}$$

By (3.1),

$$\begin{aligned} &\sum_{n=1}^{+\infty} \left| \lambda^n \int_0^{t_0} \cdots \int_0^{t_{n-1}} \left(\prod_{i=0}^{n-1} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \right. \\ &\qquad \left. \times D(t_n, x_0 + h^\lambda(t_n))dt_n \cdots dt_1 \right| \\ &\leq \sum_{n=1}^{+\infty} \lambda^n \int_0^{t_0} \cdots \int_0^{t_{n-1}} e^{f(x_0)t_0} dt_n \cdots dt_1 \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n t_0^n}{n!} e^{f(x_0)t_0} \leq e^{(\lambda + f(x_0))t_0}. \end{aligned}$$

By (3.3),

$$\begin{aligned} 0 &\leq \lambda^{N+1} \int_0^{t_0} \cdots \int_0^{t_N} \left(\prod_{i=0}^{N} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \\ &\quad \times e^{\lambda t_{N+1}} \tilde{A}^\lambda(t_{N+1})dt_{N+1} \cdots dt_1 \\ &\leq \lambda^{N+1} \int_0^{t_0} \cdots \int_0^{t_N} e^{f(x_0)t_0} e^{\lambda t_{N+1}} dt_{N+1} \cdots dt_1 \\ &= e^{f(x_0)t_0} \left(e^{\lambda t_0} - \sum_{n=0}^{N} \frac{1}{n!}(\lambda t_0)^n \right) \end{aligned}$$

and the right side tends to 0 when n goes to ∞. Therefore we obtain

$$\begin{aligned} e^{\lambda t_0}\tilde{A}^\lambda(t_0) &= D(t_0, x_0 + h^\lambda(t_0)) \\ &+ \sum_{n=1}^{+\infty} \lambda^n \int_0^{t_0} \cdots \int_0^{t_{n-1}} \left(\prod_{i=0}^{n-1} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \\ &\times D(t_n, x_0 + h^\lambda(t_n)) dt_n \cdots dt_1. \end{aligned}$$

Since

$$D(t_0, x_0 + h^\lambda(t_0)) \quad = \quad E_1[D(t_0 T_1, x_0 + h^\lambda(t_0 \hat{T}_1))] = F(1, t; h^\lambda)$$

and

$$\begin{aligned} &\int_0^{t_0} \cdots \int_0^{t_{n-1}} \left(\prod_{i=0}^{n-1} D(t_i - t_{i+1}, x_0 + h^\lambda(t_i)) \right) \\ &\quad \times D(t_n, x_0 + h^\lambda(t_n)) dt_n \cdots dt_1 \\ &= \frac{t_0^n}{n!} E_{n+1}[\prod_{i=1}^{n+1} D(t_0 T_i, x_0 + h(t_0 \hat{T}_i))] = \frac{t_0^n}{n!} F(n+1, t; h^\lambda), \end{aligned}$$

we get

$$e^{\lambda t_0}\tilde{A}^\lambda(t_0) = \sum_{n=0}^{+\infty} \lambda^n \frac{t_0^n}{n!} F(n+1, t; h^\lambda).$$

The result follows. □

By Taylor's theorem,

$$\log D(t, x_0 + h) \quad = \quad \left(D_{1,0} + \frac{1}{2} D_{1,2} h^2 \right) t + Z(t, x_0 + h) t^2$$

where

$$Z(t,x) = \int_0^1 (1-s) \left(\frac{\partial^2 D(st,x)}{\partial t^2} D(st,x) - \left(\frac{\partial D(st,x)}{\partial t} \right)^2 \right) \Big/ D(st,x)^2 ds.$$

Let

$$Z_n = \sup \left\{ \left| \frac{\partial^n Z(t,x)}{\partial t^i \partial x^{n-i}} \right| \, \middle| \, 0 \le i \le n,\, 0 \le t \le T,\, 0 \le x \le 1 \right\}.$$

LEMMA 4.2. *Let*

$$h_0^*(t) = 0$$

for $0 \leq t \leq T$. *Then*

$$\left| J^\lambda(t; h_0^*) - e^{f(x_0)t} \right| \leq C\frac{1}{\lambda}, \quad 0 \leq t \leq T,\ \lambda \geq \lambda_0, \tag{4.5}$$

$$\left| \tilde{A}^\lambda(t) - J^\lambda(t; h_0^*) \right| \leq C\frac{1}{\lambda^2}, \quad 0 \leq t \leq T,\ \lambda \geq \lambda_0, \tag{4.6}$$

$$\left| \tilde{A}^\lambda(t) - e^{f(x_0)t} \right| \leq C\frac{1}{\lambda}, \quad 0 \leq t \leq T,\ \lambda \geq \lambda_0 \tag{4.7}$$

for some constants C *and* λ_0.

PROOF. By the definition,

$$\begin{aligned} F(n,t;h) &= E_n[\exp\left(\sum_{i=1}^n \log D(tT_i, x_0 + h(tT_i))\right)] \\ &= e^{D_{1,0}t} E_n[\exp\left(\sum_{i=1}^n \frac{1}{2} D_{1,2} h(tT_i)^2 tT_i \right. \\ &\qquad \left. + Z(tT_i, x_0 + h(tT_i)) t^2 T_i^2 \right)]. \end{aligned}$$

Since $E_1[\sum_{i=1}^1 T_i^2] = 1$ and for $n \geq 2$

$$E_n[\sum_{i=1}^n T_i^2] = nE_n[T_1^2] = n! \int_{t_1+\cdots+t_{n-1}\leq 1} t_1^2 dt_{n-1}\cdots dt_1 = \frac{2}{n+1}, \tag{4.8}$$

we have

$$\begin{aligned} |e^{D_{1,0}t} - F(n,t;h_0^*)| &= e^{D_{1,0}t} \left| \exp(0) - E_n[\exp\left(\sum_{i=1}^n Z(tT_i, x_0) t^2 T_i^2\right)] \right| \\ &\leq e^{D_{1,0}t + Z_0 t^2} Z_0 t^2 E_n[\sum_{i=1}^n T_i^2] \\ &= e^{D_{1,0}t + Z_0 t^2} Z_0 t^2 \frac{2}{n+1}. \end{aligned}$$

