J. Math. Sci. Univ. Tokyo
10 (2003), 661–685.

Scattering Theory for the Coupled Klein-Gordon-Schrödinger Equations in Two Space Dimensions

By Akihiro Shimomura

Abstract. We study the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions. The scattering problem for this equation belongs to the borderline between the short range case and the long range one. We show the existence of the wave operators to this equation without any size restriction on the Klein-Gordon component of the final state.

1. Introduction

We study the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions:

$$\text{(KGS)} \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = uv, \\ \partial_t^2 v - \Delta v + v = -|u|^2. \end{cases}$$

Here u and v are complex and real valued unknown functions of $(t,x) \in \mathbb{R} \times \mathbb{R}^2$, respectively. In the present paper, we prove the existence of the wave operators to the equation (KGS) without any size restriction on the Klein-Gordon component of the final state.

A large amount of work has been devoted to the asymptotic behavior of solutions for the nonlinear Schrödinger equation and for the nonlinear Klein-Gordon equation. We consider the scattering theory for systems centering on the Schrödinger equation, in particular, the Klein-Gordon-Schrödinger, the Wave-Schrödinger and the Maxwell-Schrödinger equations. In the scattering theory for the linear Schrödinger equation, (ordinary) wave operators

2000 *Mathematics Subject Classification.* 35B40, 35P25, 35Q40.

are defined as follows. Assume that for a solution of the free Schrödinger equation with given initial data ϕ, there exists a unique time global solution u for the perturbed Schrödinger equation such that u behaves like the given free solution as $t \to \infty$. (This case is called the short range case, and otherwise we call the long range case). Then we define a wave operator W_+ by the mapping from ϕ to $u|_{t=0}$. In the long range case, ordinary wave operators do not exist and we have to construct modified wave operators including a suitable phase correction in their definition. For the nonlinear Schrödinger equation, the nonlinear wave equation and systems centering on the Schrödinger equation, we can define the wave operators and introduce the modified wave operators in the same way. According to linear scattering theory, it seems that the equation (KGS) in two space dimensions belongs to the borderline between the short range case and the long range one, because the equation (KGS) has quadratic nonlinearities, and the solutions of the free Schrödinger equation and the free Klein-Gordon equation decay as t^{-1} in L^∞ as $t \to \infty$ in two space dimensions. The Maxwell-Schrödinger equation and the Wave-Schrödinger equation in three space dimensions also belong to the same case.

There are some results of the long range scattering for nonlinear equations and systems. Ozawa [14] and Ginibre and Ozawa [4] proved the existence of modified wave operators in the borderline case for the nonlinear Schrödinger equation in one space dimension and in two and three space dimensions, respectively. Their methods were applied to the Klein-Gordon-Schrödinger equation in two space dimensions by Ozawa and Tsutsumi [15] and to the Maxwell-Schrödinger equation under the Coulomb gauge condition in three space dimensions by Tsutsumi [20]. In all results mentioned above, the restriction on the size of the final state is assumed. Furthermore in [15], the support of the Fourier transform of the Schrödinger data is restricted outside the unit disk in order to use the difference between the propagation property of the Schrödinger wave and the Klein-Gordon wave and to obtain additional time decay estimates for the nonlinear term. (See (1.4) below). In [20], the Fourier transform of the Schrödinger data vanishes in a neighborhood of the unit sphere by the same reason.

Recently Ginibre and Velo [5, 6, 7] have proved the existence of the modified wave operators for the Hartree equations with long range potentials with no restriction on the size of the final state. They decomposed the unknown function u into the complex amplitude w and the real phase φ, and solved the system for w and φ. Constructing the modified wave operators for those equations such that the domain and the range of them are same space, Nakanishi [12, 13] extended their results. Using the methods in [5, 6, 7], Ginibre and Velo showed the existence of modified wave operators for the Wave-Schrödinger equation ([8]) and for the Maxwell-Schrödinger equation under the Coulomb gauge condition ([9]) in three space dimensions with no restriction on the size of the final state. (The restriction on the support of the Fourier transform of the final state mentioned above is assumed in [8], and the vanishing asymptotic magnetic field is considered in [9]).

On the other hand, recently, the author has proved the existence of wave operators for the two dimensional Klein-Gordon-Schrödinger equation in [17], and the modified wave operators to the three dimensional Wave-Schrödinger equation in [16] and to the three dimensional Maxwell-Schrödinger equations under the Coulomb and the Lorentz gauge conditions in [18] for small scattered states without any restrictions on the support of the Fourier transform of them. Furthermore combining idea of [8] with that of [16], Ginibre and Velo [10] have proved the existence of modified wave operators for the three dimensional Wave-Schrödinger equation with restrictions on neither size of the scattered states nor the support of the Fourier transform of them.

In the present paper, we prove the existence of the wave operators for the equation (KGS) without any size restriction on the Klein-Gordon component of the final state. The proof is mainly based on choice of a suitable asymptotic profile and construction a solution for the equation (KGS) which approaches the asymptotic profile under no size restriction on the Klein-Gordon component of the final state. By using the energy method, for a given asymptotic profile satisfying suitable conditions, we solve the final value problem to the equation (KGS) such that the difference between the exact solution for that equation and the asymptotic profile decay more rapidly than the derivatives of it as in [19] (see Proposition 2.1). That difference decays as $O(t^{-k})$ $(1 < k < 2)$ as $t \to \infty$ in L^2, though the decay rate of that difference is order t^{-1} in [15, 17]. Because of this difficulty, the support of the Fourier transform of the Schrödinger data is restricted outside the unit disk as in [15] (see (1.4) below). To find a suitable asymptotic profile, we choose a second correction term for the Schrödinger component and the third one for the Klein-Gordon component. Furthermore for the Schrödinger component, the method of phase correction is applied to handle slowly decaying terms caused by the second correction terms.

Before stating our main result, we introduce some notations.

Notations. We use the following symbols:

$$\partial_0 = \partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \quad \text{for } j = 1, 2,$$
$$\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \quad \text{for a multi-index } \alpha = (\alpha_1, \alpha_2),$$
$$\nabla = (\partial_1, \partial_2), \quad \Delta = \partial_1^2 + \partial_2^2,$$

for $t \in \mathbb{R}$ and $x = (x_1, x_2) \in \mathbb{R}^2$.

Let

$$L^q \equiv L^q(\mathbb{R}^2) = \left\{ \psi \colon \|\psi\|_{L^q} = \left(\int_{\mathbb{R}^2} |\psi(x)|^q \, dx \right)^{1/q} < \infty \right\} \text{ for } 1 \le q < \infty,$$
$$L^\infty \equiv L^\infty(\mathbb{R}^2) = \left\{ \psi \colon \|\psi\|_{L^\infty} = \operatorname{ess.sup}_{x \in \mathbb{R}^2} |\psi(x)| < \infty \right\}.$$

We use the L^2-scalar product

$$(\varphi, \psi) \equiv \int_{\mathbb{R}^2} \varphi(x)\overline{\psi(x)} \, dx.$$

$\mathcal{S}$ denotes the set of rapidly decreasing functions on $\mathbb{R}^2$. Let $\mathcal{S}'$ be the set of tempered distributions on $\mathbb{R}^2$. For $w \in \mathcal{S}'$, we denote the Fourier transform of w by $\hat{w}$. For $w \in L^1(\mathbb{R}^n)$, $\hat{w}$ is represented as

$$\hat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-ix\cdot\xi} \, dx.$$

For $s, m \in \mathbb{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{ \psi \in \mathcal{S}' \colon \|\psi\|_{H^{s,m}} \equiv \|(1 + |x|^2)^{m/2}(1 - \Delta)^{s/2}\psi\|_{L^2} < \infty \}.$$

H^s denotes $H^{s,0}$. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{ \psi \in L^p \colon \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty \right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\| \cdot \|_{H^k}$ and $\| \cdot \|_{W_2^k}$ are equivalent.

