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*On the Precise Laplace Approximation for Large Deviations of Markov Chain The Nondegenerate Case**

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Abstract. Let L_n be the empirical measure of a uniformly ergodic nonreversible Markov chain on a compact metric space and Φ be a smooth functional. This paper gives a precise asymptotic evaluation of the form $E(\exp(n\Phi(L_n)))$ up to order $1 + o(1)$, in the case the Hessian of $J - \Phi$ is nondegenerate, where J is the rate function of the large deviations of empirical measure.

1. Introduction and Main Result

Let E be a compact metric space with Borel σ-algebra $\mathcal{E}$. Let $C(E)$ denote the Banach space of continuous $\mathbf{R}$-valued functions on E, equipped with supremum norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$. Let $\mathcal{M}(E)$ denote the set of signed measures on $(E, \mathcal{E})$ with finite total variations, equipped with the total variation norm $\|\cdot\|_{\rm var}$, and let $\mathcal{M}_1(E)$ and $\mathcal{M}_0(E)$ be the set of probability measures on $(E, \mathcal{E})$ and the set of all signed measures on $(E, \mathcal{E})$ with total measure 0, respectively. We also consider the weak*-topology, sometimes. Note that $\mathcal{M}_1(E)$ with the Prohorov metric $dist(\cdot, \cdot)$ is a compact space. Let $\mathbf{N}$ denote the set of non-negative integers.

Let $\Omega \equiv E^{\mathbf{N}}$. For each $n \geq 0$, let $X_n : \Omega \to E$ be the map given by $X_n = \omega(n)$. Let $\mathcal{F}$ be the σ-algebra on Ω generated by $\{X_n\}_{n\geq 0}$, $\mathcal{F}_k^h$ denotes the sub-σ-field generated by $\{X_j\}_{k\leq j\leq h}$. We denote $\mathcal{F}_0^k$ by $\mathcal{F}_k$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \{X_n\}, P_x)$ be a homogeneous Markov chain on E with transition probability $\Pi(x, dy)$ that satisfies $P_x(X_0 = x) = 1$ for all $x \in E$. The

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linear operator Π on $C(E)$ is given by

$$\Pi f(x) = \int_E f(y)\Pi(x,dy), \quad f \in C(E).$$

First, we assume the following,

A.1 *There exists a Π-invariant measure $\mu \in \mathcal{M}_1(E)$ with supp$\mu = E$, and there exists a continuous positive function $\pi : E \times E \to (0,\infty)$ such that $\Pi(x,dy) = \pi(x,y)\mu(dy)$.*

Let $L_n : \Omega \to \mathcal{M}_1(E), n \geq 1$, be the empirical measures,*i.e.*,

$$L_n = \frac{1}{n}\sum_{k=0}^{n-1}\delta_{X_k},$$

where δ_x is the Dirac measure centered in x. Under our assumptions, the following large deviation principle holds for the empirical measure L_n. (*c.f.* Deuschel-Stroock [3]).

PROPOSITION 1.1.

(1) $\limsup_{n\to\infty}\frac{1}{n}\log P_x\left(L_n \in F|X_{n-1} = y\right) \leq -\inf\{J(\nu), \nu \in F\}$ *for any $x, y \in E$ and any closed set $F \subset \mathcal{M}_1(E)$.*

(2) $\liminf_{n\to\infty}\frac{1}{n}\log P_x\left(L_n \in G|\, X_{n-1} = y\right) \geq -\inf\{J(\nu), \nu \in G\}$ *for any $x, y \in E$ and any open set $G \subset \mathcal{M}_1(E)$.*

Here the rate function $J : \mathcal{M}_1 \to [0,\infty]$ is given by

$$J(\nu) = \sup\left\{-\int_E \log\frac{\Pi u}{u}d\nu, u \in C(E), u \geq 1\right\}, \quad \nu \in \mathcal{M}_1(E).$$

Let $\Phi : \mathcal{M}(E) \to \mathbf{R}$ be a bounded and three times continuously Fréchet differentiable function with respect to norm $\|\cdot\|_{\text{var}}$ satisfying the following:

A.2 *There exist functions $\Phi^{(1)} \in C(\mathcal{M}_1(E) \times E, \mathbf{R}), \Phi^{(2)} \in C(\mathcal{M}_1(E) \times E \times E, \mathbf{R})$, and $\Phi^{(3)} \in C(\mathcal{M}_1(E) \times E \times E \times E, \mathbf{R})$, such that* *for any* $\nu \in \mathcal{M}_1(E), R_1, R_2, R_3 \in \mathcal{M}(E)$,

$$\begin{aligned} D\Phi(\nu)(R_1) &= \int_E \Phi^{(1)}(\nu, x) R_1(dx), \\ D^2\Phi(\nu)(R_1, R_2) &= \int_E \int_E \Phi^{(2)}(\nu, x, y) R_1(dx) R_2(dy), \\ D^3\Phi(\nu)(R_1, R_2, R_3) &= \int_E \int_E \int_E \Phi^{(3)}(\nu, x, y, z) R_1(dx) R_2(dy) R_3(dz). \end{aligned}$$

Then by Donsker-Varadhan [3], we have that

$$\lim_{n\to\infty} \frac{1}{n} \log E^{P_x} \left[\exp\left(n\Phi(L_n)\right)\right] = \sup\{\Phi(\nu) - J(\nu) : \nu \in \mathcal{M}_1(E)\}$$

for any $x \in E$. Write the constant in the right hand above as b_Φ, for the sake of simplicity. In this paper, we give a more precise evaluation of $E^{P_x}\left[\exp\left(n\Phi(L_n)\right)\right]$.

In the case of continuous time Markov processes, some precise evaluations have been obtained by Kusuoka-Tamura [7] for symmetric case, and by Bolthausen-Deuschel-Tamura [1] for a non-symmetric case, both under some "Central Limit Theorem Assumption", also, by Kusuoka-Liang [5] without the"Central Limit Theorem Assumption".

Define

$$K_\Phi \equiv \{\nu \in \mathcal{M}_1(E) : \Phi(\nu) - J(\nu) = b_\Phi\}.$$

It is not difficult to prove that K_Φ is non-void and compact in $\mathcal{M}_1(E)$, since $J(\nu)$ is a good convex rate function (*c.f.* Deuschel-Stroock [3, Theorem 4.1.43]). We also assume the following,

A.3 *There exists a unique element in* K_Φ, *i.e.,* $K_\Phi = \{\nu_0\}$.

For any $V \in C(E)$, we define the operator $\Pi^V : C(E) \to C(E)$, by

$$\Pi^V f(x) = e^{V(x)} \int_E \Pi(x, dy) f(y), \quad f \in C(E). \tag{1.1}$$

Then we have the following simplified Feynman-Kac formula

$$(\Pi^V)^n f(x) = E^{P_x}\left[f(X_n) \exp\left(\sum_{k=0}^{n-1} V(X_k)\right)\right], \quad f \in C(E).$$

Let $\Lambda(V)$ be the logarithmic spectral radius of Π^V, given by

$$\Lambda(V) = \lim_{n\to\infty} \frac{1}{n} \log \|(\Pi^V)^n\|_{op},$$

where $\|\cdot\|_{op}$ denotes the operator norm of bounded linear operator in $C(E)$. It is trivial that $|\Lambda(V)| \le \|V\|_\infty$. For each $n \ge 1$, $\exp(\Lambda(V)n)$ is the spectral radius of $(\Pi^V)^n$. By Deuschel-Stroock [3, Corollary 4.1.36], it follows that

$$\Lambda(V) = \sup\left\{\int_E V(x)\nu(dx) - J(\nu) : \nu \in \mathcal{M}_1(E)\right\}, \tag{1.2}$$

and

$$J(\nu) = \sup\left\{\int_E V(x)\nu(dx) - \Lambda(V) : V \in C(E)\right\}.$$

From the assumption A.1, Π^V is a compact operator with positive kernel function $\pi^V(x,y) = e^{V(x)}\pi(x,y)$. By the Perron-Frobenius argument, we see that there exists a positive $h^V \in C(E)$ such that

$$e^{-\Lambda(V)}\Pi^V h^V = h^V,$$

and it is uniquely determined up to a constant. Now, by Kolmogorov extension theorem, we can define a set of probability measures $Q_x^V, x \in E$, on $(\Omega, \mathcal{F})$ such that

$$Q_x^V(A) = \frac{e^{-n\Lambda(V)}}{h^V(x)} E^{P_x}\left[1_A h^V(X_n)\exp(\sum_{k=0}^{n-1} V(X_k))\right]$$

for all $x \in E, n \in \mathbf{N}$ and $A \in \mathcal{F}_n$. Let Q^V be the corresponding bounded linear operator on $C(E)$, *i.e.*, $Q^V f(x) = E^{Q_x^V}[f(X_1)]$, then Q^V has strictly positive continuous transition density function $\tilde{q}^V$ with respect to μ, given by

$$\tilde{q}^V(x,y) = \frac{e^{-\Lambda(V)}}{h^V(x)}\pi^V(x,y)h^V(y) \quad \text{for any} \quad x, y \in E.$$

Let $(\Pi^V)^*$ be the $L^2(d\mu)$-adjoint operator of Π^V in $C(E)$. We see in the same way as above that there exists a unique strictly positive $l^V \in C(E)$ such that $(\Pi^V)^* l^V = e^{\Lambda(V)} l^V$ and $\int_E l^V d\mu = 1$. Now, h^V is uniquely determined if we require $\int_E l^V h^V d\mu = 1$. Let $d\mu^V = h^V l^V d\mu$. Then $\{Q^V\}$ is μ^V-invariant.

We are now ready to define a new Markov chain with invariant measure ν_0. Let $V^{\nu_0} = D\Phi(\nu_0)(\delta_x - \nu_0) + \Phi(\nu_0)$. Let h denote the unique properly normalized eigenfunction of $\Pi^{V^{\nu_0}}$, *i.e.* $h = h^{V^{\nu_0}}$. We will show in Lemma 2.5 below that $b_\Phi = \Lambda(V^{\nu_0})$ and ν_0 is an invariant measure of $(Q_x^{V^{\nu_0}})$. Let us denote $(Q_x^{V^{\nu_0}})$ by Q_x, $h^{V^{\nu_0}}$ by h,*etc.*, for the sake of simplicity.

