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Systems of Renewal Equations on the Line

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Abstract. Systems of integral equations of renewal type on the whole line are considered. Using Banach-algebraic techniques, we study the asymptotic properties of the solutions and give rather general estimates of the rates of convergence.

1. Introduction

Consider the system of renewal equations

$$z_i(x) = g_i(x) + \sum_{j=1}^{n} \int_{\mathbb{R}} z_j(x-u)\, F_{ij}(du), \qquad i = 1, \dots, n, \tag{1}$$

where $g_i \in L_1(\mathbb{R})$, $i = 1, \dots, n$, and $\mathbf{F} = (F_{ij})$ is a matrix of nonnegative finite measures on $\mathbb{R}$. Denote by $\mathbf{a} = (a_1, \dots, a_n)^T$ the column vector with coordinates $a_1, \dots, a_n$. In matrix notation, the system (1) takes the form

$$\mathbf{z}(x) = \mathbf{g}(x) + \mathbf{F} * \mathbf{z}(x), \tag{2}$$

where, by definition, $(\mathbf{F} * \mathbf{z}(x))_i$ is the sum on the right-hand side of (1).

The objectives of the present paper are (i) to obtain a Stone-type decomposition for the matrix renewal measure on the whole line (see Definition 2 in Section 3) and (ii) to apply the resulting decomposition to the study of the asymptotic properties of the solution to (2), deriving rather general submultiplicative rates of convergence for $\mathbf{z}(x)$ as $x \to \pm\infty$. The results obtained will generalize those of [29], where the one-dimensional ($n = 1$) case was considered. It should be noted that the proofs from the one-dimensional case by no means automatically apply to the case of systems

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of renewal equations, and, as pointed out in [5, Chapter 8]: " ... new and powerful methods are required to handle the questions of asymptotic behavior of the solutions."

Systems of renewal equations on the *whole* line do not seem to have been considered previously. As far as results on the half-axis are concerned, the reader is referred to [5, 8, 9, 3, 22, 23, 4, 10, 28].

2. Preliminaries

Let $\mathbf{B}$ be a square matrix of order n and let $\mu_1, \ldots, \mu_n$ be its eigenvalues. The maximum of the $|\mu_j|$ is called the *spectral radius* of $\mathbf{B}$. By the Perron-Frobenius theorem [18, Theorem 9.2.1], every nonnegative irreducible matrix $\mathbf{B}$ has a positive eigenvalue r of multiplicity 1, which is equal to its spectral radius; moreover, there exist positive right and left eigenvectors corresponding to r. If we drop the irreducibility condition, then the following assertion holds [18, Theorem 9.3.1]: Every nonnegative matrix $\mathbf{B}$ has a real eigenvalue r, equal to its spectral radius, and there exists a normalized nonnegative right eigenvector corresponding to r.

We shall need some knowledge from the theory of Banach algebras of measures. In what follows, a key role will be played by Banach algebras of measures with submultiplicative weight functions.

DEFINITION 1. A function $\varphi(x)$, $x \in \mathbb{R}$, is called *submultiplicative* if $\varphi(x)$ is finite, positive, Borel-measurable and $\varphi(0) = 1$, $\varphi(x+y) \leq \varphi(x)\,\varphi(y)$ $\forall x, y \in \mathbb{R}$.

A few examples of such functions on $\mathbb{R}_+ := [0, \infty)$: $\varphi(x) = (1+x)^r$, $r > 0$; $\varphi(x) = \exp(cx^\alpha)$ with $c > 0$ and $\alpha \in (0,1)$; $\varphi(x) = \exp(rx)$ with $r \in \mathbb{R}$. Moreover, if $R(x)$, $x \in \mathbb{R}_+$, is a positive, ultimately nondecreasing regularly varying function at infinity with a nonnegative exponent α (i.e., $R(tx)/R(x) \to t^\alpha$ for $t > 0$ as $x \to \infty$ [12, Section VIII.8]), then there exist a nondecreasing submultiplicative function $\varphi(x)$ and a point $x_0 \in (0, \infty)$ such that $c_1 R(x) \leq \varphi(x) \leq c_2 R(x)$ for all $x \geq x_0$, where c_1 and c_2 are some positive constants [25, Proposition]. The product of a finite number of submultiplicative functions is again a submultiplicative function.

It is well known [15, Section 7.6] that

$$(3) \quad -\infty < r_1 := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} = \sup_{x<0} \frac{\log\varphi(x)}{x} \\ \leq \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_2 < \infty$$

and $M(h) := \sup_{|x|\leq h} \varphi(x) < \infty\ \forall h > 0$.

Consider the collection $S(\varphi)$ of all complex-valued measures κ such that $\|\kappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\,|\kappa|(dx) < \infty$; here $|\kappa|$ stands for the total variation of κ. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures, the product of two elements ν and κ of $S(\varphi)$ is defined as their convolution $\nu * \kappa$ [15, Section 4.16]. The unit element of $S(\varphi)$ is the Dirac measure δ, i.e., the measure of unit mass concentrated at the origin.

Define the Laplace transform of a measure κ as $\hat{\kappa}(s) := \int_{\mathbb{R}} \exp(sx)\,\kappa(dx)$. Relation (3) implies that the Laplace transform of any $\kappa \in S(\varphi)$ converges absolutely with respect to $|\kappa|$ for all s in the strip

$$\Pi(r_1, r_2) := \{s \in \mathbb{C} : r_1 \leq \Re s \leq r_2\}.$$

Let ν be a finite complex-valued measure. Denote by $T\nu$ the σ-finite measure with the density $v(x;\nu) := \nu((x,\infty))$ for $x \geq 0$ and $v(x;\nu) := -\nu((-\infty,x])$ for $x < 0$. In case $\int_{\mathbb{R}} |x|\,|\nu|(dx) < \infty$, $T\nu$ is a finite measure whose Laplace transform is given by $(T\nu)^\wedge(s) = [\hat{\nu}(s) - \hat{\nu}(0)]/s$, $\Re s = 0$, the value $(T\nu)^\wedge(0)$ being defined by continuity as $\int_{\mathbb{R}} x\,\nu(dx) < \infty$.

The absolutely continuous part of an arbitrary distribution F will be denoted by F_c and its singular component, by F_s: $F_s = F - F_c$.

If $\mathbf{F}$ is a matrix of nonnegative measures, then $\mathbf{F}_s$ will denote the matrix whose entries are the singular components of the corresponding elements of $\mathbf{F}$. Similar conventions will be tacitly made for integrals, inequalities and the like, where matrices and vectors are involved.

3. A Stone-Type Decomposition

We first recall Stone's decomposition of the renewal measure in the one-dimensional case. Let F be a probability distribution on $\mathbb{R}$ with $\mu =$

$\int_{\mathbb{R}} x\,F(dx) \in (0, +\infty]$, and let $H = \sum_{n=0}^{\infty} F^{n*}$ be the corresponding renewal measure; here $F^{1*} := F$, $F^{(n+1)*} := F * F^{n*}$, $n \geq 1$, and $F^{0*} := \delta$. Suppose F is *spread-out*, i.e., for some $m \geq 1$, F^{m*} has a nonzero absolutely continuous component. Stone [30] showed that then there exists a decomposition $H = H_1 + H_2$, where H_2 is a finite measure and H_1 is absolutely continuous with bounded continuous density $h(x)$ such that $\lim_{x\to+\infty} h(x) = \mu^{-1}$ and $\lim_{x\to-\infty} h(x) = 0$.

Consider the renewal equation

$$z(t) = g(t) + \int_{\mathbb{R}} z(t-y)\,F(dy) =: g(t) + z * F(t), \tag{4}$$

where $g \in L_1(\mathbb{R})$ and F is a spread-out probability distribution on $\mathbb{R}$ with positive mean μ. The function $z(t) := g * H(t) + c$ is clearly a solution to (4); here c is any constant. So the asymptotic properties of the solutions to (4) are essentially those of the convolution $g * H(t)$. As pointed out in [2] and implemented in [1], Stone's decomposition allows us to obtain an elegant proof of the following statement:

$$g * H(t) \to \begin{cases} \mu^{-1} \int_{\mathbb{R}} g(x)\,dx & \text{as } t \to +\infty, \\ 0 & \text{as } t \to -\infty, \end{cases} \tag{5}$$

provided that $g(x)$ is bounded and $\lim_{|x|\to\infty} g(x) = 0$.

Define the convolution of matrix measures as follows. If $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are two arbitrary matrices of complex-valued measures, then $\boldsymbol{\mu} * \boldsymbol{\nu} := \left(\sum_{k=1}^{n} \mu_{ik} * \nu_{kj}\right)$, provided the usual convolutions on the right-hand side make sense. Let $\mathbf{I}$ be the unit matrix.

Definition 2. A *matrix renewal measure* is a matrix of measures $\mathbf{H} := \sum_{k=0}^{\infty} \mathbf{F}^{k*}$; here $\mathbf{F}^{1*} := \mathbf{F}$, $\mathbf{F}^{(k+1)*} := \mathbf{F}^{k*} * \mathbf{F} = \mathbf{F} * \mathbf{F}^{k*}$, $\mathbf{F}^{0*} := \delta\mathbf{I}$.