For $s > 0$, we define the homogeneous Sobolev spaces $\dot{H}^s$ by the completion of $\mathcal{S}$ with respect to the norm

$$\|w\|_{\dot{H}^s} \equiv \|(-\Delta)^{s/2} w\|_{L^2}. \tag{1.1}$$

$\dot{H}^s$ is a Banach space with the norm (1.1) for $s > 0$.

Let Y and Z be two Banach spaces with the norms $\| \cdot \|_Y$ and $\| \cdot \|_Z$, respectively. We define

$$\|w\|_{Y \cap Z} \equiv \|w\|_Y + \|w\|_Z,$$

for $w \in Y \cap Z$. Then $Y \cap Z$ is a Banach space with the norm $\| \cdot \|_{Y \cap Z}$. We use the following notation:

$$[z; Y, k](t) \equiv \sup_{\tau \geq t} (\tau^k \|z(\tau)\|_Y),$$

for a Y-valued function z of $t \in \mathbb{R}$.

We set for $t \in \mathbb{R}$,

$$\begin{gathered} U(t) \equiv e^{\frac{it}{2}\Delta}, \quad \Omega \equiv (1-\Delta)^{1/2}, \quad \omega \equiv (-\Delta)^{1/2} \\ K(t) \equiv \Omega^{-1} \sin \Omega t, \quad \dot{K}(t) \equiv \cos \Omega t, \\ \mathcal{L} \equiv i\partial_t + \frac{1}{2}\Delta, \quad \mathcal{K} \equiv \partial_t^2 - \Delta + 1, \quad \Box \equiv \partial_t^2 - \Delta. \end{gathered}$$

C denotes various constants, and they may differ from line to line, when it does not cause any confusion.

Let $(u_+, v_+, \dot{v}_+)$ be a final state. u_+ and $(v_+, \dot{v}_+)$ are the Schrödinger and the Klein-Gordon components, respectively. We introduce the following asymptotic profiles:

$$u_a = u_0 + u_1, \tag{1.2}$$

$$v_a = v_0 + v_1 + v_2, \tag{1.3}$$

where

$$
\begin{aligned}
u_0(t,x) =&(U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}u_+)(x)\\
=&\frac{1}{it}e^{i|x|^2/2t-iS(t,x/t)}\hat{u}_+\left(\frac{x}{t}\right)\\
u_1(t,x) =& \left(U(t)e^{-i|\cdot|^2/2t}e^{-iS(t,-i\nabla)}\frac{i|\cdot|^2}{2t}u_+\right)(x)\\
=&-\frac{1}{it}e^{i|x|^2/2t-iS(t,x/t)}\frac{i}{2t}\Delta\hat{u}_+\left(\frac{x}{t}\right)\\
S(t,x) =& \frac{1}{t}|\hat{u}_+(x)|^2,\\
v_0(t,x) =& (\dot{K}(t)v_+)(x)+(K(t)\dot{v}_+)(x),\\
v_1(t,x) =& -\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2,\\
v_2(t,x) =& -\frac{1}{t^3}\operatorname{Im}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}\Delta u_+\left(\frac{x}{t}\right)\right).
\end{aligned}
$$

The functions u_0 and v_0 are principal terms of the asymptotic profiles u_a and v_a, respectively. Note that u_0 is an approximate solution for the free Schrödinger equation and v_0 is the solution for the free Klein-Gordon-equation.

Throughout this paper, we assume that the space dimension is two.

The main result is as follows.

THEOREM. *Let $u_+ \in H^{2,8}$, $v_+ \in H^{4,3}$ and $\dot{v}_+ \in H^{3,3}$. Assume that*

$$\operatorname{supp}\hat{u}_+ \subset \{\xi \in \mathbb{R}^2; |\xi| \geq 1+a\} \tag{1.4}$$

for some $a > 0$, and that $\|u_+\|_{H^{2,8}}$ is sufficiently small. Let $1 < k < 2$. Then the equation (KGS) *has a unique solution (u,v) satisfying*

$$
\begin{aligned}
&u \in C(\mathbb{R};H^2), \quad v \in C(\mathbb{R};H^2)\cap C^1(\mathbb{R};H^1),\\
&\sup_{t\geq 2}(t^k\|u(t)-u_a(t)\|_{L^2} + t\|u(t)-u_a(t)\|_{\dot{H}^2}) < \infty,\\
&\sup_{t\geq 2}[t^k(\|v(t)-v_a(t)\|_{H^1} + \|\partial_t v(t)-\partial_t v_a(t)\|_{L^2})\\
&\qquad + t(\|v(t)-v_a(t)\|_{\dot{H}^1\cap\dot{H}^2} + \|\partial_t v(t)-\partial_t v_a(t)\|_{\dot{H}^1})] < \infty.
\end{aligned}
$$

In particular,

$$\begin{aligned}\|u(t) - U(t)u_+\|_{H^2} + \|v(t) - v_0(t)\|_{H^2}\\ + \|\partial_t v(t) - \partial_t v_0(t)\|_{H^1} \to 0,\end{aligned}$$

as $t \to +\infty$.

Furthermore for the equation (KGS)*, the wave operator*

$$W_+ \colon (u_+, v_+, \dot{v}_+) \mapsto (u(0), v(0), \partial_t v(0))$$

is well-defined.

A similar result holds for negative time.

Remark 1.1. In Theorem, no size restriction on the Klein-Gordon component $(v_+, \dot{v}_+)$ of the final state is assumed. On the other hand, we restrict the size of the Schrödinger component u_+ of the final state and the support of the Fourier transform $\hat{u}_+$ of it.

Remark 1.2. It is well-known that the equation (KGS) is globally well-posed in $C(\mathbb{R}; H^2) \oplus [C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; H^1)]$ (see Bachelot [1], Baillon and Chadam [2], Fukuda and Tsutsumi [3] and Hayashi and von Wahl [11]).

Remark 1.3. The restriction on the size of $\|u_+\|_{H^{2,8}}$ is independent of $a > 0$ introduced in (1.4), because we construct a solution (u, v) for the equation (KGS) on the time interval $[T, \infty)$ for sufficiently large $T > 0$, which depends on $a > 0$ and suitable norms of the final state, and extend it to $\mathbb{R}$ by the global well-posedness for the equation (KGS). (Note that the size of the final state depends on a in Ozawa and Tsutsumi [15]).

