J. Math. Sci. Univ. Tokyo
10 (2003), 537–553.

On an Ergodic Property of Diffusion Semigroups on Euclidean Space

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Abstract. Consider a class of uniform elliptic diffusion processes on Euclidean spaces. We estimate transition densities and their derivatives uniformly with respect to the starting points. We use these results to prove an ergodicity of ∇P_t^c (see (1.2) and Theorem 1.4) under certain conditions. This is useful in precise estimate of large deviation principles.

1. Introduction

Let N be an integer, $W = \{w \in C([0,\infty); \mathbf{R}^N) : w(0) = 0\}$, and μ the standard Wiener measure on W.

Let $\alpha_{ij} \in C_b^\infty(\mathbf{R}^N)$, $i, j = 1, \cdots, N$, and assume that there exist $c_1, c_2 > 0$ such that

$$c_1 \sum_{i=1}^N \xi_i^2 \le \sum_{j=1}^N (\sum_{i=1}^N \alpha_{ij}(x)\xi_i)^2 \le c_2 \sum_{i=1}^N \xi_i^2, \qquad \text{for any } x, \xi \in \mathbf{R}^N.$$

Let $\beta_i \in C^\infty(\mathbf{R}^N)$, $i = 1, \cdots, N$. We assume the following through the paper.

(A-1) There exists a $c_3 > 0$ such that

$$\sum_{i,j=1}^N \xi_i \xi_j \nabla_i \beta_j(x) \le c_3 \sum_{i=1}^N \xi_i^2, \qquad \text{for any } x, \xi \in \mathbf{R},$$

where $\nabla_i = \frac{\partial}{\partial x_i}$, $i = 1, \cdots, N$.

Let us consider the following stochastic differential equation (SDE).

$$(1.1) \qquad \begin{cases} dX_i(t,x) = \sum_{j=1}^N \alpha_{ij}(X(t,x))dw_j(t) + \beta_i(X(t,x))dt, \\ \qquad\qquad i = 1, \cdots, N, \\ X(0,x) = (X_1(0,x), \cdots, X_N(0,x)) = x \in \mathbf{R}^N. \end{cases}$$

1991 *Mathematics Subject Classification.* 60J60, 37A25.
Key words: ergodicity, diffusion process.

Our first main result is the following.

THEOREM 1.1. *There exists a modification* $X : [0,\infty)\times\mathbf{R}^N\times W \to \mathbf{R}^N$ *of the solution of (1.1) satisfying the following.*
(1) $X(t,\cdot,w) : \mathbf{R}^N \to \mathbf{R}^N$ *is a smooth function for any* $t \in [0,\infty)$ *and any* $w \in W$.
(2) $\frac{\partial^\gamma}{\partial x^\gamma}X(\cdot,\cdot,w)$ *is continuous on* $[0,\infty)\times\mathbf{R}^N$ *for any* $w \in W$ *and multi index* γ.
(3) $\sup_{x\in\mathbf{R}^N}\sum_{i,j=1}^N E^\mu[\sup_{t\in[0,T]}|\nabla_i X^j(t,x)|^p] < \infty$, *for any* $p > 1, T > 0$.

Let $C_b(\mathbf{R}^N)$ denote the set of bounded continuous functions defined on $\mathbf{R}^N$. We may regard $C_b(\mathbf{R}^N)$ as a Banach space with norm $\| f \|_\infty = \sup_{x\in\mathbf{R}^N}|f(x)|$, $f \in C_b(\mathbf{R}^N)$. For any $c \in C_b(\mathbf{R}^N)$, we define the semi-group P_t^c, $t \in [0,\infty)$, on $C_b(\mathbf{R}^N)$ by

$$(1.2) \quad (P_t^c f)(x) = E^\mu[\exp(\int_0^t c(X(s,x))ds)f(X(t,x))], \qquad f \in C_b(\mathbf{R}^N).$$

In case of $c = 0$, we denote P_t^c by P_t. Then we have the following result essentially due to Kusuoka-Stroock [4].

THEOREM 1.2. *There exists a strictly positive smooth function p defined on* $(0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N$ *such that* $(P_t f)(x) = \int_{\mathbf{R}^N} p(t,x,y)f(y)dy, f \in C_b(\mathbf{R}^N)$, *and*

$$\sup\{|\frac{\partial^\gamma}{\partial y^\gamma}p(t,x,y)| + |\frac{\partial}{\partial x_i}\frac{\partial^\gamma}{\partial y^\gamma}p(t,x,y)|;\ x,y \in \mathbf{R}^N, |y| \le r\} < \infty$$

for any $t > 0$, *multi index* γ *and* $r > 0$.

Now let us introduce the following assumption.
(A-2) $\sup\{\mu(|X(t,x)| \ge n); x \in \mathbf{R}^N\} \to 0$ as $n \to \infty$ for any $t > 0$.

THEOREM 1.3. *If there exists an increasing convex function* $\varphi : [0,\infty) \to \mathbf{R}$ *such that* $\varphi(s) \to \infty$ *as* $s \uparrow \infty$, $\int^\infty \frac{ds}{\varphi(s)} < \infty$, *and that*

$$x\cdot\beta(x) \le -\varphi(|x|^2), \qquad x \in \mathbf{R}^N,$$

then (A-2) *is satisfied.*

Let $C_b^1(\mathbf{R}^N)$ denote the set of continuously differentiable functions f such that f itself and its derivatives $\nabla_i f$, $i = 1, \cdots, N$, are bounded.