In this section, we shall obtain an abstract Banach-algebraic version of Stone's decomposition for the matrix renewal measure $\mathbf{H}$. Here we will assume that $\mathbf{l} \int_{\mathbb{R}} x\,\mathbf{F}(dx)\mathbf{r} \in (0, +\infty)$. By choosing specific Banach algebras of measures, we can apply the result to the study of the rates of convergence in a multidimensional analog of (5). The case $\mathbf{l} \int_{\mathbb{R}} x\,\mathbf{F}(dx)\mathbf{r} = +\infty$ does not seem to admit a Banach-algebraic treatment and will be dealt with by traditional methods in a forthcoming paper.

Let $S(r_1, r_2)$ be the Banach algebra $S(\varphi)$ with $\varphi(x) = \max(e^{r_1 x}, e^{r_2 x})$, where $r_1 \le 0 \le r_2$. We shall also consider Banach algebras $\mathcal{A}$ of measures such that (i) $\mathcal{A} \subset S(r_1, r_2)$ and (ii) each homomorphism $\mathcal{A} \mapsto \mathbb{C}$ is the restriction to $\mathcal{A}$ of some homomorphism $S(r_1, r_2) \mapsto \mathbb{C}$. Property (ii) can be restated as follows: Each maximal ideal M of $\mathcal{A}$ is of the form $M_1 \cap \mathcal{A}$, where M_1 is a maximal ideal of $S(r_1, r_2)$. It follows from the general theory of Banach algebras that if $\nu \in \mathcal{A}$ is invertible in $S(r_1, r_2)$, then $\nu^{-1} \in \mathcal{A}$. There is already a sizeable pool of Banach algebras of measures satisfying the above properties (for an incomplete list, see, e.g., [7, 11, 20, 13, 24]). In some cases, it is the asymptotic behavior of $\nu((x, x+h])$ that characterizes the elements of $\mathcal{A}$; otherwise, a distinguishing feature of the elements of $\mathcal{A}$ is their specific density behavior (or tail behavior). Finally, one can take as $\mathcal{A}$ the Banach algebra $S(\varphi)$ with a general submultiplicative $\varphi(x)$. Denote by L the restriction of Lebesgue measure to $[0, \infty)$.

THEOREM 1. *Let $\mathbf{F}$ be an $n \times n$-matrix whose elements are finite nonnegative measures on $\mathbb{R}$. Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible with spectral radius $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue* 1 *of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l} \int_{\mathbb{R}} x\, \mathbf{F}(dx)\mathbf{r} \in (0, +\infty)$, $\varrho[(\mathbf{F}^{m*})_s^{\wedge}(r_i)] < 1$, $i = 1, 2$, for some integer $m \ge 1$, and that in the strip $\Pi(r_1, r_2)$ there are no nonzero roots of the characteristic equation*

$$\det(\mathbf{I} - \hat{\mathbf{F}}(s)) = 0. \tag{6}$$

Let $\mathcal{A}$ be a Banach algebra with properties (i) *and* (ii)*. Suppose, finally, that $\mathbf{F}$, $T\mathbf{F} \in \mathcal{A}$. Then the matrix renewal measure $\mathbf{H} = \sum_{n=0}^{\infty} \mathbf{F}^{n*}$ admits a Stone-type decomposition $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$, where $\mathbf{H}_2 \in \mathcal{A}$ and $\mathbf{H}_1 = \mathbf{r}\mathbf{l}L/\mu + rT\mathbf{H}_2$ for some $r > r_2$. If, in addition, $T^2\mathbf{F} \in \mathcal{A}$, then $\mathbf{H}_1 - \mathbf{r}\mathbf{l}L/\mu \in \mathcal{A}$.*

PROOF. We shall split the proof into several lemmas. Denote by $\hat{\mathbf{M}}(s)$ the adjugate matrix of $\mathbf{I} - \hat{\mathbf{F}}(s)$. We note that $\det(\mathbf{I} - \hat{\mathbf{F}}(s))$ is a linear combination of products of n factors which are elements of $\mathbf{I} - \hat{\mathbf{F}}(s)$. Hence $\det(\mathbf{I} - \hat{\mathbf{F}}(s))$ is the Laplace transform of some real-valued measure, say α. By the same reason, the entries of $\hat{\mathbf{M}}(s)$ are the Laplace transforms of real-valued measures belonging to $\mathcal{A}$, and we will denote by $\mathbf{M} = (M_{ij})$ the corresponding matrix of measures.

LEMMA 1. *Under the hypotheses of Theorem* 1, $T\alpha \in \mathcal{A}$ *and* $T\mathbf{M} \in \mathcal{A}$, *i.e.*, $TM_{ij} \in \mathcal{A}$ *for all* i, j.

PROOF OF LEMMA 1. It suffices to prove that if ν and κ are any two measures such that $\int_{\mathbb{R}} |x|e^{\gamma x}\,|\nu|(dx) < \infty$, $\int_{\mathbb{R}} |x|e^{\gamma x}\,|\kappa|(dx) < \infty$ and $\nu, \kappa, T\nu, T\kappa \in \mathcal{A}$, then $T\nu * \kappa \in \mathcal{A}$. But this follows immediately from the equality $T(\nu * \kappa) = (T\nu) * \kappa + \hat{\nu}(0)T\kappa$ implied by

$$\frac{\hat{\nu}(s)\hat{\kappa}(s) - \hat{\nu}(0)\hat{\kappa}(0)}{s} = \frac{[\hat{\nu}(s) - \hat{\nu}(0)]\hat{\kappa}(s)}{s} + \frac{\hat{\nu}(0)[\hat{\kappa}(s) - \hat{\kappa}(0)]}{s}, \qquad \Re s = 0.$$

Lemma 1 is proved. □

Let $r > r_2$ and $d(s) := (s - r)\det(\mathbf{I} - \hat{\mathbf{F}}(s))/s = (s - r)\hat{\alpha}(s)/s$, $s \in \Pi(r_1, r_2)$, where $d(0) := -r\hat{\alpha}'(0)$. By Lemma 1, $d(s)$ is the Laplace transform of the measure $D := \alpha - rT\alpha \in \mathcal{A}$ since

$$d(s) = \hat{\alpha}(s) - r\hat{\alpha}(s)/s = \hat{\alpha}(s) - r(T\alpha)^{\wedge}(s) = \hat{D}(s). \tag{7}$$

LEMMA 2. *Under the hypotheses of Theorem* 1, D *is invertible in* $\mathcal{A}$.

PROOF OF LEMMA 2. It suffices to prove invertibility in $S(r_1, r_2)$. Let $\mathcal{M}$ be the space of maximal ideals of the Banach algebra $S(r_1, r_2)$. Each $M \in \mathcal{M}$ induces a homomorphism $h : S(r_1, r_2) \to \mathbb{C}$ and M is the kernel of h. Denote by $\nu(M)$ the value of h at $\nu \in S(r_1, r_2)$. An element $\nu \in S(r_1, r_2)$ has an inverse if and only if $\nu(M) \neq 0$ for all $M \in \mathcal{M}$.

The space $\mathcal{M}$ is split into two sets: $\mathcal{M}_1$ is the set of those maximal ideals which do not contain the collection $L(r_1, r_2)$ of all absolutely continuous measures from $S(r_1, r_2)$, and $\mathcal{M}_2 = \mathcal{M} \setminus \mathcal{M}_1$. If $M \in \mathcal{M}_1$, then the homomorphism induced by M is of the form $h(\nu) = \hat{\nu}(s_0)$, where $r_1 \leq \Re s_0 \leq r_2$. In this case, $M = \{\nu \in S(r_1, r_2) : \hat{\nu}(s_0) = 0\}$ [15, Chapter IV, Section 4]. If $M \in \mathcal{M}_2$, then $\nu(M) = 0$ $\forall \nu \in L(r_1, r_2)$.

We now show that $D(M) \neq 0$ for each $M \in \mathcal{M}$, thus establishing the existence of $D^{-1} \in S(r_1, r_2)$. Actually, if $M \in \mathcal{M}_1$, then, for some $s_0 \in \{r_1 \leq \Re s \leq r_2\}$, we have $D(M) = \hat{D}(s_0) \neq 0$. Now let $M \in \mathcal{M}_2$. By Theorem 1 of [20],

$$h(\nu) = \int_{\mathbb{R}} \chi(x, \nu) \exp(\beta x)\, \nu(dx), \qquad \nu \in S(r_1, r_2), \tag{8}$$

where β is a real number such that $r_1 \le \beta \le r_2$, and the function $\chi(x, \nu)$ of two variables is a generalized character; here we mention only one property of a generalized character to be used later: $|\nu| - \operatorname{ess\,sup}_{x\in\mathbb{R}} |\chi(x, \nu)| \le 1$. By the multiplicative property of the functional $\nu \to \nu(M)$, $\nu \in S(r_1, r_2)$, we have $\mathbf{F}(M)^m = \mathbf{F}^{m*}(M) = (\mathbf{F}^{m*})_s(M)$. Denote $\boldsymbol{\Theta} = (\Theta_{ij}) := (\mathbf{F}^{m*})_s$. By (8), we have, for some $\beta \in [r_1, r_2]$,

$$|\Theta_{ij}(M)| = \left| \int_{\mathbb{R}} \chi(x, \Theta_{ij}) \exp(\beta x)\, \Theta_{ij}(dx) \right| \le \hat{\Theta}_{ij}(\beta). \tag{9}$$