Let Q be the operator on $C(E)$ corresponding to $\{Q_x\}_{x\in E}$. Then ν_0 is an invariant measure of Q, Q has continuous strictly positive density function $q(x,y)$ with respect to ν_0, and $q(x,y)$ satisfies $\|q_n(x,\cdot)-1\|_\infty \to 0$ exponentially fast as $n\to\infty$ uniformly in x, where $q_n(x,y)$ is given by $q_1(x,y) = q(x,y), q_{n+1}(x,y) = \int q(x,z)q_n(z,y)\nu_0(dz), n\geq 1$. Therefore we can define a $g(x,y)\in C(E\times E)$ given by

$$g(x,y) = \sum_{n=1}^{\infty}\left(q_n(x,y)-1\right).$$

Define the linear operator $G : C(E) \to C(E)$ by $Gf(x) = \int_E g(x,y)f(y)\nu_0(dy)$. Let G^* be the dual operator of G in $L^2(d\nu_0)$, *i.e.* $G^*f(x) = \int_E g(y,x)f(y)\nu_0(dy)$ for any $f\in C(E)$. Let $\overline{G} = P + G + G^*$, where P is defined by $Pf(x) = f(x) - \int_E f d\nu_0$. We also need the following operators. For $f\in C(E\times E)$, let

$$\begin{aligned}
&((\overline{G}\otimes\overline{G})f))(x_1,x_2) = f(x_1,x_2)\\
&+\int_E\int_E (g(x_1,y_1)+g(y_1,x_1))(g(x_2,y_2)+g(y_2,x_2))\\
&\qquad\qquad \times f(y_1,y_2)\nu_0(dy_1)\nu_0(dy_2)\\
&+\int_E (g(x_1,y_1)+g(y_1,x_1))f(y_1,x_2)\nu_0(dy_1)\\
&\quad+\int_E (g(x_1,y_2)+g(y_2,x_2))f(x_1,y_2)\nu_0(dy_2).
\end{aligned}$$

Define $\overline{G}_x \equiv \overline{G}\otimes I$ and $\overline{G}_y \equiv I\otimes\overline{G}$ by

$$((\overline{G}\otimes I)f)(x,y) = f(x,y) + \int_E (g(x,z)+g(z,x))f(z,y)\nu_0(dz),$$

$$((I\otimes\overline{G})f)(x,y) = f(x,y) + \int_E (g(z,y)+g(y,z))f(x,z)\nu_0(dz),$$

G_x^*, G_x, G_y, G_y^* are defined similarly, where I is the identity operator.

Let $B(f,g) \equiv \int_E f\overline{G}g d\nu_0, f,g \in C(E)$. Note that $B(f,f) \geq 0$ for any $f \in C(E)$. Actually, we have that

$$0 \leq E^{Q_{\nu_0}} \left\{ \left(\sqrt{n} \int_E f(x)(L_n - \nu_0)(dx) \right)^2 \right\} \to B(f,f) \quad \text{as} \quad n \to \infty.$$

Let $V_0 = \{f \in C(E) : B(f,f) = 0\}$ and $\widetilde{C}(E) = C(E)/V_0$. Then B is an inner product on $\widetilde{C}(E)$. Also let $\widetilde{H}$ be the completion of $\widetilde{C}(E)$ under the Hilbert norm induced by B. Since $\widetilde{C}(E) \hookrightarrow \widetilde{H}$, there is a natural map $T : C(E) \to \widetilde{H}$. Let H be the dual space of $\widetilde{H}$, and T^* be the adjoint operator of T which is a mapping from H to $\mathcal{M}(E)$. We can easily show that T^* is one to one, then H can be regarded as a subset of $\mathcal{M}(E)$ with norm $\|\overline{G}f d\nu_0\|_H^2 = \int_E f\overline{G}f d\nu_0$.

We will prove that all of the eigenvalues of $D^2\Phi(\nu_0)|_{H\times H}$ are less than or equal to 1 in Section 2 (see Proposition 2.6). In this paper, we assume the following nondegeneracy assumption.

A.4 *All of the eigenvalues of* $D^2\Phi(\nu_0)\Big|_{H\times H}$ *are smaller than 1.*

In addition, we assume

A.5 *For any* $\delta > 0$*, there exist a constant* $\varepsilon > 0$ *and a symmetric continuous function* $K_\delta : E \times E \to \mathbf{R}$ *such that* $\sup_{x,y\in E} |K_\delta(x,y)| \leq \delta$*, and*

$$\left|D^3\Phi(R)(\nu - \nu_0, \nu - \nu_0, \nu - \nu_0)\right| \leq \int_E \int_E K_\delta(x,y)(\nu - \nu_0)(dx)(\nu - \nu_0)(dy)$$

for any $R \in \mathcal{M}_1(E)$ *with* $dist(R,\nu_0) < \varepsilon$ *and any* $\nu \in \mathcal{M}_1(E)$ *with* $dist(\nu,\nu_0) < \varepsilon$*.*

Now, we can state our main theorem.

THEOREM 1.2. *Under the assumptions A.1—A.5, we have that for any* $x, y \in E$*,*

$$\begin{aligned}
&\lim_{n\to\infty} e^{-nb_\Phi} E^{P_x} \left[\exp(n\Phi(L_n)) \,\middle|\, X_{n-1} = y \right] \\
&= \frac{h(x)}{h(y)} \cdot \exp\left(\frac{1}{2} \int_E \Phi^{(2)}(\nu_0, u, u)\nu_0(du) \right. \\
&\qquad \left. + \int_E \int_E g(u,v)\Phi^{(2)}(\nu_0, u, v)\nu_0(du)\nu_0(dv) \right) \\
&\qquad \times det_2 \left(I_H - D^2\Phi(\nu_0)\right)^{-1/2}.
\end{aligned}$$

REMARK 1.1. The det_2 appeared in the Theorem above is the transformed determinant defined by $det_2(I-A) = \prod_j (1-\lambda_j)e^{-\lambda_j}$, where $\{\lambda_k\}_{k\in N}$ is the set of eigenvalues of A. This is well-defined as long as A is a Hilbert-Schmidt operator. It is easy that $D^2\Phi(\nu_0) \mid_{H\times H}$ is a Hilbert-Schmidt operator. This fact and the assumption A.4 ensure that $det_2\left(I_H - D^2\Phi(\nu_0)\right)^{-1/2}$ is well-defined.

The rest of this paper is organized as following. We give a precise form of the nondegeneracy assumption A.4 in Section 2. In Section 3, we give a forward-backward martingale decomposition. By means of it, we establish the exponential integrability of related partial sums processes. The proof of Theorem 1.2 is given in Section 4.

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2. Perturbations

In this section, we first use spectral theory for compact linear operators (see Dunford-Schwartz [4] for the details) to find the asymptotic behavior of $\Lambda(V), h^V, l^V$ when V is close to 0, then use this to give the precise statement of the nondegeneracy assumption. In this section, $C(E)$ denotes the space of complex-valued functions defined on E.

In the following, we assume that the Markov chain with semigroup Π satisfies the assumption A.1, and $V \in C(E)$ is a real-valued function and satisfies $\int_E V d\mu = 0$.

As stated in Section 1, by Perron-Frobenius argument, for $\varepsilon \in \mathbf{R}$, $e^{\Lambda(\varepsilon V)}$ is the principal eigenvalue of both the operator $\Pi^{\varepsilon V}$ and its adjoint operator $(\Pi^{\varepsilon V})^*$, and is a simple eigenvalue of both of them. So there exist a unique positive function $h^{\varepsilon V} \in C(E)$ and a probability measure $\nu^{\varepsilon V}$ on E such that

$$\Pi^{\varepsilon V} h^{\varepsilon V} = e^{\Lambda(\varepsilon V)} h^{\varepsilon V}, \quad (\Pi^{\varepsilon V})^* \nu^{\varepsilon V} = e^{\Lambda(\varepsilon V)} \nu^{\varepsilon V}, \quad \text{and} \quad \int_E h^{\varepsilon V} d\nu^{\varepsilon V} = 1.$$

Let

$$l^{\varepsilon V}(x) = e^{-\Lambda(\varepsilon V)} \int_E e^{\varepsilon V(y)} \pi(y,x) \nu^{\varepsilon V}(dy), \tag{2.1}$$

then we see that $l^{\varepsilon V} \in C(E), l^{\varepsilon V} > 0$ and $d\nu^{\varepsilon V} = l^{\varepsilon V} d\mu$.

Also, the projection operator to the eigenspace corresponding to $e^{\Lambda(\varepsilon V)}$ can be expressed as $E^{\varepsilon V} : C(E) \to C(E)$,

$$E^{\varepsilon V} f(x) = h^{\varepsilon V}(x) \int_E f(y) l^{\varepsilon V}(y) \mu(dy).$$

(So $E^{\varepsilon V} 1 = h^{\varepsilon V}$). As mentioned before, $d\mu^{\varepsilon V} = h^{\varepsilon V} l^{\varepsilon V} d\mu$ is the invariant probability measure of $Q^{\varepsilon V}$. For the sake of simplicity, we denote $\Lambda(\varepsilon V), h^{\varepsilon V}, l^{\varepsilon V}, E^{\varepsilon V}, \ \mu^{\varepsilon V}$ by $\Lambda(\varepsilon), h^{\varepsilon}, l^{\varepsilon}, E^{\varepsilon}, \ \mu^{\varepsilon}$, respectively. Note that $\Lambda(0) = 0, h^0 = l^0 = 1$, hence $E^0 = \langle \cdot \rangle_\mu$ and $\mu^0 = \mu$. Define $G_0 : C(E) \to C(E)$ by $G_0 f = \sum_{k=1}^{\infty} (\Pi^k f - \langle f \rangle_\mu)$ and let G_0^* be the adjoint operator of G_0 in $L^2(d\mu)$.

Let $F(w; z) = (zI - \Pi^{wV})^{-1}, w, z \in \mathbf{C}$. Then we have the following Proposition.

PROPOSITION 2.1. *There exist positive constants $r > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in \mathbf{R}$, $|\varepsilon| \le \varepsilon_0$, we have that $\sigma(\Pi^{\varepsilon V}) \cap \{z : |z-1| < r\} = \{e^{\Lambda(\varepsilon)}\}$ and*

$$E^{\varepsilon} = \frac{1}{2\pi i} \oint_{|z-1|=r} F(\varepsilon; z) dz.$$

PROOF. By Perron-Frobenius argument, there exists a constant $r > 0$ such that

$$\sigma(\Pi) \setminus \{1\} \subset \{z; |z| < 1 - 3r\}.$$

By Dunford-Schwartz ([4, p. 585, Lemma 3 and p. 587, Theorem 9]), we see that for this $r > 0$, there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in \mathbf{R}, |\varepsilon| \le \varepsilon_0$, we have that $|e^{\Lambda(\varepsilon)} - 1| < r$ and

$$\sigma(\Pi^{\varepsilon V}) \setminus \{e^{\Lambda(\varepsilon)}\} \subset S(\sigma(\Pi) \setminus \{1\}, r) \subset \{z; |z| < 1 - 2r\}.$$

where $S(\sigma(\Pi) \setminus \{1\}, r)$ means the r-neighborhood of $\sigma(\Pi) \setminus \{1\}$. Therefore, let $U = \{z; |z-1| < r\}$, then U is an open set with smooth boundary, $e^{\Lambda(\varepsilon)} \in U$, and $\left(\sigma(\Pi^{\varepsilon V}) \setminus \{e^{\Lambda(\varepsilon)}\}\right) \cap \overline{U} = \emptyset$. Therefore, by the definition of the spectral projections, we have

$$E^{\varepsilon} = \frac{1}{2\pi i} \oint_{\partial U} (zI - \Pi^{\varepsilon V})^{-1} dz = \frac{1}{2\pi i} \oint_{|z-1|=r} F(\varepsilon; z) dz. \ \square$$

PROPOSITION 2.2. *There exists a constant $\varepsilon_0 > 0$ such that for any $w \in \mathbf{C}, |w| \le \varepsilon_0$, there exist bounded operators $R_1(w;z)$ in $C(E)$ satisfying the following*

$$F(w;z) = F(0;z) + wF_1(0;z) + \frac{w^2}{2}F_2(0;z) + R_1(w;z),$$

where

$$F_1(0;z) = F(0;z)V\Pi F(0;z),$$
$$F_2(0;z) = F(0;z)V^2\Pi F(0;z) + 2F(0;z)V\Pi F(0;z)V\Pi F(0;z),$$

and $\sup_{z:|z-1|=r} ||R_1(w;z)||_{op} = O(|w|^3)$ *as* $|w| \to 0$.