THEOREM 1.4. *Assume* (A-1) *and* (A-2). *Then we have the following.* (1) *For any* $c \in C_b(\mathbf{R}^N)$ *and* $t > 0$, *the linear operator* P_t^c *defined on* $C_b(\mathbf{R}^N)$ *is compact. Moreover, there exist an* $h^c \in C_b(\mathbf{R}^N)$, *a probability measure* ν^c *on* $\mathbf{R}^N$ *and constants* $\lambda^c \in \mathbf{R}$, $\varepsilon > 0$ *such that* $\inf\{h^c(x);\ x \in \mathbf{R}^N\} > 0$ *and*

$$\begin{aligned}&\| \exp(-\lambda^c t) P_t^c f - (\int_{\mathbf{R}^N} \frac{f}{h^c} d\nu^c) h^c \|_\infty \\ &\qquad \le \varepsilon^{-1} \exp(-\varepsilon t) \| f \|_\infty, \qquad t \ge 0,\ f \in C_b(\mathbf{R}^N).\end{aligned}$$

(2) *Further, if* $c \in C_b^1(\mathbf{R}^N)$, *then* $h^c \in C_b^1(\mathbf{R}^N)$, *and there exits a constant* $\delta > 0$ *such that*

$$\begin{aligned}&\sum_{i=1}^N \| \exp(-\lambda^c t) \nabla_i (P^c f) - (\int_{\mathbf{R}} \frac{f}{h^c} d\nu^c) \nabla_i h^c \|_\infty \\ \le\ & \delta^{-1} \exp(-\delta t)(\| f \|_\infty + \sum_{i=1}^N \| \nabla_i f \|_\infty), \qquad t \ge 0,\ f \in C_0^\infty(\mathbf{R}^N).\end{aligned}$$

The problem of Theorem 1.1 for the case of bounded coefficients has been discussed, for example, by Ikeda-Watanabe [2]. Kusuoka-Stroock [3] [4] considered the similar question as Theorem 1.2, but under different conditions. We use them to prove Theorem 1.4. With the help of Theorem 1.4, it is easy to get an estimate of the derivative of Green operators which is useful in precise estimates of large deviation principles. This problem has been motivated by a work in [5].

The organization of the paper is as follows: In Section 2 we give the proof of Theorem 1.1. In Section 3 we define a new semi-group and use it to prove Theorem 1.2. In Section 4 we give the proof of Theorem 1.3. And the proof of Theorem 1.4 is given in Section 5.

2. Proof of Theorem 1.1

We give a proof of Theorem 1.1 in this section. First notice that by (A-1) we have

$$(2.1) \qquad (x - y) \cdot (\beta(x) - \beta(y))$$

$$= \sum_{i,j=1}^{N} (x_j - y_j)(x_i - y_i) \int_0^1 dt \nabla_i \beta_j (y + t(x-y)) \leq c_3 |x-y|^2.$$

PROPOSITION 2.1. *There exists a version $X(t,x)$ of the solution of (1.1) such that $X(\cdot,\cdot,w) : [0,\infty) \times \mathbf{R}^N \to \mathbf{R}^N$ is continuous for all $w \in W$.*

PROOF. Let $c_4 = c_3 + \sum_{i,j,k=1}^{N} \| \nabla_k \alpha_{ij} \|_\infty < \infty$. It follows from Ito's formula and (2.1) that

$$\begin{aligned} & |X(t,x) - X(t,y)|^2 \\ \leq\ & |x-y|^2 + c_4 \int_0^t |X(s,x) - X(s,y)|^2 ds \\ & +2 \sum_{i,j=1}^{N} \int_0^t (X_i(s,x) - X_i(s,y))(\alpha_{ij}(X(s,x)) - \alpha_{ij}(X(s,y)))dw_j(s). \end{aligned}$$

Let $\tau_n = \inf\{t > 0; |X(t,x) - X(t,y)| > n\}$, $n \geq 1$. For any $p > 2$, we have by Doob's inequality that

$$\begin{aligned} & E^\mu[\sup\{|X(s,x) - X(s,y)|;\ s \in [0, t \wedge \tau_n]\}^{2p}] \\ \leq\ & 3^p \Big(|x-y|^{2p} + c_4{}^p t^{p-1} \int_0^{t\wedge\tau_n} E^\mu |X(s,x) - X(s,y)|^{2p} ds \\ & + (\frac{p}{p-1})^p E^\mu \Big[\Big| \sum_{i,j=1}^{N} \int_0^{t\wedge\tau_n} (X_i(s,x) - X_i(s,y)) \\ & \qquad \times (\alpha_{ij}(X(s,x)) - \alpha_{ij}(X(s,y)))dw_j(s) \Big|^p \Big] \Big). \end{aligned}$$

Therefore, by Burkholder's inequality and Hölder's inequality, there exists a $c_5 > 0$ depending only on C, T and p such that

$$\begin{aligned} & E^\mu \Big[\sup\{|X(s,x) - X(s,y)|;\ s \in [0, t\wedge\tau_n]\}^{2p} \Big] \\ \leq\ & c_5 \Big(|x-y|^{2p} + \int_0^t E^\mu[\sup\{|X(s,x) - X(s,y)|;\ s \in [0,u]\}^{2p}] du \Big) \end{aligned}$$

for any $t \in [0,T], x, y \in \mathbf{R}^N, n \geq 1$. Letting $n \to \infty$ and using Gronwall's inequality, we get that for any $p \in (2,\infty)$ and $T > 0$, there is a $c_6 > 0$ satisfying

$$E^\mu[\sup\{|X(s,x) - X(s,y)|;\ s \in [0,t]\}^{2p}] \leq c_6 |x-y|^{2p}, \qquad x, y \in \mathbf{R}^N.$$

This and Kolomogorov's theorem (c.f., *e.g.*, Stroock-Varadhan [8] or Revuz-Yor [7]) imply our assertion. □

PROOF OF THEOREM 1.1. By Proposition 2.1, we may and do assume that the solution $X(t,x)$ is continuous with respect to (t,x) for $\mu-a.s.w$. Let $\tau_{r,n}(w) = \inf\{t > 0; \max_{|x|\le r} |X(t,x,w)| \ge n\}, r,n \ge 1$. Then $\tau_{r,n}(w) \uparrow \infty$ as $n \to \infty$ for $\mu - a.s.w$. Choose $\beta_{n,i} \in C_b^\infty(\mathbf{R}^N)$, $i = 1, \cdots, N$, such that $\beta_{n,i}(x) = \beta_i(x)$ for $|x| \le n+1$ and $i = 1, \cdots, N$. Let $X_n(t,x)$ be the solution of the SDE

$$\begin{cases} dX_{n,i}(t,x) = \sum_{j=1}^N \alpha_{ij}(X_n(t,x))dw_j(t) + \beta_{n,i}(X_n(t,x))dt, \\ \qquad\qquad i = 1, \cdots, N, \\ X_n(0,x) = x \in \mathbf{R}^N. \end{cases}$$

We recall $X_n(t,x)$ has a modification such that $X_n(t,\cdot,w) : \mathbf{R}^N \to \mathbf{R}^N$ is a smooth function for any $t \in [0,\infty)$ and $w \in W$, and $\frac{\partial^\gamma}{\partial x^\gamma} X_n(\cdot,\cdot,w) :$ $[0,\infty) \times \mathbf{R}^N$ is continuous for any $w \in W$ and multi-index γ (see Ikeda-Watanabe [2]).