Choose $\lambda \in [0, 1]$ such that $\beta = \lambda r_1 + (1 - \lambda) r_2$. Since $\hat{\Theta}_{ij}(\xi)$ is a convex function of ξ, we have $\hat{\Theta}_{ij}(\beta) \le \lambda \hat{\Theta}_{ij}(r_1) + (1 - \lambda)\hat{\Theta}_{ij}(r_2)$. If $\mathbf{A}$ and $\mathbf{B}$ are matrices such that $|\mathbf{A}| \le \mathbf{B}$, then $\varrho(\mathbf{A}) \le \varrho(\mathbf{B})$ [16, Theorem 8.1.18]. Also, it is easily verified that if $\mathbf{B} = \lambda \mathbf{B}_1 + (1 - \lambda)\mathbf{B}_2$, $\lambda \in [0, 1]$, then $\varrho(\mathbf{B}) \le \lambda \varrho(\mathbf{B}_1) + (1 - \lambda)\varrho(\mathbf{B}_2)$. It follows from (9) that

$$\varrho[\boldsymbol{\Theta}(M)] \le \varrho[\hat{\boldsymbol{\Theta}}(\beta)] \le \lambda \varrho[\hat{\boldsymbol{\Theta}}(r_1)] + (1 - \lambda)\varrho[\hat{\boldsymbol{\Theta}}(r_2)] < 1.$$

Consequently, $\varrho[\mathbf{F}(M)^m] < 1$. It follows that $\varrho[\mathbf{F}(M)]$, being equal to the m-th root of $\varrho[\mathbf{F}(M)^m]$, is also less than 1. Since $T\alpha \in L(r_1, r_2)$, (7) implies $D(M) = \alpha(M) = \det(\mathbf{I} - \mathbf{F}(M)) \ne 0$. So $D(M) \ne 0$ for all $M \in \mathcal{M}$. This means that $\exists D^{-1} \in S(r_1, r_2)$. Lemma 2 is proved. □

Consider the auxiliary matrix

$$\mathbf{q}(s) := \frac{s}{s - r}[\mathbf{I} - \hat{\mathbf{F}}(s)]^{-1}, \qquad s \in \Pi(r_1, r_2), \tag{10}$$

where $\mathbf{q}(0) := -\hat{\mathbf{M}}(0)/[r\hat{\alpha}'(0)]$.

Lemma 3. *Under the hypotheses of Theorem* 1, $\mathbf{q}(s)$ *is the Laplace transform matrix of some* $\mathbf{Q} \in \mathcal{A}$.

Proof of Lemma 3. It follows from (10) that $\mathbf{q}(s) = [1/d(s)]\hat{\mathbf{M}}(s)$. Consequently, by Lemma 2, $\mathbf{q}(s) = \hat{\mathbf{Q}}(s)$, where $\mathbf{Q} = D^{-1} * \mathbf{M} \in \mathcal{A}$, the entries of $\mathbf{Q}$ being the convolutions of D^{-1} with the corresponding elements of $\mathbf{M}$. Lemma 3 is proved. □

For $s \in \Pi(r_1, r_2) \setminus \{0\}$, we have

$$[\mathbf{I} - \hat{\mathbf{F}}(s)]^{-1} = \frac{s-r}{s}\hat{\mathbf{Q}}(s) = -\frac{r\hat{\mathbf{Q}}(0)}{s} - \frac{r[\hat{\mathbf{Q}}(s) - \hat{\mathbf{Q}}(0)]}{s} + \hat{\mathbf{Q}}(s). \tag{11}$$

We will show that (11) implies the following representation for $\mathbf{H}$:

$$\mathbf{H} = r\hat{\mathbf{Q}}(0)L - rT\mathbf{Q} + \mathbf{Q} = -\frac{\hat{\mathbf{M}}(0)}{\hat{\alpha}'(0)}L - rT\mathbf{Q} + \mathbf{Q}. \tag{12}$$

To this end, we shall need some knowledge from the theory of tempered distributions and more lemmas. Denote by $\mathcal{S}_1$ the space of rapidly decreasing functions in $\mathbb{R}$ and by $\mathcal{S}_1'$ the dual space (the space of tempered distributions) [21, Chapter 7]. It is well known that if β is a σ-finite nonnegative measure such that

$$\int_{\mathbb{R}} \frac{\beta(dx)}{(1+x^2)^k} < \infty \tag{13}$$

for some integer $k > 0$, then $\beta \in \mathcal{S}_1'$; here the tempered distribution β is defined by

$$\beta(\phi) := \int_{\mathbb{R}} \phi(x)\, \beta(dx), \qquad \phi \in \mathcal{S}_1.$$

LEMMA 4. *Let β be a σ-finite nonnegative measure such that $\beta([x, x+1]) \le c < \infty$ $\forall x \in \mathbb{R}$. Then* (13) *holds with $k = 1$.*

PROOF OF LEMMA 4. We have

$$\int_{\mathbb{R}} \frac{\beta(dx)}{1+x^2} \le \sum_{k=-\infty}^{-1} \frac{\beta([k, k+1])}{1+(k+1)^2} + \sum_{k=0}^{\infty} \frac{\beta([k, k+1])}{1+k^2} \le 2c \sum_{k=0}^{\infty} \frac{1}{1+k^2} < \infty. \ \square$$

LEMMA 5. *Let $\mathbf{F}$ be an $n \times n$-matrix whose elements are finite nonnegative measures on $\mathbb{R}$. Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1, \dots, l_n)$ and $\mathbf{r} = (r_1, \dots, r_n)^T$ corresponding to the eigenvalue* 1 *of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l}\int_{\mathbb{R}} x\, \mathbf{F}(dx)\mathbf{r} \in (0, +\infty)$. Then all the entries H_{ij} of the matrix renewal measure $\mathbf{H}$ satisfy condition* (13) *for $k = 1$, so that the H_{ij} are tempered distributions.*

Proof of Lemma 5. Although Lemma 5 is intuitive, especially in the light of a corresponding one-dimensional result [12, Chapter XI, Section 9, Theorem 1], its rigorous proof is rather involved. If $\mathbf{F}$ is concentrated on $[0,\infty)$, then the asymptotic behavior of $\mathbf{H}((x,x+h])$ as $x\to+\infty$ is described in [9] for *nonlattice* $\mathbf{F}$. A distinctive feature of *lattice* matrices $\mathbf{F}$ is the fact that all $\mathbf{F}^{k*}$ are concentrated on sets of Lebesgue measure zero. In the case under consideration, the above property of lattice matrices is violated since $\varrho[(\mathbf{F}^{m*})_s(\mathbb{R})]<1$ for some $m\geq 1$. Hence $\mathbf{F}$ is nonlattice. Suppose $\mathbf{F}$ is concentrated on $(0,\infty)$. Then, by Theorem 2.1 of [9], there exists $\lim_{x\to+\infty}\mathbf{H}((x,x+h]):=\mathbf{A}(h)<\infty$ for all $h>0$. In particular, this means that

$$\left(\sup_{x\in\mathbb{R}} H_{ij}((x,x+h])\right):=\mathbf{K}(h)<\infty. \tag{14}$$

We will show that this inequality is also valid when $\mathbf{F}$ is concentrated on $\mathbb{R}$. Suppose, for the time being, that $\mathbf{F}(\mathbb{R})$ ia a primitive [16, Definition 8.5.0] stochastic matrix, i.e., $\sum_{j=1}^n F_{ij}(\mathbb{R})=1\ \forall i$. We shall use a basic factorization identity [19, (2.9)] without describing the probabilistic meaning of its components, since it is of no importance for our purposes. We have

$$\mathbf{I}-z\hat{\mathbf{F}}(s)=[\mathbf{I}-\hat{\mathbf{G}}_-(z,s)][\mathbf{I}-\hat{\mathbf{G}}_+(z,s)],\qquad z\in(0,1],\quad \Re s=0,$$

where $\hat{\mathbf{G}}_\pm(z,s)]$, as functions of s, are matrices of Laplace transforms of nonnegative finite measures. The measures involved are concentrated on $(0,\infty)$ and $(-\infty,0]$, respectively, and they continuously depend on z. Moreover, $\varrho[\hat{\mathbf{G}}_\pm(z,0)]\leq z$. Hence, for $z\in(0,1)$, we have

$$\sum_{k=0}^{\infty} z^k\hat{\mathbf{F}}(s)^k=\sum_{k=0}^{\infty} z^k\hat{\mathbf{G}}_+(z,s)^k[\mathbf{I}-\hat{\mathbf{G}}_-(z,s)]^{-1}. \tag{15}$$

