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# *More Homological Approach to Composition of Subfactors*

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*Dedicated to Professor Susan Montgomery on her sixtieth birthday*

**Abstract.** By using homological tools in [M4], we refine in a slightly generalized form the result by Izumi and Kosaki [IK] which classifies outer actions of a matched pair of finite groups on the hyperfinite $II_1$ factor. An analogous result on outer weak actions is proved. We naturally connect outer (weak) actions with Galois extensions for Hopf (or Kac) algebras or coquasi-Hopf algebras.

## Introduction

Interactions among different research areas in mathematics are undoubtedly welcome. This paper would hopefully contribute to such interactions between Hopf algebras and operator algebras.

Almost at the same time that Chase and Sweedler founded the algebraic study of Hopf algebras in the 1960s, George Kac, an operator algebraist, reached almost the same notion as a Hopf algebra, more precisely what is nowadays called a Kac algebra, and achieved pioneering works. A finite-dimensional Kac algebra is precisely a finite-dimensional $C^*$-Hopf algebra. For a matched pair $(F, G)$ of finite groups, Kac [K] especially classified those Kac algebra extensions $\hat{G} \to \mathcal{H} \to \mathbb{C}F$ of the group algebra $\mathbb{C}F$ by the dual group algebra $\hat{G}$ which are associated to $(F, G)$, by proving an isomorphism

$$\text{Opext}(\mathbb{C}F, \hat{G}) \simeq H^2(\text{Tot}\, E^{\cdot\cdot}) \tag{1}$$

between the group of extensions modulo equivalence and the 2nd total cohomology of some double complex $E^{\cdot\cdot}$ (with our notation). This result due to Kac has been extended and applied mainly by Hopf algebraists including the author [M1,3,4] and Peter Schauenburg [P1,2]; see also [VV]. A matched

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pair $(F, G)$ is accompanied by actions $\triangleleft : G \times F \to G$, $\triangleright : G \times F \to F$ of permutations so that the cartesian product $F \times G$ forms a group, $F \bowtie G$, under the product $(a, x)(b, y) = (a(x \triangleright b), (x \triangleleft b)y)$ [T].

As was stated by Ocneanu and proved by Szymański [S], Longo [L] and others, an irreducible inclusion $\mathcal{P} \supset \mathcal{Q}$ of $II_1$ factors with finite index is of depth 2 if and only if it corresponds, roughly speaking, to a finite-dimensional Kac algebra. Izumi and Kosaki [IK] realized this correspondence for inclusions of the form

$$\mathcal{R} \rtimes_\alpha F \supset \mathcal{R}^{(\beta, G)} \tag{2}$$

which arise from outer actions $(\alpha, \beta)$ of a fixed matched pair $(F, G)$ on a factor $\mathcal{R}$. By definition [IK], an *outer action* $(\alpha, \beta)$ of $(F, G)$ on $\mathcal{R}$ is a pair of homomorphisms $\alpha : F \to \mathrm{Aut}(\mathcal{R})$, $\beta : G \to \mathrm{Aut}(\mathcal{R})$ such that $(a, x) \mapsto \alpha_a \beta_x$ induces a monomorphism $F \bowtie G \to \mathrm{Out}(\mathcal{R})$. Let $\mathcal{R}$ be the hyperfinite $II_1$ factor. Assuming that the action $\triangleleft : G \times F \to G$ in the matched pair $(F, G)$ is trivial, Izumi and Kosaki [IK] proved that there is a natural bijection

$$\mathrm{Out}((F, G), \mathcal{R}) / \sim_c \; \simeq H^2(\mathrm{Tot}\, E^{\cdot\cdot}), \tag{3}$$

where the left-hand side denotes the set of outer actions $(\alpha, \beta)$ modulo cocycle conjugation $\sim_c$; see also [HS]. By using homological tools in [M4], we will refine this result (Theorem 3.6), removing the assumption that $\triangleleft$ is trivial. We define (Definition 3.7) an *outer weak action* $(\alpha, \beta)$ of $(F, G)$, by relaxing the requirement for outer actions so that $\alpha$ can be just a map. As an analogue of (3), we will prove a bijection (Theorem 3.10)

$$\mathrm{Out}_w((F, G), \mathcal{R}) / \sim_{wc} \; \simeq H^2(\mathrm{Tot}\, D^{\cdot\cdot}) \tag{4}$$

between the set of outer weak actions $(\alpha, \beta)$ modulo weak cocycle conjugation $\sim_{wc}$ and the 2nd total cohomology of some double complex $D^{\cdot\cdot}$. $H^2(\mathrm{Tot}\, D^{\cdot\cdot})$ is easier to compute than $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$, and is isomorphic to the group $\mathrm{Opext}''(\mathbb{C}F, \hat{G})$ of coquasi-Hopf algebra extensions [M4]. As the dual notion of quasi-Hopf algebra due to Drinfeld, coquasi-Hopf algebras generalize Hopf algebras in that in their monoidal categories of comodules, the associativity constraints may be non-trivial.

There are known some equivalent ways of connecting irreducible inclusions of depth 2 with Kac algebras; for example in [IK], multiplicative unitaries are used for this connection. The most direct seems, as in [KN], to give to each inclusion a structure of Hopf-Galois extension for the corresponding Kac algebra. We will give (Theorem 4.1) to the inclusion (2) such a structure for the Kac algebra which corresponds to $(\alpha, \beta)$ via (3) and then (1). An analogous result (Proposition 4.3) relating with (4) will be also proved; this seems useful even for the original correspondence. In fact the result is applied to finding explicitly the irreducible inclusions of depth 2 which correspond to the Kac algebras $\mathcal{B}_{4m}$ defined in [M2], which includes Kac and Paljutkin's algebra of dimension 8 (Proposition 5.6).

Our results giving (3), (4) largely depend on Jones' classification [J] of outer actions of a finite group on $\mathcal{R}$. We begin with reproducing some results from [J] in the form suited for us, especially introducing *formal twisted crossed products*; see the paragraph following Example 1.2.

## 1. Jones' Classification of Outer Actions of a Finite Group

Throughout let $\mathcal{R}$ be a factor. Let $\mathbb{U}$ denote the group of unitaries in $\mathcal{R}$. The center of $\mathbb{U}$ is thus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

Let $\Gamma$ be a finite group; the identity element in $\Gamma$ will be denoted by $e$. By a *twisted action* of $\Gamma$ on $\mathcal{R}$, we mean a map $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{R})$ such that $\alpha_e = \mathrm{id}$, the identity map, and the composite

$$\pi\alpha : \Gamma \to \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R}) / \mathrm{Int}(\mathcal{R})$$

with the natural projection $\pi : \mathrm{Aut}(\mathcal{R}) \to \mathrm{Out}(\mathcal{R})$ is a group homomorphism. A twisted action $\alpha$ is called an *action* if it is a homomorphism. It is said to be *outer* if the composite $\pi\alpha$ is a monomorphism.

Given a twisted action $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{R})$, there exits a map $\nu : \Gamma \times \Gamma \to \mathbb{U}$ such that

$$\nu(g, e) = 1 = \nu(e, g), \tag{5}$$

$$\alpha_g \alpha_h = \mathrm{Ad}(\nu(g, h))\alpha_{gh}, \tag{6}$$

where $g, h \in \Gamma$. We have imposed the requirement (5) and also $\alpha_e = \mathrm{id}$, in order to have normalized (abelian or non-abelian) cohomologies. Such normalization gives no influence on our results, but is often convenient; see the remark following Theorem 2.1. One sees that

$$\delta\nu(g, h, l) = \alpha_g(\nu(h, l))\nu(g, hl)\nu(gh, l)^*\nu(g, h)^* \tag{7}$$

gives a 3-cocycle $\delta\nu : \Gamma\times\Gamma\times\Gamma \to \mathbb{T}$ in the normalized standard complex for computing the group cohomology $H^n(\Gamma, \mathbb{T})$ with coefficients in the trivial left $\Gamma$-module $\mathbb{T}$. Define the *obstruction* $\mathrm{Obs}(\alpha)$ of $\alpha$ to be the cohomology class of $\delta\nu$ in $H^3(\Gamma, \mathbb{T})$; this is independent of the choice of $\nu$, since $\nu$ is unique up to multiple by a 2-cochain $\Gamma \times \Gamma \to \mathbb{T}$. The obstruction $\mathrm{Obs}(\alpha)$ vanishes if and only if $\alpha$ together with an appropriate 2-cocycle $\nu' : \Gamma\times\Gamma \to \mathbb{U}$ form a twisted crossed product of $\Gamma$ over $\mathcal{R}$; see below.

We remark that to change the order of the four terms in the product of the right-hand side of (7), only cyclic permutations are allowed.

It will be convenient to formally think as if $\nu$ as above were a cochain in the standard complex for computing the *imaginary* group cohomology $H^n(\Gamma, \mathbb{U})$, which does not exist since $\mathbb{U}$ is not a *real* $\Gamma$-module under $\alpha$. The 3-cocycle $\delta\nu$ given by (7) is formally thought as the coboundary of $\nu$. We call a map $\nu : \Gamma \times \Gamma \to \mathbb{U}$ a *cochain* (*associated to* $\alpha$) if it satisfies (5) (and (6)).

Two twisted actions $\alpha, \alpha' : \Gamma \to \mathrm{Aut}(\mathcal{R})$ are said to be *outer conjugate* and denoted by $\alpha \sim \alpha'$, if there exist $\theta \in \mathrm{Aut}(\mathcal{R})$ and unitaries $\{u_g\}_{g\in\Gamma}$ with $u_e = 1$ such that

$$\theta\alpha'_g\theta^{-1} = \mathrm{Ad}(u_g)\alpha_g \quad (g \in \Gamma).$$

If $\alpha \sim \alpha'$, then $\mathrm{Obs}(\alpha) = \mathrm{Obs}(\alpha')$.

Let $\mathrm{Out}(\Gamma, \mathcal{R})$ denote the set of all outer twisted actions of $\Gamma$ on $\mathcal{R}$.

Theorem 1.1 (Jones [J]). *Suppose $\mathcal{R}$ is the hyperfinite $II_1$ factor. Then,* Obs *induces a bijection*

$$\mathrm{Out}(\Gamma, \mathcal{R})/\sim\ \xrightarrow{\sim} H^3(\Gamma, \mathbb{T})$$

*from the set of all outer conjugacy classes in* $\mathrm{Out}(\Gamma, \mathcal{R})$ *onto* $H^3(\Gamma, \mathbb{T})$.

*Example* 1.2 (Connes [C]). Let $\mathcal{R}$ be as above. Suppose $\Gamma = \mathbb{Z}_n = \langle g\rangle$, the cyclic group of order $n$, generated by $g$. Then Jones' bijection just given factors through the bijection

$$\mathrm{Out}(\Gamma, \mathcal{R})/\sim\ \xrightarrow{\sim} \mu_n, \quad \text{class of } \alpha \mapsto \gamma(\alpha_g),$$

where $\mu_n$ denotes the group of $n$th roots of 1 in $\mathbb{C}$. The $n$th root $\gamma(\alpha_g)$ of 1, called the *Connes obstruction* of $\alpha_g$, is given by

$$\alpha_g(u) = \gamma(\alpha_g)u, \quad \text{where } u \in \mathbb{U} \text{ with } \alpha_g^n = \mathrm{Ad}(u).$$

By [C, Remark 6.8], $\mathrm{Obs}(\alpha)$ is the class of the 3-cocycle $\kappa$ which is defined for $0 \leq i, j, k < n$ by

$$\kappa(g^i, g^j, g^k) = \gamma(\alpha_g)^{i\eta(j,k)}, \quad \text{where } \eta(j,k) = \begin{cases} 0 & \text{if } j+k < n \\ 1 & \text{if } j+k \geq n. \end{cases}$$

Let $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{R})$ be a twisted action. Let $\nu : \Gamma \times \Gamma \to \mathbb{U}$ be a cochain. We define the *formal twisted crossed product* $\mathcal{R}(\alpha, \nu)$ as the free left $\mathcal{R}$-module consisting of all sums $\sum_{g \in \Gamma} r_g \rtimes g$ ($r_g \in \mathcal{R}$) that is given the bilinear product

$$(r \rtimes g)(s \rtimes h) = r\alpha_g(s)\nu(g,h) \rtimes gh.$$

This has the identity element $1 \rtimes e$, but is not necessarily associative. $\nu$ is associated to $\alpha$ if and only if the partial associativity

$$[(r \rtimes g)(s \rtimes h)](t \rtimes e) = (r \rtimes g)[(s \rtimes h)(t \rtimes e)] \tag{8}$$

holds true for $r, s, t \in \mathcal{R}$, $g, h \in \Gamma$. If this is the case, the product is associative (so that $\mathcal{R}(\alpha, \nu)$ is a *real* twisted crossed product) if and only if $\delta\nu = 1$, the trivial 3-cocycle with constant value 1; such $\nu$ is called a 2-*cocycle associated to* $\alpha$. $\mathcal{R}(\alpha, \nu)$ is a $\Gamma$-graded (non-associative) algebra with $g$-component $\mathcal{R} \rtimes g$. One will see that $\mathcal{R}(\alpha, \nu)$ is a *real* algebra in some monoidal category if $\nu$ is associated to $\alpha$; see Remark 4.4 (1).