PROOF. By spectral theory, the resolvent function $R(z;\Pi) = (zI - \Pi)^{-1} = F(0,z)$ is analytic in $\rho(\Pi) \supset \{z; |z-1| = r\}$ and so $\sup_{\{z;|z-1|=r\}} ||F(0;z)||_{op} < \infty$. Also, since V is bounded,

$$e^{wV} - 1 = wV + \frac{w^2}{2}V^2 + r_2(w) = wV + r_3(w)$$

with $r_2(w), r_3(w) \in C(E)$ and $||r_2(w)||_\infty = O(|w|^3)$, $||r_3(w)||_\infty = O(w^2)$ as $|w| \to 0$. Therefore, by spectral theory (*c.f.*, Dunford-Schwartz [4, p.585, Corollary 2]) for perturbations,

$$\begin{aligned} F(w;z) &= R(z;e^{wV}\Pi) = F(0;z)\sum_{n=0}^{\infty}((e^{wV} - I)\Pi F(0;z))^n \\ &= F(0;z) + F(0;z)(wV + \frac{w^2}{2}V^2 + r_2(w))\Pi F(0;z) \\ &\quad + F(0;z)(wV + r_3(w))\Pi F(0;z)(wV + r_3(w))\Pi F(0;z) \\ &\quad + R_4(w;z), \end{aligned}$$

with operators $R_4(w,z)$ on $C(E)$ satisfying $\sup_{z:|z-1|=r} ||R_4(w;z)||_{op} = O(|w|^3)$ as $|w| \to 0$. This gives us our assertion. □

For any real valued function $f \in C(E)$ with $\int_E f d\mu = 0$, let J^f be the rate function corresponding to $\{Q^f\}$, *i.e.*, let $\widetilde{\Pi}^f$ denote transition operator of Q^f and let

$$J^f(\nu) = \sup\left\{-\int_E \log\frac{\widetilde{\Pi}^f u}{u}d\nu; u \in C(E), u \geq 1\right\}, \quad \nu \in \mathcal{M}_1(E).$$

We give the following proposition, which will be used later.

PROPOSITION 2.3. *For any real-valued function $f \in C(E)$ with $\int_E f d\mu = 0$ and any $\nu \in \mathcal{M}_1(E)$,*

$$J^f(\nu) = J(\nu) - \int_E f d\nu + \Lambda(f).$$

PROOF. For any $V \in C(E;\mathbf{R})$, it is obvious by definition that for any $g \in C(E)$ and any $x \in E$,

$$\begin{aligned}
[(\widetilde{\Pi}^f)^V]^n g(x) &= E^{Q_x^f}\left[g(X_n)\exp\left(\sum_{k=0}^{n-1} V(X_k)\right)\right] \\
&= e^{-n\Lambda(f)}\frac{1}{h^f(x)} \\
&\quad \times E^{P_x}\left[g(X_n)\exp\left(\sum_{k=0}^{n-1}(V+f)(X_k)\right)h^f(X_n)\right] \\
&= e^{-n\Lambda(f)}\frac{h^{V+f}(x)}{h^f(x)}\Pi^{V+f}\left(\frac{gh^f}{h^{V+f}}\right)(x).
\end{aligned}$$

Therefore, the logarithmic spectral radius of V corresponding to Q^f is

$$\Lambda^f(V) = \lim_{n\to\infty}\frac{1}{n}\log||[(\widetilde{\Pi}^f)^V]^n||_{op} = \Lambda(V+f) - \Lambda(f).$$

Therefore, by (1.2), we get that for any $\nu \in \mathcal{M}_1(E)$,

$$\begin{aligned}
& J(\nu) - \int_E f d\nu + \Lambda(f) \\
= \ & \sup\{\int_E V d\nu - \Lambda(V) - \int_E f d\nu + \Lambda(f); V \in C(E,\mathbf{R})\} \\
= \ & \sup\{\int_E \widetilde{V} d\nu - \Lambda(\widetilde{V}+f) + \Lambda(f); \widetilde{V} \in C(E,\mathbf{R})\} \\
= \ & \sup\{\int_E V d\nu - \Lambda^f(V); V \in C(E,\mathbf{R})\} \\
= \ & J^f(\nu). \ \Box
\end{aligned}$$

Now, we are able to prove the following perturbation results.

PROPOSITION 2.4. *For any $V \in C(E)$, define $\Lambda(\varepsilon)$, h^ε, and μ^ε as before. Then there are $r_5(\varepsilon) \in C(E)$ with $\|r_5(\varepsilon)\|_\infty = o(\varepsilon)$ as $\varepsilon \to 0$, satisfying the following,*

(1) $\Lambda(\varepsilon) - \frac{\varepsilon^2}{2}\langle V, (I + 2\Pi G_0)V\rangle_\mu = O(\varepsilon^3)$ *as* $\varepsilon \to 0$,

(2) $h^\varepsilon = 1 + \varepsilon G_0 V + r_5(\varepsilon)$,

(3) $J(\mu^\varepsilon) - \frac{\varepsilon^2}{2}\langle V, (I + 2\Pi G_0)V\rangle_\mu = o(\varepsilon^2)$ *as* $\varepsilon \to 0$.

PROOF. By Proposition 2.1,

$$h^\varepsilon = E^\varepsilon 1 = \frac{1}{2\pi i}\oint_{|z-1|=r} F(\varepsilon; z)1dz,$$

so

$$\langle h^\varepsilon\rangle_\mu = \langle E^\varepsilon 1\rangle_\mu = \frac{1}{2\pi i}\oint_{|z-1|=r}\langle F(\varepsilon; z)1\rangle_\mu dz,$$

and

$$\begin{aligned} e^{\Lambda(\varepsilon)}\langle h^\varepsilon\rangle_\mu &= \langle \Pi^{\varepsilon V} E^\varepsilon 1\rangle_\mu = \frac{1}{2\pi i}\oint_{|z-1|=r}\langle \Pi^{\varepsilon V} F(\varepsilon; z)1\rangle_\mu dz \\ &= \frac{1}{2\pi i}\oint_{|z-1|=r} z\langle F(\varepsilon; z)1\rangle_\mu dz. \end{aligned}$$

We calculate the two integrations above, and the ratio will give us $e^{\Lambda(\varepsilon)}$.

From the property of spectral projections, since $e^{\Lambda(0)} = 1$, we have

$$I - \Pi = (I - \Pi)(I - E^0).$$

It is easy that

$$G_0(I - \Pi)(I - E^0) = (I - \Pi)G_0(I - E^0) = I - E^0,$$

so if we let $\widetilde{G_0} = G_0\Big|_{Image(I-E^0)}$, then $\widetilde{G_0}^{-1}$ exists and is equal to $(I - \Pi)\Big|_{Image(I-E^0)}$. Therefore,

$$(zI - \Pi) = (z-1)E^0 + \widetilde{G_0}^{-1}(I + (z-1)\widetilde{G_0})(I - E^0).$$

Since $(I+(z-1)\widetilde{G_0})^{-1}$ is holomorphic around $z=1$, and $\widetilde{G_0}$, E^0, and Π are all commutative with each other,

$$\begin{aligned} F(0;z) &= (zI-\Pi)^{-1} \\ &= (z-1)^{-1}E^0 + G_0(I+(z-1)G_0)^{-1}(I-E^0) \\ &= (z-1)^{-1}E^0 + G_0(I-E^0) \\ &\quad + \sum_{n=1}^{\infty}(-1)^n(z-1)^n G_0^{n+1}(I-E^0). \end{aligned} \tag{2.2}$$

Therefore, noting $E^0V=0$, we have $F(0;z)1=(z-1)^{-1}$, and $F_1(0;z)1=(z-1)^{-1}G_0V+\sum_{n=1}^{\infty}(-1)^n(z-1)^{n-1}G_0^{n+1}V$, where $F_1(0;z)$ is as in Proposition 2.2. By residue theorem, this implies that

$$\frac{1}{2\pi i}\oint_{|z-1|=r} F(0;z)1dz = 1, \tag{2.3}$$

and

$$\frac{1}{2\pi i}\oint_{|z-1|=r} F_1(0;z)1dz = G_0V. \tag{2.4}$$

Also, by (2.2) and the fact that $\langle G_0\cdot\rangle_\mu=0$, we have $\int_E F(0,z)fd\mu = (z-1)^{-1}\int_E fd\mu$ for any $f\in C(E)$. So we have

$$\langle F_2(0,z)1\rangle_\mu = (z-1)^{-2}\langle V^2\rangle_\mu + 2(z-1)^{-2}\langle V\Pi F(0,z)V\rangle_\mu.$$

By (2.2) again, this implies that

$$\frac{1}{2\pi i}\oint_{|z-1|=r}\langle F_2(0;z)1\rangle_\mu dz = -2\langle V\Pi G_0^2V\rangle_\mu.$$

This and (2.3), (2.4), accompanied with Proposition 2.1 and Proposition 2.2, give us that

$$\langle E^\varepsilon 1\rangle_\mu = \frac{1}{2\pi i}\oint_{|z-1|=r}\langle F(\varepsilon;z)1\rangle_\mu dz = 1-\varepsilon^2\langle V\Pi G_0^2V\rangle_\mu + O(\varepsilon^3). \tag{2.5}$$

In the same way, we have

$$\frac{1}{2\pi i}\oint_{|z-1|=r}(z-1)\langle F(\varepsilon;z)1\rangle_\mu dz = \frac{\varepsilon^2}{2}\left(\langle V^2\rangle_\mu + 2\langle V\Pi G_0V\rangle_\mu\right) + O(\varepsilon^3).$$

Therefore,

$$(2.6)\quad \begin{aligned} e^{\Lambda(\varepsilon)}\langle E^\varepsilon 1\rangle_\mu &= \langle E^\varepsilon 1\rangle_\mu + \frac{1}{2\pi i}\oint_{|z-1|=r}(z-1)\langle F(\varepsilon;z)1\rangle_\mu dz \\ &= \langle E^\varepsilon 1\rangle_\mu + \frac{\varepsilon^2}{2}\left(\langle V^2\rangle_\mu + 2\langle V\Pi G_0 V\rangle_\mu\right) + O(\varepsilon^3). \end{aligned}$$

Divide (2.6) by (2.5), and we get

$$e^{\Lambda(\varepsilon)} = 1 + \frac{\varepsilon^2}{2}\left(\langle V^2\rangle_\mu + 2\langle V\Pi G_0 V\rangle_\mu\right) + O(\varepsilon^3),$$

which gives us our first assertion. Moreover, by (2.3) and (2.4),

$$h^\varepsilon = E^\varepsilon 1 = 1 + \varepsilon G_0 V + r_5(\varepsilon)$$

with $r_5(\varepsilon) \in C(E)$ and $||r_5(\varepsilon)||_\infty = o(\varepsilon)$ as $\varepsilon \to 0$, which is our second assertion.