The matrix $\hat{\mathbf{F}}_-(s):=\mathbf{I}-\hat{\mathbf{G}}_-(1,s)$ is invertible in the sense that $[\mathbf{I}-\hat{\mathbf{G}}_-(1,s)]^{-1}$ is the matrix of Laplace transforms of finite (and even nonnegative) measures. Let $\mathbf{F}_-^{-1}$ be the corresponding matrix of measures, i.e., $\hat{\mathbf{F}}_-^{-1}(s)=[\mathbf{I}-\hat{\mathbf{G}}_-(1,s)]^{-1}$. Passing in (15) from Laplace transforms to measures and letting $z\uparrow 1$, we obtain

$$\mathbf{H}(A)=\mathbf{H}_1*\mathbf{F}_-^{-1}(A),\qquad A\in\mathcal{B}, \tag{16}$$

where $\mathcal{B}$ is the σ-algebra of all Borel subsets of $\mathbb{R}$ and $\mathbf{H}_1$ is the matrix renewal measure induced by $\mathbf{F}_+(A) := \mathbf{G}_+(1, A)$, $A \in \mathcal{B}$. Notice that $\mathbf{H}_1$ is concentrated on $[0, \infty)$. In order to apply Theorem 2.1 of [9] to describe the asymptotic behavior of $\mathbf{H}_1((x, x+h])$, we have to show that $\mathbf{F}_+$ satisfies all the necessary requirements, namely, (i) $\varrho[\mathbf{F}_+(\mathbb{R})] = 1$, (ii) $\mathbf{F}_+(\mathbb{R})$ is irreducible, and (iii) $\mathbf{F}_+$ is nonlattice. In the notation of [6], $\mathbf{F}_+(\mathbb{R}) = M_+(0)$. Hence, by Lemma 2 (a) of [6], $\mathbf{F}_+(\mathbb{R})$ is a stochastic matrix, so that $\varrho[\mathbf{F}_+(\mathbb{R})] = 1$. Next, as pointed out in [6, the proof of Theorem 2], a Markov chain with transition matrix $Q = \mathbf{F}_+(\mathbb{R})$ can have only one class of essential states, which in the language of matrix theory means that $\mathbf{F}_+(\mathbb{R})$ is irreducible. Finally, the nonlattice property of $\mathbf{F}_+$ follows from the fact that $(\mathbf{F}_+^{k*})_s(\mathbb{R}) \neq \mathbf{F}_+^{k*}(\mathbb{R})$ for some $k \geq 1$. To prove this, we argue by contradiction. Suppose $(\mathbf{F}_+^{k*})_s(\mathbb{R}) = \mathbf{F}_+^{k*}(\mathbb{R})$ $\forall k$, and let $A \in \mathcal{B}$ be a set of Lebesgue measure zero such that all $(\mathbf{F}_+^{k*})_s$, $k = 1, 2, \dots$ are concentrated on A. Since $\mathbf{r} := (1, \dots, 1)^T$ is a right invariant vector of $\mathbf{F}_+(\mathbb{R})$, we have

$$\mathbf{H}_1(A)\mathbf{r} = \sum_{k=0}^{\infty} \mathbf{F}_+^{k*}(A)\mathbf{r} = \sum_{k=0}^{\infty} (\mathbf{F}_+^{k*})_s(\mathbb{R})\mathbf{r} = (\infty, \dots, \infty)^T. \tag{17}$$

Denote by $\mathbf{F}_-$ a matrix of finite measures such that $\hat{\mathbf{F}}_-(s) = \mathbf{I} - \hat{\mathbf{G}}_-(1, s)$. In view of (16), we have

$$\begin{aligned} \mathbf{H}_1(A) = \mathbf{H} * \mathbf{F}_-(A) &= \left(\sum_{k=0}^{\infty} \mathbf{F}^{mk*} * \sum_{k=0}^{m-1} \mathbf{F}^{k*} * \mathbf{F}_- \right)(A) \\ &\leq \sum_{k=0}^{\infty} \left(\mathbf{F}^{mk*} \right)_s (\mathbb{R}) \sum_{k=0}^{m-1} \mathbf{F}^{k*}(\mathbb{R}) |\mathbf{F}_-|(\mathbb{R}) \\ &\leq \sum_{k=0}^{\infty} \left[(\mathbf{F}^{m*})_s (\mathbb{R}) \right]^k \sum_{k=0}^{m-1} \mathbf{F}^{k*}(\mathbb{R}) |\mathbf{F}_-|(\mathbb{R}) < \infty. \end{aligned} \tag{18}$$

The last inequality follows from $\varrho[(\mathbf{F}_+^{m*})_s(\mathbb{R})] < 1$. Actually,

$$\left[\mathbf{I} - (\mathbf{F}^{m*})_s (\mathbb{R}) \right]^{-1} = \sum_{k=0}^{\infty} \left[(\mathbf{F}^{m*})_s (\mathbb{R}) \right]^k < \infty.$$

The contradiction between (17) and (18) proves that $\mathbf{F}_+$ is nonlattice. Applying [9, Theorem 2.1] to $\mathbf{F}_+$, we see that (14) is valid for $\mathbf{H}_1$ with some constant matrix $\mathbf{K}_1(h)$. It follows from (16) that

$$\sup_{x\in\mathbb{R}} H_{ij}((x,x+h]) \le \sum_{k=1}^{n} [\mathbf{K}_1(h)]_{ik}[\mathbf{F}_-^{-1}(\mathbb{R})]_{kj}, \tag{19}$$

i.e., $\mathbf{H}$ satisfies (14). Let now $\mathbf{F}(\mathbb{R})$ be a primitive but not necessarily stochastic matrix. We set $\mathbf{J} := \operatorname{diag}(r_1,\dots,r_n)$, $\mathbf{G} := \mathbf{J}^{-1}\mathbf{F}\mathbf{J}$, and $\mathbf{H}_G := \sum_{k=0}^{\infty}\mathbf{G}^{k*}$. Since $\mathbf{G}(\mathbb{R}) = \mathbf{J}^{-1}\mathbf{F}(\mathbb{R})\mathbf{J}$ is a primitive stochastic matrix, $\mathbf{H}_G$ satisfies (19). But then

$$\mathbf{H}((x,x+h]) = \mathbf{J}^{-1}\mathbf{H}_G((x,x+h])\mathbf{J} \le \mathbf{J}^{-1}\mathbf{K}_G(h)\mathbf{J} < \infty. \tag{20}$$

Finally, let $\mathbf{F}(\mathbb{R})$ be an irreducible matrix with imprimitivity index $d > 1$ [14]. Rearranging, if necessary, the same rows and columns of $\mathbf{F}(\mathbb{R})^d$, we can decompose it into d primitive block matrices of spectral radius 1. Without loss of generality. we may assume that such a decomposition has already been achieved:

$$\mathbf{F}(\mathbb{R})^d = \begin{pmatrix} \mathbf{F}^{(1)}(\mathbb{R}) & 0 & \dots & 0 \\ 0 & \mathbf{F}^{(2)}(\mathbb{R}) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{F}^{(d)}(\mathbb{R}) \end{pmatrix},$$

where $\mathbf{F}^{(i)}(\mathbb{R})$ are primitive matrices. It follows that all the block matrices $\mathbf{F}^{(i)}$ are of the same type as $\mathbf{F}$ itself; namely, there exists an integer $q \ge 1$ such that $\varrho[(\mathbf{F}^{(i)q*})_s(\mathbb{R})] < 1$ $\forall i$ [28, the proof of Lemma 2]. Put $\mathbf{H}^{(0)} := \sum_{k=0}^{\infty}\mathbf{F}^{kd*}$ and $\mathbf{H}^{(j)} := \mathbf{F}^{j*} * \mathbf{H}^{(0)}$, $j = 1,\dots,d-1$. It is clear that $\mathbf{H} = \sum_{j=0}^{d-1}\mathbf{H}^{(j)}$ and

$$\mathbf{H}^{(0)} = \begin{pmatrix} \mathbf{H}_1^{(0)} & 0 & \dots & 0 \\ 0 & \mathbf{H}_2^{(0)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \mathbf{H}_d^{(0)} \end{pmatrix},$$

where $\mathbf{H}_i^{(0)}$ are the matrix renewal measures induced by $\mathbf{F}^{(i)}$, $i = 1,\dots,d$. Since each of the $\mathbf{H}_i^{(0)}$ satisfies a relation similar to (14), the same is true for $\mathbf{H}^{(0)}$. Next, relations of type (19) with $\mathbf{F}_{-}^{-1}$ replaced by $\mathbf{F}^{j*}$ hold for the $\mathbf{H}^{(j)}$. Finally, "summing up" the corresponding inequalities for $\mathbf{H}^{(j)}$ over j from 0 to $d-1$, we obtain (14) in the general case. To complete the proof of Lemma 5, it now remains to apply Lemma 4. □

Denote by $\mathcal{F}(u)$ the Fourier transform of $u \in \mathcal{S}_1'$: $\mathcal{F}(u)(\phi) := u(\mathcal{F}(\phi))$, $\phi \in \mathcal{S}_1$, where

$$\mathcal{F}(\phi)(t) := (2\pi)^{-1/2} \int_{\mathbb{R}} \phi(x) \exp(-itx)\, dx, \qquad t \in \mathbb{R}.$$

Let ν be a σ-finite measure defining an element in $\mathcal{S}_1'$. For arbitrary $a \in \mathbb{R}$, we set $\nu_a(A) := \nu(A-a)$, $A \in \mathcal{B}$. Define the element $\Delta_a \nu \in \mathcal{S}_1'$ by $\Delta_a \nu := \nu - \nu_a$. Then $\mathcal{F}(\Delta_a \nu) = [1-\exp(-iat)]\mathcal{F}(\nu)$. If ν and κ are any two measures which define tempered distributions and for which the convolution $\nu * \kappa$ makes sense, then obviously $\Delta_a(\nu * \kappa) = \nu * (\Delta_a \kappa) = (\Delta_a \nu) * \kappa$.

LEMMA 6 [26, Lemma 3]. *Let ν be a finite measure. Then the tempered distribution $\mathcal{F}(\Delta_a T\nu)$ may be identified with the function*

$$(2\pi)^{-1/2}(1-e^{-iax})[\hat{\nu}(-ix) - \hat{\nu}(0)]/(-ix), \qquad x \in \mathbb{R}.$$

It is also clear that $\Delta_a L$ is Lebesgue measure on the interval $[0,a]$ and, therefore, the tempered distribution $\mathcal{F}(\Delta_a L)$ can be identified with the function $[1-\exp(-iat)]/[it(2\pi)^{1/2}]$, $t \in \mathbb{R}$.