We briefly explain how to construct the approximate solution $(u_a, v_a) = (u_0 + u_1, v_0 + v_1 + v_2)$ for large time to the equation (KGS). (We explain the details in Section 3). In order to construct a solution (u, v) which approaches the profile (u_a, v_a) as $t \to \infty$ without any size restrictions on the Klein-Gordon component $(v_+, \dot{v}_+)$ of the final state, we have to find a profile (u_a, v_a) such that the functions $\mathcal{L}u_a - u_a v_a$ and $\mathcal{K}v_a + |u_a|^2$, which are the errors of the approximation (u_a, v_a) for the equation (KGS), decay as t^{-3} in H^2 and H^1, respectively. (See Proposition 2.1 below). We can calculate

$$\mathcal{L}u_a - u_a v_a = -u_a v_0 + (\mathcal{L}u_a - u_a v_1) - u_a v_2. \tag{1.5}$$

$$\begin{aligned} \mathcal{K}v_a + |u_a|^2 =& \mathcal{K}v_0 + (\mathcal{K}v_1 + |u_0|^2) \\ &+ (\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u_0} u_1)) + |u_1|^2. \end{aligned} \tag{1.6}$$

We define the principal term v_0 of v_a by the free solution for the Klein-Gordon equation. Then the first term in the right hand side of (1.6) vanishes. We define $u_a = u_0 + u_1$ by the asymptotics of the free solution $U(t)u_+$ for the Schrödinger equation with a phase correction. We do not determine a phase function S explicitly in this step. The first term $u_a v_0$ in the right hand side of (1.5) decays as t^{-1} in L^2. To overcome this difficulty, assuming the support restriction (1.4) for $\hat{u}_+$ as in [15], we use the difference between the propagation property of the waves u_a and v_0 and we obtain additional time decay rate $O(t^{-3})$ in H^2 of this term. We consider the second and the third terms in the right hand side of (1.6). Since $|u_0|^2$ and $\bar{u}_0 u_1$ decay as t^{-1} and t^{-2} in L^2, we construct the second correction term v_1 and the third one v_2 of v_a such that $\mathcal{K}v_1 + |u_0|^2$ and $\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u_0} u_1)$ decay as t^{-3} in H^1. We consider the second term in the right hand side of (1.5). We note that the function $u_a v_1$ in the right hand side of (1.5) decays as t^{-2} in L^2. Because v_1 is the product of t^{-2} and a function of x/t, we regard v_1 as a potential. Therefore we can apply the phase correction method to $u_a v_1$, and we determine the phase function S explicitly such that $\mathcal{L}u_a - u_a v_1$ decays as t^{-3} (faster than $u_a v_1$) in H^2. The other terms in the right hand sides of (1.5) and (1.6) decay as t^{-3} in H^2 and H^1, respectively.

The outline of this paper is as follows. In Section 2, we solve the final value problem for the equation (KGS) for the asymptotic profile satisfying suitable conditions (see Proposition 2.1). In Section 3, we determine an asymptotic profile satisfying the assumptions of above final value problem.

2. The Final Value Problem

In this section, we solve the final value problem, that is, the Cauchy problem at infinity, for the equation (KGS) of general form. Namely, for an asymptotic profile (A, B) satisfying suitable assumptions, we construct a unique solution (u, v) which approaches (A, B) as $t \to \infty$.

Let (A, B) be an asymptotic profile. Here A and B are complex and real valued, respectively. We introduce the following functions:

$$R_1[A, B] = \mathcal{L}A - AB, \tag{2.1}$$

$$R_2[A, B] = \mathcal{K}B + |A|^2. \tag{2.2}$$

Proposition 2.1. *Assume that there exist positive constants δ, L_0, L_1 and L_2 such that for $t \geq 1$,*

$$\begin{aligned}
\|A(t)\|_{W^2_\infty} &\leq \delta t^{-1}, \\
\|B(t)\|_{W^2_\infty} &\leq L_0 t^{-1}, \\
\|R_1[A, B](t)\|_{H^2} &\leq L_1 t^{-3}, \\
\|R_2[A, B](t)\|_{H^1} &\leq L_2 t^{-3},
\end{aligned}$$

and assume that $\delta > 0$ is sufficiently small. Let $1 < k < 2$. Then there exists a constant $T \geq 1$, depending only on δ, L_0, L_1 and L_2, such that the equation (KGS) *has a unique solution (u, v) satisfying*

$$u \in C([T, \infty); H^2), \quad v \in C([T, \infty); H^2) \cap C^1([T, \infty); H^1), \tag{2.3}$$

$$\sup_{t \geq T} (t^k \|u(t) - A(t)\|_{L^2} + t\|u(t) - A(t)\|_{\dot{H}^2}) < \infty, \tag{2.4}$$

$$\begin{aligned}
&\sup_{t \geq T} [t^k (\|v(t) - B(t)\|_{H^1} + \|\partial_t v(t) - \partial_t B(t)\|_{L^2}) \\
&\qquad + t(\|v(t) - B(t)\|_{\dot{H}^1 \cap \dot{H}^2} + \|\partial_t v(t) - \partial_t B(t)\|_{\dot{H}^1})] < \infty.
\end{aligned} \tag{2.5}$$

Remark 2.1. In Proposition 2.1, the asymptotic profile (A, B) is not determined explicitly. In Section 3, we construct the asymptotic profile satisfying the assumptions of Proposition 2.1.

Remark 2.2. In Proposition 2.1, we do not restrict the size of the positive constants L_0, L_1 and L_2, though the smallness on the size of the constant $\delta > 0$ is assumed.

Remark 2.3. By the global well-posedness of the equation (KGS), the solution (u, v) on the time interval $[T, \infty)$ for the equation (KGS) obtained in Proposition 2.1 can be extended all times.

We consider the following final value problem:

$$\begin{cases} i\partial_t w + \dfrac{1}{2}\Delta w = wz + wB + Az - R_1[A,B], \\ \partial_t^2 z - \Delta z + z = -|w|^2 - 2\,\mathrm{Re}(w\bar{A}) - R_2[A,B] \end{cases} \tag{2.6}$$

with the condition

$$\begin{cases} \|w(t)\|_{H^2} \to 0, \quad \text{as } t \to \infty, \\ \|z(t)\|_{H^2} + \|\partial_t z(t)\|_{H^1} \to 0, \quad \text{as } t \to \infty. \end{cases} \tag{2.7}$$

Remark 2.4. If we put $w = u - A$ and $z = v - B$, then the system (KGS) is equivalent to the system (2.6). Hence we solve the equation (2.6) instead of the equation (KGS).

Let $T > 0$. We introduce the following function space:

$$\begin{aligned} X_T = \{(w,z);\ w \in C([T,\infty);H^2),\ &z \in C([T,\infty);H^2), \\ &\partial_t z \in C([T,\infty);H^1), \\ &[w;L^2,k](T) + [\Delta w;L^2,1](T) \\ &+ [z;H^1,k](T) + [\partial_t z;L^2,k](T) \\ &+ [\nabla z;H^1,1](T) + [\nabla\partial_t z;L^2,1](T) < \infty\}. \end{aligned}$$

We solve the equation (2.6) in the space X_T. The proof of the existence argument in Proposition 2.1 is based on the energy estimates for the equation (2.6) and the compactness argument. The proof of the uniqueness argument is rather easy.

Proof of Proposition 2.1. To solve the final value problem (2.6)–(2.7), we consider the final value problem of the following regularized equation:

$$\begin{cases} i\partial_t w_{a,b} + \dfrac{1}{2}\Delta w_{a,b} = (1+bt)^{-5}\rho_a * [(\rho_a * w_{a,b})(\rho_a * z_{a,b}) \\ \qquad\qquad + (\rho_a * w_{a,b})(\rho_a * B) + (\rho_a * A)(\rho_a * z_{a,b}) \\ \qquad\qquad - \rho_a * R_1[A,B]], \\ \partial_t^2 z_{a,b} - \Delta z_{a,b} + z_{a,b} \\ \qquad = -(1+bt)^{-5}\rho_a * [|\rho_a * w_{a,b}|^2 \\ \qquad\quad - 2\mathrm{Re}((\rho_a * w_{a,b})(\overline{\rho_a * A})) - \rho_a * R_2[A,B]] \end{cases} \tag{2.8}$$

with the condition

$$
\begin{cases} \|w_{a,b}(t)\|_{H^2} \to 0, \quad \text{as } t \to \infty, \\ \|z_{a,b}(t)\|_{H^2} + \|\partial_t z_{a,b}(t)\|_{H^1} \to 0, \quad \text{as } t \to \infty \end{cases} \tag{2.9}
$$

for $0 < a < 1$ and $0 < b < 1$. Here $\rho_a(x) = a^{-3}\rho(x/a)$ for $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\|\rho\|_{L^1} = 1$ and $\rho(x) = \rho(-x)$.