Uniqueness of the solution of the SDE implies that

$$\begin{aligned} &\mu(\{w \in W;\ X(t,x) = X_n(t,x) \\ &\qquad \text{for any } (t,x) \in [0,T] \times \mathbf{R}^N \text{ with } |x| \le r\}) \\ \ge\ &\mu(\tau_{r,n} \ge T) \to 1, \qquad \text{as } n \to \infty \end{aligned}$$

for any $T > 0$ and $r \ge 1$. This implies (1) and (2) of Theorem 1.1.

We next turn to (3). It is easy to see that

$$\begin{aligned} d\nabla_i X_j(t,x) &= \sum_{k,\ell=1}^N \nabla_i X_k(t,x) \nabla_k \alpha_{j\ell}(X(t,x)) dw_\ell(t) \\ &\quad + \sum_{k=1}^N \nabla_i X_k(t,x) \nabla_k \beta_j(X(t,x)) dt \end{aligned}$$

and $\nabla_i X_j(0,x) = \delta_{ij}, i,j = 1, \cdots, N$, for μ-*a.s.* w. Therefore,

$$\sum_{i,j=1}^N |\nabla_i X_j(t,x)|^2$$

$$
\begin{aligned}
\leq\ & N + 2\sum_{i,j,k,\ell=1}^{N}\int_0^t \nabla_i X_j(s,x)\nabla_i X_k(s,x)\nabla_k\alpha_{j\ell}(X(s,x))dw_\ell(s)\\
&+\sum_{i,j,\ell=1}^{N}\int_0^t (\sum_{k=1}^{N}\nabla_i X_k(s,x)\nabla_k\alpha_{j\ell}(X(s,x)))^2 ds\\
&+c_3\int_0^t\sum_{i,j=1}^{N}|\nabla_i X_j(s,x)|^2 ds.
\end{aligned}
$$

Use the same argument as in the proof of Proposition 2.1, we obtain (3) of Theorem 1.1. □

3. Semigroup $\{Q_t\}$ and Proof of Theorem 1.2

In this section, we introduce a new semigroup $\{Q_t\}_{t\geq 0}$ and use it to prove Theorem 1.2.

Let $E = C_b(\mathbf{R})^{1+N}$. E is a Banach space with norm $\| g \|_E = \sum_{i=0}^N \| g_i \|_\infty$, $g = (g_0, g_1, \cdots, g_N) \in E$. For each $c \in C_b^1(\mathbf{R}^N)$, let

$$M^c(t,x) = \exp(\int_0^t c(X(s,x))ds), \qquad (t,x) \in [0,\infty) \times \mathbf{R}^N,$$

and $Y_{ij}^c : [0,\infty) \times \mathbf{R}^N \times W \to \mathbf{R}$, $i,j = 0,1,\cdots,N$, be such that

$$
\begin{aligned}
&Y_{00}^c(t,x) = M^c(t,x), && Y_{0i}^c(t,x) = 0,\\
&Y_{i0}^c(t,x) = \nabla_i M^c(t,x), && Y_{ij}^c(t,x) = \nabla_i X_j(t,x) M^c(t,x),\\
& && \qquad i,j = 1,\cdots,N.
\end{aligned}
$$

Let us define the operator Q_t^c, $t \geq 0$, by

$$(Q_t^c g)_i(x) = \sum_{j=0}^N E^\mu[Y_{ij}^c(t,x) g_j(X(t,x))], \qquad i = 0,1,\cdots,N,\ x \in \mathbf{R}^N,$$

for $g = (g_0, g_1, \cdots, g_N) \in E$. In the case that $c = 0$, we denote Q_t^c by Q_t, and Y_{ij}^c by Y_{ij}. Since $\sup_{x\in\mathbf{R}^N} E^\mu[|Y_{ij}^c(t,x)|^2] < \infty$, $i,j = 0,1,\cdots,N$, and $X(t,x)$, $Y_{ij}^c(t,x)$, $i,j = 0,1,\cdots,N$, are continuous in x for μ-*a.s.*, we have that Q_t^c is a bounded linear operator on E for each $t \geq 0$.

Let $a_{ij}^c \in C(\mathbf{R}^N)$, $i,j = 0,1,\cdots,N$, be given by

$$
\begin{aligned}
&a_{00}^c(x) = c(x), && a_{i0}^c(x) = (\nabla_i c)(x),\\
&a_{0j}^c(x) = 0, && a_{ij}^c(x) = c(x)\delta_{ij}, \qquad i,j = 1,\cdots,N,
\end{aligned}
$$

and let A^c be the bounded linear operator given by

$$(A^c g)_i(x) = \sum_{j=0}^{N} a_{ij}^c(x) g_j(x),$$
$$i = 0, 1, \cdots, N,\ x \in \mathbf{R}^N,\ g = (g_0, g_1, \cdots, g_N) \in E.$$

PROPOSITION 3.1. *Let $c \in C_b^1(\mathbf{R}^N)$. Then we have the following.*
(1) *$\{Q_t^c;\ t \geq 0\}$ is a semigroup*
(2) *For any $f \in C_b^1(\mathbf{R}^N)$ and $t \geq 0$, $P_t^c f \in C_b^1(\mathbf{R}^N)$ and*

$$(3.1) \qquad (P_t^c f, \nabla_1(P_t^c f), \cdots, \nabla_N(P_t^c f)) = Q_t^c((f, \nabla_1 f, \cdots, \nabla_N f))$$

(3) *For any $t > 0$ and $g \in E$,*

$$Q_t^c g = Q_t g + \int_0^t Q_{t-s}^c A^c Q_s g\, ds.$$

PROOF. Notice that $dM^c(t,x) = c(X(t,x))M^c(t,x)dt$ and

$$\begin{aligned} d\nabla_i M^c(t,x) &= \Big(c(X(t,x))\nabla_i M^c(t,x) \\ &\qquad + (\sum_{k=1}^{N} \nabla_i X_k(t,x)(\nabla_k c)(X(t,x)))M^c(t,x)\Big)dt. \end{aligned}$$