We now turn to (11). Put $s = -ix$, $x \in \mathbb{R}$, multiply both sides by $(2\pi)^{-1/2}(1-e^{-iax})\phi(x)$, $\phi \in \mathcal{S}_1$, and then integrate the resulting equality over the whole line $\mathbb{R}$. This yields

$$\begin{aligned} (21) \quad & (2\pi)^{-1/2} \int_{\mathbb{R}} (1-e^{-iax})[\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \phi(x)\, dx \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} r\hat{\mathbf{Q}}(0) \frac{1-e^{-iax}}{ix} \phi(x)\, dx \\ &\quad - (2\pi)^{-1/2} \int_{\mathbb{R}} (1-e^{-iax}) \frac{r[\hat{\mathbf{Q}}(-ix) - \hat{\mathbf{Q}}(0)]}{-ix} \phi(x)\, dx \\ &\quad + (2\pi)^{-1/2} \int_{\mathbb{R}} (1-e^{-iax}) \hat{\mathbf{Q}}(-ix) \phi(x)\, dx. \end{aligned}$$

LEMMA 7. *Under the hypotheses of Theorem* 1, *the left-hand side of* (21) *is equal to* $\mathcal{F}(\Delta_a \mathbf{H})(\phi)$.

PROOF OF LEMMA 7. Let $z \in (0,1)$. Define the matrix measure $\mathbf{H}_z$ by

$$\mathbf{H}_z(A) := \sum_{k=0}^{\infty} z^k \mathbf{F}^{k*}(A), \qquad A \in \mathcal{B}.$$

Since $\varrho[z\mathbf{F}(\mathbb{R})] < 1$, the measure $\mathbf{H}_z$ is finite and $\hat{\mathbf{H}}_z(s) = [\mathbf{I} - z\hat{\mathbf{F}}(s)]^{-1}$, $\Re s = 0$. It follows that the tempered distribution $\mathcal{F}(\mathbf{H}_z)$ may be identified with the function $(2\pi)^{-1/2}[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1}$, $x \in \mathbb{R}$. Clearly, $\mathbf{H}_z \to \mathbf{H}$ in $\mathcal{S}_1'$ as $z \to 1-$, and hence $\Delta_a \mathbf{H}_z \to \Delta_a \mathbf{H}$ as $z \to 1-$. Next, $\mathcal{F}(\Delta_a \mathbf{H}_z) \to \mathcal{F}(\Delta_a \mathbf{H})$ as $z \to 1-$. Now to complete the proof of Lemma 7, it suffices to justify the passage to the limit as $z \to 1-$ in

$$\mathcal{F}(\Delta_a \mathbf{H}_z)(\phi) = (2\pi)^{-1/2} \int_{\mathbb{R}} (1 - e^{-iax})[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} \phi(x)\, dx. \tag{22}$$

Subtracting the left-hand side of (21) from the right-hand side of (22) and multiplying both sides of the resulting equation by $(2\pi)^{1/2}$, we have

$$\begin{aligned} \mathbf{A}(z) &:= \int_{\mathbb{R}} (1 - e^{-iax}) \big\{ [\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} - [\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \big\} \phi(x)\, dx \\ &= \int_{\mathbb{R}} (1 - e^{-iax})(z-1)\hat{\mathbf{F}}(-ix)[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1} \\ &\qquad\qquad \times [\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} \phi(x)\, dx. \end{aligned} \tag{23}$$

As $z \to 1-$, the integrand converges to zero for all $x \neq 0$. Now, in order to prove $\lim_{z\to 1-} \mathbf{A}(z) = \mathbf{0}$, it remains to find an integrable majorant. By Theorem 8.6.1 of [16], $\sum_{j=0}^{k} \mathbf{F}(\mathbb{R})^j / k \to \mathbf{rl}$ as $k \to \infty$. Applying the Tauberian theorem for power series [12, Chapter XIII, Section 5, Theorem 5], we have, as $z \to 1-$,

$$|(z-1)\hat{\mathbf{F}}(-ix)[\mathbf{I} - z\hat{\mathbf{F}}(-ix)]^{-1}| \le \mathbf{F}(\mathbb{R})(1-z) \sum_{j=0}^{\infty} z^j \mathbf{F}(\mathbb{R})^j \to \mathbf{rl}. \tag{24}$$

It follows that the left-hand side of (24) is bounded for all $x \in \mathbb{R}$ and for all $z < 1$ sufficiently close to 1. Let us bound the expression

$$\mathbf{B}(x) := (1 - e^{-iax})[\mathbf{I} - \hat{\mathbf{F}}(-ix)]^{-1} = \frac{(1 - e^{-iax})\hat{\mathbf{M}}(-ix)}{\hat{\alpha}(-ix)}.$$

In a sufficiently small neighborhood of zero, we have

$$(25) \qquad |\mathbf{B}(x)| \le \frac{|ax||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)x + o(x)|} \le \frac{|ax||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)||x| - |o(x)|} \le \frac{2|a||\mathbf{M}|(\mathbb{R})}{|\hat{\alpha}'(0)|}.$$

It remains to bound $\mathbf{B}(x)$ outside a neighborhood of zero. We have

$$(26) \qquad |\mathbf{B}(x)| \le 2|\mathbf{M}|(\mathbb{R}) \Big/ \inf_{|x| \ge \varepsilon} |\hat{\alpha}(-ix)|.$$

The function $\hat{\alpha}(-ix)$, $x \in \mathbb{R}$, is continuous. By assumption, $|\hat{\alpha}(-ix)| \ne 0$ $\forall x \ne 0$. We will show that $\liminf_{|x|\to\infty} |\hat{\alpha}(-ix)| > 0$, thus establishing

$$(27) \qquad \inf_{|x| \ge \varepsilon} |\hat{\alpha}(-ix)| > 0.$$

We argue by contradiction. Assume that

$$(28) \qquad \liminf_{|x|\to\infty} |\hat{\alpha}(-ix)| = 0.$$

Then there exists a sequence $\{x_k\}$ such that $|x_k| \to \infty$, $\hat{\alpha}(-ix_k) \to 0$ and $\hat{\mathbf{F}}(-ix_k) \to \mathbf{G}$ as $k \to \infty$, where $\mathbf{G}$ is a numerical matrix with $\varrho(\mathbf{G}) \ge 1$ since $\det(\mathbf{I} - \mathbf{G}) = \lim_{k\to\infty} \hat{\alpha}(-ix_k) = 0$. Let m be the number appearing in the statement of Theorem 1. Then $\hat{\mathbf{F}}(-ix_k)^m \to \mathbf{G}^m$ as $k \to \infty$. By the Riemann-Lebesgue lemma, $\lim_{k\to\infty} \Big\{\hat{\mathbf{F}}(-ix_k)^m - [(\mathbf{F}^{m*})_s]^\wedge(-ix_k)\Big\} = \mathbf{0}$, and hence $|\mathbf{G}| \le (\mathbf{F}^{m*})_s(\mathbb{R})$, whence $\varrho(\mathbf{G}^m) \le \varrho[(\mathbf{F}^{m*})_s(\mathbb{R})] < 1$. On the other hand, $\varrho(\mathbf{G}^m) = [\varrho(\mathbf{G})]^m \ge 1$. The contradiction shows that the assumption (28) is false. It follows from (24)–(27) that the integrand in (23) is bounded entrywise by a matrix of functions $\phi(x)\mathbf{K}$, where $\mathbf{K} > \mathbf{0}$ is a constant matrix. Lemma 7 is proved. □

LEMMA 8. *Let* $\mathbf{F}$ *be an* $n \times n$*-matrix whose elements are finite non-negative measures on* $\mathbb{R}$*. Suppose that the matrix* $\mathbf{F}(\mathbb{R})$ *is irreducible and*

$\varrho[\mathbf{F}(\mathbb{R})] = 1$. *Choose positive left and right eigenvectors* $\mathbf{l} = (l_1, \ldots, l_n)$ *and* $\mathbf{r} = (r_1, \ldots, r_n)^T$ *corresponding to the eigenvalue* 1 *of* $\mathbf{F}(\mathbb{R})$. *Assume that* $\mu := \mathbf{l} \int_{\mathbb{R}} x\, \mathbf{F}(dx)\mathbf{r} \in (0, +\infty)$. *Then, for any fixed* $h > 0$,

$$\mathbf{H}((x, x + h]) \to \mathbf{0} \qquad \text{as } x \to -\infty.$$

Proof of Lemma 8. This is done by retracing the proof of Lemma 5 and applying, wherever necessary, the dominated convergence theorem. First, suppose $\mathbf{F}(\mathbb{R})$ is a primitive stochastic matrix. We have $\lim_{x\to-\infty} \mathbf{H}_1((x - y, x - y + h]) = \mathbf{0}$ $\forall y \in \mathbb{R}$ since $\mathbf{H}_1$ is concentrated on $(0, \infty)$. By dominated convergence, the assertion of the lemma now follows from (16) and from (14) with $\mathbf{H}$ replaced by $\mathbf{H}_1$. If $\mathbf{F}(\mathbb{R})$ is primitive but not necessarily stochastic, then the assertion of the lemma is implied by the equality in (20). Consider now the general case. By the above, $\lim_{x\to-\infty} \mathbf{H}^{(0)}((x, x + h]) = \mathbf{0}$. Applying again the dominated convergence theorem, we have $\lim_{x\to-\infty} \mathbf{H}^{(j)}((x, x+h]) = \mathbf{0}$, $j = 1, \ldots, d-1$. and hence $\mathbf{H}((x, x + h]) \to \mathbf{0}$ as $x \to -\infty$. Lemma 8 is proved. □