Using the contraction mapping principle, we easily see that for any $0 < a, b < 1$, there exists a constant $\tilde{T}_{a,b} > 0$ such that the equation (2.8) has a unique solution $(w_{a,b}, z_{a,b})$ satisfying

$$
w_{a,b} \in \bigcap_{j=1}^{\infty} C^2([\tilde{T}_{a,b}, \infty); H^j), \tag{2.10}
$$

$$
z_{a,b} \in \bigcap_{j=1}^{\infty} C^2([\tilde{T}_{a,b}, \infty); H^j), \tag{2.11}
$$

$$
\sup_{t \ge \tilde{T}_{a,b}} \left[(1+bt)^4 \sum_{|\alpha|+j \le 2} \|\partial_x^\alpha \partial_t^j w_{a,b}(t)\|_{L^2} \right] < \infty, \tag{2.12}
$$

$$
\sup_{t \ge \tilde{T}_{a,b}} \left[(1+bt)^4 \sum_{|\alpha|+j \le 2} \|\partial_x^\alpha \partial_t^j z_{a,b}(t)\|_{L^2} \right] < \infty. \tag{2.13}
$$

Since the initial value problem of the equation (2.8) is time globally solvable, we can extend the above solution $(w_{a,b}, z_{a,b})$ to the time interval $[0, \infty)$. We note that we do not assume the smallness of δ, L_0, L_1 and L_2 here.

We set

$$
\begin{aligned} F_{a,b}(t) \equiv & [w_{a,b}; L^2, k](t) + [\Delta w_{a,b}; L^2, 1](t) \\ & + [z_{a,b}; H^1, k](t) + [\partial_t z_{a,b}; L^2, k](t) \\ & + [\nabla z_{a,b}; H^1, 1](t) + [\nabla \partial_t z_{a,b}; L^2, 1](t) \end{aligned} \tag{2.14}
$$

In order to estimate $F_{a,b}$ independent of a and b, we have to derive the various a priori estimates of $w_{a,b}$ and $z_{a,b}$ independent of a and b. Since the detailed proof for the equation (2.8) is rather complicated and the regularizing factors $\rho_a *$ and $(1+bt)^{-5}$ cause no trouble, we describe only the formal calculations for the equation (2.6) as in [19].

Let $T \geq 1$ be a constant determined later, and let (w, z) be the solution for the equation (2.6) on $[T, \infty)$, which are smooth and decay rapidly enough as $t \to \infty$. For $t \geq T$, we put

$$\begin{aligned} F(t) \equiv & [w; L^2, k](t) + [\Delta w; L^2, 1](t) + [z; H^1, k](t) \\ & + [\partial_t z; L^2, k](t) + [\nabla z; H^1, 1](t) + [\nabla \partial_t z; L^2, 1](t) \end{aligned}$$

To estimate $F(T)$, we derive the various a priori estimates for w and z. Throughout the proof of this proposition, we set

$$L = \max\{L_0, L_1, L_2\}.$$

We first evaluate w and Δw. Let $t \geq T$. From the equality

$$-\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 = -\operatorname{Im}(A(t)z(t) + R_1[A, B](t), w(t)),$$

we obtain

$$\begin{aligned} -\frac{d}{dt}\|w(t)\|_{L^2} \leq & \|A(t)\|_{L^\infty}\|z(t)\|_{L^2} + \|R_1[A, B](t)\|_{L^2} \\ \leq & \delta t^{-k-1}[z; L^2, k](T) + Lt^{-3}. \end{aligned}$$

Integrating over the interval $[t, \infty)$, we see

$$\|w(t)\| \leq \delta t^{-k}[z; L^2, k](T) + Lt^{-2},$$

and hence we have

$$\begin{aligned} [w; L^2, k](T) \leq & \delta[z; L^2, k](T) + LT^{-(2-k)} \\ \leq & \delta F(T) + LT^{-(2-k)}. \end{aligned} \tag{2.15}$$

By operating Δ both side of the first equation in the system (2.6), we have

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\|\Delta w(t)\|_{L^2}^2 = & -\operatorname{Re}(\partial_t \Delta w(t), \Delta w(t)) \\ = & -\operatorname{Im}(2\nabla w(t) \cdot \nabla(z(t) + B(t)) + w(t)\Delta(z(t) + B(t)) \\ & + \Delta(A(t)z(t)) - \Delta R_1[A, B](t), \Delta w(t)). \end{aligned}$$

By the above equality, Hölder's inequality and the Sobolev embedding theorem, we see

$$
\begin{aligned}
&-\frac{d}{dt}\|\Delta w(t)\|_{L^2}\\
\le& C(\|\nabla w(t)\|_{L^4}\|\nabla z(t)\|_{L^4}+\|w(t)\|_{L^\infty}\|\Delta z(t)\|_{L^2}\\
&+\|\nabla w(t)\|_{L^2}\|\nabla B(t)\|_{L^\infty}+\|w(t)\|_{L^2}\|\Delta B(t)\|_{L^\infty}\\
&+\|A(t)\|_{W^2_\infty}\|z(t)\|_{H^2}+\|\Delta R_1[A,B](t)\|_{L^2})\\
\le& C(\|\omega^{3/2}w(t)\|_{L^2}\|\omega^{3/2}z(t)\|_{L^2}+\|w(t)\|_{H^{3/2}}\|\Delta z(t)\|_{L^2}\\
&+\|\nabla w(t)\|_{L^2}\|\nabla B(t)\|_{L^\infty}+\|w(t)\|_{L^2}\|\Delta B(t)\|_{L^\infty}\\
&+\|A(t)\|_{W^2_\infty}\|z(t)\|_{H^2}+\|\Delta R_1[A,B](t)\|_{L^2})\\
\le& C([w;L^2,k](T)^{1/4}[\Delta w;L^2,1](T)^{3/4}\\
&\quad\times[z;H^1,k](T)^{1/2}[\nabla z;H^1,1](T)^{1/2}t^{-3k/4-5/4}\\
&+([w;L^2,k](T)+[w;L^2,k](T)^{1/4}[\Delta w;L^2,1](T)^{3/4})\\
&\quad\times[\nabla z;H^1,1](T)t^{-k/4-9/4}\\
&+L([w;L^2,k](T)+[w;L^2,k](T)^{1/2}[\Delta w;L^2,1](T)^{1/2})t^{-k/2-3/2}\\
&+\delta([z;H^1,k](T)+[\nabla z;H^1,1](T))t^{-2}+Lt^{-3}).
\end{aligned}
$$

Integrating over the interval $[t,\infty)$, we have

$$
\begin{aligned}
\|\Delta w(t)\|_{L^2}\le& C(F(T)^2(t^{-3k/4-1/4}+t^{-k/4-5/4})\\
&+F(T)(Lt^{-k/2-1/2}+\delta t^{-1})+Lt^{-2}).
\end{aligned}
$$

Noting $1<k<2$, we obtain

$$
\begin{aligned}
[\Delta w;L^2,1](T)\le& C(F(T)^2T^{-(k-1)/4}\\
&+F(T)(LT^{-(k-1)/2}+\delta)+LT^{-1}).
\end{aligned}
\tag{2.16}
$$