Therefore we get

$$
\begin{aligned}
|e^{D_{1,0}t} - J^{\lambda}(t;h_0^*)| &\leq e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}|e^{D_{1,0}t} - F(n+1,t;h_0^*)| \\
&\leq 2e^{D_{1,0}t+Z_0t^2}Z_0\frac{t}{\lambda}e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^{n+1}}{(n+1)!}\frac{n+1}{n+2} \\
&\leq 2e^{D_{1,0}t+Z_0t^2}Z_0\frac{t}{\lambda}
\end{aligned}
$$

and then yields (4.5). For $x_0 + h_0$, $x_0 + h_1 \in \mathcal{V}[0,T]$,

$$
\begin{aligned}
(4.9)\quad & |F(n,t;h_1) - F(n,t;h_0)| \\
&\leq e^{D_{1,0}t}E_n[\exp\left(\sum_{i=1}^{n}\frac{1}{2}|D_{1,2}|h_0(tT_i)^2 \vee h_1(tT_i)^2tT_i + Z_0t^2T_i^2\right) \\
&\quad \left|\sum_{i=1}^{n}\frac{1}{2}D_{1,2}(h_1(tT_i)^2 - h_0(tT_i)^2)tT_i \right. \\
&\quad \left. + (Z(tT_i,x_0+h_1(tT_i)) - Z(tT_i,x_0+h_0(tT_i)))t^2T_i^2\right|] \\
&\leq e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}E_n[\sum_{i=1}^{n}\frac{|D_{1,2}|}{2}|h_1(tT_i)^2 - h_0(tT_i)^2|tT_i \\
&\quad + Z_1|h_1(tT_i) - h_0(tT_i)|t^2T_i^2].
\end{aligned}
$$

By Theorem 2.1 we get

$$
\begin{aligned}
& |F(n,t;h^{\lambda}) - F(n,t;h_0^*)| \\
&\quad \leq \quad e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}E_n[\sum_{i=1}^{n}\frac{1}{2}|D_{1,2}|h^{\lambda}(tT_i)^2tT_i + Z_1|h^{\lambda}(tT_i)|t^2T_i^2] \\
&\quad \leq \quad e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}\left(\frac{1}{2}|D_{1,2}|\frac{C_0^2}{\lambda^2}t + Z_1\frac{C_0}{\lambda}t^2\frac{2}{n+1}\right)
\end{aligned}
$$

for some constant C_0. Therefore we get

$$
\begin{aligned}
& |J^{\lambda}(t;h^{\lambda}) - J^{\lambda}(t;h_0^*)| \\
&\quad \leq e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}|F(n+1,t;h^{\lambda}) - F(n+1,t;h_0^*)|
\end{aligned}
$$

$$\leq e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}\left(\frac{1}{2}|D_{1,2}|\frac{C_0^2}{\lambda^2}t+2Z_1\frac{C_0}{\lambda^2}te^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^{n+1}}{(n+1)!}\frac{n+1}{n+2}\right)$$
$$\leq e^{(D_{1,0}+\frac{1}{2}|D_{1,2}|)t+Z_0t^2}\left(\frac{1}{2}|D_{1,2}|C_0+2Z_1\right)\frac{C_0}{\lambda^2}t.$$

By Lemma 4.1 we get (4.6). (4.7) is an immediate consequence of (4.5) and (4.6). □

Let

$$G(z,\psi)=\psi+\sum_{i=2}^{\infty}\sum_{k=2}^{i}\gamma_{i,k}z^{k-1}\psi^{i-k}$$

where $\gamma_{i,k}$, $2\leq k\leq i$ are constants. We seek a formal power series of

$$\Psi(z)=\sum_{j=1}^{\infty}h_j^*z^j$$

such that

$$G(z,\Psi(z))=0.$$

The solution of this problem is given by solving the equations in terms of the coefficients of

$$\Psi(z)+\sum_{i=2}^{\infty}\sum_{k=2}^{i}\gamma_{i,k}z^{k-1}\Psi(z)^{i-k}=0.$$

These equations are of the form

$$h_1^*=-\gamma_{2,2},$$
$$h_j^*=-P_j(\gamma_{i,k},h_l^*:2\leq k\leq i\leq j+1,l\leq j-1),\quad j\geq 2$$

where P_j is a polynomial with positive integer coefficients. Therefore we can solve recursively for the coefficients $h_j^*, j\geq 1$ and they are uniquely determined.

Replacing $\gamma_{i,k}$ by

$$\frac{D_{k,i-k+1}}{D_{1,2}(\alpha_1+\beta_0)}\left(\begin{pmatrix} i\\ k\end{pmatrix}\frac{\alpha_k}{i!}+\begin{pmatrix} i-1\\ k-1\end{pmatrix}\frac{\beta_{k-1}}{(i-1)!}\right).$$

Since h_j^* depends on $\alpha_1,\dots,\alpha_{j+1},\beta_0,\dots,\beta_j$, we denote h_j^* by

$$h_j^*(\vartheta_j)$$

where $\vartheta_j=(\alpha_1,\dots,\alpha_{j+1},\beta_0,\dots,\beta_j)$.

Lemma 4.3. *For all $n\in\boldsymbol{N}$ if there exist $\tilde{M}_m:[0,\infty)\times[0,T]\to\boldsymbol{R}$ for $0\le m\le n+1$ satisfying*

$$\begin{aligned}(4.10)\quad (\min(1,e^{f(1)t})-1)L_m(\lambda t) &\le \tilde{M}_m(\lambda,t)\\ &\le (e^{f(x_0)t}-1)L_m(\lambda t),\quad 0\le t\le T,\end{aligned}$$

there exists a constant C_n such that for $0<t\le T,\ \lambda>0$,

$$(4.11)\quad |h_n^*(\theta_n(\lambda,t))|\le C_n,$$

$$(4.12)\quad \left|\sum_{i=2}^{n+1}\sum_{k=1}^{i}\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))\frac{1}{\lambda^k}\left(\sum_{j=1}^{n}\frac{h_j^*(\theta_j(\lambda,t))}{\lambda^j}\right)^{i-k}\right|\le\frac{C_n}{\lambda^{n+2}}$$

where

$$\Gamma_{i,k}(\alpha_0,\beta_0)=D_{k,i-k+1}\left(\begin{pmatrix} i\\ k\end{pmatrix}\frac{\alpha_0}{i!}+\begin{pmatrix} i-1\\ k-1\end{pmatrix}\frac{\beta_0}{(i-1)!}\right),$$

$$\theta_n(\lambda,t)=(\tilde{M}_1(\lambda,t),\dots,\tilde{M}_{n+1}(\lambda,t),L_0(\lambda t),\dots,L_n(\lambda t)).$$

Proof. By (4.10),

$$\begin{aligned}\tilde{M}_1(\lambda,t)+L_0(\lambda t) &\ge (\min(1,e^{f(1)t})-1)L_1(\lambda t)+L_0(\lambda t)\\ &\ge \min(1,e^{f(1)t})L_1(\lambda t)\ge 0.\end{aligned}$$