We return to the proof of Theorem 1. It follows from (11) and Lemmas 3–7 that $\mathcal{F}(\Delta_a\mathbf{H}) = r\hat{\mathbf{Q}}(0)\mathcal{F}(\Delta_a L) - \mathcal{F}(r\Delta_a T\mathbf{Q}) + \mathcal{F}(\Delta_a\mathbf{Q})$. Passing over from Fourier transforms to their inverse images, we obtain

$$\Delta_a\mathbf{H} = r\hat{\mathbf{Q}}(0)\Delta_a L - \Delta_a rT\mathbf{Q} + \Delta_a\mathbf{Q}. \tag{29}$$

Let $\mathcal{D}(\mathbb{R})$ be the space of all infinitely differentiable functions with compact supports. Any tempered distribution is completely determined by its values at functions $\phi \in \mathcal{D}(\mathbb{R})$ since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}_1$ [21, Theorem 7.10]. Let ϕ be an arbitrary element of $\mathcal{D}(\mathbb{R})$ whose support is contained in a finite interval $[c, d]$. Apply both sides of (29) to ϕ. The left-hand side becomes $\int_{\mathbb{R}} \phi(x)\, \mathbf{H}(dx) - \int_{\mathbb{R}} \phi(x+a)\, \mathbf{H}(dx)$. The latter integral is bounded in absolute value by the matrix

$$\mathbf{H}([c - a, d - a]) \max\{|\phi(x)| : x \in [c, d]\},$$

which, by Lemma 8, tends to the null matrix $\mathbf{0}$ as $a \to +\infty$. Next,

$$\int_{\mathbb{R}} r\hat{\mathbf{Q}}(0)\phi(x)\, \Delta_a L(dx) = r\hat{\mathbf{Q}}(0) \int_0^a \phi(x)\, dx \to r\hat{\mathbf{Q}}(0) \int_0^\infty \phi(x)\, dx$$

as $a \to +\infty$ and

$$\Delta_a rT\mathbf{Q}(\phi) = \int_{\mathbb{R}} \phi(x)\, rT\mathbf{Q}(dx) - \int_{\mathbb{R}} \phi(x+a)\, rT\mathbf{Q}(dx);$$

the latter integral is bounded in absolute value by

$$rT\mathbf{Q}([c-a, d-a]) \max\{|\phi(x)| : x \in [c,d]\} \to \mathbf{0} \qquad \text{as } a \to +\infty$$

since $\mathbf{Q}$ is a matrix of finite measures and

$$|T\mathbf{Q}([c-a, d-a])| = \left| \int_{c-a}^{d-a} \mathbf{Q}((-\infty, x])\, dx \right| \le (d-c)|\mathbf{Q}|((-\infty, d-a]).$$

Finally,

$$\Delta_a \mathbf{Q}(\phi) = \int_{\mathbb{R}} \phi(x)\, \mathbf{Q}(dx) - \int_{\mathbb{R}} \phi(x+a)\, \mathbf{Q}(dx).$$

The latter integral is bounded in absolute value by

$$\mathbf{Q}([c-a, d-a]) \max\{|\phi(x)| : x \in [c,d]\} \to \mathbf{0} \qquad \text{as } a \to +\infty$$

since $\mathbf{Q}$ is a matrix of finite measures. Summing up, we arrive at the equality $\mathbf{H}(\phi) = r\hat{\mathbf{Q}}(0)L(\phi) - rT\mathbf{Q}(\phi) + \mathbf{Q}(\phi)$ $\forall \phi \in \mathcal{D}(\mathbb{R})$, whence (12) follows.

LEMMA 9. *Let* $\mathbf{F}$ *be an* $n \times n$*-matrix whose elements are finite non-negative measures on* $\mathbb{R}$*. Suppose that the matrix* $\mathbf{F}(\mathbb{R})$ *is irreducible and* $\varrho[\mathbf{F}(\mathbb{R})] = 1$*. Assume that* $\hat{\mathbf{F}}'(0) := (\hat{F}'_{ij}(0))$ *is a matrix with finite entries. Choose positive left and right eigenvectors* $\mathbf{l} = (l_1, \dots, l_n)$ *and* $\mathbf{r} = (r_1, \dots, r_n)^T$ *corresponding to the eigenvalue* 1 *of* $\hat{\mathbf{F}}(0)$ *in such a way that* $\mathbf{lr} = 1$*. Then*

$$-\frac{\hat{\mathbf{M}}(0)}{\hat{\alpha}'(0)} = \frac{\mathbf{rl}}{\mathbf{l}\hat{\mathbf{F}}'(0)\mathbf{r}}. \tag{30}$$

PROOF OF LEMMA 9. We shall suitably modify the arguments employed in the proofs of Lemmas 8 and 6 in [17]. Unlike [17], here we shall use the fact that, because of the irreducibility of $\hat{\mathbf{F}}(0)$, $\sum_{j=1}^{k} \hat{\mathbf{F}}(q)^j/k \to \mathbf{rl}$

as $k \to \infty$ [16, Theorem 8.6.1], and we will also replace the column vector $\mathbf{e}$ with unit coordinates by the right eigenvector $\mathbf{r}$. We have $\hat{\mathbf{M}}(0) = c\mathbf{r}\mathbf{l}$, where

$$c = \lim_{z \to 1-} \det(\mathbf{I} - z\hat{\mathbf{F}}(0))/(1 - z) > 0.$$

Indeed, for $0 < z < 1$, the adjugate matrix $\hat{\mathbf{M}}_z(0)$ of $\mathbf{I} - z\hat{\mathbf{F}}(0)$ is equal to

$$\det(\mathbf{I} - z\hat{\mathbf{F}}(0)) \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k = \frac{\det(\mathbf{I} - z\hat{\mathbf{F}}(0))}{1 - z}(1 - z) \sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k. \tag{31}$$

The characteristic polynomial $f(\lambda)$ of $\hat{\mathbf{F}}(0)$ is equal to $\det(\lambda\mathbf{I} - \hat{\mathbf{F}}(0)) = \prod_{j=1}^{k}(\lambda - \lambda_j)^{m_j}$, where λ_j are the eigenvalues of $\hat{\mathbf{F}}(0)$ and m_j are their multiplicities, $j = 1, \ldots, k$, $\sum_{j=1}^{k} m_j = n$, the eigenvalue $\lambda_1 = 1$ having multiplicity 1. Hence

$$\det(\mathbf{I} - z\hat{\mathbf{F}}(0)) = z^n f(1/z) = z^n \prod_{j=1}^{k}(1/z - \lambda_j)^{m_j} = (1 - z) \prod_{j=2}^{k}(1 - z\lambda_j)^{m_j}.$$

Thus, $c := \prod_{j=2}^{k}(1 - \lambda_j)^{m_j} > 0$ since $\lambda_j \neq 1\ \forall j > 1$ and if λ_j is a complex eigenvalue, then $\exists \lambda_i = \overline{\lambda_j}$ with $m_i = m_j$, whence $(1-\lambda_j)^{m_j}(1-\lambda_i)^{m_i} = |1-\lambda_j|^{2m_j}$. By the Tauberian theorem for power series, $(1-z)\sum_{k=0}^{\infty} z^k \hat{\mathbf{F}}(0)^k \to \mathbf{r}\mathbf{l}$ as $z \to 1-$ [12, Chapter XIII, Section 5, Theorem 5]. Therefore, it follows from (31) that

$$\hat{\mathbf{M}}(0) = \lim_{z \to 1-} \hat{\mathbf{M}}_z(0) = c\mathbf{r}\mathbf{l}. \tag{32}$$

Differentiating $[\mathbf{I} - \hat{\mathbf{F}}(s)]\hat{\mathbf{M}}(s) = \hat{\alpha}(s)\mathbf{I}$ at $s = 0$, we obtain $-\hat{\mathbf{F}}'(0)\hat{\mathbf{M}}(0) + [\mathbf{I} - \hat{\mathbf{F}}(0)]\hat{\mathbf{M}}'(0) = \hat{\alpha}'(0)\mathbf{I}$. Multiply both sides of the last equality first by $\mathbf{l}$ from the left and then by $\mathbf{r}$ from the right. Taking into account (32) and $\mathbf{l}\mathbf{r} = 1$, we arrive at

$$-\mathbf{l}\hat{\mathbf{F}}'(0)\hat{\mathbf{M}}(0)\mathbf{r} = -\mathbf{l}\hat{\mathbf{F}}'(0)c\mathbf{r} = \hat{\alpha}'(0). \tag{33}$$

Now (30) follows from (32) and (33). Lemma 9 is proved. $\square$

By Lemma 9, $r\hat{\mathbf{Q}}(0) = -\hat{\mathbf{M}}(0)/\hat{\alpha}'(0) = \mathbf{rl}/\mu$. In order to complete the proof of Theorem 1, it now remains to put $\mathbf{H}_1 := (\mathbf{rl}/\mu)L - rT\mathbf{Q}$ and $\mathbf{H}_2 := \mathbf{Q}$ (see (12)).

REMARK 1. In the case $r_1 = r_2 = 0$, the requirement that in $\Pi(r_1, r_2)$ there be no nonzero roots of (6) is superfluous as is shown by the following lemma.