We next estimate z and ∇z. Let $t\ge T$. By the energy estimate, Hölder's inequality and the Sobolev embedding theorem, we see

$$
\begin{aligned}
&\|z(t)\|_{H^1}+\|\partial_t z(t)\|_{L^2}\\
\le& C\int_t^\infty \||w(s)|^2+2\,\mathrm{Re}(w(s)\overline{A(s)})+R_2[A,B](s)\|_{L^2}\,ds
\end{aligned}
$$

$$\begin{aligned}
&\le C\int_t^\infty (\|w(s)\|_{L^4}^2 + \|w(s)\|_{L^2}\|A(s)\|_{L^\infty} + \|R_2[A,B](s)\|_{L^2})\,ds \\
&\le C\int_t^\infty (\|\omega^{1/2}w(s)\|_{L^2}^2 + \|w(s)\|_{L^2}\|A(s)\|_{L^\infty} + \|R_2[A,B](s)\|_{L^2})\,ds \\
&\le C\int_t^\infty ([w;L^2,k](T)^{3/2}[\Delta w;L^2,1](T)^{1/2}s^{-3k/2-1/2} \\
&\qquad + \delta[w;L^2,k](T)s^{-k-1} + Ls^{-3})\,ds \\
&\le C([w;L^2,k](T)^{3/2}[\Delta w;L^2,1](T)^{1/2}t^{-(3k/2-1/2)} \\
&\quad + \delta[w;L^2,k](T)t^{-k} + Lt^{-2}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[z;H^1,k](T) + [\partial_t z;L^2,k](T) \\
&\quad \le C([w;L^2,k](T)^{3/2}[\Delta w;L^2,1](T)^{1/2}T^{-(k-1)/2} \\
&\qquad + \delta[w;L^2,k](T) + LT^{-(2-k)}) \\
&\quad \le C(F(T)^2T^{-(k-1)/2} + \delta F(T) + LT^{-(2-k)}).
\end{aligned} \tag{2.17}$$

In the same way as above, we have

$$\begin{aligned}
&\|\nabla z(t)\|_{H^1} + \|\nabla\partial_t z(t)\|_{L^2} \\
&\quad \le C([w;L^2,k](T)[\Delta w;L^2,1](T)t^{-k} + \delta([w;L^2,k](T) \\
&\qquad + [w;L^2,k](T)^{1/2}[\Delta w;L^2,1](T)^{1/2})t^{-(k+1)/2} + Lt^{-2}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&[\nabla z;H^1,1](T) + [\nabla\partial_t z;L^2,1](T) \\
&\qquad \le C(F(T)^2T^{-(k-1)} + \delta F(T) + LT^{-1}).
\end{aligned} \tag{2.18}$$

From the estimates (2.15)–(2.18), we see

$$\begin{aligned}
F(T) \le\, & C((1+L)T^{-(k-1)/4} + LT^{-(2-k)} + \delta) \\
&\times (F(T)^2 + F(T) + 1).
\end{aligned} \tag{2.19}$$

The above proof of (2.19) is rather formal. But exactly in the same way as above, we can show that there exists a constant $C>0$ independent of a and b such that

$$\begin{aligned}
F_{a,b}(T) \le\, & C((1+L)T^{-(k-1)/4} + LT^{-(2-k)} + \delta) \\
&\times (F_{a,b}(T)^2 + F_{a,b}(T) + 1).
\end{aligned} \tag{2.20}$$

where $F_{a,b}$ is defined by (2.14). Note the behavior of the positive function $f(p) = p/(p^2 + p + 1)$ for $p \geq 0$. In particular, f has the maximum $1/3$ at $p = 1$. We also remark that according to (2.10)–(2.13), $F_{a,b}(t) \to 0$ as $t \to \infty$. Therefore recalling $1 < k < 2$, we see that if $\delta > 0$ is sufficiently small and $T \geq 1$, depending only on δ and L, is sufficiently large such that

$$C((1+L)T^{-(k-1)/4} + LT^{-(2-k)} + \delta) \leq \frac{1}{3},$$

where the constant C appears in the estimate (2.20), then

$$F_{a,b}(T) \leq 1. \tag{2.21}$$

Here we note that the estimate (2.21) is independent of a and b. If $a \to 0$ and $b \to 0$, then the estimate (2.21) and the standard compactness argument show that there exists a solution $(w, z) \in X_T$ for the equation (2.6) for sufficiently small $\delta > 0$ and sufficiently large $T \geq 1$.

It remains to prove the uniqueness. Let $\delta > 0$ be sufficiently small and let $T \geq 1$ be sufficiently large as above. Let (w_1, z_1) and (w_2, z_2) be solutions for the equation (2.6) in X_T. They satisfy the equation

$$\begin{cases} i\partial_t(w_1 - w_2) + \dfrac{1}{2}\Delta(w_1 - w_2) = (w_1 - w_2)z_1 + (w_1 - w_2)B \\ \qquad\qquad\qquad\qquad + w_2(z_1 - z_2) + A(z_1 - z_2), \\ \partial_t^2(z_1 - z_2) - \Delta(z_1 - z_2) + (z_1 - z_2) \\ \qquad = -(|w_1| + |w_2|)(|w_1| - |w_2|) - 2\,\mathrm{Re}((w_1 - w_2)\bar{A}). \end{cases} \tag{2.22}$$

Let $t \geq T$.

In the same way as in the estimate (2.15), we have

$$\begin{aligned} -&\frac{d}{dt}\|w_1(t) - w_2(t)\|_{L^2} \\ \leq& \|(w_2(t) + A(t))(z_1(t) - z_2(t))\|_{L^2} \\ \leq& (\|w_2(t)\|_{L^\infty} + \|A(t)\|_{L^\infty})\|z_1(t) - z_2(t)\|_{L^2} \\ \leq& C(\|w_2(t)\|_{H^{3/2}} + \|A(t)\|_{L^\infty})\|z_1(t) - z_2(t)\|_{L^2} \\ \leq& C([w_2; L^2, k](T) + [w_2; L^2, k](T)^{1/4}[\Delta w_2; L^2, 1](T)^{3/4}) \\ &\quad \times [z_1 - z_2; H^1, k](T)t^{-5k/4-3/4} + \delta[z_1 - z_2; H^1, k](T)t^{-k-1}. \end{aligned}$$

Integrating above inequality over the interval $[t,\infty)$, we obtain

$$\begin{aligned}
&\|w_1(t)-w_2(t)\|_{L^2}\\
&\quad\le C\{([w_2;L^2,k](T)+[w_2;L^2,k](T)^{1/4}[\Delta w_2;L^2,1](T)^{3/4})t^{-(5k/4-1/4)}\\
&\qquad+\delta t^{-k}\}[z_1-z_2;H^1,k](T).
\end{aligned}$$

This implies

$$\begin{aligned}
&[w_1-w_2;L^2,k](T)\\
&\quad\le C\{([w_2;L^2,k](T)+[w_2;L^2,k](T)^{1/4}[\Delta w_2;L^2,1](T)^{3/4})\\
&\qquad\times T^{-(k-1)/4}+\delta\}[z_1-z_2;H^1,k](T).
\end{aligned}\tag{2.23}$$