By (3.7) we have for $2\le k\le i\le n+1$,

$$\begin{aligned}&\left|\frac{\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))}{\Gamma_{2,1}(\tilde{M}_1(\lambda,t),L_0(\lambda t))}\right|\\ &\quad\le\frac{|\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))|}{|D_{1,2}|\min(1,e^{f(1)t})L_1(\lambda t)}\end{aligned}$$

$$
\begin{aligned}
&\le \left|\frac{D_{k,i-k+1}}{D_{1,2}}\right| \left(\begin{pmatrix} i \\ k \end{pmatrix} \frac{(e^{f(x_0)t}-1)k!}{i!\min(1,e^{f(1)t})} \right. \\
&\quad \left. + \begin{pmatrix} i-1 \\ k-1 \end{pmatrix} \frac{(k-1)!}{(i-1)!\min(1,e^{f(1)t})} \right) \\
&= \frac{|D_{k,i-k+1}|e^{f(x_0)t}}{|D_{1,2}|(i-k)!\min(1,e^{f(1)t})}.
\end{aligned}
$$

Since $h_n^*(\theta_n(\lambda,t))$ is a polynomial of

$$
\frac{\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))}{\Gamma_{2,1}(\tilde{M}_1(\lambda,t),L_0(\lambda t))}, \quad 2 \le k \le i \le n+1,
$$

we get (4.11). By $\Gamma_{i,1}=0$ for $i \ge 3$ and the definition of h_n^*,

$$
\begin{aligned}
&\left| \sum_{i=2}^{n+1}\sum_{k=1}^{i} \frac{\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))}{\Gamma_{2,1}(\tilde{M}_1(\lambda,t),L_0(\lambda t))} \frac{1}{\lambda^{k-1}} \left(\sum_{j=1}^{n} \frac{h_j^*(\theta_j(\lambda,t))}{\lambda^j} \right)^{i-k} \right| \\
&= \left| \sum_{j=1}^{n} \frac{h_j^*(\theta_j(\lambda,t))}{\lambda^j} + \sum_{i=2}^{n+1}\sum_{k=2}^{i} \frac{\Gamma_{i,k}(\tilde{M}_k(\lambda,t),L_{k-1}(\lambda t))}{\Gamma_{2,1}(\tilde{M}_1(\lambda,t),L_0(\lambda t))} \frac{1}{\lambda^{k-1}} \right. \\
&\quad \left. \times \left(\sum_{j=1}^{n} \frac{h_j^*(\theta_j(\lambda,t))}{\lambda^j} \right)^{i-k} \right| \\
&\le \frac{C_n}{\lambda^{n+1}}
\end{aligned}
$$

for some constant C_n and then we get (4.12). □

LEMMA 4.4. *Suppose that there exist $M_{n,N}(\lambda,t)$ satisfying*

$$
\begin{aligned}
(\min(1,e^{f(1)t})-1)L_n(\lambda t) &\le M_{n,N}(\lambda,t) \\
&\le (e^{f(x_0)t}-1)L_n(\lambda t), \quad 0 \le t \le T,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
&|M_n(\lambda,t) - M_{n,N}(\lambda,t)| \\
&\quad \le C\frac{L_0(\lambda t)}{\lambda^N}, \quad 0 \le t \le T, \quad 1 \le n \le N+1
\end{aligned} \tag{4.14}
$$

for some constant C *and* $N \geq 1$. *Let*

$$\theta_{n,N}(\lambda,t) \;=\; (M_{1,N}(\lambda,t),\dots,M_{n+1,N}(\lambda,t),L_0(\lambda t),\dots,L_n(\lambda t)).$$

Then for $1 \leq n \leq N$ *there exist* C_n *and* λ_n *such that*

(4.15) $|h_n^*(\theta_{n,N}(\lambda,t))| \leq C_n, \quad 0 < t \leq T,\ \lambda \geq \lambda_n,$

$$\left|h^\lambda(t) - \sum_{i=1}^{n} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i}\right| \leq C_n \frac{1}{\lambda^{n+1}}, \quad 0 < t \leq T,\ \lambda \geq \lambda_n, \tag{4.16}$$

$$\left|\frac{\partial A^\lambda}{\partial x}(t, x_0 + \sum_{i=1}^{n} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i})\right| \leq C_n \frac{1}{\lambda^{n+2}}, \quad 0 < t \leq T,\ \lambda \geq \lambda_n, \tag{4.17}$$

$$\begin{aligned}\left|\tilde{A}^\lambda(t) - A^\lambda(t, x_0 + \sum_{i=1}^{n} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i})\right| \\ \leq C_n \frac{1}{\lambda^{2n+3}}, \quad 0 < t \leq T,\ \lambda \geq \lambda_n.\end{aligned} \tag{4.18}$$

PROOF. (4.15) is an immediate consequence of Lemma 4.3. We use the mathematical induction to prove (4.16), (4.17) and (4.18). By Theorem 2.1, (4.1) and (4.2),

$$\begin{aligned}&\left|\sum_{n=2}^{N_0}\sum_{k=1}^{n} \frac{1}{\lambda^k} h^\lambda(t)^{n-k}\Gamma_{n,k}(M_k(\lambda,t), L_{k-1}(\lambda t))\right| \\ &\quad = \left|\frac{\partial A^\lambda(t, x_0 + h^\lambda(t))}{\partial x}\right. \\ &\qquad \left. - \sum_{n=1}^{N_0}\sum_{k=1}^{n} \frac{h^\lambda(t)^{n-k}}{\lambda^k}\Gamma_{n,k}(M_k(\lambda,t), L_{k-1}(\lambda t))\right| \\ &\quad \leq \frac{\tilde{C}_{N_0} L_0(\lambda t)}{\lambda^{N_0+1}}\end{aligned} \tag{4.19}$$

for some constant $\tilde{C}_{N_0}$. Substituting 2 for N_0,

$$\begin{aligned}&\left|\frac{1}{\lambda} h^\lambda(t) D_{1,2}\left(M_1(\lambda,t) + L_0(\lambda t)\right) + \frac{1}{\lambda^2} D_{2,1}\left(\frac{M_2(\lambda,t)}{2} + L_1(\lambda t)\right)\right| \\ &\quad \leq \frac{\tilde{C}_2 L_0(\lambda t)}{\lambda^3}.\end{aligned}$$