By (2.2) and the definition of $F_1(0,z)$, the coefficient of the term $(z-1)^{-1}$ in the expansion of $F_1(0,z)$ around $z=1$ is $E^0 V\Pi G_0(I-E^0) + G_0(I-E^0)V\Pi E^0 = E^0 V\Pi G_0 + G_0 V\Pi E^0$. So by Proposition 2.1 and Proposition 2.2,

$$\begin{aligned} E^\varepsilon &= \frac{1}{2\pi i}\oint_{|z-1|=r} F(\varepsilon;z)dz \\ &= E^0 + \varepsilon(E^0 V\Pi G_0 + G_0 V\Pi E^0) + R_8(\varepsilon), \end{aligned}$$

where $R_8(\varepsilon)$ are operators on $C(E)$ satisfying $||R_8(\varepsilon)||_{op} = o(\varepsilon)$ as $\varepsilon \to 0$. Therefore, for any $f \in C(E)$,

$$\begin{aligned} E^\varepsilon f &= \int_E f d\mu + \varepsilon\Big(\int_E V\Pi G_0 f d\mu + \int_E f d\mu G_0 V\Big) + R_8(\varepsilon)f \\ &= (1+\varepsilon G_0 V)\int_E f d\mu + \varepsilon\int_E V\Pi G_0 f d\mu + R_8(\varepsilon)f \\ &= h^\varepsilon \int_E f d\mu + \varepsilon\int_E V\Pi G_0 f d\mu + R_9(\varepsilon)f, \end{aligned}$$

where $R_9(\varepsilon)$ are operators on $C(E)$ satisfying $||R_9(\varepsilon)||_{op} = o(\varepsilon)$ as $\varepsilon \to 0$. Comparing this with

$$E^\varepsilon f = h^\varepsilon \int_E l^\varepsilon f d\mu,$$

we can get that $l^\varepsilon d\mu = (1+\varepsilon G_0^*\Pi^* V)d\mu + r_6(\varepsilon)d\mu$, where $r_6(\varepsilon)d\mu \in C(E)^*$ with $||r_6(\varepsilon)d\mu||_{C(E)^*} = o(\varepsilon)$ as $\varepsilon \to 0$.

Therefore, there exist $r_7(\varepsilon) \in C(E)$ with $||r_7(\varepsilon)d\mu||_{C(E)^*} = o(\varepsilon)$ as $\varepsilon \to 0$ such that $d\mu^\varepsilon = h^\varepsilon l^\varepsilon d\mu = (1+\varepsilon(G_0 + G_0^*\Pi^*)V + r_7(\varepsilon))d\mu$.

Note that $G_0 = \Pi G_0 + I - E^0$ and $\langle V\rangle_\mu = 0$. Therefore, by Proposition 2.3, we have

$$\begin{aligned} J(\mu^\varepsilon) &= \varepsilon\langle V\rangle_{\mu^\varepsilon} - \Lambda(\varepsilon) \\ &= \varepsilon\langle V, \varepsilon(I+2\Pi G_0)V) + r_7(\varepsilon)\rangle_\mu - \frac{\varepsilon^2}{2}\langle V, (I+2\Pi G_0)V\rangle_\mu + o(\varepsilon^2) \\ &= \frac{\varepsilon^2}{2}\langle V, (I+2\Pi G_0)V\rangle_\mu + o(\varepsilon^2). \ \square \end{aligned}$$

Let $\Phi : \mathcal{M}_1(E) \to \mathbf{R}$ be smooth in the sense of assumption A.2. For $\nu \in \mathcal{M}_1(E)$, the first derivative of Φ at ν is denoted by $D\Phi(\nu)$. Define

$$V^\nu(x) = D\Phi(\nu)(\delta_x - \nu) + \Phi(\nu), \qquad x \in E.$$

Lemma 2.5. *$\mu^{V^{\nu_0}} = \nu_0$.i.e. Q is ν_0 -invariant, and $b_\Phi = \Lambda(V^{\nu_0})$.*

Proof. We use the method of Bolthausen-Deuschel-Tamura [2].

For any $V \in C(E)$, let J^V be the rate function corresponding to (Q^V), then by Proposition 2.3,

$$J^V(\nu_0) = J(\nu_0) - \int_E V d\nu_0 + \Lambda(V).$$

It is well known that $J^V(\nu_0) = 0$ if and only if $\nu_0 = \mu^V$.

Now, from the definition of ν_0, ν_0 maximizes $\Phi - J$, so by the convexity of J, we have that for any $t \in (0,1)$ and any $\nu \in \mathcal{M}_1(E)$,

$$\begin{aligned} \Phi(\nu_0) - J(\nu_0) &\geq \Phi(t\nu + (1-t)\nu_0) - J(t\nu + (1-t)\nu_0) \\ &\geq \Phi(t\nu + (1-t)\nu_0) - tJ(\nu) - (1-t)J(\nu_0), \end{aligned}$$

therefore,

$$\frac{\Phi(t\nu + (1-t)\nu_0) - \Phi(\nu_0)}{t} \leq J(\nu) - J(\nu_0).$$

The left hand side converges to $D\Phi(\nu_0)(\nu-\nu_0) = \int_E V^{\nu_0} d\nu - \int_E V^{\nu_0} d\nu_0$ as $t \to 0$. So we have

$$J^{V^{\nu_0}}(\nu_0) = J(\nu_0) - \int_E V^{\nu_0} d\nu_0 \le J(\nu) - \int_E V^{\nu_0} d\nu = J^{V^{\nu_0}}(\nu) \tag{2.7}$$

for any $\nu \in \mathcal{M}_1(E)$. Therefore, ν_0 minimizes $J^{V^{\nu_0}}$, and hence $J^{V^{\nu_0}}(\nu_0) = 0$. This implies that $\nu_0 = \mu^{V^{\nu_0}}$.

Also, by the definition of b_Φ, (2.7), and (1.2), we have that

$$\begin{aligned} b_\Phi &= \int_E V^{\nu_0} d\nu_0 - J(\nu_0) \\ &= \sup\left\{ \int_E V^{\nu_0} d\mu - J(\mu), \mu \in \mathcal{M}_1(E) \right\} = \Lambda(V^{\nu_0}). \ \square \end{aligned}$$

Apply Proposition 2.4 to $\{Q_x\} = \{Q_x^{V^{\nu_0}}\}$, and we get the following proposition.

PROPOSITION 2.6.

$$D^2\Phi(\nu_0)(\overline{G} f d\nu_0, \overline{G} f d\nu_0) \le \langle f, \overline{G} f \rangle_{\nu_0}$$

for any $f \in C(E)$ *with* $\int_E f d\nu_0 = 0$.

PROOF. Take any $f \in C(E)$ with $\int_E f d\nu_0 = 0$ and fix it for a while. First, by Proposition 2.4 (3) applied to $\{Q_x\} = \{Q_x^{V^{\nu_0}}\}$ with invariant measure ν_0, we have

$$J^{V^{\nu_0}}(\nu_0^{\varepsilon f}) = \frac{\varepsilon^2}{2} \int_E f(I+2G) f d\nu_0 + o(\varepsilon^2) = \frac{\varepsilon^2}{2} \int_E f \overline{G} f d\nu_0 + o(\varepsilon^2). \tag{2.8}$$

Also, by the proof of Proposition 2.4 applied to ν_0,

$$d\nu_0^{\varepsilon f} = (1 + \varepsilon \overline{G} f + r_7'(\varepsilon)) d\nu_0 \tag{2.9}$$

with $|r_7'(\varepsilon)| = o(\varepsilon)$.

Since ν_0 maximizes $\Phi(\nu) - J(\nu) = \Phi(\nu) - J^{V^{\nu_0}}(\nu) + \int V^{\nu_0} d\nu - \Lambda(V^{\nu_0}) = \Phi(\nu) - D\Phi(\nu_0)(\nu) - J^{V^{\nu_0}}(\nu) - C_{\nu_0}$, we have

$$\Phi(\nu_0^{\varepsilon f}) - D\Phi(\nu_0)(\nu_0^{\varepsilon f}) - J^{V^{\nu_0}}(\nu_0^{\varepsilon f}) \le \Phi(\nu_0) - D\Phi(\nu_0)(\nu_0) - J^{V^{\nu_0}}(\nu_0)$$

for any $\varepsilon \in \mathbf{R}$. That is,

$$\Phi(\nu_0^{\varepsilon f}) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu_0^{\varepsilon f} - \nu_0) \le J^{V^{\nu_0}}(\nu_0^{\varepsilon f}) - J^{V^{\nu_0}}(\nu_0) \tag{2.10}$$

for any $\varepsilon \in \mathbf{R}$. By (2.9), the left hand side is equal to $\frac{\varepsilon^2}{2} D^2\Phi(\nu_0)(\overline{G} f d\nu_0, \overline{G} f d\nu_0) + o(\varepsilon^2)$ as $\varepsilon \to 0$. Also, the right hand side is equal to $\frac{\varepsilon^2}{2}\int_E f\overline{G} f d\nu_0 + o(\varepsilon^2)$ by (2.8). This gives us our assertion. □

3. Lemmas

In this section, we prove the forward-backward martingale decomposition for partial sums processes and establish the exponential integrability for the partial sums processes by means of it.

Before discuss the related partial sums process, let us first establish an inequality for martingale differences.

Lemma 3.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{A}_n$ be a sequence of nondecreasing σ-sub-algebras of $\mathcal{A}$. Let $\{d_k, \mathcal{A}_k; k \ge 1\}$ be a martingale difference, and assume that $\sup_k \|d_k\|_\infty \le C$ for some constant $C > 0$. For any $\varepsilon > 0$ and $\lambda \in \mathbf{R}$, if $e^{|\lambda| C} - 1 - |\lambda| C \le \frac{1+\varepsilon}{2} C^2\lambda^2$ is satisfied, then*

$$E\left[\exp\left(\sum_{k=1}^n \left(\lambda d_k - \frac{(1+\varepsilon)\lambda^2}{2} E[d_k^2 | \mathcal{A}_{k-1}]\right)\right)\right] \le 1 \tag{3.1}$$

holds for all $n \in N$.