DEFINITION 1.3. Let $\alpha, \alpha'$ be twisted actions, and let $\nu, \nu'$ be cochains. We write

$$\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha', \nu'),$$

if there exist $\theta \in \mathrm{Aut}(\mathcal{R})$ and unitaries $\{u_g\}_{g \in \Gamma}$ with $u_e = 1$, such that

$$\theta\alpha'_g\theta^{-1} = \mathrm{Ad}(u_g)\alpha_g, \tag{9}$$

$$\theta(\nu'(g,h))u_{gh} = u_g\alpha_g(u_h)\nu(g,h) \tag{10}$$

for $g, h \in \Gamma$.

The conditions (9), (10) are equivalent to that the $\Gamma$-graded isomorphism $\mathcal{R}(\alpha', \nu') \xrightarrow{\sim} \mathcal{R}(\alpha, \nu)$ given by $r \rtimes g \mapsto \theta(r)u_g \rtimes g$ preserves the product. Hence, $\simeq$ gives an equivalence relation among formal twisted crossed products.

Proposition 1.4. *Two twisted actions $\alpha, \alpha'$ are outer conjugate to each other if and only if there exist cochains $\nu, \nu'$ associated to $\alpha, \alpha'$, respectively, such that $\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha', \nu')$.*

Proof. The 'if' part is trivial. For the 'only if' part suppose we are given $\theta$, $\{u_g\}_{g\in\Gamma}$ satisfying (9). Choose a cochain $\nu$ associated to $\alpha$, and define $\nu'$ by the formula (10). Then we have $\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha', \nu')$. Since $\alpha$ and $\nu$ satisfy (8), $\alpha'$ and $\nu'$ do, too. Hence $\nu'$ is associated to $\alpha'$. $\square$

Proposition 1.5. *Let $\nu, \nu'$ be cochains associated to twisted actions $\alpha, \alpha'$, respectively.*

*(1) $\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha', \nu')$ implies $\delta\nu = \delta\nu'$.*

*(2) The converse holds true, if $\mathcal{R}$ is the hyperfinite $II_1$ factor, and $\alpha$ and $\alpha'$ are outer.*

Proof. (1) Let $\theta \in \mathrm{Aut}(\mathcal{R})$, and let $\{u_g\}_{g\in\Gamma}$ be unitaries with $u_e = 1$. Notice that the cochains $\mu, \mu'$ defined by

$$\mu(g,h) = u_g \alpha_g(u_h)\nu(g,h)u_{gh}^*, \quad \mu'(g,h) = \theta(\nu'(g,h))$$

are associated to the twisted actions $\beta, \beta'$ defined by

$$\beta_g = \mathrm{Ad}(u_g)\alpha_g, \quad \beta'_g = \theta\alpha'_g\theta^{-1},$$

respectively. Since we see by direct computations that $\delta\mu = \delta\nu$, $\delta\mu' = \delta\nu'$, it follows that if (9) and (10) are satisfied, $\delta\nu = \delta\nu'$.

(2) Suppose $\delta\nu = \delta\nu'$ under the assumptions given in Part 2. Then, $\alpha \sim \alpha'$ by Theorem 1.1. By Proposition 1.4, there exists some $\omega : \Gamma \times \Gamma \to \mathbb{T}$ such that $\mathcal{R}(\alpha, \omega\nu) \simeq \mathcal{R}(\alpha', \nu')$. One sees by Part 1 that $\omega$ is a 2-cocycle.

It remains to prove that $\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha, \omega\nu)$. This is proved by the idea which proves [IK, Lemma 2.7], in the special case when $\alpha$ is an outer action and $\nu = 1$. In fact, choose then an arbitrary outer action $\alpha^\circ$, and let $(\pi, X)$ be a finite-dimensional projective representation corresponding to $\omega$, so that

$$\pi(g)\pi(h) = \omega(g,h)\pi(gh) \quad (g, h \in \Gamma).$$

Let $\mathcal{N} = B(X)$. We have unitaries $u_g = 1 \otimes \pi(g)$ in $\mathcal{R} \otimes \mathcal{N}$ $(\simeq \mathcal{R})$ such that

$$u_g(\alpha^\circ \otimes \mathrm{id})(u_h) = u_g u_h = \omega(g,h)u_{gh}.$$

Hence, $\mathcal{R} \otimes \mathcal{N}(\alpha^\circ \otimes \mathrm{id}, 1) \simeq \mathcal{R} \otimes \mathcal{N}(\alpha^\circ \otimes 1, \omega)$, which implies $\mathcal{R}(\alpha, 1) \simeq \mathcal{R}(\alpha, \omega)$ since any two outer actions (in particular, $\alpha^\circ \otimes \mathrm{id}$ and $\alpha$) are conjugate by Jones [J].

To prove in general, let $\alpha^\circ$ be as above. Then, $\nu \otimes \omega$ is associated to the $*$-automorphism $\alpha \otimes \alpha^\circ$ of $\mathcal{R} \otimes \mathcal{R}$ ($\simeq \mathcal{R}$), and $\delta(\nu \otimes \omega) = \delta(\omega\nu)$. By the results in the preceding paragraphs, we have a 2-cocycle $\eta : \Gamma \times \Gamma \to \mathbb{T}$ such that

$$\mathcal{R}(\alpha, \omega\nu) \simeq \mathcal{R} \otimes \mathcal{R}(\alpha \otimes \alpha^\circ, \nu \otimes \omega\eta) \simeq \mathcal{R} \otimes \mathcal{R}(\alpha \otimes \alpha^\circ, \nu \otimes 1).$$

Since $\omega$ is arbitrary, $\mathcal{R}(\alpha, \nu) \simeq \mathcal{R}(\alpha, \omega\nu)$, as desired. $\square$

## 2. The Cohomology of a Matched Pair of Groups

Let $(F, G)$ be a matched pair of groups [T]. Thus $F$ and $G$ are (possibly infinite) groups, and there are given actions of permutations

$$G \overset{\triangleleft}{\leftarrow} G \times F \overset{\triangleright}{\rightarrow} F$$

(one from the right and the other from the left) such that

$$xy \triangleleft a = (x \triangleleft (y \triangleright a))(y \triangleleft a), \quad x \triangleright ab = (x \triangleright a)((x \triangleleft a) \triangleright b),$$

where $a, b \in F$, $x, y \in G$. These conditions are equivalent to that the cartesian product $F \times G$ forms a group, $F \bowtie G$, under the product

$$(a, x)(b, y) = (a(x \triangleright b), (x \triangleleft b)y).$$

The group $\Gamma = F \bowtie G$ *exactly factorizes* into the subgroups $F = F \times \{e\}$, $G = \{e\} \times G$ in the sense that

$$\Gamma = FG, \quad F \cap G = \{e\}. \tag{11}$$

Therefore we will write $ax$ for the element $(a, x)$ in $F \bowtie G$.

The notion of a matched pair was implicit in Kac's paper [K]. But, his notion was somewhat irregular in that the two group actions were both supposed to be from the left. Kac [K] also constructed some double complex to obtain such a cohomology that describes Kac (or Hopf) algebra extensions. His construction was reproduced in [M1, Appendix] in such a modified way that specializes the construction of the Singer cohomology [Si] which describes Hopf algebra extensions in a more general form. This construction in [M1] was further modified by [M4] in such a way that is suited to describe explicit homotopy equivalences between relevant complexes. We are going to reproduce some of the results in [M4], which are suited for right modules, modifying in such a way that is suited for left modules.

For a group $\Gamma$, $\Gamma$-modules will mean as usual modules over the integral group ring $\mathbb{Z}\Gamma$. We define a double complex of left $F \bowtie G$-modules,

$$
C_{\cdot\cdot} = \begin{array}{ccccc}
\vdots & & \vdots & & \\
\downarrow & & \downarrow & & \\
C_{01} & \xleftarrow{\partial} & C_{11} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial'} & & \\
C_{00} & \xleftarrow{\partial} & C_{10} & \longleftarrow & \cdots,
\end{array}
$$

as follows. $C_{pq}$ denotes the free left $F \bowtie G$-module with basis $(x_1, \dots, x_q; a_1, \dots, a_p)$, where $e \neq a_i \in F$, $e \neq x_i \in G$. This is also regarded as the quotient module of the free left $F \bowtie G$-module over the set $G^q \times F^p$, the cartesian product of $q$ copies of $G$ and $p$ copies of $F$, by the submodule generated by those elements $(x_1, \dots, x_q; a_1, \dots, a_p)$ in which some $a_i$ or $x_i$ equals $e$. The horizontal and vertical differentials, $\partial$ and $\partial'$, are the $F \bowtie G$-linear maps given by

$$
\begin{aligned}
&(-1)^q \partial(x_1, \dots, x_q; a_1, \dots, a_p) \\
&= x_1 \cdots x_q \triangleright a_1 (x_1 \triangleleft (x_2 \cdots x_q \triangleright a_1), \dots, \\
&\qquad\qquad x_{q-1} \triangleleft (x_q \triangleright a_1), x_q \triangleleft a_1; a_2, \dots, a_p) \\
&\quad + \sum_{i=1}^{p-1} (-1)^i (x_1, \dots, x_q; a_1, \dots, a_i a_{i+1}, \dots, a_p) \\
&\quad + (-1)^p (x_1, \dots, x_q; a_1, \dots, a_{p-1}),
\end{aligned}
$$

$$
\begin{aligned}
&\partial'(x_1, \dots, x_q; a_1, \dots, a_p) \\
&= x_1 (x_2, \dots, x_q; a_1, \dots, a_p) \\
&\quad + \sum_{i=1}^{q-1} (-1)^i (x_1, \dots, x_i x_{i+1}, \dots, x_q; a_1, \dots, a_p)
\end{aligned}
$$

$$+ (-1)^q (x_1, \dots, x_{q-1}; x_q \triangleright a_1, (x_q \triangleleft a_1) \triangleright a_2, \dots, \\ (x_q \triangleleft a_1 \cdots a_{p-1}) \triangleright a_p).$$

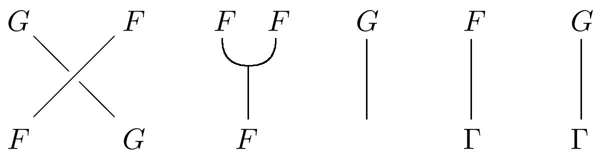
The lowest horizontal complex $C_{\cdot 0}$ and the leftmost vertical complex $C_{0 \cdot}$ coincide with the normalized bar resolutions of $\mathbb{Z}$, the trivial left modules over $F$ and $G$, respectively. The rows and columns in $C_{\cdot\cdot}$ are all exact, so that the total complex $\operatorname{Tot} C_{\cdot\cdot}$ gives a non-standard free $F \bowtie G$-resolution of $\mathbb{Z}$ if we define an augmentation $\varepsilon : C_{00} = \mathbb{Z}(F \bowtie G) \to \mathbb{Z}$ by $\varepsilon(ax) = 1$. Hence it is homotopy equivalent to the normalized bar $F \bowtie G$-resolution of $\mathbb{Z}$, which we denote by

$$B_{\cdot} = 0 \leftarrow B_0 \xleftarrow{\delta} B_1 \xleftarrow{\delta} B_2 \xleftarrow{\delta} \cdots .$$

To give explicit homotopy equivalences,

$$\Pi_{\cdot} : B_{\cdot} \to \operatorname{Tot} C_{\cdot\cdot}, \quad \Phi_{\cdot} : \operatorname{Tot} C_{\cdot\cdot} \to B_{\cdot},$$

we set $\Gamma = F \bowtie G$, and let the diagrams

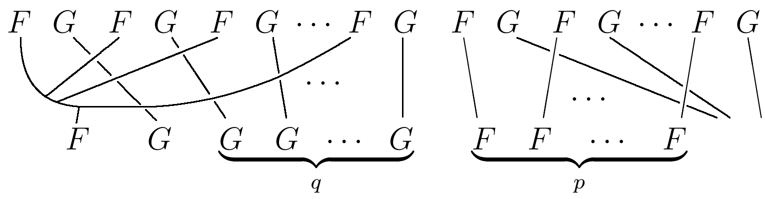
(12) 


stand respectively for the map (in fact a bijection)

$$\times : G \times F \to F \times G, \quad \times(x, a) = (x \triangleright a, x \triangleleft a),$$

the product $F \times F \to F$, the trivial map $G \to \{e\}$ and the inclusions $F \hookrightarrow \Gamma$, $G \hookrightarrow \Gamma$. We thus omit to write $\{e\}$ in diagrams. In the following we fix $n > 0$, and suppose $n = p + q$ with $p, q \geq 0$.