By (4.14) and the definition of h_1^*, we have

$$\begin{aligned}&\left|\frac{1}{\lambda^2}h_1^*(\theta_{1,N}(\lambda,t))D_{1,2}\left(M_1(\lambda,t)+L_0(\lambda t)\right)+\frac{1}{\lambda^2}D_{2,1}\left(\frac{M_2(\lambda,t)}{2}+L_1(\lambda t)\right)\right|\\&\quad\le\left|\frac{h_1^*(\theta_{1,N}(\lambda,t))D_{1,2}}{\lambda^2}\left(M_{1,N}(\lambda,t)+L_0(\lambda t)\right)\right.\\&\qquad\left.+\frac{D_{2,1}}{\lambda^2}\left(\frac{M_{2,N}(\lambda,t)}{2}+L_1(\lambda t)\right)\right|+C_1\frac{L_0(\lambda t)}{\lambda^{N+2}}\\&\quad=C_1\frac{L_0(\lambda t)}{\lambda^{N+2}}\end{aligned}$$

for some constant C_1. Therefore we get

$$\begin{aligned}&\left|\frac{1}{\lambda}\left(h^\lambda(t)-\frac{h_1^*(\theta_{1,N}(\lambda,t))}{\lambda}\right)D_{1,2}\left(M_1(\lambda,t)+L_0(\lambda t)\right)\right|\\&\quad\le\quad\left|\frac{1}{\lambda}h^\lambda(t)D_{1,2}\left(M_1(\lambda,t)+L_0(\lambda t)\right)+\frac{1}{\lambda^2}D_{2,1}\left(\frac{M_2(\lambda,t)}{2}+L_1(\lambda t)\right)\right|\\&\qquad+\left|\frac{1}{\lambda^2}h_1^*(\theta_{1,N}(\lambda,t))D_{1,2}\left(M_1(\lambda,t)+L_0(\lambda t)\right)\right.\\&\qquad\left.+\frac{1}{\lambda^2}D_{2,1}\left(\frac{M_2(\lambda,t)}{2}+L_1(\lambda t)\right)\right|\\&\quad\le\quad C_1'\frac{L_0(\lambda t)}{\lambda^3}\end{aligned}$$

for some constant C_1'. By (3.6) and (3.7),

$$0\ \le\ \frac{L_n(\lambda t)}{M_1(\lambda,t)+L_0(\lambda t)}\le\frac{L_n(\lambda t)}{L_0(\lambda t)}\le n! \tag{4.20}$$

and then

$$\left|h^\lambda(t)-\frac{h_1^*(\theta_{1,N}(\lambda,t))}{\lambda}\right|\ \le\ \frac{C_1'L_0(\lambda t)}{|D_{1,2}|\left(M_1(\lambda,t)+L_0(\lambda t)\right)}\frac{1}{\lambda^2}\le\frac{C_1'}{|D_{1,2}|}\frac{1}{\lambda^2}.$$

When

$$|h^\lambda(t)-h(t)|\le\frac{C_1'L_0(\lambda t)}{|D_{1,2}|\left(M_1(\lambda,t)+L_0(\lambda t)\right)}\frac{1}{\lambda^2},$$

we have

$$\begin{aligned}\left|\frac{\partial A^\lambda(t,x_0+h(t))}{\partial x}\right| &= \left|\frac{\partial A^\lambda(t,x_0+h(t))}{\partial x}-\frac{\partial A^\lambda(t,x_0+h^\lambda(t))}{\partial x}\right|\\ &\le \left|\frac{h(t)-h^\lambda(t)}{\lambda}D_{1,2}\left(M_1(\lambda,t)+L_0(\lambda t)\right)\right|+C_1''\frac{1}{\lambda^3}\\ &\le (C_1'+C_1'')\frac{1}{\lambda^3}\end{aligned}$$

for some constant C_1''. Then we get

$$\left|\frac{\partial A^\lambda}{\partial x}(t,x_0+\frac{h_1^*(\theta_{1,N}(\lambda,t))}{\lambda})\right| \le (C_1'+C_1'')\frac{1}{\lambda^3}$$

and

$$\begin{aligned}&\left|A^\lambda(t,x_0+h^\lambda(t))-A^\lambda(t,x_0+\frac{h_1^*(\theta_{1,N}(\lambda,t))}{\lambda})\right|\\ &\le \left|h^\lambda(t)-\frac{h_1^*(\theta_{1,N}(\lambda,t))}{\lambda}\right|\\ &\quad\times\sup\left\{\left|\frac{\partial A^\lambda(t,x_0+h)}{\partial x}\right|\,\middle|\,|h^\lambda(t)-h|\le\frac{C_1'L_0(\lambda t)}{|D_{1,2}|\left(M_1(\lambda,t)+L_0(\lambda t)\right)}\frac{1}{\lambda^2}\right\}\\ &\le \frac{C_1'(C_1'+C_1'')}{|D_{1,2}|}\frac{1}{\lambda^5}.\end{aligned}$$

The assertion holds for $n=1$.

Suppose that the assertion holds for $n\le N_1\le N-1$. Substituting N_1+2 for N_0 in (4.19),

$$\left|\sum_{n=2}^{N_1+2}\sum_{k=1}^{n}\frac{1}{\lambda^k}h^\lambda(t)^{n-k}\Gamma_{n,k}(M_k(\lambda,t),L_{k-1}(\lambda t))\right|\le\frac{\tilde{C}_{N_1+2}L_0(\lambda t)}{\lambda^{N_1+3}}.$$

By Lemma 4.3 and (4.14), we have

$$\left|\sum_{n=2}^{N_1+2}\sum_{k=1}^{n}\frac{1}{\lambda^k}\left(\sum_{i=1}^{N_1+1}\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i}\right)^{n-k}\Gamma_{n,k}(M_k(\lambda,t),L_{k-1}(\lambda t))\right|$$

$$
\begin{aligned}
&\leq \left| \sum_{n=2}^{N_1+2} \sum_{k=1}^{n} \frac{1}{\lambda^k} \left(\sum_{i=1}^{N_1+1} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i} \right)^{n-k} \Gamma_{n,k}(M_{k,N}(\lambda,t), L_{k-1}(\lambda t)) \right| \\
&\quad + C_{N_1} \frac{L_0(\lambda t)}{\lambda^{N+2}} \\
&\leq C'_{N_1} \frac{L_0(\lambda t)}{\lambda^{N_1+3}}
\end{aligned}
$$

for some constant C_{N_1} and C'_{N_1}. By the induction hypothesis (4.16), we obtain

$$
\begin{aligned}
&\left| \frac{1}{\lambda} \left(h^\lambda(t) - \sum_{i=1}^{N_1+1} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i} \right) D_{1,2}\left(M_1(\lambda,t) + L_0(\lambda t) \right) \right| \\
&\leq \left| \sum_{n=2}^{N_1+2} \sum_{k=1}^{n} \frac{1}{\lambda^k} h^\lambda(t)^{n-k} \Gamma_{n,k}(M_k(\lambda,t), L_{k-1}(\lambda t)) \right| \\
&\quad + \left| \sum_{n=2}^{N_1+2} \sum_{k=1}^{n} \frac{1}{\lambda^k} \left(\sum_{i=1}^{N_1+1} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i} \right)^{n-k} \Gamma_{n,k}(M_k(\lambda,t), L_{k-1}(\lambda t)) \right| \\
&\quad + \left| \sum_{n=3}^{N_1+2} \sum_{k=1}^{n} \frac{1}{\lambda^k} \left(h^\lambda(t)^{n-k} - \left(\sum_{i=1}^{N_1+1} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i} \right)^{n-k} \right) \right. \\
&\qquad \left. \times \Gamma_{n,k}(M_k(\lambda,t), L_{k-1}(\lambda t)) \right| \\
&\leq C''_{N_1} \frac{L_0(\lambda t)}{\lambda^{N_1+3}}
\end{aligned}
$$