In the same way as in the estimate (2.17), we see

$$\begin{aligned}
&\|z_1(t)-z_2(t)\|_{H^1}\\
&\quad\le C\int_t^\infty(\|w_1(s)\|_{L^\infty}+\|w_2(s)\|_{L^\infty}+\|A(s)\|_{L^\infty})\\
&\qquad\times\|w_1(s)-w_2(s)\|_{L^2}\,ds\\
&\quad\le C\int_t^\infty(\|w_1(s)\|_{H^{3/2}}+\|w_2(s)\|_{H^{3/2}}+\|A(s)\|_{L^\infty})\\
&\qquad\times\|w_1(s)-w_2(s)\|_{L^2}\,ds\\
&\quad\le C\int_t^\infty\{([w_1;L^2,k](T)+[w_1;L^2,k](T)^{1/4}[\Delta w_1;L^2,1](T)^{3/4}\\
&\qquad+[w_2;L^2,k](T)+[w_2;L^2,k](T)^{1/4}[\Delta w_2;L^2,1](T)^{3/4})s^{-5k/4-3/4}\\
&\qquad+\delta t^{-k-1}\}[w_1-w_2;L^2,k](T)\,ds\\
&\quad\le C\{([w_1;L^2,k](T)+[w_1;L^2,k](T)^{1/4}[\Delta w_1;L^2,1](T)^{3/4}\\
&\qquad+[w_2;L^2,k](T)+[w_2;L^2,k](T)^{1/4}[\Delta w_2;L^2,1](T)^{3/4})t^{-(5k/4-1/4)}\\
&\qquad+\delta t^{-k}\}[w_1-w_2;L^2,k](T).
\end{aligned}$$

This implies

$$
(2.24)\quad
\begin{aligned}
&[z_1 - z_2; H^1, k](T) \\
\le & C\{([w_1; L^2, k](T) + [w_1; L^2, k](T)^{1/4}[\Delta w_1; L^2, 1](T)^{3/4} \\
&\quad + [w_2; L^2, k](T) + [w_2; L^2, k](T)^{1/4}[\Delta w_2; L^2, 1](T)^{3/4}) \\
&\quad \times T^{-(k-1)/4} + \delta\}[w_1 - w_2; L^2, k](T).
\end{aligned}
$$

The estimates (2.23) and (2.24) yield

$$
\begin{aligned}
&[w_1 - w_2; L^2, k](T) + [z_1 - z_2; H^1, k](T) \\
&\le C\{([w_1; L^2, k](T) + [w_1; L^2, k](T)^{1/4}[\Delta w_1; L^2, 1](T)^{3/4} \\
&\qquad + [w_2; L^2, k](T) + [w_2; L^2, k](T)^{1/4}[\Delta w_2; L^2, 1](T)^{3/4})T^{-(k-1)/4} \\
&\qquad + \delta\}([w_1 - w_2; L^2, k](T) + [z_1 - z_2; H^1, k](T)).
\end{aligned}
$$

Since $1 < k < 2$ and since $[w_j; L^2, k](T)$ and $[\Delta w_j; L^2, 1](T)$ $(j = 1, 2)$ do not increase with respect to T, if $\delta > 0$ is sufficiently small and $T \ge 1$ is sufficiently large, then

$$[w_1 - w_2; L^2, k](T) + [z_1 - z_2; H^1, k](T) \le 0.$$

From this, we see that $(w_1, z_1) = (w_2, z_2)$. Therefore if $\delta > 0$ is sufficiently small and $T \ge 1$ is sufficiently large, then the solution for the equation (2.6) is unique in X_T.

Recalling Remark 2.4, we see that if $\delta > 0$ is sufficiently small and $T \ge 1$, which depends only on δ and L, is sufficiently large, then there exists a unique solution (u, v) for the equation (KGS) satisfying the conditions (2.3)–(2.5). This completes the proof of this proposition. □

3. Asymptotics and Proof of Theorem

In this section, by constructing an asymptotic profile (u_a, v_a) satisfying the assumptions of Proposition 2.1 under suitable conditions on the final state, we prove Theorem. Let $(u_+, v_+, \dot{v}_+)$ be a final state.

We find an asymptotic profile of the form $(u_a, v_a) = (u_0 + u_1, v_0 + v_1 + v_2)$. u_0 and v_0 are the principal terms of u_a and v_a, respectively. $(u_0 \gg u_1$,

$v_0 \gg v_1 \gg v_2$). It is natural to expect that (u_0, v_0) is the free profile or the modified free profile. Let R_1 and R_2 be defined by (2.1) and (2.2), respectively. Then

$$\begin{aligned} R_1[u_a, v_a] =& \mathcal{L}u_a - u_a v_a \\ =& - u_a v_0 + (\mathcal{L}u_a - u_a v_1) - u_a v_2. \end{aligned} \tag{3.1}$$

$$\begin{aligned} R_2[u_a, v_a] =& \mathcal{K}v_a + |u_a|^2 \\ =& \mathcal{K}v_0 + (\mathcal{K}v_1 + |u_0|^2) \\ & + (\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u_0}u_1)) + |u_1|^2. \end{aligned} \tag{3.2}$$

We set

$$v_0(t,x) = (\dot{K}(t)v_+)(x) + (K(t)\dot{v}_+)(x).$$

v_0 is a solution of the free Klein-Gordon equation with initial data $(v_+, \dot{v}_+)$. Namely, the first term $\mathcal{K}v_0$ in the right hand side of the equation (3.2) vanishes. The time decay estimates of v_0 (Lemmas 3.1 and 3.2 below) are well-known. (See, e.g., Lemmas 2.2 and 2.3 in Ozawa and Tsutsumi [15]).

LEMMA 3.1. *There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\|v_0(t)\|_{H^2} \leq \|v_+\|_{H^2} + \|\dot{v}_+\|_{H^1},$$
$$\|v_0(t)\|_{W^2_\infty} \leq C(\|v_+\|_{H^{4,2}} + \|\dot{v}_+\|_{H^{3,2}})t^{-1}.$$

LEMMA 3.2. *Let $a > 0$. There exists a constant M'_a depending on a such that for $t \geq 1$,*

$$\sum_{|\alpha|\leq 2} \|\partial^\alpha v_0(t)\|_{L^\infty(|x|\geq(1+a)t)} \leq M'_a(\|v_+\|_{H^{4,3}} + \|\dot{v}_+\|_{H^{3,3}})t^{-3}.$$

REMARK 3.1. According to Lemma 3.2, we see that v_0 decays more rapidly with respect to t outside the light cone.

We consider the second term $\mathcal{K}v_1 + |u_0|^2$ in the right hand side of the equation (3.2). Because u_0 is the modified free profile for the Schrödinger equation, we may consider that $|u_0|^2$ behaves like $t^{-2}|\hat{u}_+(x/t)|^2$, and

$\||u_0(t)|^2\|_{L^2}$ decays as $O(t^{-1})$. This is not sufficient to satisfy the assumption on R_2 of Proposition 2.1. In order to obtain improved time decay estimates of R_2, we choose the second correction term v_1 of v_a such that $\mathcal{K}v_1 + t^{-2}|\hat{u}_+(x/t)|^2$ decays faster than $t^{-2}|\hat{u}_+(x/t)|^2$. We put

$$v_1(t,x) = -\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2,$$

Then

$$\mathcal{K}v_1(t,x) + \frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2 = -\Box\left(\frac{1}{t^2}\left|\hat{u}_+\left(\frac{x}{t}\right)\right|^2\right).$$

By a direct calculation, we have the following lemma.