As a variant of the Alexander-Whitney map, $\Pi_n : B_n \to \operatorname{Tot}_n C_{\cdot\cdot} = \bigoplus_{p+q=n} C_{pq}$ is the $\Gamma$-linear map whose $(p, q)$-component is given by

$(\Pi_n)_{p,q} =$  .

For example,

$$\Pi_2(ax, by) = ((a, x \triangleright b), a(x; b), a(x \triangleright b)(x \triangleleft b, y)).$$

Let $S_{q,p}$ denote the set of $(q,p)$-shuffles, that is, those permutations $t$ on $\{1, \ldots, n\}$ such that $t(1) < \cdots < t(q)$, $t(q+1) < \cdots < t(n)$. For each $t \in S_{q,p}$, let $t_*$ denote such a braid diagram of $n$ strings coming out of

$$\underbrace{G \ \cdots \ G}_{q}\underbrace{F \ \cdots \ F}_{p}$$

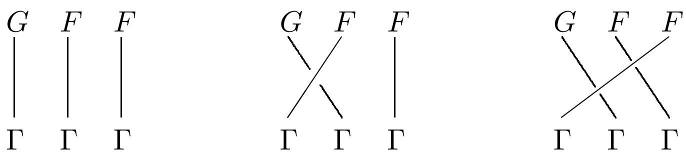
and terminating into

$$\underbrace{\Gamma \ \cdots\cdots \ \Gamma}_{n}$$

that is determined by the following rules: (i) The string which comes out of $G$ or $F$ in the $i$th place, counted from the left, terminates into $\Gamma$ in the $t(i)$th place; (ii) Any crossing in $t_*$ must be of the same form as the first diagram in (12). We understand that $t_*$ defines a $\Gamma$-linear map $C_{pq} \to B_n$, giving values of the basis elements. Now, $\Phi_n : \mathrm{Tot}_n\, C_{\cdot\cdot} \to B_n$ is the $\Gamma$-linear map given by

$$\Phi_n|_{C_{pq}} = \sum_{t \in S_{q,p}} (\operatorname{sgn} t) t_*.$$

By definition, $\operatorname{sgn} t = (-1)^{l(t)}$, where $l(t)$, the length of $t$, coincides with the number of crossings in $t_*$. For example, since the $(1,2)$-shuffles gives the diagrams



,

we have

$$\Phi_3(x; a, b) = (x, a, b) - (x \triangleright a, x \triangleleft a, b) + (x \triangleright a, (x \triangleleft a) \triangleright b, x \triangleleft ab),$$

where $x \in G$, $a, b \in F$. When $n = 0$, we define $\Pi_0$ and $\Phi_0$ both to be the identity map on $\mathbb{Z}\Gamma$.

Theorem 2.1 ([M4, Thm. 1.8]). *$\Pi.: B. \to \mathrm{Tot}\, C..$ and $\Phi.: \mathrm{Tot}\, C.. \to B.$ are $F \bowtie G$-linear homotopy equivalences which induce the identity map on $\mathbb{Z}$, such that $\Pi.\Phi. = \mathrm{id}$.*

It only holds that $\Pi.\Phi. = \mathrm{id}$ since the complexes we consider are normalized.

Regard $\mathbb{T}$ as a trivial left module over $\Gamma = F \bowtie G$. The group $\mathrm{Hom}_\Gamma(C_{pq}, \mathbb{T})$ of $\Gamma$-linear maps $C_{pq} \to \mathbb{T}$ is naturally identified with the multiplicative group $\mathrm{Map}_+(G^q \times F^p, \mathbb{T})$ of those maps $f: C_{pq} \to \mathbb{T}$ which are normalized in the sense that $f(x_1, \ldots, x_q; a_1, \ldots, a_p) = 1$ whenever any $a_i$ or $x_i$ equals $e$. We define a double cochain complex by $C^{\cdot\cdot} = C^{\cdot\cdot}(\mathbb{T}) := \mathrm{Hom}_\Gamma(C.., \mathbb{T})$, which looks like:

$$
\begin{array}{ccccccccc}
C^{\cdot\cdot} & = & C^{\cdot\cdot}(\mathbb{T}) & & & & & & \\
 & & \vdots & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \mathrm{Map}_+(G^2,\mathbb{T}) & \longrightarrow & \vdots\ \ddots & & & & \\
 & = & \uparrow{\scriptstyle \partial'} & & \uparrow & & & & \\
 & & \mathrm{Map}_+(G,\mathbb{T}) & \xrightarrow{\partial} & \mathrm{Map}_+(G\times F,\mathbb{T}) & \longrightarrow & \vdots\ \ddots & & \\
 & & \uparrow{\scriptstyle \partial'} & & \uparrow{\scriptstyle \partial'} & & \uparrow & & \\
 & & \mathbb{T} & \xrightarrow{\partial} & \mathrm{Map}_+(F,\mathbb{T}) & \xrightarrow{\partial} & \mathrm{Map}_+(F^2,\mathbb{T}) & \longrightarrow & \cdots \quad .
\end{array}
$$

Up to sign of differentials, this double complex coincides with $D^{\cdot\cdot}$ in [M3, p. 173] (with $k^\times$ replaced by $\mathbb{T}$), and with $C^{\cdot\cdot}(\mathbb{T})$ in [M4, Sect. 2].

The cochain complex $B^{\cdot} = B^{\cdot}(\mathbb{T}) := \mathrm{Hom}_\Gamma(B., \mathbb{T})$ is the normalized standard complex for computing $H^n(\Gamma, \mathbb{T})$, which looks like:

$$B^{\cdot} = B^{\cdot}(\mathbb{T}) = \mathbb{T} \xrightarrow{\delta} \mathrm{Map}_+(F \bowtie G, \mathbb{T}) \xrightarrow{\delta} \mathrm{Map}_+((F \bowtie G)^2, \mathbb{T}) \to \cdots \quad .$$

Corollary 2.2. *$\Pi.$ and $\Phi.$ induce homotopy equivalences,*

$$\Pi^{\cdot} : \mathrm{Tot}\, C^{\cdot\cdot} \to B^{\cdot}, \quad \Phi^{\cdot} : B^{\cdot} \to \mathrm{Tot}\, C^{\cdot\cdot},$$

*such that* $\Phi^{\cdot}\Pi^{\cdot} = \mathrm{id}$.

Remove from $C^{\cdot\cdot}$ the leftmost vertical complex and then the lowest horizontal one, to obtain the following complexes.

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
D^{\cdot\cdot} = D^{\cdot\cdot}(\mathbb{T}) = & \mathrm{Map}_+(G \times F, \mathbb{T}) & \longrightarrow & \mathrm{Map}_+(G \times F^2, \mathbb{T}) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 & \mathrm{Map}_+(F, \mathbb{T}) & \longrightarrow & \mathrm{Map}_+(F^2, \mathbb{T}) & \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & \\
E^{\cdot\cdot} = E^{\cdot\cdot}(\mathbb{T}) = & \mathrm{Map}_+(G^2 \times F, \mathbb{T}) & \longrightarrow & \mathrm{Map}_+(G^2 \times F^2, \mathbb{T}) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 & \mathrm{Map}_+(G \times F, \mathbb{T}) & \longrightarrow & \mathrm{Map}_+(G \times F^2, \mathbb{T}) & \longrightarrow \cdots .
\end{array}$$

We can regard $E^{\cdot\cdot} \subset D^{\cdot\cdot} \subset C^{\cdot\cdot}$. However, we count the total dimension in $D^{\cdot\cdot}$ and $E^{\cdot\cdot}$ so that $\mathrm{Tot}^n D^{\cdot\cdot}$ (resp., $\mathrm{Tot}^n E^{\cdot\cdot}$) equals the direct sum of $\mathrm{Map}_+(G^q \times F^p, \mathbb{T})$ with $p + q = n + 1$, where $p > 0$, $q \geq 0$ (resp., $p > 0$, $q > 0$); as a consequence, it is their *second* total cohomology groups, as is common in various extension theories, that are isomorphic to groups of (coquasi-)Hopf algebra extensions (see Section 4). We have thus

$$\mathrm{Tot}\, E^{\cdot\cdot}[1] \subset \mathrm{Tot}\, D^{\cdot\cdot}[1] \subset \mathrm{Tot}\, C^{\cdot\cdot},$$

where [1] stands for +1 dimension shift. The inclusions induce homomorphisms

$$H^n(\mathrm{Tot}\, E^{\cdot\cdot}) \to H^n(\mathrm{Tot}\, D^{\cdot\cdot}) \to H^{n+1}(F \bowtie G, \mathbb{T}). \tag{13}$$

Since the quotient complex of $\mathrm{Tot}\, C^{\cdot\cdot}$ by $\mathrm{Tot}\, D^{\cdot\cdot}[1]$ is the standard complex for computing $H^n(G, \mathbb{T})$, we have a long exact sequence,

$$\begin{aligned} \cdots \to H^{n-1}(\mathrm{Tot}\, D^{\cdot\cdot}) &\to H^n(F \bowtie G, \mathbb{T}) \\ &\to H^n(G, \mathbb{T}) \to H^n(\mathrm{Tot}\, D^{\cdot\cdot}) \to \cdots \quad . \end{aligned} \tag{14}$$

The analogous long exact sequence involving $H^n(\mathrm{Tot}\, E^{\cdot\cdot})$ is the so-called Kac exact sequence; see [K, (3,14)], [M1, Appendix] and also [M3,4], [P1,2].

Replacing the coefficients $\mathbb{T}$ by $\mathbb{C}^\times = \mathbb{C}\backslash\{0\}$, we have double complexes $D^{\cdot\cdot}(\mathbb{C}^\times)$, $E^{\cdot\cdot}(\mathbb{C}^\times)$. It follows by the universal coefficient theorem that for any finite group $\Gamma$, the natural homomorphisms $H^n(\Gamma, \mathbb{T}) \to H^n(\Gamma, \mathbb{C}^\times)$, $n > 0$, are isomorphisms. This together with (14) and the Kac exact sequences prove the following.

Proposition 2.3. *If $F$ and $G$ are finite, we have natural isomorphisms*

$$\begin{aligned} H^n(\mathrm{Tot}\, D^{\cdot\cdot}(\mathbb{T})) &\simeq H^n(\mathrm{Tot}\, D^{\cdot\cdot}(\mathbb{C}^\times)), \\ H^n(\mathrm{Tot}\, E^{\cdot\cdot}(\mathbb{T})) &\simeq H^n(\mathrm{Tot}\, E^{\cdot\cdot}(\mathbb{C}^\times)) \end{aligned} \tag{15}$$

*for all $n > 0$.*

## 3. The Izumi-Kosaki Invariant and its Variant

We fix a matched pair $(F, G)$ of finite groups.