Proof. From the assumptions, we have that

$$\begin{aligned}
E\left[\exp\left(\lambda d_k\right) | \mathcal{A}_{k-1}\right] &= E\left[1 + \lambda d_k + \sum_{n=2}^\infty \frac{\lambda^n d_k^n}{n!} \,\middle|\, \mathcal{A}_{k-1}\right] \\
&\le 1 + \sum_{n=2}^\infty \frac{|\lambda|^n C^{n-2}}{n!} E\left[d_k^2 \,\middle|\, \mathcal{A}_{k-1}\right] \\
&= 1 + C^{-2} E\left[d_k^2 | \mathcal{A}_{k-1}\right]\left(e^{|\lambda| C} - 1 - |\lambda| C\right) \\
&\le 1 + \frac{(1+\varepsilon)\lambda^2}{2} E\left[d_k^2 | \mathcal{A}_{k-1}\right] \\
&\le \exp\left(\frac{(1+\varepsilon)\lambda^2}{2} E[d_k^2 | \mathcal{A}_{k-1}]\right), \quad k \ge 1.
\end{aligned}$$

Thus, we have

$$E\left[\exp\left(\lambda d_k - \frac{(1+\varepsilon)\lambda^2}{2}E[d_k^2|\mathcal{A}_{k-1}]\right)\,\Big|\,\mathcal{A}_{k-1}\right] \leq 1.$$

Therefore

$$\begin{aligned} & E\left[\exp\left(\sum_{k=1}^{n}\left(\lambda d_k - \frac{(1+\varepsilon)\lambda^2}{2}E[d_k^2|\mathcal{A}_{k-1}]\right)\right)\right] \\ = \ & E\left[\exp\left(\sum_{k=1}^{n-1}\left(\lambda d_k - \frac{(1+\varepsilon)\lambda^2}{2}E[d_k^2|\mathcal{A}_{k-1}]\right)\right)\right. \\ & \left.\times E\left[\exp\left(\lambda d_n - \frac{(1+\varepsilon)\lambda^2}{2}E[d_n^2|\mathcal{A}_{n-1}]\right)\,\Big|\,\mathcal{A}_{n-1}\right]\right] \\ \leq \ & E\left[\exp\left(\sum_{k=1}^{n-1}\left(\lambda d_k - \frac{(1+\varepsilon)\lambda^2}{2}E[d_k^2|\mathcal{A}_{k-1}]\right)\right)\right]. \end{aligned}$$

This gives us our assertion by induction. □

REMARK 3.1. Let $g(y) = \sup_{x\in(0,y]} x^{-2}(e^x - 1 - x), y > 0, g(0) = 1/2$. For $\varepsilon > 0$, define

$$\gamma(\varepsilon) = 1 \wedge \sup\left\{y \in [0,1] : g(y) \leq \frac{1+\varepsilon}{2}\right\} > 0.$$

If $\lambda \in \mathbf{R}$ and $C > 0$ satisfy $|\lambda|C \leq \gamma(\varepsilon)$, then (3.1) holds.

The following Lemma is a consequence of simple integration, we omit its proof.

LEMMA 3.2. *Let X be a random variable, $A \in \mathcal{F}$, and $0 < \alpha < 1/2$, suppose that $E(\exp(\xi X), A) \leq e^{\alpha\xi^2}$ for all $\xi \in \mathbf{R}$. Then*

$$E\left[\exp(\frac{1}{2}X^2), A\right] \leq \left(\frac{1}{1-2\alpha}\right)^{1/2}.$$

From now on, let Q and $\{Q_x\}_{x\in E}$ be as defined in Section 1. By Lemma 2.5, we have that ν_0 is the unique $\{Q_x\}$-invariant probability measure.

LEMMA 3.3. *Let $f \in C(E)$ with $\int_E f(x)\nu_0(dx) = 0$ and let $u = Gf + f$. Then we have $(I - Q)u = f$. Let $\{X_k, k \geq 1\}$ be the Markov chain corresponding to Q_x. Let $d_k = u(X_k) - Qu(X_{k-1})$, then $\{d_k, k \geq 1\}$ is a martingale difference and we have the following forward decomposition*

$$\sum_{k=0}^{n-1} f(X_k) = u(X_0) - u(X_n) + \sum_{k=1}^{n} d_k. \tag{3.2}$$

PROOF. The first assertion is easy, actually,

$$\begin{aligned}(I-Q)u &= (I-Q)f + (I-Q)Gf \\ &= (I-Q)f + \lim_{n\to\infty} \sum_{k=1}^{n} (Q^k f - Q^{k+1} f) \\ &= (I-Q)f + Qf - \lim_{n\to\infty} Q^{n+1} f = f.\end{aligned}$$

With this in hand, by the definition of d_k, we have that

$$\begin{aligned}u(X_n) &= u(X_0) + \sum_{k=1}^{n} \Big(u(X_k) - Qu(X_{k-1}) \Big) \\ &\quad + \sum_{k=1}^{n} \Big(Qu(X_{k-1}) - u(X_{k-1}) \Big) \\ &= u(X_0) + \sum_{k=1}^{n} d_k - \sum_{k=0}^{n-1} f(X_k).\end{aligned}$$

The fact that $\{d_k\}_{k\geq 1}$ is a martingale difference is trivial, since by Markov property, we have $E[u(X_k) - Qu(X_{k-1}) | \mathcal{F}_{k-1}] = 0$. This completes the proof of our Lemma. □

In the remainder of this paper, let $A_\varepsilon = \{dist(L_n, \nu_0) \leq \varepsilon\}$ for any $\varepsilon > 0$.

LEMMA 3.4. *Let $f \in C(E)$ with $\int_E f d\nu_0 = 0$ and $\|f\|_{H^*} \leq 1$. Then for any $\alpha < 1$, there exists an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,*

$$\sup_{x,y\in E} \sup_{n\geq 1} E^{Q_x} \left[\exp\left(\frac{\alpha}{2n} \left(\sum_{k=0}^{n-1} f(X_k) \right)^2 \right), A_\varepsilon \,\middle|\, X_{n-1} = y \right] < \infty.$$

Proof. Since

$$\frac{1}{n}\left|\left(\sum_{k=0}^{n-1} f(X_k)\right)^2 - \left(\sum_{k=1}^{n-2} f(X_k)\right)^2\right| \leq \frac{4n-4}{n}\|f\|_\infty^2 \leq 4\|f\|_\infty^2,$$

we have that for any $x, y \in E$,

$$\begin{aligned} & E^{Q_x}\left[\exp\left(\frac{\alpha}{2n}\left(\sum_{k=0}^{n-1} f(X_k)\right)^2\right), A_\varepsilon \,\middle|\, X_{n-1} = y\right] \\ \leq \;& q_n(x,y)^{-1} C e^{8\|f\|_\infty^2} E^{Q_{\nu_0}}\left[\exp\left(\frac{\alpha}{2n}\left(\sum_{k=0}^{n-1} f(X_k)\right)^2\right), A_\varepsilon\right] \end{aligned}$$

where $C \equiv \sup_{x,y,x',y'}\{q(x,x')q(y',y)\} < \infty$.

By (3.2), we have that

$$\frac{1}{n}\left(\sum_{k=1}^{n} f(X_k)\right)^2 \leq \frac{2\|u\|_\infty}{n} + \frac{2}{n}\Big(u(X_n) - u(X_0)\Big)\cdot\sum_{k=1}^{n} d_k + \frac{1}{n}\left(\sum_{k=1}^{n} d_k\right)^2.$$

From the boundedness of u and d_k, it is sufficient to prove

$$\sup_{n\geq 1} E^{Q_{\nu_0}}\left[\exp\left(\frac{\alpha}{2n}\left(\sum_{k=1}^{n} d_k\right)^2\right), A_\varepsilon\right] < \infty. \tag{3.3}$$

Since $\alpha < 1$, we can find an $\varepsilon_1 > 0$ such that $\alpha' \equiv \alpha(1+\varepsilon_1) < 1$. Let $C' = (1 + \|Q\|_{op})\|u\|$, then $\|d_k\| \leq C'$ for all $k \geq 1$. By Remark 3.1, there exists a constant $\lambda_0 = \lambda_0(\varepsilon_1, C') > 0$ such that for all $|\lambda| \leq \lambda_0$,

$$E^{Q_{\nu_0}}\left[\exp\left(\lambda\sum_{k=1}^{n} d_k - \frac{(1+\varepsilon_1)\lambda^2}{2}\sum_{k=1}^{n} E^{Q_{\nu_0}}[d_k^2|\mathcal{F}_{k-1}]\right)\right] \leq 1. \tag{3.4}$$

Take $\delta \in (0, (\frac{1}{\alpha'} - 1) \wedge (\frac{\lambda_0}{4\alpha}))$ and $n_0 > [4\|u\|_\infty/\delta]$. Then there exists a constant $\varepsilon_2 > 0$ such that for any $n \geq n_0$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} d_k\right| \leq \frac{2\|u\|_\infty}{n} + \left|\frac{1}{n}\sum_{k=1}^{n} f(X_k)\right| \leq \delta \quad \text{on} \quad A_{\varepsilon_2}.$$

Let $g = Q(u^2) - (Qu)^2$, by Markov property, we have

$$g(X_{k-1}) = E^{Q_{\nu_0}}\left[d_k^2 \,\middle|\, \mathcal{F}_{k-1}\right], \quad k \geq 1.$$

On the other hand, since Q is ν_0-invariant, we get by Lemma 3.3 that

$$\begin{aligned}
\int_E g d\nu_0 &= (u,u)_{L^2(d\nu_0)} - (Qu, Qu)_{L^2(d\nu_0)} \\
&= 2(u,(I-Q)u)_{L^2(d\nu_0)} - ((I-Q)u,(I-Q)u)_{L^2(d\nu_0)} \\
&= 2(f+Gf,f)_{L^2(d\nu_0)} - (f,f)_{L^2(d\nu_0)} = (f,\overline{G}f)_{L^2(d\nu_0)} \\
&= \|f\|_{H^*}^2 \leq 1.
\end{aligned}$$

Since g is bounded, there exists an $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} g(X_k) - \int_E g d\nu_0\right| < \delta \quad \text{on} \quad A_{\varepsilon_3}.$$

So

$$\left|\sum_{k=1}^{n} E^{Q_{\nu_0}}\left[d_k^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right| \leq n(1+\delta) \qquad \text{on} \qquad A_{\varepsilon_3}.$$

This accompanied with (3.4) gives us that for any $|\xi| \leq \sqrt{\frac{n}{\alpha}}\lambda_0$,

$$E^{Q_{\nu_0}}\left[\exp\left(\xi\sqrt{\frac{\alpha}{n}}\sum_{k=1}^{n} d_k\right), A_{\varepsilon_3}\right] \leq \exp\left(\xi^2\frac{(1+\delta)\alpha'}{2}\right), \quad n \geq 1.$$

Also, if $|\xi| \geq \sqrt{\frac{n}{\alpha}}\lambda_0$, then

$$\begin{aligned}
E^{Q_{\nu_0}}\left[\exp\left(\xi\sqrt{\frac{\alpha}{n}}\sum_{k=1}^{n} d_k\right), A_{\varepsilon_3}\right] &\leq \exp\left\{|\xi|\sqrt{\frac{n}{\alpha}}\delta n\right\} \\
&\leq \exp\left\{\xi^2\alpha\delta/\lambda_0\right\} \leq e^{\xi^2/4}.
\end{aligned}$$

Therefore, we have our assertion by Lemma 3.2. □

For $\delta > 0$, let $\mathcal{V}_\delta$ be the collection of all symmetric, bilinear functions $V(x,y) \in C(E^2, \mathbf{R})$ satisfying the following two conditions,

(1) $\int_E V(x,y)\nu_0(dy) = 0$ for any $x \in E$,

(2) $\sup_{x,y} |V(x,y)| \le \delta$.