for some constant C''_{N_1}. By (4.20),

$$
\begin{aligned}
\left| h^\lambda(t) - \sum_{i=1}^{N_1+1} \frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i} \right| &\leq \frac{C''_{N_1} L_0(\lambda t)}{|D_{1,2}| \left(M_1(\lambda,t) + L_0(\lambda t) \right)} \frac{1}{\lambda^{N_1+2}} \\
&\leq \frac{C''_{N_1}}{|D_{1,2}|} \frac{1}{\lambda^{N_1+2}}.
\end{aligned}
$$

When

$$
|h^\lambda(t) - h(t)| \leq \frac{C''_{N_1}}{|D_{1,2}|} \frac{1}{\lambda^{N_1+2}},
$$

we have

$$\begin{aligned}\left|\frac{\partial A^\lambda(t,x_0+h(t))}{\partial x}\right| &= \left|\frac{\partial A^\lambda(t,x_0+h(t))}{\partial x}-\frac{\partial A^\lambda(t,x_0+h^\lambda(t))}{\partial x}\right| \\ &\leq \left|\sum_{n=2}^{N_1+2}\sum_{k=1}^{n}\frac{1}{\lambda^k}\left(h^\lambda(t)^{n-k}-h(t)^{n-k}\right)\Gamma_{n,k}(M_k(\lambda,t),L_{k-1}(\lambda t))\right| \\ &\quad +C'''_{N_1}\frac{1}{\lambda^{N_1+3}} \leq C''''_{N_1}\frac{1}{\lambda^{N_1+3}}\end{aligned}$$

for some constants C'''_{N_1} and C''''_{N_1}. Then we get

$$\left|\frac{\partial A^\lambda}{\partial x}(t,x_0+\sum_{i=1}^{N_1+1}\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i})\right| \leq C''''_{N_1}\frac{1}{\lambda^{N_1+3}}$$

and

$$\begin{aligned}&\left|A^\lambda(t,x_0+h^\lambda(t))-A^\lambda(t,x_0+\sum_{i=1}^{N_1+1}\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i}))\right| \\ &\leq \left|h^\lambda(t)-\sum_{i=1}^{N_1+1}\frac{h_i^*(\theta_{i,N}(\lambda,t))}{\lambda^i}\right| \\ &\qquad \times\sup\left\{\left|\frac{\partial A^\lambda(t,x_0+h)}{\partial x}\right| \,\middle|\, |h^\lambda(t)-h|\leq\frac{C''_{N_1}}{|D_{1,2}|}\frac{1}{\lambda^{N_1+2}}\right\} \\ &\leq \frac{C''_{N_1}C''''_{N_1}}{|D_{1,2}|}\frac{1}{\lambda^{2(N_1+1)+3}}.\end{aligned}$$

The assertion holds for N_1+1 and then it holds for all $1\leq n\leq N$. □

LEMMA 4.5. *Suppose that λ is sufficiently large and $\tilde{h}(t)$ satisfies*

$$0\leq x_0+\tilde{h}(t)\leq 1,\quad 0\leq t\leq T,$$
$$|h^\lambda(t)-\tilde{h}(t)|\leq C\frac{1}{\lambda^{N+1}},\quad 0\leq t\leq T$$

for some constant C. Let

$$\tilde{M}_n(\lambda,t)=\int_0^{\lambda t}u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda};\tilde{h})-1\right)du.$$

Then

$$(\min(1, e^{f(1)t}) - 1)L_n(\lambda t) \leq \tilde{M}_n(\lambda, t) \leq (e^{f(x_0)t} - 1)L_n(\lambda t), \quad 0 \leq t \leq T, \tag{4.21}$$

$$|M_n(\lambda, t) - \tilde{M}_n(\lambda, t)| \leq C_n \frac{L_0(\lambda t)}{\lambda^{N+2}}, \quad 0 \leq t \leq T \tag{4.22}$$

for some constant C_n.

Proof. By (4.3), we get (4.21). By (4.8) and (4.9),

$$\begin{aligned}
&|F(n, t; h^\lambda) - F(n, t; \tilde{h})| \\
&\leq \; e^{(D_{1,0} + \frac{1}{2}|D_{1,2}|)t + Z_0 t^2} \\
&\quad \times E_n[\sum_{i=1}^{n} \frac{1}{2}|D_{1,2}||h^\lambda(tT_i)^2 - \tilde{h}(tT_i)^2|tT_i + Z_1|h^\lambda(tT_i) - \tilde{h}(tT_i)|t^2T_i^2] \\
&\leq \; C_1 \left(\frac{1}{\lambda^{N+2}} + \frac{1}{\lambda^{N+1}} \frac{t^2}{n+1} \right)
\end{aligned}$$

for some constant C_1. Therefore we get

$$\begin{aligned}
|J^\lambda(t; h^\lambda) - J^\lambda(t; \tilde{h})| &\leq e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} |F(n+1, t; h^\lambda) - F(n+1, t; \tilde{h})| \\
&\leq \frac{C_1}{\lambda^{N+2}} + \frac{C_1}{\lambda^{N+1}} \frac{t}{\lambda} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} \frac{n+1}{n+2} \\
&\leq \frac{C_1(1+T)}{\lambda^{N+2}}
\end{aligned}$$

and then

$$\begin{aligned}
&|M_n(\lambda, t) - \tilde{M}_n(\lambda, t)| \\
&\leq \int_0^{\lambda t} u^n e^{-u} \left| J^\lambda(t - \frac{u}{\lambda}; h^\lambda) - J^\lambda(t - \frac{u}{\lambda}; \tilde{h}) \right| du \\
&\leq L_n(\lambda t) \frac{C_1(1+T)}{\lambda^{N+2}} \leq L_0(\lambda t) \frac{n! C_1(1+T)}{\lambda^{N+2}}.
\end{aligned}$$

The result follows. □

Proof of Theorem 2.2. We use the mathematical induction. Let

$$M_{n,1}(\lambda,t) \;=\; \left(e^{f(x_0)t}-1\right)L_n(\lambda t).$$

By Lemma 4.2,

$$\begin{aligned}
&|M_n(\lambda,t)-M_{n,1}(\lambda,t)| \\
\le\;& \left|\int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda};h^\lambda)-1\right)du\right. \\
&\left.-\int_0^{\lambda t} u^n e^{-u}\left(e^{f(x_0)(t-\frac{u}{\lambda})}-1\right)du\right| \\
&+\left|\int_0^{\lambda t} u^n e^{-u}\left(e^{f(x_0)(t-\frac{u}{\lambda})}-1\right)du-\left(e^{f(x_0)t}-1\right)L_n(\lambda t)\right| \\
\le\;& C_1\frac{L_n(\lambda t)+L_{n+1}(\lambda t)}{\lambda}
\end{aligned}$$

for some constant C_1. By Lemma 4.4 the assertion holds for $N=1$.