Definition 3.1 ([IK, Def. 2.1]). A pair $(\alpha, \beta)$ of actions (or homomorphisms) $\alpha : F \to \mathrm{Aut}(\mathcal{R})$, $\beta : G \to \mathrm{Aut}(\mathcal{R})$ is called an *action* (resp., *outer action*) of the matched pair $(F, G)$, if the map $\alpha\beta : F \bowtie G \to \mathrm{Aut}(\mathcal{R})$ defined by

$$(\alpha\beta)_{ax} = \alpha_a \beta_x \quad (a \in F, \quad x \in G) \tag{16}$$

is a twisted action (resp., outer twisted action). Two actions $(\alpha, \beta)$, $(\alpha', \beta')$ of $(F, G)$ are said to be *cocycle conjugate* and denoted by $(\alpha, \beta) \sim_c (\alpha', \beta')$, if there exist $\theta \in \mathrm{Aut}(\mathcal{R})$ and systems $\{u_a\}_{a \in F}$, $\{v_x\}_{x \in G}$ of unitaries with $u_e = v_e = 1$, such that

$$u_{ab} = u_a \alpha_a(u_b), \tag{17}$$

$$v_{xy} = v_x \beta_x(v_y), \tag{18}$$

$$\theta \alpha'_a \theta^{-1} = \mathrm{Ad}(u_a)\alpha_a, \tag{19}$$

$$\theta \beta'_x \theta^{-1} = \mathrm{Ad}(v_x)\beta_x. \tag{20}$$

Let $(\alpha, \beta)$ be an action of the matched pair $(F, G)$. Let $\nu : F \bowtie G \times F \bowtie G \to \mathbb{U}$ be a cochain associated to $\alpha\beta$. Since

$$\begin{aligned}(\alpha\beta)_{ax}(\alpha\beta)_{by} &= \alpha_a\beta_x\alpha_b\beta_y = \alpha_a \operatorname{Ad}(\nu(x,b))\alpha_{x\triangleright b}\beta_{x\triangleleft b}\beta_y \\ &= \operatorname{Ad}(\alpha_a(\nu(x,b)))(\alpha\beta)_{axby},\end{aligned}$$

we can choose $\nu$ so as

$$\nu(ax, by) = \alpha_a(\nu(x,b)), \tag{21}$$

where $a, b \in F$, $x, y \in G$.

Definition 3.2. Such $\nu$ is said to be *reduced.*

We can formally think the complexes $C^{\cdot\cdot}(\mathbb{U})$, $B^{\cdot}(\mathbb{U})$ with coefficients in $\mathbb{U}$ on which $F \bowtie G$ acts by $\alpha\beta$, and also $\Pi^{\cdot} : \operatorname{Tot} C^{\cdot\cdot}(\mathbb{U}) \to B^{\cdot}(\mathbb{U})$. Then, $\nu$ is reduced if and only if $\nu = \Pi^2(\nu_1)$, where we formally define

$$\nu_1 = \nu|_{G\times F} \ \text{ in } \ C^{11}(\mathbb{U}). \tag{22}$$

Proposition 3.3. *Suppose $\nu$ is a reduced cochain associated to $\alpha\beta$.*
*(1) The 3-cocycle $\delta\nu$ (see (7)) is given by*

$$\delta\nu(ax, by, cz) = \alpha_{x\triangleright b}\beta_{x\triangleleft b}(\nu(y,c))\nu(x,b)^*\nu(x, b(y\triangleright c))\alpha_{x\triangleright b}(\nu((x\triangleleft b)y, c))^*.$$

*(2) $\Phi^3(\delta\nu)$ is in $\operatorname{Tot}^2 E^{\cdot\cdot}$. If we write $\Phi^3(\delta\nu) = (\sigma, \tau)$, where $\sigma : G \times F^2 \to \mathbb{T}$, $\tau : G^2 \times F \to \mathbb{T}$, then*

$$\begin{aligned}\sigma(x; a, b) &= \nu(x,a)^*\nu(x,ab)\alpha_{x\triangleright a}(\nu(x\triangleleft a, b))^* \\ \tau(x, y; a) &= \nu(x, y\triangleright a)\nu(xy, a)^*\beta_x(\nu(y,a)).\end{aligned}$$

*(3) $\Pi^3(\sigma, \tau) = \delta\nu$.*

Proof. (1), (2) Straightforward.
(3) This follows since we compute

$$\begin{aligned}\Pi^3(\sigma,\tau)(ax, by, cz) &= \sigma(x; b, y\triangleright c)\tau(x\triangleleft b, y; c) \\ &= \sigma(x; b, y\triangleright c)\alpha_{x\triangleright b}(\tau(x\triangleleft b, y; c)) \\ &= \nu(x,b)^*\nu(x, b(y\triangleright c))\alpha_{x\triangleright b}(\nu(x\triangleleft b, y\triangleright c))^* \\ &\quad \alpha_{x\triangleright b}(\nu(x\triangleleft b, y\triangleright c))\alpha_{x\triangleright b}(\nu((x\triangleleft b)y, c))^* \\ &\quad \alpha_{x\triangleright b}\beta_{x\triangleleft b}(\nu(y,c)) \\ &= \delta\nu(ax, by, cz). \ \square\end{aligned}$$

Since by Part 2 above, we can formally suppose that $\sigma = \partial\nu_1$, $\tau = \partial'\nu_1$, we write

$$(\sigma, \tau) = \boldsymbol{\partial}\nu_1, \tag{23}$$

the formal coboundary in $\mathrm{Tot}\, E^{\cdot\cdot}(\mathbb{U})$. Part 3 states

$$\Pi^3(\boldsymbol{\partial}\nu_1) = \delta\nu. \tag{24}$$

Let $(\alpha, \beta), (\alpha', \beta')$ be actions of $(F, G)$ on $\mathcal{R}$.

Proposition 3.4. *$(\alpha, \beta) \sim_c (\alpha', \beta')$ if and only if there exist reduced cochains $\nu, \nu'$ associated to $\alpha\beta, \alpha'\beta'$, respectively, such that $\mathcal{R}(\alpha\beta, \nu) \simeq \mathcal{R}(\alpha'\beta', \nu')$.*

Proof. 'If'. Suppose an isomorphism $\mathcal{R}(\alpha'\beta', \nu') \xrightarrow{\sim} \mathcal{R}(\alpha\beta, \nu)$, $r \rtimes ax \mapsto \theta(r) w_{ax} \rtimes ax$ is given by $\theta$ and $\{w_{ax}\}_{a,x}$. Notice that $\nu$ and $\nu'$, being reduced, are both trivial on $F \times F$ and $G \times G$. Let $u_a = w_a$, $v_x = w_x$. Then, (17)–(20) follow from (9), (10), and hence $(\alpha, \beta) \sim_c (\alpha', \beta')$.

'Only if'. Suppose $\theta$, $\{u_a\}_{a \in F}$ and $\{v_x\}_{x \in G}$ satisfy (17)–(20). Choose a reduced cochain $\nu'$ associated to $\alpha'\beta'$. By the proof of Proposition 1.4 there exists uniquely a cochain $\nu$ associated to $\alpha\beta$, such that the bijection

$$\mathcal{R}(\alpha'\beta', \nu') \xrightarrow{\sim} \mathcal{R}(\alpha\beta, \nu), \quad r \rtimes ax \mapsto \theta(r) u_a \alpha_a(v_x) \rtimes ax$$

preserves the product.

It remains to prove $\nu$ is reduced. Notice from (10) that $\nu|_{F \times F}$, $\nu|_{F \times G}$ and $\nu|_{G \times G}$ are all trivial. In this case, $\mathcal{R}(\alpha\beta, \nu)$ satisfies the partial associativities

$$[(r \rtimes a)(s \rtimes x)](t \rtimes by) = (r \rtimes a)[(s \rtimes x)(t \rtimes by)], \tag{25}$$

$$[(r \rtimes ax)(s \rtimes b)](t \rtimes y) = (r \rtimes ax)[(s \rtimes b)(t \rtimes y)], \tag{26}$$

if and only if the equations

$$\nu(ax, by) = \alpha_a(\nu(x, by))\nu(a, (x \triangleright b)(x \triangleleft b)y), \tag{27}$$

$$\nu(ax, by) = \nu(ax, b)\nu(a(x \triangleright b)(x \triangleleft b), y) \tag{28}$$

hold, where $a, b \in F$, $x, y \in G$, $r, s, t \in \mathcal{R}$. It is easy to see (21) implies (27), (28). To prove the converse, let $b = e$ in (27), $a = e$ in (28), and $x = e$ in (28). Then we have

$$\nu(ax, y) = 1, \quad \nu(x, by) = \nu(x, b), \quad \nu(a, by) = 1,$$

which together with (27) imply (21).

The result implies that $\mathcal{R}(\alpha'\beta', \nu')$ satisfies (25), (26), and so $\mathcal{R}(\alpha\beta, \nu)$ does, too. Hence $\nu$, satisfying (27), (28) and hence (21), is reduced. □

Proposition 3.5. *Let $\nu, \nu'$ be reduced cochains associated to $\alpha\beta, \alpha'\beta'$, respectively.*

*(1) $\mathcal{R}(\alpha\beta, \nu) \simeq \mathcal{R}(\alpha'\beta', \nu')$ implies $\boldsymbol{\partial}\nu_1 = \boldsymbol{\partial}\nu'_1$.*

*(2) The converse holds true, if $\mathcal{R}$ is the hyperfinite $II_1$ factor, and $(\alpha, \beta)$ and $(\alpha', \beta')$ (or namely, $\alpha\beta$ and $\alpha'\beta'$) are outer.*

Proof. This follows from Proposition 1.5, since by (24), $\boldsymbol{\partial}\nu_1 = \boldsymbol{\partial}\nu'_1$ if and only if $\delta\nu = \delta\nu'$. □

Given an action $(\alpha, \beta)$ of $(F, G)$ on $\mathcal{R}$, we choose a reduced cochain $\nu$ associated to $\alpha\beta$, and define the *Izumi-Kosaki invariant* $\mathrm{ik}(\alpha, \beta)$ to be the cohomology class of $\boldsymbol{\partial}\nu_1$ in $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$; see (23). This is independent of the choice of $\nu$. By Propositions 3.4 and 3.5 (1), $(\alpha, \beta) \sim_c (\alpha', \beta')$ implies $\mathrm{ik}(\alpha, \beta) = \mathrm{ik}(\alpha', \beta')$.

Let $\mathrm{Out}((F, G), R)$ denote the set of outer actions of the matched pair $(F, G)$ on $\mathcal{R}$. The following generalizes Izumi and Kosaki [IK, Thm. 2.5], removing their assumption that the action $\triangleleft : G \times F \to G$ associated to $(F, G)$ is trivial.

Theorem 3.6. *Suppose $\mathcal{R}$ is the hyperfinite $II_1$ factor. Then,* ik *induces a bijection*

$$\mathrm{Out}((F, G), \mathcal{R}) / \sim_c \xrightarrow{\sim} H^2(\mathrm{Tot}\, E^{\cdot\cdot})$$

*from the set of all cocycle conjugacy classes in* $\mathrm{Out}((F, G), \mathcal{R})$ *onto* $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$.

Proof. *Injectivity.* Let $(\alpha, \beta), (\alpha', \beta')$ be outer actions. Let $\nu, \nu'$ be reduced cochains associated to $\alpha\beta$, $\alpha'\beta'$, respectively. Suppose $\boldsymbol{\partial}\nu_1$ and $\boldsymbol{\partial}\nu'_1$

are cohomologous in $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$. Then for some cochain $\eta : G \times F \to \mathbb{T}$, $(\boldsymbol{\partial}\nu_1)(\boldsymbol{\partial}\eta) = \boldsymbol{\partial}\nu_1'$ in $\mathrm{Tot}\, E^{\cdot\cdot}$. Replace $\nu$ with $\nu\Pi^2(\eta)$. Then, $\boldsymbol{\partial}\nu_1 = \boldsymbol{\partial}\nu_1'$. By Propositions 3.5 (2) and 3.4, $\mathcal{R}(\alpha\beta, \nu) \simeq \mathcal{R}(\alpha'\beta', \nu')$ and $(\alpha, \beta) \sim_c (\alpha', \beta')$.