Let $a_{ijk} \in C(\mathbf{R}^N)$, $i, j, k = 0, 1, \cdots, N$, be given by

$$a_{000}(x) = a_{00k}(x) = a_{i00}(x) = a_{i0k}(x) = a_{0j0}(x) = a_{0jk}(x) = 0,$$
$$a_{ij0}(x) = \nabla_i \beta_j(x), \qquad a_{ijk}(t,x) = \nabla_i \alpha_{jk}(x), \qquad i, j, k = 1, \cdots, N.$$

Then

$$\begin{aligned} dY_{ij}^c(t,x) &= \sum_{k=0}^{N}\sum_{\ell=1}^{N} Y_{ik}^c(t,x) a_{kj\ell}(X(t,x)) dw_\ell(t) \\ &\qquad + \sum_{k=0}^{N} Y_{ik}^c(t,x)(a_{kj0}(X(t,x)) + a_{kj}^c(X((t,x)))dt, \\ Y_{ij}^c(0,x) &= \delta_{ij}, \qquad i, j = 0, 1, \cdots, N. \end{aligned}$$

Define $\theta : [0,\infty)\times W \to W$ by $\theta(s,w)(t) = w(t+s) - w(s)$, $s,t \in [0,\infty)$, $w \in W$. Then we have from the uniqueness of the solution of SDE that

$$Y^c_{ij}(t+s,x,w) = \sum_{k=0}^{N} Y^c_{ik}(t,x,w)Y^c_{kj}(s,X(t,x,w),\theta(t,w)) \tag{3.2}$$

This implies assertion (1).

For any $x, v \in \mathbf{R}^N$ and $f \in C^1_b(\mathbf{R}^N)$, we have

$$\begin{aligned}
&(P^c_t f)(x+v) - (P^c_t f)(x)\\
= \;& \sum_{i=1}^{N} E^{\mu}[\int_0^1 Y^c_{i0}(t,x+sv)f(X(t,x+sv))ds]\\
&\quad + \sum_{i,j=1}^{N} E^{\mu}[\int_0^1 Y^c_{ij}(t,x+sv)\nabla_j f(X(t,x+sv))ds].
\end{aligned}$$

Since $\sum_{i,j}\sup_x E[|Y^c_{ij}(t,x)|^2] < \infty$, this gives us assertion (2).

Let $(Y^{-1}_{ij}(t,x))^N_{i,j=0}$ denote the inverse matrix of $(Y^0_{ij}(t,x))^N_{i,j=0}$. Then by Ito's formula, $d(\sum_{k=0}^N Y^c_{ik}(t,x)Y^{-1}_{kj}(t,x)) = \sum_{k,\ell=0}^N Y^c_{ik}(t,x)a^c_{k\ell}(X(t,x))\times Y^{-1}_{\ell j}(t,x)dt$. Therefore, by (3.2),

$$\begin{aligned}
&Y^c_{ij}(t,x,w) - Y^0_{ij}(t,x,w)\\
= \;& \sum_{k,\ell=0}^{N}\int_0^t Y^c_{ik}(s,x,w)a^c_{k\ell}(X(s,x,w))Y^0_{\ell j}(t-s,X(s,x,w),\theta(s,w))ds.
\end{aligned}$$

This implies assertion (3). □

PROPOSITION 3.2. *There exist* $q_{ij} \in C^\infty((0,\infty)\times\mathbf{R}^N\times\mathbf{R}^N)$, $i,j = 0,1,\cdots,N$, *such that*

$$\begin{aligned}
&q_{0j} = 0, j = 1,\cdots,N,\\
&\sup\{|\frac{\partial^\gamma}{\partial y^\gamma}q_{ij}(t,x,y)|;\ x,y\in\mathbf{R}^N \text{ with } |y|\le r\} < \infty
\end{aligned} \tag{3.3}$$

for any $t > 0$, $r \ge 1$ *and multi-index* γ, *and*

$$\begin{aligned}
(Q_t g)_i(x) \;=\; &\sum_{j=0}^{N}\int_{\mathbf{R}^N} q_{ij}(t,x,y)g_j(y)dy,\\
&\quad t>0,\ g=(g_0,g_1,\cdots,g_N)\in E.
\end{aligned} \tag{3.4}$$

Proof. The proof is similar to that of Kusuoka-Stroock [4, Theorem 4.5], so we give only a sketch.

Let us take an arbitrary $r \geq 1$ and fix it for a while. Choose a $\beta' \in C_0^\infty(\mathbf{R}^N; \mathbf{R}^N)$ such that $\beta'(x) = \beta(x)$ for any $x \in \mathbf{R}^N$ with $|x| \leq r+3$. Let $X'(t,x)$ be the solution of the SDE

$$\begin{cases} dX_i'(t,x) = \sum_{j=1}^N \alpha_{ij}(X'(t,x))dw_j(t) + \beta_i'(X'(t,x))dt, \qquad i = 1, \cdots, N, \\ X'(0,x) = x \in \mathbf{R}^N. \end{cases}$$

By Theorem 1.1, we may and do assume that $X'(t,x)$ is smooth in x. Let $\tau'(w,x) = \inf\{t \geq 0;\ |X'(t,x)| = r+3\}$. By using the uniqueness of the SDE's for $X(t,x)$ and $\nabla_i X_j(t,x)$, $i,j = 1, \cdots, N$, we have that $X(t,x,w) = X'(t,x,w)$ and $\nabla_i X_j(t,x) = \nabla_i X_j'(t,x)$, $t \leq \tau'(x,w)$, for any $x \in \mathbf{R}^N$ with $|x| \leq r+1$. Define $Y_{ij}' : [0,\infty) \times \mathbf{R}^N \times W \to \mathbf{R}$, $i,j = 0,1,\cdots,N$, by $Y_{ij}'(t,x) = \delta_{ij}$ if $i = 0$ or $j = 0$, and $Y_{ij}'(t,x) = \nabla_i X_j'(t,x)$ if $i \neq 0$ and $j \neq 0$. Let Q_t', $t > 0$ be the bounded linear operator on E given by

$$\begin{aligned} (Q_t' g)_i(x) \quad &= \quad \sum_{j=0}^N E^\mu[Y_{ij}'(t,x) g_j(X'(t,x)), t < \tau'(x)], \\ &\qquad g = (g_0, g_1, \cdots, g_N) \in E. \end{aligned}$$