LEMMA 3.5. *There exists a $\delta_0 \in (0,1)$ such that*

$$\sup_{V\in\mathcal{V}_{\delta_0}} \sup_{x,y\in E} \sup_{n\ge 0} E^{Q_x}\left[\exp\left(n\int_E\int_E V(x,y)L_n(dx)L_n(dy)\right)\,\Big|\, X_{n-1}=y\right]$$
$$= C_0 < \infty.$$

PROOF. Notice that

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} V(X_k,X_j) - \frac{1}{n}\sum_{k=1}^{n-2}\sum_{j=1}^{n-2} V(X_k,X_j)\right| \le \frac{4n-4}{n}\|V\|_\infty \le 4\|V\|_\infty.$$

Let $C = \sup_{x,y,x',y'}\{q(x,x')q(y',y)\}$, then for any $x,y\in E$,

$$\begin{aligned}
& E^{Q_x}\left[\exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} V(X_k,X_j)\right)\,\Big|\, X_{n-1}=y\right]\\
\le\ & q_n(x,y)^{-1}E^{Q_{\nu_0}}\\
& \times\left[q(x,X_1)q(X_{n-2},y)\exp\left(\frac{1}{n}\sum_{k=1}^{n-2}\sum_{j=1}^{n-2} V(X_k,X_j)+4\|V\|_\infty\right)\right]\\
\le\ & q_n(x,y)^{-1}Ce^{8\|V\|_\infty}E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} V(X_k,X_j)\right)\right].
\end{aligned}$$

Since $\int_E V(x,y)\nu_0(dy) = 0$ for any $x\in E$, by Lemma 3.3, there exists a $U_1(x,y)\in C(E^2,\mathbf{R})$ such that

$$((I-Q_x)U_1)(x,y) = V(x,y).$$

Let $d_k = \sum_{j=0}^{k-1}\Big(U_1(X_k,X_j) - (Q_xU_1)(X_{k-1},X_j)\Big)$, then $\{d_k,\mathcal{F}_k, k\ge 1\}$ is a martingale difference and we have the following decomposition

$$\sum_{j=0}^{n-1}\sum_{k=j}^{n-1} V(X_k,X_j) = \sum_{j=0}^{n-1}\Big(U_1(X_j,X_j) - U_1(X_n,X_j)\Big) + \sum_{k=1}^{n} d_k. \tag{3.5}$$

On the other hand, there exists a function $U(x,y) \in C(E^2, \mathbf{R})$ satisfying $\int_E U(x,y)\nu_0(dy) = 0$ for any $x \in E$ such that

$$((I - Q_y^*)U)(x,y) = U_1(x,y),$$

where Q_y^* is the adjoint operator of Q_y in $L^2(d\nu_0)$.

As in Lemma 3.3, we have the following backward decomposition

$$\begin{aligned} U(X_k, X_0) - U(X_k, X_k) &= \sum_{j=1}^{k} \big(U(X_k, X_{j-1}) - (Q_y^* U)(X_k, X_j)\big) \\ &\quad + \sum_{j=1}^{k} U_1(X_k, X_j), \end{aligned}$$

and

$$\begin{aligned} &(Q_x U)(X_{k-1}, X_0) - (Q_x U)(X_{k-1}, X_k) \\ &= \sum_{j=1}^{k} \big((Q_x U)(X_{k-1}, X_{j-1}) - (Q_y^* Q_x U)(X_{k-1}, X_j)\big) \\ &\quad + \sum_{j=1}^{k} (Q_x U_1)(X_{k-1}, X_j). \end{aligned}$$

For $1 \le j \le k$, let

$$\begin{aligned} d_j^{(k)} = &\Bigg[\Big(U(X_k, X_{j-1}) - (Q_y^* U)(X_k, X_j)\Big) \\ &- \Big((Q_x U)(X_{k-1}, X_{j-1}) - (Q_y^* Q_x U)(X_{k-1}, X_j)\Big)\Bigg], \end{aligned} \tag{3.6}$$

and

$$Z_k = \sum_{j=1}^{k} d_j^{(k)}.$$

Then we have that

$$\begin{aligned} d_k + Z_k = &U(X_k, X_0) - U(X_k, X_k) \\ &- ((Q_x U)(X_{k-1}, X_0) - (Q_x U)(X_{k-1}, X_k)). \end{aligned} \tag{3.7}$$

Let $C_1 = (1+\|Q^*\|_{op})(1+\|Q\|_{op}), C_2 = \|I+G^*\|_{op}\|I+G\|_{op}$, and choose $2\delta_0 = (32C_1C_2)^{-1}\wedge\gamma(1)(128C_1C_2)^{-1}\wedge\gamma(1)((1+\|Q\|_{op})(\|I+G\|_{op}))^{-1}$, where $\gamma(1)$ is defined as in Remark 3.1.

Since $V \in \mathcal{V}_{\delta_0}$, we get from (3.5) that

$$\left|\sum_{k=0}^{n-1}\sum_{j=0}^{n-1} V(X_k, X_j) - 2\sum_{k=1}^{n} d_k\right| \le 2C_1\delta_0 n.$$

Therefore, it is sufficient to prove that

$$\sup_{V\in\mathcal{V}_{2\delta_0}} \sup_{n\ge1} E^{Q_{v_0}}\left[\exp\left(\frac{1}{2n}\sum_{k=1}^{2n} d_k\right)\right] < \infty.$$

By our assumptions and the definition of d_k, we have that $\sup_{1\le k\le n}\left\|\frac{d_k}{n}\right\| \le (1+\|Q\|_{op})(\|I+G\|_{op})\delta$. By Remark 3.1, since $(1+\|Q\|_{op})(\|I+G\|_{op})\delta \le \gamma(1)$, we have that

$$E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n}\sum_{k=1}^{n} d_k - \frac{1}{n^2}\sum_{k=1}^{n} E^{Q_{\nu_0}}[d_k^2|\mathcal{F}_{k-1})\right)\right] \le 1,$$

therefore, by Schwartz inequality, we have that

$$E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{2n}\sum_{k=1}^{n} d_k\right)\right] \le E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n^2}\sum_{k=1}^{n} E^{Q_{\nu_0}}[d_k^2|\mathcal{F}_{k-1}]\right)\right]^{1/2}.$$

Since U and Q_xU are bounded, by (3.7), it is sufficient to prove that

$$\sup_{V\in\mathcal{V}_{2\delta_0}} \sup_{n\ge1} E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n^2}\sum_{k=1}^{n} E^{Q_{\nu_0}}\left[Z_k^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right)\right] < \infty. \tag{3.8}$$

By virtue of Jensen's inequality, we get

$$\begin{aligned} & E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n^2}\sum_{k=1}^{n} E^{Q_{\nu_0}}\left[Z_k^2 \,\middle|\, \mathcal{F}_{k-1}\right]\right)\right] \\ \le\ & \frac{1}{n}\sum_{k=1}^{n} E^{Q_{\nu_0}}\left(\exp\left(\frac{1}{n}E^{Q_{\nu_0}}\left(Z_k^2 \,\middle|\, \mathcal{F}_{k-1}\right)\right)\right) \\ \le\ & \frac{1}{n}\sum_{k=1}^{n} E^{Q_{\nu_0}}\left[\exp\left(\frac{1}{n}Z_k^2\right)\right]. \end{aligned}$$

By Remark 3.1, we have that

$$E^{Q_{\nu_0}}\left[\exp\left(\lambda Z_k - \lambda^2 E^{Q_{\nu_0}}[Z_k^2|\mathcal{F}_j^n]\right)\right] \le 1 \tag{3.9}$$

holds for any $|\lambda| \le \frac{\gamma(1)}{32}$. From the definition of $d_j^{(k)}$, we have that $\max_{1\le j\le k} \|d_j^{(k)}\|_\infty \le C_1C_2\delta_0$, hence

$$E^{Q_{\nu_0}}[Z_k^2|\mathcal{F}_j^n] \le n(C_1C_2\delta)^2 \le \frac{n}{32^2},$$

therefore,

$$E^{Q_{\nu_0}}\left[\exp\left(\xi\sqrt{\frac{2}{n}}Z_k\right)\right] \le \exp\left(\frac{2\xi^2}{32^2}\right) \quad \text{for any} \quad |\xi| < \frac{\gamma(1)\sqrt{n}}{32}.$$

On the other hand, if $|\xi| \ge \frac{\gamma(1)\sqrt{n}}{32}$, from the fact that $\left|\sqrt{\frac{2}{n}}Z_k\right| \le \sqrt{n}C_1C_2\delta_0$, we have

$$E^{Q_{\nu_0}}\left[\exp\left(\xi\sqrt{\frac{2}{n}}Z_k\right)\right] \le E^{Q_{\nu_0}}\left[\exp\left(\xi^2\frac{16C_1C_2\delta_0}{\gamma(1)}\right)\right] \le e^{\xi^2/4}.$$

Now, our assertion follows from Lemma 3.2. □

LEMMA 3.6. *Let $V \in C(E\times E,\mathbf{R})$. Assume that V is symmetric and satisfies $\int_E V(x,y)\nu_0(dy) = 0$ for any $x \in E$. Then for any $\varepsilon > 0$, there exist $N \in \mathbf{N}$, $f_n, g_n \in C(E), n = 1,2,\cdots,N$, such that*

$$\int_E f_k(x)\nu_0(dx) = 0, \int_E g_k(y)\nu_0(dy) = 0,$$

and

$$V(x,y) = \sum_{k=1}^N \{f_k(x)g_k(y) + f_k(y)g_k(x)\} + V'(x,y) \tag{3.10}$$

where $V'(x,y) \in C(E\times E,\mathbf{R})$ is symmetric and satisfies $\int_E V'(x,y)\nu_0(dy) = 0$ for any $x \in E$ and $\sup_{x,y}|V'(x,y)| \le \varepsilon$.