*Surjectivity.* Let $(\sigma, \tau) \in Z^2(\mathrm{Tot}\, E^{\cdot\cdot})$, and define $\phi = \Pi^3(\sigma, \tau) \in Z^3(B^{\cdot})$. By Theorem 1.1, there exists a twisted action $\tilde{\alpha}$ of $F \bowtie G$ together with an associated cochain $\mu$ such that $\delta\mu = \phi$. Since $\phi$ is trivial on $F \times F \times F$ and $G \times G \times G$, Theorem 1.1 and Proposition 1.4 allow us to suppose $\tilde{\alpha} = \alpha\beta$, where $\alpha$ and $\beta$ are actions of $F$ and $G$, respectively, so that $(\alpha, \beta)$ is an outer action of $(F, G)$. By the same reason we may suppose $\mu$ is trivial on $F \times F$ and $G \times G$. Nevertheless, $\mu$ may not be reduced. But, we can write $\nu = \omega\mu$, where $\nu$ is reduced and $\omega$ is a $\mathbb{T}$-valued 2-cochain. Since $\omega$ is necessarily trivial on $F \times F$ and $G \times G$, $\Phi^2(\omega)$ is in $\mathrm{Tot}^1 E^{\cdot\cdot}$. We see $\delta\nu = (\delta\omega)\phi$. It follows by applying $\Phi^3$ that $\boldsymbol{\partial}\nu_1 = \boldsymbol{\partial}\Phi^2(\omega)(\sigma, \tau)$, since $\Phi^3(\delta\omega) = \boldsymbol{\partial}\Phi^2(\omega)$, a coboundary in $\mathrm{Tot}\, E^{\cdot\cdot}$. This proves that the cohomology classes $[\boldsymbol{\partial}\nu_1] = \mathrm{ik}(\alpha, \beta)$ and $[(\sigma, \tau)]$ in $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$ coincide. □

To obtain an analogous result (Theorem 3.11), we modify Definition 3.1 as follows.

Definition 3.7. A pair $(\alpha, \beta)$ of a map $\alpha : F \to \mathrm{Aut}(\mathcal{R})$ with $\alpha_e = \mathrm{id}$ and a homomorphism $\beta : G \to \mathrm{Aut}(\mathcal{R})$ is called a *weak action* (resp., *outer weak action*) of the matched pair, if the map $\alpha\beta : F \bowtie G \to \mathrm{Aut}(\mathcal{R})$ defined by (16) is a twisted action (resp., outer twisted action). Two weak actions $(\alpha, \beta), (\alpha', \beta')$ of $(F, G)$ are said to be *weakly cocycle conjugate* and denoted by $(\alpha, \beta) \sim_{wc} (\alpha', \beta')$, if there exist $\theta \in \mathrm{Aut}(\mathcal{R})$ and systems $\{u_a\}_{a \in F}$, $\{v_x\}_{x \in G}$ of unitaries with $u_e = v_e = 1$, such that (18), (19) and (20) are satisfied.

Let $(\alpha, \beta)$ be a weak action of the matched pair $(F, G)$. A cochain $\nu$ associated to $\alpha\beta$ can be chosen so that

$$\nu(ax, by) = \alpha_a(\nu(x, b))\nu(a, x \triangleright b). \tag{29}$$

Such $\nu$ is said to be *reduced* in this generalized case. Define

$$\nu_0 = \nu|_{F \times F}, \quad \nu_1 = \nu|_{G \times F}. \tag{30}$$

We can formally suppose that $(\nu_0, \nu_1)$ is in $\mathrm{Tot}^1 D^{\cdot\cdot}(\mathbb{U})$, and $\nu = \Pi^2(\nu_0, \nu_1)$ in $B^2(\mathbb{U})$.

Proposition 3.8. *Suppose $\nu$ is a reduced cochain associated to $\alpha\beta$.*
*(1) The 3-cocycle $\delta\nu$ (see (7)) is given by*

$$\begin{aligned}\delta\nu(ax,by,cz) &= \alpha_{a(x\triangleright b)}\beta_{x\triangleleft b}(\nu(y,c))\nu(a,x\triangleright b)^* \\ &\quad \alpha_a(\nu(x,b))^*\alpha_a\beta_x(\nu(b,y\triangleright c)) \\ &\quad \alpha_a(\nu(x,b(y\triangleright c)))\nu(a,x\triangleright b(y\triangleright c))\nu(a(x\triangleright b),(x\triangleleft b)y\triangleright c)^* \\ &\quad \alpha_{a(x\triangleright b)}(\nu((x\triangleleft b)y,c))^*.\end{aligned}$$

*(2) $\Phi^3(\delta\nu)$ is in $\mathrm{Tot}^2 D^{\cdot\cdot}$. If we write $(\omega,\sigma,\tau)=\Phi^3(\delta\nu)$, where $\omega : F\times F\times F\to\mathbb{T}$, $\sigma : G\times F\times F\to\mathbb{T}$ and $\tau : G\times G\times F\to\mathbb{T}$, then*

$$\begin{aligned}\omega(a,b,c) &= \delta\nu_0(a,b,c) = \alpha_a(\nu(b,c))\nu(a,bc)\nu(ab,c)^*\nu(a,b)^*, \\ \sigma(x;a,b) &= \nu(x,a)^*\beta_x(\nu(a,b))\nu(x,ab)\nu(x\triangleright a,(x\triangleleft a)\triangleright b)^* \\ &\quad \alpha_{x\triangleright a}(\nu(x\triangleleft a,b))^*, \\ \tau(x,y;a) &= \beta_x(\nu(y,a))\nu(x,y\triangleright a)\nu(xy,a)^*.\end{aligned}$$

*(3) $\Pi^3(\omega,\sigma,\tau)=\delta\nu$.*

Proof. (1), (2) Straightforward.
(3) This follows since we compute

$$\begin{aligned}\Pi^3(\omega,\sigma,\tau)(ax,by,cz) &= \tau(x\triangleleft b,y;c)\sigma(x;b,y\triangleright c)\omega(a,x\triangleright b,(x\triangleleft b)y\triangleright c) \\ &= \alpha_{a(x\triangleright b)}(\tau(x\triangleleft b,y;c))\nu(a,x\triangleright b)^*\alpha_a(\sigma(x;b,y\triangleright c)) \\ &\quad \omega(a,x\triangleright b,(x\triangleleft b)y\triangleright c)\nu(a,x\triangleright b) \\ &= \alpha_{a(x\triangleright b)}(\nu((x\triangleleft b)y,c))^* \\ &\quad \alpha_{a(x\triangleright b)}\beta_{x\triangleleft b}(\nu(y,c))\nu(a,x\triangleright b)^*\alpha_a(\nu(x,b))^* \\ &\quad \alpha_a\beta_x(\nu(b,y\triangleright c))\alpha_a(\nu(x,b(y\triangleright c))) \\ &\quad \nu(a,(x\triangleright b)((x\triangleleft b)y\triangleright c)) \\ &\quad \nu(a(x\triangleright b),(x\triangleleft b)y\triangleright c)^* \\ &= \delta\nu(ax,by,cz). \ \square\end{aligned}$$

With the notation as above we write

$$(\omega,\sigma,\tau)=\boldsymbol{\partial}(\nu_0,\nu_1), \tag{31}$$

since we can formally suppose that this equation holds in $\mathrm{Tot}\, D^{\cdot\cdot}(\mathbb{U})$. Part 3 above states

$$\Pi^3(\boldsymbol{\partial}(\nu_0, \nu_1)) = \delta\nu. \tag{32}$$

Proposition 3.9. *Two weak actions $(\alpha, \beta), (\alpha', \beta')$ of $(F, G)$ on $\mathcal{R}$ are weakly cocycle conjugate if and only if there exist reduced cochains $\nu, \nu'$ associated to $\alpha\beta, \alpha'\beta'$, respectively, such that $\mathcal{R}(\alpha\beta, \nu) \simeq \mathcal{R}(\alpha'\beta', \nu')$.*

Proof. Modify the proof of Proposition 3.4. We only remark that such a cochain $\nu$ associated to $\alpha\beta$ that is trivial on $F \times G$ and $G \times G$ is reduced if and only if $\mathcal{R}(\alpha\beta, \nu)$ satisfies the partial associativities (25), (26). □

Proposition 3.10. *Let $(\alpha, \beta), (\alpha', \beta')$ be weak actions of the matched pair $(F, G)$ on $\mathcal{R}$. Let $\nu, \nu'$ be reduced cochains associated to $\alpha\beta, \alpha'\beta'$, respectively.*

*(1) $\mathcal{R}(\alpha\beta, \nu) \simeq \mathcal{R}(\alpha'\beta', \nu')$ implies $\boldsymbol{\partial}(\nu_0, \nu_1) = \boldsymbol{\partial}(\nu'_0, \nu'_1)$; see (31).*

*(2) The converse holds true, if $\mathcal{R}$ is the hyperfinite $II_1$ factor, and $(\alpha, \beta)$ and $(\alpha', \beta')$ (or namely, $\alpha\beta$ and $\alpha'\beta'$) are outer.*

Proof. By (32), $\boldsymbol{\partial}(\nu_0, \nu_1) = \boldsymbol{\partial}(\nu'_0, \nu'_1)$ if and only if $\delta\nu = \delta\nu'$. Hence this proposition follows from Proposition 1.5. □

Given a weak action $(\alpha, \beta)$ of $(F, G)$, choose a reduced cochain $\nu$ associated to $\alpha\beta$, and define $\mathrm{ik}_w(\alpha, \beta)$ to be the cohomology class of $\boldsymbol{\partial}(\nu_0, \nu_1)$ in $H^2(\mathrm{Tot}\, D^{\cdot\cdot})$. This is independent of the choice of $\nu$. By Propositions 3.9 and 3.10 (1), $(\alpha, \beta) \sim_{wc} (\alpha', \beta')$ implies $\mathrm{ik}_w(\alpha, \beta) = \mathrm{ik}_w(\alpha', \beta')$.

Let $\mathrm{Out}_w((F, G), \mathcal{R})$ denote the set of outer weak actions of the matched pair $(F, G)$ on $\mathcal{R}$. An easy modification of the proof of Theorem 3.6, using Propositions 3.9 and 3.10, proves the following.

Theorem 3.11. *Suppose $\mathcal{R}$ is the hyperfinite $II_1$ factor. Then,* $\mathrm{ik}_w$ *induces a bijection*

$$\mathrm{Out}_w((F, G), \mathcal{R})/\sim_{wc} \xrightarrow{\sim} H^2(\mathrm{Tot}\, D^{\cdot\cdot})$$

*from the set of all weak cocycle conjugacy classes in* $\mathrm{Out}_w((F,G),\mathcal{R})$ *onto* $H^2(\mathrm{Tot}\,D^{\cdot\cdot})$.

One sees from (24), (32) that the following diagram commutes, where the vertical arrows in the left-hand side are given by $(\alpha,\beta)\mapsto(\alpha,\beta)\mapsto\alpha\beta$, and those in the right-hand sides are as in (13).

$$\begin{array}{ccc}
\mathrm{Out}((F,G),\mathcal{R}) & \xrightarrow{\mathrm{ik}} & H^2(\mathrm{Tot}\,E^{\cdot\cdot}) \\
\downarrow & & \downarrow \\
\mathrm{Out}_w((F,G),\mathcal{R}) & \xrightarrow{\mathrm{ik}_w} & H^2(\mathrm{Tot}\,D^{\cdot\cdot}) \\
\downarrow & & \downarrow \\
\mathrm{Out}(F\bowtie G,\mathcal{R}) & \xrightarrow{\mathrm{Obs}} & H^3(F\bowtie G,\mathbb{T})
\end{array}$$

## 4. Hopf-Galois Extensions Arising from (Weak) Actions of a Matched Pair

We continue to fix a matched pair $(F,G)$ of finite groups.

A finite-dimensional Kac algebra is precisely a finite-dimensional $C^*$-Hopf algebra. We denote by $\mathbb{C}F$ the group algebra of $F$, which forms a Kac algebra with $a^*=a^{-1}$, where $a\in F$. We denote by $\hat{G}$, though often denoted by $(\mathbb{C}G)^*$, $\mathbb{C}^G$ or $l^\infty(G)$, the dual Kac algebra of $\mathbb{C}G$. The dual basis of the basis $x$ $(\in G)$ in $\mathbb{C}G$ will be denoted by $e_x$, so that $(e_x)^*=e_x$ in $\hat{G}$.