Suppose that it holds for $N\le N_1$. Let $h^\lambda_{N_1}(t)=v^\lambda_{N_1}(T-t)-x_0$ and

$$M_{n,N_1+1}(\lambda,t)=\int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda};h^\lambda_{N_1})-1\right)du.$$

By Lemma 4.5

$$\begin{aligned}
&(\min(1,e^{f(1)t})-1)L_n(\lambda t)\le M_{n,N_1+1}(\lambda,t) \\
&\qquad\le(e^{f(x_0)t}-1)L_n(\lambda t),\quad 0\le t\le T, \\
&|M_n(\lambda,t)-M_{n,N_1+1}(\lambda,t)| \\
&\qquad\le C_{N_1+1}\frac{1}{\lambda^{N_1+2}},\quad 0\le t\le T,\ \ 1\le n\le N_1+2
\end{aligned}$$

for some constant C_{N_1+1}. By Lemma 4.4 the assertion holds for $N=N_1+1$ and then the result follows. □

Remark 4.1. By the arguments before Lemma 4.3 we have shown how to determine h^*_n recursively. Further we have shown how to construct $M_{n,N}$ successively by the proof of Theorem 2.2. Therefore we can construct v^λ_N successively. In the following corollaries we give v^λ_1 and v^λ_2.

Corollary 4.1. *Let*

$$\begin{aligned} h_1^*(\alpha_1,\alpha_2,\beta_0,\beta_1) &= -\frac{D_{2,1}(\alpha_2/2+\beta_1)}{D_{1,2}(\alpha_1+\beta_0)}, \\ M_{n,1}(\lambda,t) &= \left(e^{f(x_0)t}-1\right) L_n(\lambda t), \\ \theta_{1,1}(\lambda,t) &= (M_{1,1}(\lambda,t), M_{2,1}(\lambda,t), L_0(\lambda t), L_1(\lambda t)). \end{aligned}$$

Let v_1^λ *be given by*

$$\begin{aligned} v_1^\lambda(T-t) &= x_0 + \frac{h_1^*(\theta_{1,1}(\lambda,t))}{\lambda} \\ &= x_0 - \sigma^2 x_0(1-x_0)(1-2x_0) \\ &\quad \times \frac{\left(e^{f(x_0)t}-1\right) L_2(\lambda t)/2 + L_1(\lambda t)}{\left(e^{f(x_0)t}-1\right) L_1(\lambda t) + L_0(\lambda t)} \frac{1}{\lambda} \end{aligned}$$

for $0 < t \le T$ *and* $v_1^\lambda(T) = x_0$. *Then* v_1^λ *satisfies (2.6), (2.7) and (2.8) for* $N = 1$.

Note that h_1^* and $(x_0 - 1/2)$ have same signs. Also when we invest for a short period, $h_1^*(\theta_{1,1}(\lambda,t))/\lambda$ becomes negligible since

$$\lim_{t\to 0}\left|\frac{h_1^*(\theta_{1,1}(\lambda,t))}{\lambda}\right| \le \lim_{t\to 0}\left|\frac{D_{2,1}}{D_{1,2}}\right| \frac{(e^{f(x_0)t}-1)L_2(\lambda t)/2 + L_1(\lambda t)}{L_0(\lambda t)}\frac{1}{\lambda} = 0.$$

Corollary 4.2. *Let*

$$\begin{aligned} & h_2^*(\alpha_1,\alpha_2,\alpha_3,\beta_0,\beta_1,\beta_2) \\ & \quad = -\frac{D_{2,2}\left(\frac{\alpha_2}{2}+\beta_1\right) h_1^*(\alpha_1,\alpha_2,\beta_0,\beta_1) + D_{3,1}\left(\frac{\alpha_3}{3!}+\frac{\beta_2}{2}\right)}{D_{1,2}(\alpha_1+\beta_0)}, \end{aligned}$$

$$\begin{aligned} M_{n,3}(\lambda,t) &= \int_0^{\lambda t} u^n e^{-u}\left(J^\lambda(t-\frac{u}{\lambda}; h_1^\lambda) - 1\right) du, \\ h_1^\lambda(t) &= v_1^\lambda(T-t) - x_0, \\ \theta_{1,3}(\lambda,t) &= (M_{1,3}(\lambda,t), M_{2,3}(\lambda,t), L_0(\lambda t), L_1(\lambda t)), \\ \theta_{2,3}(\lambda,t) &= (M_{1,3}(\lambda,t), M_{2,3}(\lambda,t), M_{3,3}(\lambda,t), L_0(\lambda t), L_1(\lambda t), L_2(\lambda t)). \end{aligned}$$

Let v_2^λ *be given by*

$$v_2^\lambda(T-t) = x_0 + \sum_{i=1}^{2} \frac{h_i^*(\theta_{i,3}(\lambda,t))}{\lambda^i}$$

for $0 < t \leq T$ *and* $v_2^\lambda(T) = x_0$. *Then* v_2^λ *satisfies (2.6), (2.7) and (2.8) for* $N = 2$.

5. Limit of the Value Function

In the previous section, we have shown that the optimal strategy converges to x_0 when λ tends to ∞. In this section we show the limit of the value function.