LEMMA 10. *Let $\mathbf{F}$ be an $n \times n$-matrix whose elements are finite nonnegative measures on $\mathbb{R}$. Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Assume that $\varrho[(\mathbf{F}^{m*})_s(\mathbb{R})] < 1$ for some integer $m \geq 1$. Then* $\det(\mathbf{I} - \hat{\mathbf{F}}(s)) \neq 0$ *for all $s \neq 0$ such that $\Re s = 0$.*

PROOF OF LEMMA 10. Assume the contrary, i.e., $\det(\mathbf{I} - \hat{\mathbf{F}}(s_0)) = 0$ for some $s_0 \neq 0$ with $\Re s_0 = 0$. This means that 1 is an eigenvalue of $\hat{\mathbf{F}}(s_0)$. By Theorem 8.4.5 of [16], there exist numbers $\theta_1, \dots, \theta_n \in \mathbb{R}$ such that $\hat{\mathbf{F}}(s_0) = \mathbf{D}\mathbf{F}(\mathbb{R})\mathbf{D}^{-1}$, where $\mathbf{D} = \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$, and hence $\hat{\mathbf{F}}(s_0)^m = \mathbf{D}\mathbf{F}(\mathbb{R})^m\mathbf{D}^{-1}$. By assumption, $\mathbf{F}^{m*} = \mathbf{F}^{(1)} + \mathbf{F}^{(2)}$, where $\mathbf{F}^{(1)}$ is a nonzero matrix of absolutely continuous nonnegative measures, so that

$$\mathbf{F}(\mathbb{R})^m = \mathbf{D}^{-1}\hat{\mathbf{F}}^{(1)}(s_0)\mathbf{D} + \mathbf{D}^{-1}\hat{\mathbf{F}}^{(2)}(s_0)\mathbf{D},$$

which is impossible by the following reason. Suppose $F_{jk}^{(1)}(\mathbb{R}) \neq 0$. Then

$$\left|\hat{F}_{jk}^{(1)}(s_0)\right| = \left|\int_{\mathbb{R}} e^{s_0 x}\, F_{jk}^{(1)}(dx)\right| < F_{jk}^{(1)}(\mathbb{R})$$

and hence

$$\begin{aligned}[\mathbf{D}^{-1}\hat{\mathbf{F}}(s_0)^m\mathbf{D}]_{jk} &= \left|e^{-i\theta_j}\hat{F}_{jk}^{(1)}(s_0)e^{i\theta_k} + e^{-i\theta_j}\hat{F}_{jk}^{(2)}(s_0)e^{i\theta_k}\right| \\ &\leq \left|\hat{F}_{jk}^{(1)}(s_0)\right| + \left|\hat{F}_{jk}^{(2)}(s_0)\right| < F_{jk}^{(1)}(\mathbb{R}) + F_{jk}^{(2)}(\mathbb{R}) = (\mathbf{F}^{m*})_{jk}(\mathbb{R}).\end{aligned}$$

This contradiction proves Lemma 10. □

4. Solutions and Rates of Convergence

Consider first the simplest case $\mathcal{A} = S(0,0)$, i.e., $\mathcal{A}$ is the algebra of finite measures. Suppose $\mathbf{g} \in L_1(\mathbb{R})$. It is clear that under the hypotheses of Theorem 1 with $T^2\mathbf{F} \in \mathcal{A}$, $\mathbf{z}(x) := \mathbf{H} * \mathbf{g}(x)$ is a solution to (2); moreover, if $\mathbf{c}$ is an arbitrary right eigenvector of $\mathbf{F}(\mathbb{R})$ corresponding to the eigenvalue 1, then $\mathbf{z}(x) + \mathbf{c}$ is also a solution. So the asymptotic properties of solutions to (2) are those of the convolution $\mathbf{H} * \mathbf{g}(x)$. The decomposition of $\mathbf{H}$ provided by Theorem 1 allows us to write $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_i := \mathbf{H}_i * \mathbf{g}$, $i = 1, 2$. It follows that $\mathbf{z}_2 \in L_1(\mathbb{R})$ and $\mathbf{z}_1$ is a continuous function such that $\mathbf{z}_1(x) \to \mathbf{rl} \int_{\mathbb{R}} \mathbf{g}(y)\, dy/\mu$ as $x \to +\infty$ and $\mathbf{z}_1(x) \to (0, \dots, 0)^T$ as $t \to -\infty$. If, in addition, $\mathbf{g}(x)$ is bounded and $\lim_{|x|\to\infty} \mathbf{g}(x) = (0, \dots, 0)^T$, then $\lim_{|x|\to\infty} \mathbf{z}_2(x) = (0, \dots, 0)^T$, and the solution $\mathbf{z}$ has the following property:

$$\mathbf{z}(x) \to \begin{cases} \mathbf{rl} \int_{\mathbb{R}} \mathbf{g}(y)\, dy/\mu & \text{as } x \to +\infty, \\ (0, \dots, 0)^T & \text{as } x \to -\infty. \end{cases} \tag{34}$$

In this section, we will limit ourselves to obtaining submultiplicative rates of convergence in (34) by means of the Stone-type decomposition of Theorem 1 with $\mathcal{A} := S(\varphi)$. Let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function such that $r_1 \le 0 \le r_2$. By Theorem 1 of [20], $S(\varphi)$ satisfies properties (i) and (ii) of the preceding section, and hence Theorem 1 applies. We note some nuances. Let F be a nonnegative finite measure. Relation $TF \in S(\varphi)$ implies $F \in S(\varphi)$. Actually,

$$\begin{aligned} \int_0^\infty \varphi(x) F((x,\infty))\, dx &\ge \sum_{k=0}^\infty \inf_{x\in[k,k+1)} \varphi(x) F((k+1,k+2]) \\ &\ge \frac{1}{M(1)} \sum_{k=0}^\infty \int_{k+1}^{k+2} \varphi(x)\, F(dx) = \frac{1}{M(1)} \int_1^\infty \varphi(x)\, F(dx). \end{aligned}$$

Since, obviously, $\int_0^1 \varphi(x)\, F(dx) < \infty$, we have $\int_0^\infty \varphi(x)\, F(dx) < \infty$. Similarly, $\int_{-\infty}^0 \varphi(x)\, F(dx) < \infty$. Therefore, instead of the hypotheses $\mathbf{F}$, $T\mathbf{F} \in S(\varphi)$ in Theorem 1, we may assume only $T\mathbf{F} \in S(\varphi)$. Similarly, the set of conditions $\mathbf{F}$, $T\mathbf{F}$, $T^2\mathbf{F} \in S(\varphi)$ may be replaced by $T^2\mathbf{F} \in S(\varphi)$. Suppose now that $\varphi(x)/\exp(r_1 x)$ is nonincreasing on $(-\infty, 0)$ and $\varphi(x)/\exp(r_2 x)$ is nondecreasing on $[0, \infty)$. Theorem 3 of [27] implies that if $r_1 = 0 = r_2$ and $\int_{\mathbb{R}}(1+|x|)^k\varphi(x)\,F(dx) < \infty$ for some integer $k \geq 1$, or if $r_1 < 0 = r_2$ and $\int_0^\infty (1+x)^k\varphi(x)\,F(dx) < \infty$, or if $r_1 = 0 < r_2$ and $\int_{-\infty}^0 (1+|x|)^k\varphi(x)\,F(dx) < \infty$, then $T^kF \in S(\varphi)$. If $r_1 < 0 < r_2$, then $F \in S(\varphi) \Rightarrow T^kF \in S(\varphi)\ \forall k \geq 1$ [27, Theorem 2]. Suppose now that $r_1 = 0 = r_2$. Then, instead of the hypotheses $\mathbf{F}$, $T\mathbf{F} \in S(\varphi)$ in Theorem 1, we may assume only $\mathbf{F} \in S(\varphi_1)$, where $\varphi_1(x) := (1+|x|)\varphi(x)$. Similarly, the set of conditions $\mathbf{F}$, $T\mathbf{F}$, $T^2\mathbf{F} \in S(\varphi)$ may be replaced by $\mathbf{F} \in S(\varphi_2)$, where $\varphi_2(x) := (1+|x|)^2\varphi(x)$. In the latter case, H_2 will be in $S(\varphi_1)$. Suppose $r_1 < 0 < r_2$. Then the set of conditions $\mathbf{F}$, $T\mathbf{F}$, $T^2\mathbf{F} \in S(\varphi)$ may be replaced by $\mathbf{F} \in S(\varphi)$. The intermediary cases $r_1 < 0 = r_2$ and $r_1 = 0 < r_2$ are dealt with in a similar way.