Essentially due to Kac [K], $H^2(\mathrm{Tot}\,E^{\cdot\cdot})$ is naturally isomorphic to the group $\mathrm{Opext}(\mathbb{C}F,\hat{G})$ of the equivalence classes of those Kac algebra extensions $\hat{G}\to\mathcal{H}\to\mathbb{C}F$ which are associated to the fixed matched pair $(F,G)$. For $(\sigma,\tau)\in Z^2(\mathrm{Tot}\,E^{\cdot\cdot})$, the vector space $\hat{G}\otimes\mathbb{C}F$ of tensor product forms a Hopf algebra, $\hat{G}\#_{\sigma,\tau}\mathbb{C}F$, of twisted bicrossed product with the structure described in [M3, p. 170]. This is in fact a Kac algebra in which all $1\#a$ $(a\in F)$ are unitaries, and forms in the obvious way an extension associated to $(F,G)$. The assignment $(\sigma,\tau)\mapsto\hat{G}\#_{\sigma,\tau}\mathbb{C}F$ induces a natural isomorphism $H^2(\mathrm{Tot}\,E^{\cdot\cdot})\simeq\mathrm{Opext}(\mathbb{C}F,\hat{G})$. The isomorphism (15) implies that any Hopf algebra extension is equivalent to a unique (up to equivalence) Kac algebra extension.

It follows by [IK, Remarks 1, p. 5] that given an outer action $(\alpha,\beta)$ of $(F,G)$ on $\mathcal{R}$, we have an irreducible inclusion $\mathcal{R}\rtimes_\alpha F\supset\mathcal{R}^{(\beta,G)}$ of factors of depth 2; the depth 2 condition follows also from the next theorem of ours. We will connect this inclusion with such a Kac algebra extension whose equivalence class corresponds to $\mathrm{ik}(\alpha,\beta)$. Among the known equivalent ways of connecting irreducible inclusions of $II_1$ factors of depth 2, with Kac algebras (see the Introduction), the most direct seems to give to such an inclusion a structure of Hopf-Galois extensions. For a finite-dimensional Kac algebra $\mathcal{H}$, a $*$-algebra $\mathcal{P}$ is called a *right $\mathcal{H}$-comodule algebra* if it is given a $*$-homomorphism $\rho:\mathcal{P}\to\mathcal{P}\otimes\mathcal{H}$ which is a right $\mathcal{H}$-comodule structure. In this case, $\mathcal{P}\supset\mathcal{Q}$ is called an *$\mathcal{H}$-Galois extension* [Mo, Def. 8.1.1], if $\mathcal{Q}$ equals the $*$-subalgebra $\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$ consisting of the elements $p$ such that $\rho(p)=p\otimes 1$, and the map

$$\tilde{\rho}:\mathcal{P}\otimes_{\mathcal{Q}}\mathcal{P}\to\mathcal{P}\otimes\mathcal{H},\quad \tilde{\rho}(p\otimes q)=(p\otimes 1)\rho(q) \tag{33}$$

is a bijection. It follows by Kadison and Nikshych [KN] that an irreducible inclusion of $II_1$ factors with finite index is of depth 2 if and only if it is $\mathcal{H}$-Galois for some finite-dimensional Kac algebra $\mathcal{H}$.

THEOREM 4.1. *Let $(\alpha,\beta)$ be an outer action of $(F,G)$ on $\mathcal{R}$. Choose a reduced cochain $\nu$ associated to $\alpha\beta$. Let $\mathcal{H}=\hat{G}\#_{\sigma,\tau}\mathbb{C}F$, where $(\sigma,\tau)=\boldsymbol{\partial}\nu_1$; see (23). Then, $\mathcal{R}\rtimes_\alpha F\supset\mathcal{R}^{(\beta,G)}$ is an $\mathcal{H}$-Galois extension with respect to the structure $\rho:\mathcal{R}\rtimes_\alpha F\to(\mathcal{R}\rtimes_\alpha F)\otimes\mathcal{H}$ defined by*

$$\rho(r\rtimes a)=\sum_{x\in G}(\beta_x(r)\nu(x,a)\rtimes(x\triangleright a))\otimes(e_x\# a). \tag{34}$$

PROOF. We only prove that $\rho$ preserves $*$. The remaining will be proved in Proposition 4.3 below in a generalized situation.

Since one sees $\rho(r^*\rtimes 1)=\rho(r\rtimes 1)^*$, it remains to prove that $\rho(1\rtimes a^{-1})=\rho(1\rtimes a)^*$. Since $a^{-1}e_x=e_{x\triangleleft a}a^{-1}$ $(=e_{x\triangleleft a}\# a^{-1})$ in $\mathcal{H}$, we see

$$\begin{aligned}\rho(1\rtimes a)^* &= \sum_{x,y\in G}(x\triangleright a)^{-1}\nu(x,a)^*\otimes\overline{\sigma(y;a^{-1},a)}e_y a^{-1}e_x\\ &= \sum_{y\in G}(y\triangleright a^{-1})\nu(y\triangleleft a^{-1},a)^*\otimes\overline{\sigma(y;a^{-1},a)}e_y a^{-1}.\end{aligned}$$

This equals

$$\rho(1 \rtimes a^{-1}) = \sum_{x \in G} \nu(x, a^{-1})(x \triangleright a^{-1}) \otimes e_x a^{-1},$$

since $\nu(x, a^{-1}) = \alpha_{x \triangleright a^{-1}}(\nu(x \triangleleft a^{-1}, a))^* \overline{\sigma(x; a^{-1}, a)}$, as follows from Proposition 3.3 (2). □

Remark 4.2. (1) For $(\alpha, \beta)$ as above, let $\nu'$ be another choice of a reduced cochain associated to $\alpha\beta$. This gives rises, as above, to a structure $\rho' : \mathcal{R} \rtimes_\alpha F \to (\mathcal{R} \rtimes_\alpha F) \otimes \mathcal{H}'$ with $\mathcal{H}' = \hat{G} \#_{\sigma', \tau'} \mathbb{C}F$, where $(\sigma', \tau') = \boldsymbol{\partial} \nu'_1$. Since we have $\eta : G \times F \to \mathbb{T}$ such that $\nu'_1 = \eta \nu_1$, it follows that $(\sigma', \tau') = (\sigma \partial \eta, \tau \partial' \eta)$. By [M3, Prop. 1.8], $e_x \# a \mapsto \eta(x; a) e_x \# a$ gives an equivalence $\mathcal{H}' \xrightarrow{\sim} \mathcal{H}$ of extensions, which is compatible with the structures $\rho', \rho$.

(2) Izumi and Kosaki [IK, Remarks 2, p. 5] prove that if $(\alpha, \beta)$ and $(\alpha', \beta')$ are cocycle conjugate outer actions of $(F, G)$, the inclusions $\mathcal{R} \rtimes_\alpha F \supset \mathcal{R}^{(\beta, G)}$ and $\mathcal{R} \rtimes_{\alpha'} F \supset \mathcal{R}^{(\beta', G)}$ are conjugate to each other; see also Remark 4.4 (2) below.

It is proved in [M4, Prop. 4.15] that $H^2(\mathrm{Tot}\, D^{\cdot\cdot})$ is naturally isomorphic to the group $\mathrm{Opext}''(\mathbb{C}F, \hat{G})$ of the coquasi-equivalence classes of coquasi-bialgebra extensions associated to the matched pair $(F, G)$; see [M4, Defs. 4.11, 4.12].

Let $(\omega, \sigma, \tau) \in Z^2(\mathrm{Tot}\, D^{\cdot\cdot})$. In particular, $\omega : F \times F \times F \to \mathbb{T}$ is a 3-cocycle. Denote $\bar{\omega} = \omega^{-1}$, the inverse of $\omega$. In the same way as above we construct a coalgebra, $\mathcal{H} = \hat{G} \#_{\sigma, \tau} \mathbb{C}F$, with unital, but non-associative product. Each right $\mathcal{H}$-comodule $V$ can be regarded as an $F$-graded vector space $V = \bigoplus_{a \in F} V_a$ through the canonical coalgebra epimorphism $\mathcal{H} \to \mathbb{C}F$. The tensor product $V \otimes W$ of two right $\mathcal{H}$-comodules forms a right $\mathcal{H}$-comodule along the product $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. The 2-cocycle condition in $\mathrm{Tot}\, D^{\cdot\cdot}$ assures that the right $\mathcal{H}$-comodules form such a monoidal category, $\mathcal{M}^{(\mathcal{H}, \bar{\omega})}$, in which the left and right unit constraints are trivial, and the associativity constraint $(V \otimes W) \otimes U \xrightarrow{\sim} V \otimes (W \otimes U)$ is given by

$$\bar{\omega}(a, b, c) = \overline{\omega(a, b, c)} : (V_a \otimes W_b) \otimes U_c \xrightarrow{\sim} V_a \otimes (W_b \otimes U_c),$$

where $a, b, c \in F$. Thus, $(\mathcal{H}, \bar{\omega})$ is a coquasi-bialgebra [M4, Sect. 4.2]; this is in fact a coquasi-Hopf algebra as is proved in [M3, Lemma 4.1] in the dual context. For another $(\mathcal{H}',\bar\omega') = (\hat G\#_{\sigma',\tau'}\mathbb{C}F,\bar\omega')$, $(\mathcal{H},\bar\omega)$ and $(\mathcal{H}',\bar\omega')$ are said to be *coquasi-equivalent* [M4, Def. 4.11], if the monoidal categories $\mathcal{M}^{(\mathcal{H},\bar\omega)}$ and $\mathcal{M}^{(\mathcal{H}',\bar\omega')}$ are monoidally equivalent in some specific way. The assignment $(\omega,\sigma,\tau)\mapsto(\hat G\#_{\sigma,\tau}\mathbb{C}F,\bar\omega)$ induces a natural isomorphism $H^2(\mathrm{Tot}\,D^{\cdot\cdot})\xrightarrow{\sim}\mathrm{Opext}''(\mathbb{C}F,\hat G)$.

For $(\mathcal{H},\bar\omega)$ as above, an $(\mathcal{H},\bar\omega)$-Galois extension $\mathcal{P}\supset\mathcal{Q}$ is an algebra object $\mathcal{P}$ with structure, say $\rho$, in $\mathcal{M}^{(\mathcal{H},\bar\omega)}$ such that $\mathcal{Q}=\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$ and the map $\tilde\rho$ defined by (33) is a bijection. Notice that if $\mathcal{Q}=\mathcal{P}^{\mathrm{co}\,\mathcal{H}}$, $\mathcal{Q}$ is an ordinary algebra, $\mathcal{P}$ is an ordinary $\mathcal{Q}$-bimodule, and $\mathcal{P}\otimes_{\mathcal{Q}}\mathcal{P}$ is defined in the ordinary way. This definition of Galois extensions is available for general coquasi-bialgebras.

Proposition 4.3. *Let $(\alpha,\beta)$ be a weak action of $(F,G)$ on $\mathcal{R}$. Choose a reduced cochain $\nu$ associated to $\alpha\beta$, and set $(\mathcal{H},\bar\omega)=(\hat G\#_{\sigma,\tau}\mathbb{C}F,\bar\omega)$, where $(\omega,\sigma,\tau)=\boldsymbol{\partial}(\nu_0,\nu_1)$; see (31).*

*(1) The formal twisted crossed product $\mathcal{R}(\alpha,\nu_0)$ is an algebra object in $\mathcal{M}^{(\mathcal{H},\bar\omega)}$ with the structure $\rho$ defined by the same formula as (34).*

*(2) If the action $\beta$ is outer, the inclusion $\mathcal{R}(\alpha,\nu_0)\supset\mathcal{R}^{(\beta,G)}$ is an $(\mathcal{H},\bar\omega)$-Galois extension.*

Proof. (1) By [M4, Prop. 4.20], $\mathcal{M}^{(\mathcal{H},\bar\omega)}$ is identified with the monoidal category ${}_G\mathcal{M}^F_{\omega,\sigma,\tau}$ defined by [M4, Def. 4.1]. An algebra object in ${}_G\mathcal{M}^F_{\omega,\sigma,\tau}$ is a unital, but non-associative $F$-graded algebra $A=\bigoplus_{a\in F}A_a$ with a unital, but non-associative $G$-action $G\times A\to A$, $(x,\xi)\mapsto x\cdot\xi$ such that $x\cdot 1=1$, $x\cdot A_a\subset A_{x\triangleright a}$,

$$\begin{aligned} x\cdot(y\cdot\xi) &= \tau(x,y;a)(xy)\cdot\xi,\\ x\cdot(\xi\eta) &= \sigma(x;a,b)(x\cdot\xi)((x\triangleleft a)\cdot\eta),\\ (\xi\eta)\mu &= \bar\omega(a,b,c)\xi(\eta\mu), \end{aligned} \tag{35}$$

where $\xi\in A_a$, $\eta\in A_b$, $\mu\in A_c$. In particular the last condition (35) requires the diagram

$$\begin{array}{ccc} (A \otimes A) \otimes A & \xrightarrow{\text{ass}} & A \otimes (A \otimes A) \\ \Big\downarrow{\scriptstyle m \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes m} \\ A \otimes A & & A \otimes A \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & A & \end{array}$$

to commute, where $m : A \otimes A \to A$ denotes the product and ass denotes associativity constraint in ${}_G\mathcal{M}^F_{\omega,\sigma,\tau}$ which is given by the 3-cocycle $\bar{\omega}$.