Let $B_R = \{x \in \mathbf{R}^N;\ |x| < R\}$, $R > 0$. By the same argument as in Kusuoka-Stroock [4, Section 4], there exist $q_{ij}'(t,x,\cdot) \in L^1(\mathbf{R}^N, dx)$, $i,j = 0,1,\cdots,N$, such that $q_{ij}'(t,x,\cdot)\Big|_{B_{r+2}} \in C^\infty(B_{r+2})$ for any $(t,x) \in (0,\infty) \times \mathbf{R}^N$ with $|x| \leq r+1$, $(Q_t' g)_i(x) = \sum_{j=0}^N \int_{\mathbf{R}^N} q_{ij}'(t,x,y) g_j(y) dy$ for any $g \in E$,

$$\sup\{\frac{\partial^\gamma}{\partial y^\gamma} q_{ij}'(t,x,y); x,y \in B_{r+1}\} < \infty, \qquad t > 0,$$

and

$$\begin{aligned} \sup\{\frac{\partial^\gamma}{\partial y^\gamma} q_{ij}'(t,x,y); (t,x,y) \in (0,T] \times \mathbf{R}^N \times B_r \\ \text{with } |x| = r+1\} < \infty, \qquad T > 0, \end{aligned}$$

for any multi-index γ. Moreover, by definition, we have $q_{ij}'(t,x,\cdot) = 0$ if one of the following holds: $i = 0, j \neq 0$ or $i \neq 0, j = 0$. Now define $\tau_n(x,w)$,

$n \geq 0$, and $\sigma_n(x,w)$ $n \geq 1$, $x \in \mathbf{R}^N$, inductively, by

$$\begin{aligned}
&\tau_0(x,w) = 0, \\
&\sigma_n(x,w) = \inf\{t \geq \tau_{n-1}(x,w); |X(t,x,w)| \leq r+1\}, \\
&\tau_n(x,w) = \inf\{t \geq \sigma_n(x,w); |X(t,x,w)| \geq r+3\}, \qquad n \geq 1.
\end{aligned}$$

Then we see that for any $g \in C_0(B_r)^{1+N} \subset E$,

$$\begin{aligned}
(Q_t g)_i(x) &= \sum_{n=1}^{\infty}\sum_{j=0}^{N} E^{\mu}[Y_{ij}(t,x)g_j(X(t,x)); \sigma_n(x) \leq t < \tau_n(x)] \\
&= \sum_{n=1}^{\infty}\sum_{j=0}^{N} E^{\mu}[Y_{ij}(\sigma_n(x),x)(Q'_{t-\sigma_n(x)}g)_j(X(\sigma_n(x),x)); \sigma_n(x) \leq t] \\
&= \sum_{k=0}^{N} \int_{B_r} dy g_k(y) \\
&\quad \times (\sum_{n=1}^{\infty}\sum_{j=0}^{N} E^{\mu}[Y_{ij}(\sigma_n(x),x)(q'_{jk}(t-\sigma_n(x), X(\sigma_n(x),x), y); \\
&\qquad\qquad \sigma_n(x) \leq t]).
\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2. For any $f \in C_0^{\infty}(\mathbf{R}^N)$, we have by (3.1) and (3.4) that

$$\begin{aligned}
P_t f(x) &= \int_{\mathbf{R}^N} q_{0,0}(t,x,y)f(y)dy, \\
(\nabla_i P_t f)(x) &= \int_{\mathbf{R}^N} q_{i,0}(t,x,y)f(y)dy - \sum_{j=1}^{N}\int_{\mathbf{R}^N} \frac{\partial}{\partial y_j} q_{i,j}(t,x,y)f(y)dy.
\end{aligned}$$

This combined with (3.3) yields Theorem 1.2. □

4. Proof of Theorem 1.3

We prove Theorem 1.3 in this section.

Let $c_7 = \sum_{i,j=1}^{N} \|\alpha_{ij}(x)\|_{\infty}^2$. By assumption, there exists an $s_0 > 0$ such that

$$t_0 := \int_{s_0}^{\infty} \frac{ds}{\varphi(s) - c_7} < \infty.$$

Let $a(t) \geq s_0$, $t \in (0, t_0]$, be given by

$$\int_{a(t)}^{\infty} \frac{ds}{\varphi(s) - c_7} = t.$$

Let $v(t,x) = E^{\mu}[|X(t,x)|^2]$, $(t,x) \in [0,\infty) \times \mathbf{R}^N$. Then by assumption and Ito's formula, we have

$$\frac{d}{dt} v(t,x) \leq c_7 - E^{\mu}[\varphi(|X(t,x)|^2)] \leq c_7 - \varphi(v(t,x)).$$

Let $\tau(x) = \inf\{t \geq 0;\ v(t,x) \leq s_0\}$. Then

$$\begin{array}{ll} -\frac{1}{\varphi(v(t,x)) - c_7} \frac{d}{dt} v(t,x) \geq 1, & 0 < t < \tau(x), \\ v(t,x) \leq s_0, & t > \tau(x). \end{array}$$

This implies that

$$\int_{v(t,x)}^{v(0,x)} \frac{ds}{\varphi(s) - c_7} \geq t, \qquad \text{for } 0 < t \leq \tau(x) \wedge t_0.$$

Hence $v(t,x) \leq a(t)$, $0 < t \leq \tau(x) \wedge t_0$. So

$$v(t,x) \leq a(t), \qquad t \in (0, t_0].$$

Therefore,

$$\sup_{x \in \mathbf{R}^N} \mu(|X(t,x)| > r) \leq \frac{a(t)}{r^2}, \qquad r > 0, t \in (0, t_0].$$

This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.4

Throughout this section we assume (A-1) and (A-2).

PROPOSITION 5.1. (1) *P_t^c is a compact operator on $C_b(\mathbf{R}^N)$ for any $c \in C_b(\mathbf{R}^N)$ and $t > 0$.*
(2) *Q_t^c is a compact operator on E for any $c \in C_b^1(\mathbf{R}^N)$ and $t > 0$.*

PROOF. Since the proofs are similar, we give only the proof of assertion (2). By Proposition 3.1 (3), it is sufficient to prove that Q_t is compact on E for any $t > 0$.