THEOREM 2. *Let $\varphi(x)$ be a submultiplicative function such that $r_1 \leq 0 \leq r_2$, and let $\mathbf{g}(x)$, $x \in \mathbb{R}$, be a Borel-measurable vector function such that* (a) $\mathbf{g} \in L_1(\mathbb{R})$, (b) $\mathbf{g}\cdot\varphi \in L_\infty(\mathbb{R})$, (c) $\mathbf{g}(x)\varphi(x) \to 0$ *as $|x| \to \infty$ outside a set of Lebesgue measure zero, and* (d) $\varphi(t)\int_t^\infty |\mathbf{g}(x)|\,dx \to 0$ *as $t \to +\infty$ and $\varphi(t)\int_{-\infty}^t |\mathbf{g}(x)|\,dx \to 0$ as $t \to -\infty$. Let $\mathbf{F}$ be an $n \times n$-matrix whose elements are finite nonnegative measures on $\mathbb{R}$. Suppose that the matrix $\mathbf{F}(\mathbb{R})$ is irreducible and $\varrho[\mathbf{F}(\mathbb{R})] = 1$. Choose positive left and right eigenvectors $\mathbf{l} = (l_1,\dots,l_n)$ and $\mathbf{r} = (r_1,\dots,r_n)^T$ corresponding to the eigenvalue 1 of $\mathbf{F}(\mathbb{R})$. Assume that $\mu := \mathbf{l}\int_{\mathbb{R}} x\,\mathbf{F}(dx)\mathbf{r} \in (0,+\infty)$, $\varrho[(\mathbf{F}^{m*})_s^\wedge(r_i)] < 1$, $i = 1,2$, for some integer $m \geq 1$, and that in the strip $\Pi(r_1,r_2)$ there are no nonzero roots of the characteristic equation* (6) *distinct from zero. Suppose $T^2\mathbf{F} \in S(\varphi)$. Then, as t approaches $\pm\infty$ outside a set of Lebesgue measure zero,*

$$\sup_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|\leq|\mathbf{g}|}\left|\mathbf{H}*\boldsymbol{\alpha}(t) - \frac{\mathbf{rl}}{\mu}\int_{\mathbb{R}}\boldsymbol{\alpha}(x)\,dx\right| = o\left(\frac{1}{\varphi(t)}\right)$$

and $\sup_{\boldsymbol{\alpha}:|\boldsymbol{\alpha}|\leq|\mathbf{g}|}|\mathbf{H}*\boldsymbol{\alpha}(t)| = o(1/\varphi(t))$, respectively, the $\boldsymbol{\alpha}(x)$ being Borel-measurable vector functions on $\mathbb{R}$.

PROOF. By Theorem 1 with $\mathcal{A} = S(\varphi)$, both $\mathbf{H}_1 - \mathbf{rl}L/\mu$ and $\mathbf{H}_2$ are elements of $S(\varphi)$. Choose $\tilde{\mathbf{g}} \in L_1(\mathbb{R})$ such that $\tilde{\mathbf{g}} = \mathbf{g}$ a.e., $\sup_{x\in\mathbb{R}}|\tilde{\mathbf{g}}(x)|\varphi(x) < \infty$, and $\tilde{\mathbf{g}}(x)\varphi(x) \to 0$ as $|x| \to \infty$ in the usual sense. It suffices to put $\tilde{\mathbf{g}}(x) = 0$ on $\{x \in \mathbb{R} : |\mathbf{g}(x)|\varphi(x) > \|\mathbf{g}\cdot\varphi\|_\infty\}$ and on a set, say B, of Lebesgue measure zero such that $\lim_{x\notin B,|x|\to\infty}\mathbf{g}(x)\varphi(x) = 0$;

otherwise, $\tilde{\mathbf{g}}(x) := \mathbf{g}(x)$. By Fubini's theorem, the sets

$$A_1 := \left\{ x : \left| \mathbf{H}_1 - \frac{\mathbf{rl}L}{\mu} \right| * |\tilde{\mathbf{g}}|(x) \neq \left| \mathbf{H}_1 - \frac{\mathbf{rl}L}{\mu} \right| * |\mathbf{g}|(x) \right\}$$

and $A_2 := \{x : |\mathbf{H}_2| * |\tilde{\mathbf{g}}|(x) \neq |\mathbf{H}_2| * |\mathbf{g}|(x)\}$ are both of Lebesgue measure zero. Set $A := A_1 \cup A_2$. We have

$$\varphi(t)|\mathbf{H}_2| * |\tilde{\mathbf{g}}|(t) \leq \int_{\mathbb{R}} |\mathbf{H}_2|(dx)\varphi(t-x)\varphi(x)|\tilde{\mathbf{g}}(t-x)|.$$

By dominated convergence, the right-hand side tends to zero as $|t| \to \infty$, and so does the left-hand side. Similarly, $\lim_{|t|\to\infty} \varphi(t)|\mathbf{H}_1 - \mathbf{rl}L/\mu| * |\tilde{\mathbf{g}}|(t) = 0$. Hence both $\varphi(t)|\mathbf{H}_2| * |\mathbf{g}|(t)$ and $\varphi(t)|\mathbf{H}_1 - \mathbf{rl}L/\mu| * |\mathbf{g}|(t)$ tend to zero as $|t| \to \infty$, remaining outside the set A of Lebesgue measure zero. The first assertion of the theorem now follows from the obvious inequality

$$\begin{aligned} &\left| \mathbf{H} * \boldsymbol{\alpha}(t) - \frac{\mathbf{rl}}{\mu} \int_{\mathbb{R}} \boldsymbol{\alpha}(x)\, dx \right| \\ &\qquad \leq \left| \mathbf{H}_1 - \frac{\mathbf{rl}L}{\mu} \right| * |\mathbf{g}|(t) + |\mathbf{H}_2| * |\mathbf{g}|(t) + \frac{\mathbf{rl}}{\mu} \int_t^{\infty} |\mathbf{g}(x)|\, dx \end{aligned}$$

and condition (d). The case $t \to -\infty$ is dealt with in a similar way. □

References

[1] Alsmeyer, G., *Erneuerungstheorie*, B.G. Teubner, Stuttgart, 1991.

[2] Arjas, E. Nummelin, E. and R. L. Tweedie, Uniform limit theorems for non-singular renewal and Markov renewal processes, J. Appl. Probab. **15** (1978), 112–125.

[3] Athreya, K. B. and P. E. Ney, *Branching Processes*, Springer, Berlin, 1972.

[4] Athreya, K. B. and P. E. Ney, A Markov process approach to systems of renewal equations with application to branching processes, Branching processes. Advances in Probability and Related Topics (A. Joffe and P. Ney, eds.), vol. 5, Marcel Dekker, New York, 1978, pp. 293–317.

[5] Bellman, R. and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.

[6] Borovkov, K. A., Stability theorems and estimates of the rate of convergence of the components of factorizations for walks defined on Markov chains, Theory Probab. Appl. **25** (1980), 325–334.

[7] Chover, J., Ney, P. and S. Wainger, Functions of probability measures, J. Analyse Math. **26** (1973), 255–302.

[8] Crump, K. S., On systems of renewal equations, J. Math. Anal. Appl. **30** (1970), 425–434.

[9] Crump, K. S., On systems of renewal equations: the reducible case, J. Math. Anal. Appl. **31** (1970), 517–528.

[10] Engibaryan, N. B., Renewal theorems for a system of integral equations, Mat. Sb. **189:12** (1998), 59–72 (Russian); English transl. in Sb. Math. **189** (1998).

[11] Essén, M., Banach algebra methods in renewal theory, J. Anal. Math. **26** (1973), 303–336.

[12] Feller, W., *An Introduction to Probability Theory and Its Applications*, vol. II, Wiley, New York, 1966.

[13] Frenk, J. B. G., *On Banach Algebras, Renewal Measures and Regenerative Processes*, CWI Tracts, vol. 38, Centre for Mathematics and Computer Science, Amsterdam, 1987.

[14] Gantmacher, F., *The Theory of Matrices*, vol. II, Chelsea Publishing Co., New York, 1959.

[15] Hille, E. and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, American Mathematical Society, Providence, RI, 1957.

[16] Horn, R. A. and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1986.

[17] Jelenković, P. R. and A. A. Lazar, Subexponential asymptotics of a Markov-modulated random walk with queueing applications, J. Appl. Probab. **35** (1998), 325–347.

[18] Lancaster, P., *Theory of Matrices*, Academic Press, New York, 1969.

[19] Presman, E. L., Factorization methods and boundary problems for sums of random variables given on a Markov chain, Math. USSR Izv. **3** (1969), 815–852.

[20] Rogozin, B. A. and M. S. Sgibnev, Banach algebras of measures on the line, Siberian Math. J. **21** (1980), 265–273.

[21] Rudin, W., *Functional Analysis*, McGraw-Hill, New York, 1973.

[22] Sevastyanov, B. A., *Branching Processes*, Nauka, Moscow, 1971 (Russian); German transl. B. A. Sewastjanow, *Verzweigungsprozesse*, Akademie-Verlag, Berlin, 1974 (German).

[23] Sevastyanov, B. A. and V. P. Chistyakov, Multi-dimensional renewal equations, and moments of branching processes, Theory Probab. Appl. **16** (1971), 199–214.

[24] Sgibnev, M. S., Banach algebras of measures of class $\mathcal{S}(\gamma)$, Siberian Math. J. **29** (1988), 647–655.
[25] Sgibnev, M. S., Submultiplicative moments of the supremum of a random walk with negative drift, Statist. Probab. Lett. **32** (1997), 377–383.
[26] Sgibnev, M. S., Exact asymptotic behavior in a renewal theorem for convolution equivalent distributions with exponential tails, SUT J. Math. **35** (1999), 247–262.
[27] Sgibnev, M. S., An asymptotic expansion for the distribution of the supremum of a random walk, Studia Math. **140** (2000), 41–55.
[28] Sgibnev, M. S., Stone decomposition for a matrix renewal measure on a half-line, Sbornik: Mathematics **192** (2001), 1025–1033.
[29] Sgibnev, M. S., Stone's decomposition of the renewal measure via Banach-algebraic techniques, Proc. Amer. Math. Soc. **130** (2002), 2425–2430.
[30] Stone, C., On absolutely continuous components and renewal theory, Ann. Math. Statist. **37** (1966), 271–275.
[31] Wermer, J., *Banach Algebras and Several Complex Variables*, 2nd ed., Springer, New York, 1976.

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