$\mathcal{R}(\alpha, \nu_0)$ is regarded as an object in ${}_G\mathcal{M}^F_{\omega,\sigma,\tau}$ with the original $F$-gradation together with $G$-action $x \cdot (r \rtimes a) = \beta_x(r)\nu(x, a) \rtimes (x \triangleright a)$. It is straightforward to see that this is an algebra object. See also Remark 4.4 (1) below.

(2) Let $\mathcal{P} = \mathcal{R}(\alpha, \nu_0)$, $\mathcal{Q} = \mathcal{R}^{(\beta,G)}$. Then one sees easily $\mathcal{P}^{\mathrm{co}\,\mathcal{H}} = \mathcal{Q}$. Suppose $\beta$ is outer. It is well-known, as Prof. Yamagami kindly informed to the author, that $\mathcal{R} \supset \mathcal{Q}$ is $\hat{G}$-Galois, so that the map $\mathcal{R} \otimes_{\mathcal{Q}} \mathcal{R} \to \mathcal{R} \otimes \hat{G}$ given by $r \otimes s \mapsto \sum_{x\in G} r\beta_x(s) \otimes e_x$ is a bijection. As its base extension we have a bijection

$$\mathcal{P} \otimes_{\mathcal{Q}} \mathcal{R} \xrightarrow{\sim} \mathcal{P} \otimes \hat{G}, \quad p \otimes r \mapsto \sum_{x\in G} p\beta_x(r) \otimes e_x. \tag{36}$$

Regard $\mathcal{P}\otimes_{\mathcal{Q}}\mathcal{P}$ and $\mathcal{P}\otimes\mathcal{H}$ as $F$-graded space with $a$-components $\mathcal{P}\otimes_{\mathcal{Q}}(\mathcal{R}\rtimes a)$ and $\mathcal{P} \otimes (\hat{G}\#a)$, respectively, where $a \in F$. Then $\tilde{\rho}$ is $F$-graded, whose $a$-component $(\tilde{\rho})_a$ is the composite of (36) with

$$\mathcal{P} \otimes \hat{G} \to \mathcal{P} \otimes \hat{G}, \quad p \otimes e_x \mapsto p\nu(x, a)(x \triangleright a) \otimes e_x. \tag{37}$$

Here since $(pr)q = p(rq)$ for $p, q \in \mathcal{P}$, $r \in \mathcal{R}$, we have written $prq$ for them. Since the right multiplications by $\nu(x, a)$, $1 \rtimes (x\triangleright a)$ in $\mathcal{P}$ are both bijections, the map (37) and hence $\tilde{\rho}$ are, too. □

Remark 4.4. Let the notation be as above.

(1) Write $\Gamma = F \bowtie G$. Let $\phi = \delta\nu$, a 3-cocycle. We see as above that $\mathcal{R}(\alpha\beta, \nu)$ is an algebra object in the monoidal category $\mathcal{M}^{(\Gamma,\bar{\phi})}$ of $\Gamma$-graded

spaces whose associativity constraint is given by $\bar{\phi} = \phi^{-1}$. It includes the ordinary algebra $\mathbb{C}G$ as a sub-algebra object since $\bar{\phi}$ is trivial on $G \times G \times G$. It follows that $\mathcal{R}(\alpha\beta, \nu)$ is an algebra objects in the monoidal category ${}_G\mathcal{M}_G^{(\Gamma,\bar{\phi})}$ of $\mathbb{C}G$-bimodules in $\mathcal{M}^{(\Gamma,\bar{\phi})}$. Schauenburg [P1, Thm. 3.3.5] gives a monoidal equivalence ${}_G\mathcal{M}_G^{(\Gamma,\bar{\phi})} \approx {}_G\mathcal{M}^F_{\omega,\sigma,\tau}$, under which $\mathcal{R}(\alpha\beta, \nu)$ corresponds to $\mathcal{R}(\alpha, \nu_0)$. This gives an alternative proof of Part 1 of the preceding proposition.

(2) Suppose the action $\beta$ is outer. Let $(\alpha', \beta')$ be another weak action of $(F, G)$ such that $(\alpha, \beta) \sim_{wc} (\alpha', \beta')$. Choose a reduced cochain $\nu'$ associated to $\alpha'\beta'$. We will see that if $\delta\nu'_0 = \delta\nu_0$, there is an isomorphism $\mathcal{R}(\alpha', \nu'_0) \simeq \mathcal{R}(\alpha, \nu_0)$ of non-associative algebras which induces a $*$-isomorphism $\mathcal{R}^{(\beta',G)} \simeq \mathcal{R}^{(\beta,G)}$. We may suppose $\delta\nu' = \delta\nu$ to obtain those $\theta$, $\{u_a\}_{a\in F}$, $\{v_x\}_{x\in G}$ which satisfy (18), (19), (20) and make

$$f : \mathcal{R}(\alpha', \nu'_0) \to \mathcal{R}(\alpha, \nu_0), \quad f(r \rtimes a) = \theta(r)u_a \rtimes a$$

an isomorphism. As in [IK, Remarks 2, p. 5], it follows by applying the Noether-Skolem theorem to the two $*$-homomorphisms

$$\mathbb{C}G \to \mathcal{R} \rtimes_\beta G, \quad x\ (\in G) \mapsto 1 \rtimes x,\ v_x \rtimes x,$$

we have $w \in \mathbb{U}$ such that $wv_x = \beta_x(w)$, and hence

$$\mathrm{Ad}(w)\theta\beta'_x = \mathrm{Ad}(w)\,\mathrm{Ad}(v_x)\beta_x\theta = \beta_x\,\mathrm{Ad}(w)\theta,$$

where $x \in G$. This proves that $\mathrm{Ad}(w \rtimes e)f$ is a desired isomorphism.

## 5. Examples

Let $n = 2m$ be an even natural number with $m > 1$. Let

$$D_{2n} = \langle a, x \mid a^n = 1 = x^2, \quad xa = a^{-1}x\rangle$$

denote the dihedral group of order $2n$. We have four subgroups in $D_{2n}$,

$$E = \langle ax\rangle, \quad F_1 = \langle a\rangle, \quad F_2 = \langle a^2, ax\rangle, \quad G = \langle x\rangle, \tag{38}$$

which are isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_n$, $D_n$, $\mathbb{Z}_2$, respectively. One sees that $D_{2n}$ exactly factorizes (see (11)) in two ways as $D_{2n} = F_1G = F_2G$, in which

$F_i$ are both normal. Hence we have two matched pairs, $(F_1, G)$, $(F_2, G)$, such that $F_i \bowtie G = D_{2n}$. The cohomology groups $H^2(\operatorname{Tot} D^{\cdot\cdot})$ arising from the two matched pairs are naturally isomorphic to each other by a general reason [M4, Prop. 4.6], and are computed in [M4, Prop. 5.3] so that

$$H^2(\operatorname{Tot} D^{\cdot\cdot}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_n. \tag{39}$$

On the other hand it seems incident that the two cohomology groups $H^2(\operatorname{Tot} E^{\cdot\cdot})$ are isomorphic to each other, as is seen from [M2, Props. 3.10, 3.11] (or the independent [IK, Prop. 7.5]) so that

$$H^2(\operatorname{Tot} E^{\cdot\cdot}) \simeq \mathbb{Z}_2. \tag{40}$$

Proposition 5.1. *The restriction homomorphisms from $D_{2n}$ to $E, F_1, G$ (see (38)) induce an isomorphism*

$$H^3(D_{2n}, \mathbb{T}) \xrightarrow{\sim} H^3(E, \mathbb{T}) \oplus H^3(F_1, \mathbb{T}) \oplus H^3(G, \mathbb{T}).$$

Proof. This might be known. But, this can be easily proved as follows, by using (39), (40).

Since the action $\triangleleft : G \times F_i \to G$ associated to the matched pair is trivial, we have

$$H^n(\operatorname{Tot} C^{\cdot\cdot}) \simeq H^{n-1}(\operatorname{Tot} D^{\cdot\cdot}) \oplus H^n(G, \mathbb{T}). \tag{41}$$

Choose here the matched pair $(F_1, G)$. Since $H^3(F_1, \mathbb{T}) \simeq \mathbb{Z}_n$, it follows by (39), (40) that the exact sequence

$$H^2(\operatorname{Tot} E^{\cdot\cdot}) \to H^2(\operatorname{Tot} D^{\cdot\cdot}) \to H^3(F_1, \mathbb{T}) \tag{42}$$

arising from the obvious short exact sequence of complexes is necessarily a split short exact sequence. Therefore we have a natural (split) short exact sequence

$$H^2(\operatorname{Tot} E^{\cdot\cdot}) \to H^3(D_{2n}, \mathbb{T}) \to H^3(F_1, \mathbb{T}) \oplus H^3(G, \mathbb{T}). \tag{43}$$

It remains to prove that the composite

$$H^2(\operatorname{Tot} E^{\cdot\cdot}) \to H^3(D_{2n}, \mathbb{T}) \to H^3(E, \mathbb{T}) \tag{44}$$

is an isomorphism. $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$ is generated by the cohomology class of $(\sigma,\tau) \in Z^2(\mathrm{Tot}\, E^{\cdot\cdot})$, where $\sigma : G \times F_1 \times F_1 \to \mathbb{T}$ is trivial, and $\tau : G \times G \times F_1 \to \mathbb{T}$ is given by $\tau(x,x;a^i) = (-1)^i$. Its image in $H^3(E,\mathbb{T})$ is represented by $\Pi^3(\sigma,\tau)|_{E^3}$, whose cohomology class indeed generates $H^3(E,\mathbb{T})$ since

$$\Pi^3(\sigma,\tau)(ax,ax,ax) = \tau(x,x;a) = -1.$$

This completes the proof. □

REMARK 5.2. We assert that Proposition 5.1 fails to hold if $F_1$ is replaced by $F_2$. By the proposition, $H^3(D_{2n},\mathbb{T}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_n \oplus \mathbb{Z}_2$ if $n$ is even. This implies our assertion when $m$ $(= n/2)$ is even.

Suppose $n$ is odd. We have still the exact factorization $D_{2n} = F_1 G$, while $F_2$ then equals $D_{2n}$. It follows by (41) that $H^3(D_{2n},\mathbb{T}) \simeq \mathbb{Z}_{2n}$, since we see by modifying the proof of [M4, Prop. 5.3] that $H^2(\mathrm{Tot}\, D^{\cdot\cdot})$ arising from $(F_1, G)$ is isomorphic to $\mathbb{Z}_n$. It follows that if $m$ is odd, we have the same split short exact sequences as (42), (43), with $F_1$ replaced by $F_2$. But, the composite (44) is zero since $ax$ is mapped to $e$ through the projection $F_2 G \to G$.

We suppose again that $n$ $(= 2m > 2)$ is even. In the following we also suppose $\mathcal{R}$ is the hyperfinite $II_1$ factor. Since the last proof shows that the natural homomorphisms $H^2(\mathrm{Tot}\, E^{\cdot\cdot}) \to H^2(\mathrm{Tot}\, D^{\cdot\cdot}) \to H^3(D_{2n},\mathbb{T})$ are (split) monomorphisms, we have by Theorems 3.6, 3.11 that

$$\begin{aligned} \mathrm{Out}((F_i,G),\mathcal{R})/\sim_c \;&\subset\; \mathrm{Out}_w((F_i,G),\mathcal{R})/\sim_{wc} \\ &\subset\; \mathrm{Out}(F_i \bowtie G,\mathcal{R})/\sim, \end{aligned} \tag{45}$$

where $i = 1, 2$.