Choose and fix $\varphi \in C_0^\infty(\mathbf{R}^N;[0,1])$ with $\varphi(x)=1$ for $|x|\le 1$. Let $\varphi_n(x)=\varphi(nx)$, $x\in\mathbf{R}^N$, $n\ge 1$. Let $Q_{t,n}$ be the linear operator on E given by $Q_{t,n}g=Q_t(\varphi_n^2 g)$, $g\in E$. First, we prove that $Q_{t,n}$ is compact on E for any $t>0$ and $n\ge 1$.

Fix $t>0$ and $n\ge 1$ for a while. Let $\{g^{(m)}\}_{m=1}^\infty$ be any bounded sequence in E. Then the sequence $\{(1-\triangle)^{-1}(\varphi_n g^{(m)})\}_{m=1}^\infty$ is relatively compact in E. So by taking subsequence if necessary, we may and do assume that $\{(1-\triangle)^{-1}(\varphi_n g^{(m)})\}_{m=1}^\infty$ is convergent in E. Since

$$(Q_{t,n}g^{(m)})_i(x)=\sum_{j=0}^N\int_{\mathbf{R}^N}(1-\triangle_y)(\varphi_n(y)q_{ij}(t,x,y))(1-\triangle)^{-1}(\varphi_n g^{(m)})(y)dy,$$

we get by Proposition 3.2 that $\| Q_{t,n}g^{(m)}-Q_{t,n}g^{(m')}\|_E\to 0$ as $m,m'\to\infty$. This finishes the proof of the fact that $Q_{t,n}$ is compact on E for any $t>0$ and $n\ge 1$.

Notice that

$$\begin{aligned}
&\| Q_{t,n}g-Q_tg\|_E\\
\le\ &\sum_{i,j=0}^N\sup_{x\in\mathbf{R}^N}|E^\mu[Y_{ij}^0(t,x)(1-\varphi_n(X(t,x))^2)g_j(X(t,x))]|\\
\le\ &(\sum_{i,j=0}^N\sup_{x\in\mathbf{R}^N}E^\mu[|Y_{ij}^0(t,x)|^2]^{1/2})\sup_{x\in\mathbf{R}^N}\mu(|X(t,x)|\ge n)^{1/2}\| g\|_E,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{for any } g\in E.
\end{aligned}$$

So $Q_{t,n}\to Q_t$ $(n\to\infty)$ as operators on $C_b(E)$. Hence Q_t is also compact on E. $\square$

PROPOSITION 5.2. *For $c\in C_b(\mathbf{R}^N)$ and $t>0$, there exist an $h\in C_b(\mathbf{R}^N)$, a probability measure ν in $\mathbf{R}^N$, and $\lambda_0,C,\varepsilon>0$ such that $P_t^ch=\lambda_0h$, $\int_{\mathbf{R}^N}hd\nu=1$, $\inf\{h(x);x\in\mathbf{R}^N\}>0$ and*

$$\|\lambda_0^{-n}(P_t^c)^nf-(\int_{\mathbf{R}^N}\frac{f}{h}d\nu)h\|_\infty\le C(1-\varepsilon)^n\| f\|_\infty,\qquad f\in C_b(\mathbf{R}^N).$$

PROOF. Let $c\in C_b(\mathbf{R}^N)$ and $t>0$ be given. We first prove the following.

Claim 1. If $f \in C_b(\mathbf{R}^N)$ satisfies $f \geq 0$ and $f \neq 0$, then $\inf\{(P_t^c f)(x); x \in \mathbf{R}^N\} > 0$.

Proof of Claim 1. By virtue of support theorem (c.f. Stroock-Varadhan [8]), we have that $\mu(X(t/2, x) \in U) > 0$, $x \in \mathbf{R}^N$, for any non-void open set U in $\mathbf{R}^N$. So $(P_{t/2}^c f)(x) > 0$, $x \in \mathbf{R}^N$. By (A-2), we see that there exists a $r > 0$ such that $\mu(|X(t/2, x)| \leq r) \geq 1/2$ for all $x \in \mathbf{R}^N$. Therefore,

$$\inf\{(P_t^c f)(x); x \in \mathbf{R}^N\} = \inf\{P_{t/2}^c(P_{t/2}^c(f))(x); x \in \mathbf{R}^N\} > 0.$$

This implies Claim 1.

Let B be the complex Banach space given by $B = C_b^c(\mathbf{R}^N; \mathbf{C})$ with norm $\| f \|_B = \sup_{x \in \mathbf{R}^N} |f(x)|$, $f \in B$. Then B is the complex extension of $C_b(\mathbf{R}^N)$. So the bounded linear operator P_t^c can be extended to a bounded linear operator on B. We denote this by the same symbol P_t^c. P_t^c is a compact linear operator on B, and the spectrum $\sigma(P_t^c)$ of P_t^c has no cluster point except 0. Let $\lambda_0 = \max\{|\lambda|; \lambda \in \sigma(P_t^c)\}$.

Claim 2. Suppose that $\lambda \in \sigma(P_t^c)$ with $|\lambda| = \lambda_0$ and $f \in B$ satisfies $f \neq 0$ and $P_t^c f = \lambda f$. Then $\lambda = \lambda_0$ and there exists an $a \in \mathbf{C}$ such that $f = a|f|$.

Proof of Claim 2. It is obviuos that $P_t^c(|f|) - |\lambda||f| \geq 0$. So it is sufficient to prove $P_t^c(|f|) = |\lambda||f|$. Let $h = P_t^c(|f|) \in C_b(\mathbf{R}^N)$. Then $P_t^c h - \lambda_0 h \geq 0$, and $\inf\{h(x); x \in \mathbf{R}^N\} > 0$ by Claim 1. Suppose that $P_t^c h - \lambda_0 h \neq 0$. Then by Claim 1, there exists a $\delta > 0$ such that $P_t^c(P_t^c h) \geq (\lambda_0 + \delta) P_t^c h$. Therefore we have $\limsup_{n \to \infty} \frac{1}{n} \log \| (P_t^c)^n h \|_B \geq \log(\lambda_0 + \delta)$. This contradicts the fact that $\lim_{n \to \infty} \frac{1}{n} \log \| (P_t^c)^n \|_{operator} = \log \lambda_0$. So we have $P_t^c h = \lambda_0 h$. This and Claim 1 imply $P_t^c(|f|) = |\lambda||f|$, which completes the proof of Claim 2.