PROPOSITION 5.1. *There exist* $C > 0$ *and* $\lambda_0 > 0$ *such that*

$$\left|\frac{\partial A^\lambda(t,x_0)}{\partial x}\right| \leq C\frac{1}{\lambda^2}, \quad 0 \leq t \leq T,\ \lambda \geq \lambda_0, \tag{5.1}$$

$$\left|A^\lambda(t,x_0+h^\lambda(t)) - A^\lambda(t,x_0)\right| \leq C\frac{1}{\lambda^3}, \quad 0 \leq t \leq T,\ \lambda \geq \lambda_0. \tag{5.2}$$

PROOF. By Theorem 2.1 when λ is sufficiently large, $|h^\lambda(t)| \leq C_0/\lambda$ for some constant C_0. When $|h(t)| \leq C_0/\lambda$, we have by (4.1) and (4.2)

$$\left|\frac{\partial A^\lambda(t,x_0+h(t))}{\partial x}\right| \leq C_0'\frac{1}{\lambda^2}$$

for some constant C_0'. Therefore we get

$$\left|\frac{\partial A^\lambda(t,x_0+h^\lambda(t))}{\partial x}\right| \leq C_0'\frac{1}{\lambda^2}$$

and

$$\begin{aligned}\left|A^\lambda(t,x_0+h^\lambda(t)) - A^\lambda(t,x_0)\right| &\leq |h^\lambda(t)| \sup_{|h|\leq C_0\frac{1}{\lambda}} \left|\frac{\partial A^\lambda(t,x_0+h)}{\partial x}\right| \\ &\leq C_0C_0'\frac{1}{\lambda^3}. \square\end{aligned}$$

Lemma 5.1. *There exists $C_1 > 0$ such that*

$$\left| J^{\lambda}(t; h_0^*) - \left(e^{f(x_0)t} + 2Z(0, x_0) e^{f(x_0)t} \frac{t}{\lambda} \right) \right| \le C_1 \frac{1}{\lambda^2}, \quad 0 \le t \le T.$$

Proof. In the similar way to (4.8), we have

$$\begin{aligned} E_n[\sum_{i=1}^n T_i^3] &= \frac{6}{(n+2)(n+1)}, \\ E_n[\sum_{i=1}^n T_i^4] &= \frac{24}{(n+3)(n+2)(n+1)}. \end{aligned}$$

Since

$$\begin{aligned} E_2[T_1^2 T_2^2] &= \int_0^1 t_1^2 (1-t_1)^2 dt_1 = \frac{1}{30}, \\ E_n[T_1^2 T_2^2] &= (n-1)! \int_{t_1 + \cdots + t_{n-1} \le 1} t_1^2 t_2^2 dt_{n-1} \cdots dt_1 \\ &= \frac{4}{(n+3)(n+2)(n+1)n}, \quad n \ge 3, \end{aligned}$$

we have for all $n \ge 2$

$$E_n[T_1^2 T_2^2] = \frac{4}{(n+3)(n+2)(n+1)n}.$$

Since

$$E_n[\left(\sum_{i=1}^n T_i^2 \right)^2] = n E_n[T_1^4] + (n^2 - n) E_n[T_1^2 T_2^2],$$

we get

$$E_n[\left(\sum_{i=1}^n T_i^2 \right)^2] = \frac{28n - 4}{(n+3)(n+2)(n+1)}$$

for all $n \ge 1$. Therefore we obtain

$$\begin{aligned} &\left| e^{D_{1,0}t} \left(1 + Z(0, x_0) t^2 \frac{2}{n+1} \right) - F(n, t; h_0^*) \right| \\ &= e^{D_{1,0}t} \left| 1 + Z(0, x_0) t^2 \frac{2}{n+1} - E_n[\exp\left(\sum_{i=1}^n Z(tT_i, x_0) t^2 T_i^2 \right)] \right| \end{aligned}$$

$$
\begin{aligned}
&= e^{D_{1,0}t}\left| E_n[\sum_{i=1}^n (Z(0,x_0) - Z(tT_i,x_0))t^2T_i^2 \right. \\
&\quad \left. - \left(\sum_{i=1}^n Z(tT_i,x_0)t^2T_i^2\right)^2 \int_0^1 (1-s)\exp\left(s\sum_{i=1}^n Z(tT_i,x_0)t^2T_i^2\right) ds] \right| \\
&\le e^{D_{1,0}t} Z_1 t^3 E_n[\sum_{i=1}^n T_i^3] + \frac{1}{2} e^{D_{1,0}t} Z_0^2 t^4 \exp\left(Z_0 t^2\right) E_n[\left(\sum_{i=1}^n T_i^2\right)^2] \\
&\le e^{D_{1,0}t} t^3 \left(6Z_1 + 14Z_0^2 t \exp\left(Z_0t^2\right)\right) \frac{1}{(n+2)(n+1)}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&\left| e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n}{n!} e^{D_{1,0}t}\left(1 + Z(0,x_0)\frac{2t^2}{n+2}\right) - J^\lambda(t;h_0^*)\right| \\
&\quad \le \quad e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n}{n!} \left| e^{D_{1,0}t}\left(1 + Z(0,x_0)t^2\frac{2}{n+2}\right) - F(n+1,t;h_0^*)\right| \\
&\quad \le \quad \left(6Z_1 + 14Z_0^2 t\exp\left(Z_0t^2\right)\right) e^{D_{1,0}t}\frac{t}{\lambda^2} e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^{n+2}}{(n+2)!}\frac{n+1}{n+3} \\
&\quad \le \quad \left(6Z_1 + 14Z_0^2 t\exp\left(Z_0t^2\right)\right) e^{D_{1,0}t}\frac{t}{\lambda^2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left| e^{D_{1,0}t}\left(1 + 2Z(0,x_0)\frac{t}{\lambda}\right) - e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^n}{n!} e^{D_{1,0}t}\left(1 + Z(0,x_0)\frac{2t^2}{n+2}\right)\right| \\
&\qquad \le 2Z(0,x_0)e^{D_{1,0}t}\frac{t}{\lambda}\left| 1 - e^{-\lambda t}\sum_{n=0}^\infty \frac{(\lambda t)^{n+1}}{(n+1)!}\frac{n+1}{n+2}\right| \\
&\qquad = 2Z(0,x_0)e^{D_{1,0}t}\frac{1}{\lambda^2} e^{-\lambda t}\sum_{n=1}^\infty \frac{(\lambda t)^n}{n!} \le 2Z(0,x_0)e^{D_{1,0}t}\frac{1}{\lambda^2},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left| J^\lambda(t;h_0^*) - e^{D_{1,0}t}\left(1 + 2Z(0,x_0)\frac{t}{\lambda}\right)\right| \\
&\qquad \le \left(6Z_1 t + 14Z_0^2 t^2\exp\left(Z_0t^2\right) + 2Z(0,x_0)\right) e^{D_{1,0}t}\frac{1}{\lambda^2}.
\end{aligned}
$$

The result follows. □

LEMMA 5.2. *There exists $C_1 > 0$ such that*

$$\left|\frac{\partial J^\lambda(t;h_0^*)}{\partial t} - f(x_0)e^{f(x_0)t}\right| \le C_1\frac{1}{\lambda}, \quad 0 \le t \le T.$$

PROOF. By the definition, $J^\lambda(t;h_0^*)$ satisfies

$$J^\lambda(t;h_0^*) = \lambda\int_0^t D(t-s,x_0)e^{-\lambda(t-s)}J^\lambda(s;h_0^*)ds + D(t,x_0)e^{-\lambda t}.$$