PROOF. By Stone-Weierstrass theorem, there exist an $N \in \mathbf{N}$ and functions $\tilde{f}_k, \tilde{g}_k \in C(E), k = 1, 2, \cdots, N$, and $\tilde{V}(x, y) \in C(E \times E)$ such that $|\widetilde{V}(x, y)| < \varepsilon/8$ and

$$V(x, y) = 2\left(\sum_{k=1}^{N} \tilde{f}_k(x)\tilde{g}_k(y) + \widetilde{V}(x, y)\right).$$

Replacing $\tilde{f}_k, \tilde{g}_k$ by $f_k = \tilde{f}_k - \int_E \tilde{f}_k d\nu_0, g_k = \tilde{g}_k - \int_E \tilde{g}_k d\nu_0$, we have

$$V(x, y) = 2\left(\sum_{k=1}^{N}(f_k(x)g_k(y) + \widetilde{\widetilde{V}}(x, y)\right)$$

where $\widetilde{\widetilde{V}}(\cdot, \cdot)$ is given by

$$\begin{aligned} & \widetilde{\widetilde{V}}(x, y) \\ = \quad & \tilde{V}(x, y) + \sum_{k=1}^{N} \tilde{f}_k(x) \int_E \tilde{g}_k(z)\nu_0(dz) \\ & + \sum_{k=1}^{N} \tilde{g}_k(y) \int_E \tilde{f}_k(z)\nu_0(dz) - \sum_{k=1}^{N} \int_E \tilde{f}(z)\nu_0(dz) \int_E \tilde{g}(z)\nu_0(dz) \\ = \quad & \widetilde{V}(x, y) - \int_E \widetilde{V}(x, z)\nu_0(dz) - \int_E \widetilde{V}(z, y)\nu_0(dz) \\ & - \int_E \int_E \widetilde{V}(z, w)\nu_0(dz)\nu_0(dw), \end{aligned}$$

hence $\left|\widetilde{\widetilde{V}}(x, y)\right| \leq \varepsilon/2$.

Therefore, since $V(\cdot, \cdot)$ is symmetric, we have that

$$V(x, y) = \sum_{k=1}^{n} \{f_k(x)g_k(y) + f_k(y)g_k(x)\} + V'(x, y)$$

where $V'(x, y) = \widetilde{\widetilde{V}}(x, y) + \widetilde{\widetilde{V}}(y, x)$. This completes the proof of our Lemma. □

LEMMA 3.7. *Let $f_i(x) \in C(E), i = 1, 2, \cdots, m$, let $\{a_{ij}\}_{i,j=1,2,\cdots,m}$ be a symmetric matrix, and let $V(x, y) = \sum_{i,j=1}^{m} a_{ij} f_i(x) f_j(y)$. Define the* *symmetric bilinear continuous function* $A_V : \mathcal{M}(E) \times \mathcal{M}(E) \to \mathbf{R}$ *by*

$$A_V(R_1, R_2) = \int_E \int_E V(x,y) R_1(dx) R_2(dy).$$

Suppose that all of the eigenvalues of $A_V|_{H\times H}$ *are smaller than 1. Then there exists a constant* $\varepsilon_0 > 0$ *such that for any* $x \in E$ *and any* $\varepsilon \le \varepsilon_0$,

$$\sup_{x,y\in E} \sup_{n\ge 1} E^{Q_x}\left[\exp\left(\frac{n}{2}\int_E\int_E V(x,y) L_n(dx) L_n(dy)\right), A_\varepsilon \,\middle|\, X_{n-1} = y\right] < \infty.$$

PROOF. Let $\mathcal{U}$ be the linear space spanned by $\{f_1, \cdots, f_m\}$ and let d be the dimension of $\mathcal{U}$. Denote $\widetilde{\mathcal{U}} \equiv \{\overline{G} f d\nu_0, f \in \mathcal{U}\}$, which is a subset of H. Then there exists a C.O.N.S. $\{e_1, \cdots, e_d\}$ of $\mathcal{U}$ such that

$$\langle \overline{G} e_i d\nu_0, \overline{G} e_j d\nu_0\rangle_H = \langle e_i, \overline{G} e_j\rangle_{\nu_0} = \delta_{ij}, \quad A_V(\overline{G} e_i d\nu_0, \overline{G} e_j d\nu_0) = a_i \delta_{ij},$$

and

$$V(x,y) = \sum_{i,j}^{d} c_{ij} e_i(x) e_j(x),$$

where $\{c_{ij}\}_{i,j=1,2,\cdots,d}$ is a symmetric matrix. On the other hand,

$$A_V(\overline{G} e_i d\nu_0, \overline{G} e_j d\nu_0) = \int_E\int_E V(x,y) \overline{G} e_i \nu_0(dx) \overline{G} e_j \nu_0(dy) = c_{ij} = a_i \delta_{ij}.$$

Therefore

$$V(x,y) = \sum_{i=1}^{d} a_i e_i(x) e_i(y).$$

By our assumption, there exists a constant $\varepsilon \in (0,1)$ such that $a_i \le 1 - \varepsilon$, $i = 1, \cdots, d$. Therefore

$$\sum_{k=0}^{n-1}\sum_{m=0}^{n-1} V(X_k, X_m) \le (1-\varepsilon) \sum_{i=1}^{d} \left(\sum_{k=0}^{n-1} e_i(X_k)\right)^2.$$

Since $\{x \in \mathbf{R}^d : \|x\| \le (1+\varepsilon)^{-1/2}\} = \bigcap \{\{x \in \mathbf{R}^d : (x,\xi) \le (1+\varepsilon)^{-1/2}\} : \xi \in \mathbf{R}^d, \|\xi\| = 1\}$, there exist an $N \in \mathbf{N}$ and $\xi_i = (\xi_i^1, \cdots, \xi_i^d), i = 1, 2, \cdots, N$, with $\|\xi_i\|_{\mathbf{R}^d} = 1$ such that

$$\bigcap_{i=1}^{N} \left\{x \in \mathbf{R}^d ; (x, \xi_i) \le \frac{1}{(1+\varepsilon)^{1/2}}\right\} \subset \{x \in \mathbf{R}^d : \|x\| \le 1\}.$$

This implies that

$$\|x\|^2 \le (1+\varepsilon) \max_{1\le i\le N} (x, \xi_i)^2, \quad x \in \mathbf{R}^d.$$

Define

$$\tilde{e}_i = \sum_{j=1}^{d} \xi_i^j e_j, \qquad i = 1, 2, \cdots, N,$$

then we have

$$(\tilde{e}_i \overline{G} \tilde{e}_i)_{L^2(d\nu_0)} = 1, \quad \langle \tilde{e}_i \rangle_{\nu_0} = 0, \qquad i = 1, 2, \cdots, N.$$

On the other hand,

$$\begin{aligned} \sum_{j=1}^{d} \left(\int e_j(x) dL_n(dx) \right)^2 &\le (1+\varepsilon) \max_{1\le i\le N} \sum_{j=1}^{d} \left(\int_E e_j(x) L_n(dx) \cdot \xi_j^i \right)^2 \\ &= (1+\varepsilon) \max_{1\le i\le N} \left(\int_E \tilde{e}_i L_n(dx) \right)^2, \end{aligned}$$

i.e.,

$$\sum_{j=1}^{d} \left(\sum_{k=0}^{n-1} e_j(X_k) \right)^2 \le (1+\varepsilon) \max_{1\le i\le N} \left(\sum_{k=0}^{n-1} \tilde{e}_i(X_k) \right)^2.$$

Therefore, we get by Lemma 3.4 that

$$\begin{aligned} &\sup_{n\ge 1} E^{Q_{\nu_0}} \left[\exp\left(\frac{n}{2} \int_E \int_E V(x,y) L_n(dx) L_n(dy) \right), A_\varepsilon \,\middle|\, X_{n-1} = y \right] \\ &\le N \times \sup_{n\ge 1} \max_{1\le i\le N} E^{Q_{\nu_0}} \left[\exp\left(\frac{1-\varepsilon^2}{2n} \left(\sum_{k=0}^{n-1} \tilde{e}_i(X_k) \right)^2 \right), A_\varepsilon \,\middle|\, X_{n-1} = y \right] \\ &< \infty. \ \square \end{aligned}$$

LEMMA 3.8. *For any continuous symmetric function $V \in C(E \times E, \mathbf{R})$ satisfying $\int_E V(x,y)\nu_0(dy) = 0$ for any $x \in E$, define the symmetric bilinear and continuous function $A_V : \mathcal{M}(E) \times \mathcal{M}(E) \to \mathbf{R}$ by*

$$A_V(R_1, R_2) = \int_E \int_E V(x,y) R_1(dx) R_2(dy).$$

Assume that all of the eigenvalues of $A_V|_{H\times H}$ *are smaller than 1, then there exists a constant* $\varepsilon_0 > 0$ *such that for any* $x \in E$ *and any* $\varepsilon \le \varepsilon_0$*,*

$$\sup_{x,y\in E}\sup_{n\ge 1} E^{Q_x}\left[\exp\left(\frac{n}{2}\int_E\int_E V(x,y)L_n(dx)L_n(dy)\right), A_\varepsilon \,\middle|\, X_{n-1}=y\right] < \infty.$$

PROOF. Let a_0 be the maximum eigenvalue of $A|_{H\times H}$, which is smaller than 1 by our assumption. For any $\delta > 0$ there exist a $N \in \mathbf{N}$ and $\{f_k, g_k; k = 1, 2, \cdots, N\} \subset C(E)$ such that

$$\begin{aligned} V(x,y) &= \sum_{k=1}^N \{f_k(x)g_k(y) + f_k(y)g_k(x)\} + V'(x,y) \\ &= V_1(x,y) + V'(x,y) \end{aligned}$$

where $|V'(x,y)| < \delta$ and $V_1(x,y) = \sum_{k=1}^N \{f_k(x)g_k(y) + f_k(y)g_k(x)\}$.

For any $\delta > 0$, it is easy that the operator norm of $A_{V'}\Big|_{H\times H}$ is also smaller than $\widetilde{C}^2\delta^2$. Now, $A_{V_1} = A_V - A_{V'}$, and all of the eigenvalues of $A_V\Big|_{H\times H}$ are smaller than 1 uniformly by our assumption. Therefore, by the continuity of spectral theory (*c.f.*, Dunford-Schwartz [4]), we have that there exists a $\delta_1 > 0$ such that all of the eigenvalues of $A_{V_1}\Big|_{H\times H}$ are also smaller than 1 uniformly as long as $\delta < \delta_1$. Write the maximum as $\eta_\delta < 1$ ($\eta_\delta \to a_0$ as $\delta \to 0$).

Choose constants $p > 1$ and $\delta > 0$ such that $p\eta_\delta < 1$ and $\delta \cdot q < \delta_0$, where q stands for the dual number of p and δ_0 is as in Lemma 3.5. By Hölder inequality,

$$\begin{aligned} & E^{Q_x}\left[\exp\left(\frac{n}{2}\int_E\int_E V(x,y)L_n(dx)L_n(dy)\right), A_\varepsilon \,\middle|\, X_{n-1}=y\right] \\ \le\ & E^{Q_x}\left[\exp\left(\frac{n}{2}\int_E\int_E pV_1(x,y)L_n(dx)L_n(dy)\right), A_\varepsilon \,\middle|\, X_{n-1}=y\right]^{1/p} \\ & \times E^{Q_x}\left[\exp\left(\frac{n}{2}\int_E\int_E qV'(x,y)L_n(dx)L_n(dy)\right), A_\varepsilon \,\middle|\, X_{n-1}=y\right]^{1/q}. \end{aligned}$$

This accompanied with Lemma 3.5 and Lemma 3.7 gives us our assertion. □

4. Proof of the Main Theorem

We give the proof of Theorem 1.2 in this section. Let

$$\begin{aligned}\widetilde{\Phi}(\nu) &\equiv \Phi(\nu) - \int_E \phi^{\nu_0}(y)\nu(dy)\\ &= \Phi(\nu) - \Phi(\nu_0) - D\Phi(\nu_0)(\nu - \nu_0).\end{aligned}$$

Note that

$$\begin{aligned}&e^{-\lambda n} E^{P_x}\left[\exp\left(n\Phi(L_n)\right), A \mid X_n = y\right]\\ &= \frac{h(x)}{h(y)} E^{Q_x}\left[\exp\left(n\widetilde{\Phi}(L_n)\right), A \mid X_n = y\right]\end{aligned} \tag{4.1}$$

for any $A \in \mathcal{F}_n$.