By Claims 1 and 2, we see that $\lambda_0 \in \sigma(P_t^c)$ and there exists an $h \in C_b(\mathbf{R}^N)$ such that $P_t^c h = \lambda_0 h$ and $\inf_{x \in \mathbf{R}^d} h(x) > 0$. Let $E(\lambda_0) = E(\lambda_0, P_t^c)$ be the projection operator as in Dunford-Schwartz [1, Chapter VII].

Claim 3. The dimension of the image of $E(\lambda_0)$ is one.

Proof of Claim 3. Suppose not. Then we have by Claim 2 that there exists an $f \in B$ such that $\lambda_0^{-n} \parallel (P_t^c)^n f \parallel_B \to \infty$ as $n \to \infty$. On the other hand, since $\inf_{x \in \mathbf{R}^N} h(x) > 0$, there exists a constant $c_f > 0$ such that $|f| \leq c_f h$, hence $\limsup_{n\to\infty} \lambda_0^{-n} \parallel (P_t^c)^n f \parallel_B \leq c_f \parallel h \parallel_\infty$. This makes a contradiction.

By Claim 2, we have that $\sup\{|\lambda|; \lambda \in \sigma(P_t^c) \setminus \{\lambda_0\}\} < \lambda_0$. Combining this with Claim 3, we get that there exist a bounded linear operator $S : B \to \mathbf{C}$ and $C, \varepsilon > 0$ such that

$$\parallel \lambda_0^{-n}(P_t^c)^n f - S(f)h \parallel_B \leq C(1-\varepsilon)^n \parallel f \parallel_B, \qquad f \in B.$$

We can easily see that $S(h) = 1$ and $S(f) \geq 0$ for any $f \in C_b(\mathbf{R}^N, \mathbf{R}^+)$. Moreover $S(\lambda_0^{-1} P_t^c f) = S(f)$ for $f \in B$. For $n \geq 1$, let $\varphi_n(x) = ((n - |x|) \wedge 1) \vee 0$, $x \in \mathbf{R}^N$. Then by Riesz's theorem, there exist finite measures ν_n on $\mathbf{R}^N$ such that $S(\varphi_n f) = \int f d\nu_n$ for $f \in C_b(\mathbf{R}^N)$ and $n \in \mathbf{N}$. We have by (A-2) that $(P_t^c(\varphi_n f))(x) \uparrow (P_t^c f)(x)$ as $n \to \infty$ for $x \in \mathbf{R}^N$ and $f \in C_b(\mathbf{R}^N, \mathbf{R}^+)$. Since P_t^c is compact, we have that $S(\varphi_n f) \to S(f)$ in B as $n \to \infty$ for $f \in C_b(\mathbf{R}^N)$. So there exists a finite measure ν on $\mathbf{R}^N$ such that $S(f) = \int f d\nu$ for $f \in C_b(\mathbf{R}^N)$. Re-normalize ν and h if necessary, and we get Proposition 5.2. □

Proposition 5.3. *Let $c \in C_b(\mathbf{R}^N)$. Then there exist an $h \in C_b(\mathbf{R}^N)$, a probability measure ν on $\mathbf{R}^N$ and $\eta \in \mathbf{R}$, $C, \varepsilon > 0$ (different from before) such that $P_t^c h = \exp(\eta t)h$, $t > 0$, $\int_{\mathbf{R}^N} h d\nu = 1$, $\inf\{h(x); x \in \mathbf{R}^N\} > 0$ and*

$$\parallel \exp(-\eta t)P_t^c f - (\int_{\mathbf{R}^N} \frac{f}{h} d\nu)h \parallel_\infty \leq C\exp(-\varepsilon t) \parallel f \parallel_\infty, \qquad t > 0, \ f \in C_b(\mathbf{R}^N).$$

Proof. By Proposition 5.2, for each $n \geq 0$ there exist a $h_n \in C_b(\mathbf{R}^N)$, a $\lambda_n > 0$ and a probability measure ν_n on $\mathbf{R}^N$ such that $P_{2^{-n}}^c h_n = \lambda_n h_n$, $\int_{\mathbf{R}^N} h_n d\nu_n = 1$, $\inf\{h_n(x); x \in \mathbf{R}^N\} > 0$, and $\lambda_n^{-k} P_{2^{-n}k}^c f \to (\int_{\mathbf{R}^N} \frac{f}{h_n} d\nu_n)h_n$ in $C_b(\mathbf{R}^N)$ as $k \to \infty$ for $f \in C_b(\mathbf{R}^N)$. So $\nu_n = \nu_0$, $h_n = h_0$ and $\lambda_n = \lambda_0^{2^{-n}}$ for $n \geq 1$. Let $\eta = \log \lambda_0$. Since $(P_s^c h_0)(x) \to (P_t^c h_0)(x)$ as

$s \to t$ for each $x \in \mathbf{R}^N$, we get that $P_t^c h_0 = \exp(\eta t) h_0$ for $t > 0$. Also,

$$\begin{aligned} & \| \exp(-\eta t) P_t^c f - (\int_{\mathbf{R}^N} \frac{f}{h_0} d\nu_0) h_0 \|_\infty \\ \leq \quad & \exp(2 \| c \|_\infty) \| \lambda_0^{-n} (P_1^c)^n f - (\int_{\mathbf{R}^N} \frac{f}{h_0} d\nu_0) h_0 \|_\infty \end{aligned}$$

for $t \in [n, n+1]$ and $f \in C_b(\mathbf{R}^N)$. These imply our Proposition. □

Proposition 5.3 implies (1) of Theorem 1.4. Next, we prove (2) of Theorem 1.4.

Let $E^{\mathbf{C}} = C_b(\mathbf{R}^N; \mathbf{C}^{1+N})$. Then $E^{\mathbf{C}}$ is a complex extension of the real Banach space E. Let $c \in C_b^1(\mathbf{R}^N)$ and fix it in the rest of this section. Then for every $t > 0$, Q_t^c can be extended to a compact linear operator on $E^{\mathbf{C}}$. We use the same symbol Q_t^c to denote this. Let R_n denote the linear operator $Q_{2^{-n}}^c$ on $E^{\mathbf{C}}$, $n \geq 0$. The spectrum $\sigma(R_n)$ has no cluster points except zero. Let $E(\lambda; R_n)$, $\lambda \neq 0$, denotes the spectral projection, i.e., $E(\lambda; R_n) = F(R_n)$, where F is a function such that $F = 1$ in a neighborhood of λ and $F = 0$ in a neighborhood of $\sigma(R_n) \setminus \{\lambda\}$ (c.f. Dunford-Schwartz [1, Chapter VII]). Since $R_n = R_{n+1}^2$, we have by [1] the following:

Proposition 5.4.