Lemma 3.3. *Let $k = 0, 1, 2$. There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\|\omega^k v_1(t)\|_{L^2} \leq C\|u_+\|^2_{H^{0,2}} t^{-k-1},$$
$$\sum_{|\alpha|=k} \|\partial^\alpha v_1(t)\|_{L^\infty} \leq C\|u_+\|^2_{H^{0,4}} t^{-k-2}$$
$$\left\|\mathcal{K}v_1(t) + \frac{1}{t^2}\left|\hat{u}_+\left(\frac{\cdot}{t}\right)\right|^2\right\|_{H^1} \leq C\|u_+\|^2_{H^{2,4}} t^{-3}.$$

We next consider the second term $\mathcal{L}u_a - u_a v_1$ in the right hand side of (3.1). Because u_0 is the modified free profile for the Schrödinger equation, $\|u_0(t)v_1(t)\|_{L^2}$ decays as $O(t^{-2})$. This is not sufficient to satisfy the assumption on R_1 of Proposition 2.1. In order to obtain improved time decay estimates of R_1, we choose the Schrödinger part $u_a = u_0 + u_1$ of the asymptotic profile such that $\mathcal{L}u_a - u_a v_1$ decays faster than $u_a v_1$. We use the method of phase correction. We write

$$u_a = MDe^{-iS}W_a = MDe^{-iS}(W_0 + W_1),$$

where $W_a = W_0 + W_1$ is a complex amplitude, S is a real phase and M and D are the following operators:

$$(Mf)(t,x) = e^{i|x|^2/2t}f(x), \quad (Dg)(t,x) = \frac{1}{it}g\left(t, \frac{x}{t}\right).$$

It is well-known that

$$U(t) = M(t)D(t)\mathcal{F}M(t). \tag{3.3}$$

By a direct calculation,

$$\begin{aligned} &\mathcal{L}u_a - u_a v_1 \\ &= MDe^{-iS}\Big[i\partial_t W_0 + \left(i\partial_t W_1 + \frac{1}{2t^2}\Delta W_0 \right) \\ &\quad + (\partial_t S - (D_0^{-1}v_1))W_a + \frac{1}{2t^2}\Delta W_1 \\ &\quad - \frac{i}{2t^2}(2\nabla S \cdot \nabla W_a + W_a \Delta S) - \frac{1}{2t^2}|\nabla S|^2 W_a \Big], \end{aligned} \tag{3.4}$$

where D_0 and D_0^{-1} are the following operators:

$$(D_0 g)(t,x) = g\left(t, \frac{x}{t}\right), \quad (D_0^{-1}g)(t,x) = g(t,tx).$$

In view of the relation (3.3), we put

$$W_0(t,x) = \hat{u}_+(x).$$

Since W_0 is independent of t, the first term in $[\dots]$ of the right hand side in the equality (3.4) vanishes.

Next we set

$$S(t,x) = \frac{1}{t}|\hat{u}_+(x)|^2$$

so that

$$\partial_t S(t,x) = (D_0^{-1}v_1)(t,x) = -\frac{1}{t^2}|\hat{u}_+(x)|^2.$$

Therefore the third term in $[\dots]$ of the right hand side in the equality (3.4) vanishes.

We consider the second in $[\dots]$ of the right hand side in the equality (3.4). Since the L^2-norms of $(2t)^{-2}\Delta W_0$ decays as $O(t^{-2})$, this term does not satisfy the assumptions on R_1 in Proposition 2.1. We determine

$$W_1(t,x) = -\frac{i}{2t}\Delta\hat{u}_+(x)$$

so that

$$i\partial_t W_1 + \frac{1}{2t^2}\Delta W_0 = 0.$$

Namely the second in $[\dots]$ of the right hand side in the equality (3.4) vanishes.

Finally we determine

$$\begin{gathered} u_0 = MDe^{-iS}W_0 = \frac{1}{it}e^{i|x|^2/2t - iS(t,x/t)}\hat{u}_+\left(\frac{x}{t}\right), \\ u_1 = MDe^{-iS}W_1 = -\frac{1}{it}e^{i|x|^2/2t - iS(t,x/t)}\frac{i}{2t}\Delta\hat{u}_+\left(\frac{x}{t}\right), \\ u_a = u_0 + u_1. \end{gathered}$$

Then we have

$$\begin{aligned} &\mathcal{L}u_a - u_a v_1 \\ &= MDe^{-iS}\left[\frac{1}{2t^2}\Delta W_1 - \frac{i}{2t^2}(2\nabla S\cdot\nabla W_a + W_a\Delta S)\right. \\ &\quad\left. - \frac{1}{2t^2}|\nabla S|^2 W_a\right]. \end{aligned} \tag{3.5}$$

By the definitions of the functions W_0, W_1 and S, the equality (3.5) and Hölder's inequality and the Sobolev embedding theorem, we have the following.

LEMMA 3.4. *Assume that* $\|u_+\|_{H^{2,8}} \le 1$. *There exists a constant* $C > 0$ *such that for* $t \ge 1$,

$$\begin{gathered} \|u_0(t)\|_{H^2} \le C\|u_+\|_{H^{2,4}}, \\ \|u_0(t)\|_{W^2_\infty} \le C\|u_+\|_{H^{2,6}}t^{-1}, \\ \|u_1(t)\|_{H^2} \le C\|u_+\|_{H^{2,6}}t^{-1}, \\ \|u_1(t)\|_{W^2_\infty} \le C\|u_+\|_{H^{2,8}}t^{-2}, \\ \|\mathcal{L}u_a(t) - u_a(t)v_1(t)\|_{H^2} \le C\|u_+\|_{H^{2,8}}t^{-3} \end{gathered}$$

We consider the third term $\mathcal{K}v_2 + 2\operatorname{Re}(\bar{u_0}u_1)$ in the right hand side of the equation (3.2). By the definitions of u_0 and u_1, $\|\bar{u_0}u_1\|_{L^2}$ decays as $O(t^{-2})$.

This is not sufficient to satisfy the assumption on R_2 of Proposition 2.1. In order to obtain improved time decay estimates of R_2, we choose the third correction term v_2 of v_a such that $\mathcal{K}v_2 + 2\,\mathrm{Re}(\bar{u_0}u_1)$ decays faster than $2\,\mathrm{Re}(\bar{u_0}u_1)$. We put

$$v_2(t,x) = -2\,\mathrm{Re}(\bar{u_0}u_1) = -\frac{1}{t^3}\,\mathrm{Im}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}\Delta\hat{u}_+\left(\frac{x}{t}\right)\right).$$

Then

$$\mathcal{K}v_2(t,x) + 2\,\mathrm{Re}(\bar{u_0}u_1) = -\Box\left[\frac{1}{t^3}\,\mathrm{Im}\left(\overline{\hat{u}_+\left(\frac{x}{t}\right)}\Delta\hat{u}_+\left(\frac{x}{t}\right)\right)\right].$$

By a direct calculation, we have the following lemma.

LEMMA 3.5. *Let $k = 0, 1, 2$. There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\|\omega^k v_2(t)\|_{L^2} \leq C\|u_+\|_{H^{0,4}}^2 t^{-k-2},$$
$$\sum_{|\alpha|=k} \|\partial^\alpha v_2(t)\|_{L^\infty} \leq C\|u_+\|_{H^{0,4}}^2 t^{-k-3}$$
$$\|\mathcal{K}v_2(t) + 2\,\mathrm{Re}(\overline{u_0(t)}u_1(t))\|_{H^1} \leq C\|u_+\|_{H^{2,6}}^2 t^{-3}.$$

Finally we consider the first term $u_a v_0$ in the right hand side of the equality (3.1). From Lemmas 3.1 and 3.4, $\|u_a(t)v_0(t)\|_{L^2}$ decays as $O(t^{-1})$ generally. Since this is not sufficient to satisfy the assumptions on R_1 of Proposition 2.1, we have to obtain additional time decay estimate for $u_a v_0$. Under the assumption (1.4) on the support of the Fourier transform $\hat{u}_+$ of the Schrödinger data u_+ as in [15], the improved time decay estimate follows from Lemma 3.2.