Recall from Example 1.2 the definition of the Connes obstruction $\gamma$. To each $\alpha \in \mathrm{Out}(D_{2n},\mathcal{R})$, we can assign an element $(\gamma(\alpha_{ax}),\gamma(\alpha_x),\gamma(\alpha_a))$ in $\mu_2 \times \mu_2 \times \mu_n$.

COROLLARY 5.3. *This assignment gives a bijection*

$$\mathrm{Out}(D_{2n},\mathcal{R})/\sim\; \xrightarrow{\sim} \mu_2 \times \mu_2 \times \mu_n.$$

Proof. This follows from the preceding proposition together with Example 1.2. □

Proposition 5.4. *Choose the matched pair* $(F_1, G) = (\mathbb{Z}_n, \mathbb{Z}_2)$. *Assign to each action or weak action* $(\alpha, \beta)$ *of* $(\mathbb{Z}_n, \mathbb{Z}_2)$, *an element* $\gamma(\alpha_a \beta_x)$ *in* $\mu_2$ *or* $(\gamma(\alpha_a \beta_x), \gamma(\alpha_a))$ *in* $\mu_2 \times \mu_n$. *Then we have bijections*

$$\mathrm{Out}((\mathbb{Z}_n, \mathbb{Z}_2), \mathcal{R}) / \sim_c \xrightarrow{\sim} \mu_2,$$
$$\mathrm{Out}_w((\mathbb{Z}_n, \mathbb{Z}_2), \mathcal{R}) / \sim_{wc} \xrightarrow{\sim} \mu_2 \times \mu_n.$$

Proof. An outer twisted action $\tilde{\alpha}$ of $D_{2n}$ is outer conjugate to some $\alpha\beta$, where $(\alpha, \beta)$ is an action (resp., weak action) of $(F_1, G)$, if and only if $\tilde{\alpha}|_{F_1}$ and $\tilde{\alpha}|_G$ are (resp., $\tilde{\alpha}|_G$ is) outer conjugate to action(s), if and only if $\gamma(\tilde{\alpha}_a) = \gamma(\tilde{\alpha}_x) = 1$ (resp., $\gamma(\tilde{\alpha}_x) = 1$). Hence the proposition follows from Corollary 5.3 together with (45). □

We have precisely two (up to equivalence) Kac algebra extensions associated to the matched pair $(\mathbb{Z}_n, \mathbb{Z}_2)$. The split extension is given by the dual $\hat{D}_{2n}$ of the group algebra $\mathbb{C}D_{2n}$, while the unique non-split one is given by $\hat{T}_{4m}$, where $T_{4m}$ is the dicyclic (or generalized quaternion) group of order $4m$; see [M2, Remark 3.2]. By the preceding proposition we have precisely two (up to cocycle conjugacy) outer actions $(\alpha, \beta)$ of $(\mathbb{Z}_n, \mathbb{Z}_2)$ on $\mathcal{R}$, which are distinguished according to whether $\gamma(\alpha_a \beta_x) = 1$ or $-1$. In each case, the irreducible inclusion $\mathcal{R} \rtimes_\alpha \mathbb{Z}_n \supset \mathcal{R}^{(\beta, \mathbb{Z}_2)}$ is $\hat{D}_{2n}$- or $\hat{T}_{4m}$-Galois, respectively.

Proposition 5.5. *Choose the matched pair* $(F_2, G) = (D_n, \mathbb{Z}_2)$. *Assigning to each weak action* $(\alpha, \beta)$ *of* $(D_n, \mathbb{Z}_2)$, *an element* $(\gamma(\alpha_{ax}), \gamma(\alpha_{ax}\beta_x))$ *in* $\mu_2 \times \mu_n$, *we have a bijection*

$$\mathrm{Out}_w((D_n, \mathbb{Z}_2), \mathcal{R}) / \sim_{wc} \xrightarrow{\sim} \mu_2 \times \mu_n.$$

*A weak action* $(\alpha, \beta)$ *is weakly cocycle conjugate to some action of the matched pair if and only if* $\gamma(\alpha_{ax}) = 1$ *and* $\gamma(\alpha_{ax}\beta_x) = \pm 1$.

Proof. The proof of the first half is similar to the last proof. The 'only if' part of the second half follows since $\gamma(\alpha_{a^2}) = \gamma(\alpha_{ax}\beta_x)^2$. The 'if' part follows since we know that $\mathrm{Out}((D_n, \mathbb{Z}_2), \mathcal{R}) / \sim_c$ consists of two elements. □

We have precisely two Kac algebra extensions associated to the matched pair $(D_n, \mathbb{Z}_2)$. They are explicitly described in [M2]; the split one is denoted by $\mathcal{A}_{4m}$, while the unique non-split one is denoted by $\mathcal{B}_{4m}$. When $m = 2$, $\mathcal{B}_8$ coincides with the Kac algebra due to Kac and Paljutkin [KP]. Aside from the Kac algebras constructed by Sekine [Se], $\mathcal{B}_{4m}$ are another candidate of generalizations of Kac and Paljutkin's algebra. It is known so far that they possess the following interesting properties: (i) $\mathcal{B}_{4m}$ is selfdual [CDMM]; (ii) $\mathcal{B}_{4m}$ has precisely $2m$ quasitriangular structures, none of which is triangular [Su]; (iii) There exists no $\mathcal{B}_{4m}$-Galois extension over $\mathbb{C}$ other than $\mathcal{B}_{4m}$ itself [M2] ([IK, Thm. 13.15] when $m = 2$); this implies that there exists no Hopf algebra other than $\mathcal{B}_{4m}$ whose (co)module category is monoidally equivalent to $\mathcal{B}_{4m}$'s.

For an outer action $\tilde{\alpha} : D_{2n} \to \mathrm{Aut}(\mathcal{R})$, let $\alpha = \tilde{\alpha}|_{F_2}$, $\beta = \tilde{\alpha}|_G$. Then $(\alpha, \beta)$ is an outer action of $(D_n, \mathbb{Z}_2)$, and $\mathcal{R} \rtimes_\alpha D_n \supset \mathcal{R}^{(\beta,\mathbb{Z}_2)}$ is $\mathcal{A}_{4m}$-Galois.

By Proposition 5.5 there exists an outer weak action $(\alpha, \beta)$ of $(D_n, \mathbb{Z}_2)$ such that $\gamma(\alpha_{ax}) = 1$ and $\gamma(\alpha_{ax}\beta_x) = -1$. This $(\alpha, \beta)$ is weakly cocycle conjugate to some outer action of the matched pair; it seems, however, difficult to find such an outer action in an explicit form. Since $\alpha$ is outer conjugate to some outer action of $D_n$, we can construct a twisted crossed product $\mathcal{R}(\alpha, \nu)$, where $\nu : D_n \times D_n \to \mathbb{U}$ is a 2-cocycle associated to $\alpha$. Once one finds such a twisted crossed product, it is the hyperfinite $II_1$ factor, and $\mathcal{R}(\alpha, \nu) \supset \mathcal{R}^{(\beta,\mathbb{Z}_2)}$ is an irreducible inclusion, which is $\mathcal{B}_{4m}$-Galois since $\mathrm{ik}_w(\alpha, \beta)$ is the unique non-trivial element in $H^2(\mathrm{Tot}\, E^{\cdot\cdot})$. The inclusion is independent (up to conjugacy) of the choice of $(\alpha, \beta)$ and $\nu$; see Remark 4.4 (2).

We wish to construct such a twisted crossed product explicitly by generators and relations, from two $*$-automorphisms $\chi, \theta \in \mathrm{Aut}(\mathcal{R})$ of period 2 such that the composite $\chi\theta$ has outer period $n$ with $\gamma(\chi\theta) = -1$. Notice that for the outer weak action $(\alpha, \beta)$ as above, we may suppose that $\chi = \alpha_{ax}$ and $\theta = \beta_x$ satisfy the conditions just given, and conversely two such $*$-automorphisms give rise to $(\alpha, \beta)$ as above. We have $u \in \mathbb{U}$ such that $(\chi\theta)^n = \mathrm{Ad}(u)$ with $\chi\theta(u) = -u$. As in [IK, (8.1)], we may suppose

$$\chi(u) = u^{-1}, \quad \theta(u) = -u^{-1}, \tag{46}$$

by replacing $u$ with $\xi^{-1/2}u$, where $\xi = u\chi(u)$. Notice here that $\xi \in \mathbb{T}$, since one sees

$$\mathrm{Ad}(u^{-1}) = \chi(\chi\theta)^n\chi = \chi\,\mathrm{Ad}(u)\chi = \mathrm{Ad}(\chi(u)).$$

Set $b = a^2$, $y = ax$ in $G$. Then,

$$G = D_n = \langle b, y \mid b^m = 1 = y^2, \quad yb = b^{-1}y \rangle.$$

Let $\varphi = (\chi\theta)^2$. Then,

$$\varphi(u) = u. \tag{47}$$

$\chi, \theta$ and $\varphi$ give rise to an outer weak action $(\alpha, \beta)$ of $(D_n, \mathbb{Z}_2)$ given by

$$\alpha_{b^i y^j} = \varphi^i \chi^j, \quad \beta_{x^j} = \theta^j \quad (0 \le i < m, \quad 0 \le j < 2). \tag{48}$$

By definition an $\mathcal{R}$-*ring* is a pair of an algebra $\mathcal{P}$ and an algebra homomorphism $\mathcal{R} \to \mathcal{P}$. Any twisted crossed product $\mathcal{R}(\alpha, \nu)$ is naturally an $\mathcal{R}$-ring.

Proposition 5.6. *Let $\mathcal{P}$ denote the $\mathcal{R}$-ring generated by two elements $B, Y$, and defined by the relations*

$$B^m = u, \quad Y^2 = 1, \quad YB = B^{-1}Y,$$
$$Br = \varphi(r)B, \quad Yr = \chi(r)Y \quad (r \in \mathcal{R}).$$

*There exists a 2-cocycle $\nu : D_n \times D_n \to \mathbb{U}$ associated to the $\alpha$ as in (48), such that*

$$f : \mathcal{R}(\alpha, \nu) \to \mathcal{P}, \quad f(r \rtimes b^i y^j) = rB^iY^j \quad (0 \le i < m, \quad 0 \le j < 2)$$

*gives an isomorphism of $\mathcal{R}$-rings. Hence $\mathcal{P}$ is the hyperfinite $II_1$-factor including $\mathcal{R}$, in which $B$ and $Y$ are unitaries. Moreover, $\mathcal{P} \supset \mathcal{R}^\theta$ is an irreducible inclusion which is $\mathcal{B}_{4m}$-Galois, where $\mathcal{R}^\theta = \{r \in \mathcal{R} \mid \theta(r) = r\}$.*

Proof. It is enough to prove the existence of $\nu$. An easy application of the diamond lemma [B, Prop. 7.1] proves that $B^iY^j$ ($0 \le i < m$, $0 \le j < 2$) form a left $\mathcal{R}$-free basis in $\mathcal{P}$. A point is to see that the equations (46), (47) resolve overlap ambiguities among the relations for $\mathcal{P}$; for examples, since $\varphi(u) = u$ implies that $BB^m = Bu = uB = B^mB$, the overlap ambiguity between $B^m = u$ and itself is resolved.

It follows that $\mathcal{P}$ forms a twisted crossed product of $D_n$ over $\mathcal{R}$ with $b^iy^j$-component $\mathcal{R}B^iY^j$, where $0 \le i < m$, $0 \le j < 2$. Since

$$B^iY^jr = \varphi^i\chi^j(r)B^iY^j = \alpha_{b^iy^j}(r)B^iY^j,$$

we have a desired $\nu$, which is indeed $\mathbb{U}$-valued since $u \in \mathbb{U}$. □

In a different manner, Izumi and Kosaki [IK, Thm. 8.3] give explicitly an irreducible inclusion corresponding to Kac and Paljutkin's algebra $\mathcal{B}_8$.

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