Differentiating this with respect to t,

$$\frac{\partial J^\lambda(t;h_0^*)}{\partial t} = \lambda\int_0^t \frac{\partial D(t-s,x_0)}{\partial t}e^{-\lambda(t-s)}J^\lambda(s;h_0^*)ds + \frac{\partial D(t,x_0)}{\partial t}e^{-\lambda t}.$$

By Lemma 4.2,

$$\begin{aligned}
&\left|\frac{\partial J^\lambda(t;h_0^*)}{\partial t} - D_{1,0}e^{D_{1,0}t}\right| \\
&\le \left|\lambda\int_0^t \frac{\partial D(t-s,x_0)}{\partial t}e^{-\lambda(t-s)}e^{D_{1,0}s}ds + \frac{\partial D(t,x_0)}{\partial t}e^{-\lambda t} - D_{1,0}e^{D_{1,0}t}\right| \\
&\quad + C\frac{1}{\lambda} \\
&= \left|-\int_0^t\left(-\frac{\partial^2 D(t-s,x_0)}{\partial t^2} + \frac{\partial D(t-s,x_0)}{\partial t}D_{1,0}\right)e^{-\lambda(t-s)}e^{D_{1,0}s}ds\right| \\
&\quad + C\frac{1}{\lambda} \le C'\frac{1}{\lambda}
\end{aligned}$$

for some constants C and C'. The result follows. □

PROOF OF THEOREM 2.3. By (2.5) and Lemma 4.2,

$$\left|A^\lambda(t,x) - \left(\int_0^t D(t-s,x)e^{-\lambda(t-s)}\lambda J^\lambda(s;h_0^*)ds + D(t,x)e^{-\lambda t}\right)\right| \le C\frac{1}{\lambda^2}$$

for some constant C. By the integration by parts,

$$\left| A^\lambda(t,x) - \left(J^\lambda(t;h_0^*) - \int_0^t \left(-\frac{\partial D(t-s,x)}{\partial t} J^\lambda(s;h_0^*) + D(t-s,x)\frac{\partial J^\lambda(s;h_0^*)}{\partial t} \right) e^{-\lambda(t-s)} ds \right) \right| \le C\frac{1}{\lambda^2}.$$

By Lemmas 4.2, 5.1 and 5.2,

$$\left| A^\lambda(t,x) - e^{f(x_0)t} - 2Z(0,x_0)e^{f(x_0)t}\frac{t}{\lambda} - \int_0^t \left(\frac{\partial D(t-s,x)}{\partial t} e^{f(x_0)s} - D(t-s,x)f(x_0)e^{f(x_0)s} \right) e^{-\lambda(t-s)} ds \right| \le C'\frac{1}{\lambda^2}$$

for some constant C'. Then

$$\lim_{\lambda\to\infty} A^\lambda(t,x) = e^{f(x_0)t}$$

and

$$\begin{aligned} \lim_{\lambda\to\infty} \lambda(A^\lambda(t,x) - e^{f(x_0)t}) &= 2Z(0,x_0)e^{f(x_0)t}t + \frac{\partial D(0,x)}{\partial t}e^{f(x_0)t} \\ &\quad - D(0,x)f(x_0)e^{f(x_0)t} \\ &= 2Z(0,x_0)e^{f(x_0)t}t + (f(x)-f(x_0))e^{f(x_0)t}. \end{aligned}$$

By the definition we have

$$\begin{aligned} f(x) &= f(x_0) - \frac{1}{2}\alpha(1-\alpha)\sigma^2(x-x_0)^2, \\ Z(0,x_0) &= \int_0^1 (1-s)\frac{D_{2,0}D_{0,0} - D_{1,0}^2}{D_{0,0}^2} ds = \frac{1}{2}(D_{2,0} - D_{1,0}^2) \\ &= -\frac{1}{4}\alpha(1-\alpha)x_0^2(1-x_0)^2\sigma^4. \end{aligned}$$

The results follow. □

REMARK 5.1. When $0 < x_0 < 1$ is not satisfied, the optimal strategy is trivial, regardless of liquidity of the risky asset. When $x_0 \leq 0$, the optimal strategy is $v^\lambda \equiv 0$ since $f(x) \leq f(0)$ for $0 \leq x \leq 1$. Also when $x_0 \geq 1$, the optimal strategy is $v^\lambda \equiv 1$ since $f(x) \leq f(1)$ for $0 \leq x \leq 1$. In both cases, the value function is represented by the form of (2.5) and when the asset is completely liquid,

$$V^\infty(t,x,w) = \begin{cases} w^\alpha e^{\alpha r(T-t)}, & x_0 \leq 0, \\ w^\alpha \exp\left(\left(\alpha\mu - \frac{1}{2}\alpha(1-\alpha)\sigma^2\right)(T-t)\right), & x_0 \geq 1. \end{cases}$$

Note that the optimal strategy does not depend on λ.

Suppose that $x_0 \leq 0$. Since

$$\tilde{A}^\lambda(t) = \int_0^t e^{f(0)(t-s)} e^{-\lambda(t-s)} \lambda \tilde{A}^\lambda(s) ds + e^{f(0)t} e^{-\lambda t},$$

we obtain $\tilde{A}^\lambda(t) = e^{f(0)t}$. By the integration by parts, we get

$$\begin{aligned} A^\lambda(t,x) &= e^{f(0)t} - \int_0^t \left(-\frac{\partial D(t-s,x)}{\partial t} e^{f(0)s} \right. \\ &\qquad \left. + D(t-s,x) f(0) e^{f(0)s} \right) e^{-\lambda(t-s)} ds. \end{aligned}$$

Therefore we obtain for $0 \leq t \leq T$

$$\begin{aligned} &V^\lambda(t,x,w) \to w^\alpha e^{f(0)(T-t)} = V^\infty(t,x,w), \\ &\lambda(V^\infty(t,x,w) - V^\lambda(t,x,w)) \to w^\alpha (f(0) - f(x)) e^{f(0)(T-t)} \geq 0 \end{aligned}$$

as $\lambda \to \infty$ uniformly in $0 \leq x \leq 1$.

Suppose that $x_0 \geq 1$. Similarly we get for $0 \leq t \leq T$

$$\begin{aligned} &V^\lambda(t,x,w) \to w^\alpha e^{f(1)(T-t)} = V^\infty(t,x,w), \\ &\lambda(V^\infty(t,x,w) - V^\lambda(t,x,w)) \to w^\alpha (f(1) - f(x)) e^{f(1)(T-t)} \geq 0 \end{aligned}$$

as $\lambda \to \infty$ uniformly in $0 \leq x \leq 1$.

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