LEMMA 3.6. *Let $a > 0$. Assume that the condition* (1.4) *is satisfied. Then there exists a constant $M_a > 0$ depending on a such that for $t \geq 1$,*

$$\|u_a(t)v_0(t)\|_{H^2} \leq M_a\|u_+\|_{H^{2,6}}(\|v_+\|_{H^{4,3}} + \|\dot{v}_+\|_{H^{3,3}})t^{-3}.$$

From Lemmas 3.1, 3.3, 3.4, 3.5 and 3.6, we have time decay estimates for the functions (u_a, v_a), $R_1[u_a, v_a]$ and $R_2[u_a, v_a]$.

LEMMA 3.7. *Let $a > 0$ and let M_a be the constant introduced in Lemma 3.6. Assume that the condition* (1.4) *is satisfied and that $\|u_+\|_{H^{2,8}} \leq 1$. Then there exists a constant $C > 0$ such that for $t \geq 1$,*

$$\begin{gathered}
\|u_a(t)\|_{W^2_\infty} \leq C\|u_+\|_{H^{2,8}} t^{-1}, \\
\|v_a(t)\|_{W^2_\infty} \leq C(\|u_+\|_{H^{2,8}} + \|v_+\|_{H^{4,2}} + \|\dot{v}_+\|_{H^{3,2}})t^{-1}, \\
\|R_1[u_a, v_a](t)\|_{H^2} \leq C(1 + M_a)(\|u_+\|_{H^{2,8}} + \|v_+\|_{H^{4,3}} + \|\dot{v}_+\|_{H^{3,3}})t^{-3}, \\
\|R_2[u_a, v_a](t)\|_{H^1} \leq C\|u_+\|_{H^{2,8}} t^{-3}.
\end{gathered}$$

PROOF OF THEOREM. We assume that all the assumptions of Theorem are satisfied. If we put

$$\begin{gathered}
(A, B) = (u_a, v_a), \\
\delta = C\|u_+\|_{H^{2,8}}, \\
L_0 = C(\|u_+\|_{H^{2,8}} + \|v_+\|_{H^{4,2}} + \|\dot{v}_+\|_{H^{3,2}}), \\
L_1 = C(1 + M_a)(\|u_+\|_{H^{2,8}} + \|v_+\|_{H^{4,3}} + \|\dot{v}_+\|_{H^{3,3}}), \\
L_2 = C\|u_+\|_{H^{2,8}},
\end{gathered}$$

where $C > 0$ and M_a are the constants introduced in Lemma 3.7, then the assumptions in Proposition 2.1 are satisfied. By Proposition 2.1, if $\|u_+\|_{H^{2,8}}$ is sufficiently small and if $T \geq 1$, which depends on $a > 0$, $\|u_+\|_{H^{2,8}}$, $\|v_+\|_{H^{4,3}}$ and $\|\dot{v}_+\|_{H^{3,3}}$, is sufficiently large, then there exists a unique solution (u, v) satisfying

$$\begin{aligned}
&u \in C([T, \infty); H^2), \quad v \in C([T, \infty); H^2) \cap C^1([T, \infty); H^1), \\
&\quad \sup_{t \geq T}(t^k \|u(t) - u_a(t)\|_{L^2} + t\|u(t) - u_a(t)\|_{\dot{H}^2}) < \infty, \\
&\sup_{t \geq T}[t^k(\|v(t) - v_a(t)\|_{H^1} + \|\partial_t v(t) - \partial_t v_a(t)\|_{L^2}) \\
&\qquad + t(\|v(t) - v_a(t)\|_{\dot{H}^1 \cap \dot{H}^2} + \|\partial_t v(t) - \partial_t v_a(t)\|_{\dot{H}^1})] < \infty.
\end{aligned}$$

Since the equation (KGS) is globally well-posed in $C(\mathbb{R}; H^2) \oplus [C(\mathbb{R}; H^2) \cap C^1(\mathbb{R}; H^1)]$ (see Bachelot [1], Baillon and Chadam [2], Fukuda and Tsutsumi

[3] and Hayashi and von Wahl [11]), the unique solution (u, v) on the time interval $[T, \infty)$, which is obtained above, can be extended to all times. This completes the proof of Theorem. □

Acknowledgement. The author would like to thank the referee for his valuable suggestions.

References

[1] Bachelot, A., Probléme de Cauchy pour des systèms hyperboliques semi-linéaires, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 453–478.

[2] Baillon, J. B. and J. M. Chadam, *The Cauchy problem for the coupled Schrödinger-Klein-Gordon equations*, in "Contemporary Developments in Continuum Mechanics and Partial Differential Equations", (G. M. de La Penha and L. A. Medeiros Eds), North-Holland, Amsterdam (1989), 37–44.

[3] Fukuda, I. and M. Tsutsumi, On coupled Klein-Gordon-Schrödinger equations II, J. Math. Anal. Appl. **66** (1978), 358–378.

[4] Ginibre, J. and T. Ozawa, Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \geq 2$, Comm. Math. Phys. **151** (1993), 619–645.

[5] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for some Hartree type equations I, Rev. Math. Phys. **12** (2000), 361–429.

[6] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for some Hartree type equations II, Ann. Henri Poincaré **1** (2000), 753–800.

[7] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for some Hartree type equations III, Gevrey spaces and low dimensions, J. Differential Equations **175** (2001), 415–501.

[8] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for the Wave-Schrödinger system, Ann. Henri Poincaré **3** (2002), 537–612.

[9] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for the Maxwell-Schrödinger system I. The case of vanishing asymptotic magnetic field, Comm. Math. Phys. **236** (2003), 395–448.

[10] Ginibre, J. and G. Velo, Long range scattering and modified wave operators for the Wave-Schrödinger system II, to appear in Ann. Henri Poincaré.

[11] Hayashi, N. and W. von Wahl, On the global strong solutions of coupled Klein-Gordon-Schrödinger equations, J. Math. Soc. Japan **39** (1987), 489–497.

[12] Nakanishi, K., Modified wave operators for the Hartree equation with data, image and convergence in the same space, Commun. Pure Appl. Anal. **1** (2002), 237–252.

[13] Nakanishi, K., Modified wave operators for the Hartree equation with data, image and convergence in the same space, II, Ann. Henri Poincaré **3** (2002), 503–535.
[14] Ozawa, T., Long range scattering for nonlinear Schrödinger equations in one space dimension, Comm. Math. Phys. **139** (1991), 479–493.
[15] Ozawa, T. and Y. Tsutsumi, Asymptotic behavior of solutions for the coupled Klein-Gordon-Schrödinger equations, preprint, RIMS-775 (1991); Adv. Stud. Pure Math. **23** (1994), 295–305.
[16] Shimomura, A., Modified wave operators for the coupled Wave-Schrödinger equations in three space dimensions, Discrete Contin. Dyn. Syst. **9** (2003), 1571–1586.
[17] Shimomura, A., Wave operators for the coupled Klein-Gordon-Schrödinger equations in two space dimensions, to appear in Funkcial. Ekvac.
[18] Shimomura, A., Modified wave operators for Maxwell-Schrödinger equations in three space dimensions, Ann. Henri Poincaré **4** (2003), 661–683.
[19] Shimomura, A., Scattering theory for Zakharov equations in three space dimensions with large data, to appear in Commun. Contemp. Math.
[20] Tsutsumi, Y., Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions, Comm. Math. Phys. **151** (1993), 543–576.

(Received July 8, 2003)

Department of Mathematics
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo 171-8588, Japan