(1) $\sigma(R_n) = \{\lambda^2; \lambda \in \sigma(R_{n+1})\}$, $n \geq 0$.

(2) $E(\lambda^2, R_n) = E(\lambda, R_{n+1}) + E(-\lambda, R_{n+1})$ *For any* $\lambda \in \mathbf{C} \setminus \{0\}$ *and* $n \geq 0$.

Proof of Theorem 1.4. Let h, ν and η be as in Proposition 5.3, $S_n = \{\lambda \in \sigma(R_n);\ |\lambda| \geq \exp(2^{-n}(\eta - 2))\}$, and $\#(S_n)$ the number of elements of S_n. Then by Proposition 5.4, we have $S_n = \{\lambda^2;\ \lambda \in S_{n+1}\}$, $n \geq 0$. Hence $\#(S_n)$ is non-decreasing in n. Also,

$$\sum_{\lambda \in S_0} E(\lambda; R_0) = \sum_{\lambda \in S_n} E(\lambda; R_n).$$

So $\#(S_n)$ is dominated by the dimension of $\mathrm{Im}\left(\sum_{\lambda \in S_0} E(\lambda; R_0)\right)$. Thus there exist $n_0 \geq 1$ and $M \geq 1$ such that $\#(S_n) = M$ for any $n \geq n_0$. So there exist $\lambda_{n,i}$, $n \geq n_0$, $i = 1, 2, \cdots, M$, such that $S_n = \{\lambda_{n,i};\ i = 1, \cdots, M\}$

and $\lambda_{n,i} = \lambda_{n+1,i}$, $i = 1, \cdots, M$, $n \geq n_0$. Therefore, $E_i = E(\lambda_{n,i}; R_n)$, $i = 1, 2, \cdots, M$, is independent of $n \geq n_0$. By the same argument as in the proof of Proposition 5.3, we have $Q_t^c E_i = E_i Q_t^c$ for $i = 1, \cdots, M$, and there exists a $C > 0$ such that

$$\| Q_t^c - \sum_{i=1}^{M} Q_t^c E_i \|_{operator} \leq C \exp((\eta - 1)t), \qquad t > 0.$$

Let R_t^i, $t \geq 0$, $i = 1, \cdots, M$, be the restriction of Q_t^c on $\mathrm{Im}(E_i)$. Then $\{R_t^i;\ t \geq 0\}$ is a continuous semigroup of linear operators on $\mathrm{Im}(E_i)$. Moreover, $\lambda_{n,i}$ is the unique eigenvalue of $R_{2^{-n}}^i$. So there exists an $\eta_i \in \mathbf{C}$ such that $\exp(\eta_i t)$ is the unique eigenvalue of R_t^i, $i = 1, \cdots, M$.

Now let $f \in C_b^1(\mathbf{R}^N)$. Then we have by Proposition 3.1 (2)that

$$\begin{aligned} &\int_{\mathbf{R}^N} dx(\psi_0(x) - \sum_{i=1}^{N} \nabla_i \psi_i(x))(P_t^c f)(x)dx \\ = \ &\int_{\mathbf{R}^N} dx(\psi_0(x), \cdots, \psi_N(x)) \cdot (Q_t^c(f, \nabla_1 f, \cdots, \nabla_N f))(x) \end{aligned}$$

for any $\psi_i \in C_0^\infty(\mathbf{R}^N)$ and $i = 0, 1, \cdots, N$. Therefore,

$$\begin{aligned} &\exp(-\eta t) \sum_{j=1}^{M} \int_{\mathbf{R}^N} dx(\psi_0(x), \cdots, \psi_N(x)) \cdot (R_t^j(E_j(f, \nabla_1 f, \cdots, \nabla_N f)))(x) \\ \to &(\int_{\mathbf{R}^N} \frac{f}{h} d\nu)(\int_{\mathbf{R}^N} dx(\psi_0(x) - \sum_{i=0}^{N} \nabla_i \psi_i(x))h(x)), \qquad \text{as } t \to \infty. \end{aligned}$$

So

$$\exp(-\eta t) \sum_{i=1}^{M} R_t^i(E_i(f, \nabla_1 f, \cdots, \nabla_N f)) \to (\int_{\mathbf{R}^N} \frac{f}{h} d\nu)(h, \nabla_1 h, \cdots, \nabla_N h)$$

in the sense of Schwartz' distribution. Since $\mathrm{Im}(E_i)$, $i = 1, \cdots, M$, are of finite dimensions and are linearly independent, we get that

$$\begin{aligned} &\exp(-\eta t) R_t^i(E_i(f, \nabla_1 f, \cdots, \nabla_N f)) \to 0, \qquad \text{if } \eta_i \neq \eta, \\ &\exp(-\eta t) R_t^i(E_i(f, \nabla_1 f, \cdots, \nabla_N f)) \\ &\qquad \to (\textstyle\int_{\mathbf{R}^N} \frac{f}{h} d\nu)(h, \nabla_1 h, \cdots, \nabla_N h) \quad \text{if } \eta_i = \eta. \end{aligned}$$

These imply that

$$\begin{cases} E_i(f, \nabla_1 f, \cdots, \nabla_N f) = 0, & \text{if } \eta_i \neq \eta \text{ and } Re(\eta_i) \geq \eta, \\ E_i(f, \nabla_1 f, \cdots, \nabla_N f) & \\ \qquad\qquad = (\int_{\mathbf{R}^N} \frac{f}{h} d\nu)(h, \nabla_1 h, \cdots, \nabla_N h), & \text{if } \eta_i = \eta. \end{cases}$$

So $h \in C_b^1(\mathbf{R}^N)$ and there exist $C, \delta > 0$ such that

$$\begin{aligned} & \| \exp(-\eta t) Q_t^c(f, \nabla_1 f, \cdots, \nabla_N f) - (\int_{\mathbf{R}^N} \frac{f}{h} d\nu)(h, \nabla_1 h, \cdots, \nabla_N h) \|_E \\ \leq \;& C \exp(-\delta t) \end{aligned}$$

for any $t > 0$ and $f \in C_b^1(\mathbf{R}^N)$. This completes the proof of Theorem 1.4. □

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(Received December 4, 2002)

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