Lemma 4.1. *For any $x, y \in E$ and any $\varepsilon > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log E^{Q_x}\left[\exp\left(n\widetilde{\Phi}(L_n)\right), A_\varepsilon^c \,\middle|\, X_{n-1} = y\right] < 0 \tag{4.2}$$

Proof. By the assumption A.3, Proposition 1.1 and Deuschel-Stroock [3, Exercise 2.1.24], we have

$$\begin{aligned}\text{(LHS of 4.2)} &= \limsup_{n\to\infty} \log(\exp(-nb_\Phi) E^{P_x}\left[\exp(n\Phi(L_n)); A_\varepsilon^c\right]\\ &= -(\Phi(\nu_0) - J(\nu_0)) + \limsup_{n\to\infty} E^{P_x}\left[\exp(n\Phi(L_n)); A_\varepsilon^c\right]\\ &= -(\Phi(\nu_0) - J(\nu_0)) + \sup\{\Phi(\nu) - J(\nu); \nu \in A_\varepsilon^c\} < 0,\end{aligned}$$

which implies our assertion. □

Lemma 4.2. *There exist constants $p > 1$ and $\varepsilon > 0$ such that*

$$\sup_{x,y\in E}\sup_{n\geq 1} E^{Q_x}\left[e^{pn\widetilde{\Phi}(L_n)}, A_\varepsilon \,\middle|\, X_{n-1} = y\right] < \infty. \tag{4.3}$$

Proof. By the assumption A.4, the maximum a_0 of the eigenvalue of $D^2\Phi(\nu_0)|_{H\times H}$ is less than 1, so we can find a $p > 1$ such that $a_0 \cdot p < 1$. For this p, there exists a $r > 1$ such that $a_0 \cdot p \cdot r < 1$. Let s be the dual number of r, *i.e.*, $\frac{1}{r} + \frac{1}{s} = 1$. By Hölder inequality,

$$\begin{aligned}
& E^{Q_x}\left[e^{pn\widetilde{\Phi}(L_n)}, A_\varepsilon \,\middle|\, X_{n-1} = y\right] \\
= \;& E^{Q_x}\Big[\exp\Big\{p\frac{n}{2}D^2\Phi(\nu_0)(L_n - \nu_0, L_n - \nu_0) + pnR(\nu_0, L_n - \nu_0)\Big\}, \\
& \quad A_\varepsilon | X_{n-1} = y\Big] \\
\le \;& E^{Q_x}\left[\exp\Big\{pr\frac{n}{2}D^2\Phi(L_n - \nu_0, L_n - \nu_0)\Big\}, A_\varepsilon \,\middle|\, X_{n-1} = y\right]^{1/r} \qquad (4.4) \\
& \times\; E^{Q_x}\left[\exp\Big\{ps\frac{n}{2}R(\nu_0, L_n - \nu_0)\Big\}, A_\varepsilon \,\middle|\, X_{n-1} = y\right]^{1/s}, \qquad (4.5)
\end{aligned}$$

where $R(\nu_0, \cdot)$ is the 3rd remainder of the Taylor expansion of Φ at ν_0, i.e. $R(\nu_0, \nu - \nu_0) = \widetilde{\Phi}(\nu_0) - \frac{1}{2}D^2\Phi(\nu_0)(\nu - \nu_0, \nu - \nu_0)$.

For $U(x, y) \in C(E \times E)$, define

$$\begin{aligned}
\overline{U}(x, y) &= U(x, y) - \int_E U(x, y)\nu_0(dy) - \int_E U(x, y)\nu_0(dx) \\
&\quad + \int_E\int_E U(x, y)\nu_0(dx)\nu_0(dy)
\end{aligned}$$

and

$$\widetilde{U}(R_1, R_2) = \int_E\int_E U(x, y)R_1(dx)R_2(dy).$$

Then $\widetilde{\overline{U}}(R_1, R_2) = \widetilde{U}(R_1, R_2)$ for any $R_1, R_2 \in \mathcal{M}(E)$. Therefore,

$$nD^2\Phi(\nu_0)(L_n - \nu_0, L_n - \nu_0) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{k=0}^{n-1}\overline{\Phi^{(2)}(\nu_0, \cdot, \cdot)}(X_k, X_j)$$

and all of the conditions of Lemma 3.8 are satisfied for this $\overline{\Phi^{(2)}(\nu_0; \cdot, \cdot)}$. Therefore, (4.4) is bounded for $n > 0$ if $\varepsilon > 0$ is small enough.

As for (4.5), choose $\delta \in (0, 1/2ps)$, by the assumption A.5, for this δ, there exist a constant $\varepsilon' > 0$ and a function K_δ such that K_δ satisfies all of the conditions of Lemma 3.8, and

$$\begin{aligned} n\,|R(\nu_0, L_n-\nu_0)| &\le n\int_E\int_E K_\delta(x,y)(L_n-\nu_0)(dx)(L_n-\nu_0)(dy) \\ &= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\overline{K}_\delta(X_k,X_j), \quad \text{on} \quad A_{\varepsilon'}. \end{aligned}$$

By Lemma 3.8 again, we get that (4.5) is bounded for all $n>0$ if ε is small enough. This completes the proof of our Lemma. □

LEMMA 4.3. *There exists an* $\varepsilon_0>0$ *such that for any* $\varepsilon\le\varepsilon_0$ *and any* $x,y\in E$,

$$\begin{aligned} &\lim_{n\to\infty} E^{Q_x}\left[\exp\left(n\widetilde{\Phi}(L_n)\right), A_\varepsilon \,\middle|\, X_{n-1}=y\right] \\ &= \exp\left(\frac{1}{2}\int_E \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)\Big|_{(u,u)}\nu_0(du)\right)\times det_2(I-D^2\Phi(\nu_0))^{-1/2}. \end{aligned}$$

PROOF. By the strong mixing property of $Q_{0,x}^{n-1,y}$, X_n and $\sqrt{n}(L_n-\nu_0)$ are asymptotically independent as $n\to\infty$ under $Q_{0,x}^{n-1,y}$ for any $x,y\in E$. Also,

$$\begin{aligned} &E^{Q_{0,x}^{n-1,y}}\left[\exp\left(\sqrt{-1}\sqrt{n}\int_E u(x)(L_n-\nu_0)(dx)\right)\right] \\ &\to \exp\left(-\frac{1}{2}\langle u,\overline{G}u\rangle_{\nu_0}\right) \end{aligned}$$

as $n\to\infty$ for any $u\in C(E)$.

Take a separable Hilbert space H_1 such that the set $\{\overline{G}ud\nu_0 | \int_E u\overline{G}ud\nu_0<\infty\}$ is a dense linear subspace of H_1 and the inclusion map is a Hilbert-Schmidt operator. Let W be an H_1-valued random variable such that

$$E\left[\exp(\sqrt{-1}(u,W))\right] = \exp\left(-\frac{1}{2}\langle u,\overline{G}u\rangle_{\nu_0}\right)$$

for any $u\in H_1^*$. Write the distribution of W as γ.

From the central limit theorem for Hilbert space random variables, the law of $\sqrt{n}(L_n - \nu_0)$ under $Q_{0,x}^{n-1,y}$ converges weakly to γ as $n \to \infty$ on H_1.

As shown before, $D^2\Phi(\nu_0)(\cdot,\cdot)\Big|_{H\times H}$ is a Hilbert-Schmidt function. Let λ_m and $\overline{G}e_m d\nu_0, m = 1, 2, \cdots$, be the eigenvalues and the corresponding eigenvectors of $D^2\Phi(\nu_0)(\cdot,\cdot)\Big|_{H\times H}$. By the central limit theorem and ergodic theorem,

$$n\int_E\int_E \sum_{m=1}^N \lambda_m e_m(x)e_m(y)L_n(dx)L_n(dy) - \int_E \sum_{m=1}^N \lambda_m e_m(x)\overline{G}e_m(x)L_n(dx)$$

converges to $\sum_{m=1}^N \lambda_m((e_m, W)^2 - 1)$ in distribution under Q_x for any $N \in \mathbf{N}$ and any $x \in E$. Also,

$$E^{Q_x}\Bigg(\Bigg[\Bigg\{n\int_E\int_E \Phi^{(2)}(\nu_0,x,y)L_n(dx)L_n(dy) - \int_E \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)\Big|_{(x,x)} L_n(dx)\Bigg\} - \Bigg\{n\int_E\int_E \sum_{k=1}^N \lambda_m e_m(x)e_m(y)L_n(dx)L_n(dy) - \int_E \sum_{k=1}^N \lambda_m e_m(x)\overline{G}e_m(x)L_n(dx)\Bigg\}\Bigg]^2\Bigg) \to 0$$

as $N \to \infty$ uniformly for $n \in \mathbf{N}$. Therefore, let $: D^2\Phi(\nu_0)(W,W) :$ be the $L^2(d\gamma)$-limit of $\sum_{m=1}^N \lambda_m((e_m, W)^2 - 1)$, which is well-defined since $D^2\Phi(\nu_0)(\cdot,\cdot)|_{H\times H}$ is a Hilbert-Schmidt type function, then

$$n\int_E\int_E \Phi^{(2)}(\nu_0,x,y)L_n(dx)L_n(dy) - \int_E \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}L_n(du)$$

converges to $: D^2\Phi(\nu_0)(W,W) :$ under $Q_{0,x}^{n-1,y}$ as $n \to \infty$. It is easy that

$$\int_E \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}L_n(du) \to \int_E \overline{G}_x\Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}\nu_0(du) \quad Q_{0,x}^{n-1,y}-\text{a.s.}$$

as $n \to \infty$.

Also, as in the proof of Lemma 4.2, we have by the assumption (A.5) and Lemma 3.5 that for any $\delta > 0$,

$$
\begin{aligned}
& E^{Q_{0,x}^{n-1,y}}\left[\frac{\delta_0}{\delta} n\,|R(\nu_0, L_n - \nu_0)|\right] \\
& \leq E^{Q_{0,x}^{n-1,y}}\left(n\frac{\delta_0}{\delta}\int_E\int_E K_\delta(x,y)(L_n-\nu_0)(dx)(L_n-\nu_0)(dy)\right) \\
& \leq E^{Q_{0,x}^{n-1,y}}\left(\exp\left(\frac{\delta_0}{\delta} n\int_E\int_E K_\delta(x,y)(L_n-\nu_0)(dx)(L_n-\nu_0)(dy)\right)\right) \\
& + E^{Q_{0,x}^{n-1,y}}\left(\exp\left(\frac{\delta_0}{\delta} n\int_E\int_E -K_\delta(x,y)(L_n-\nu_0)(dx)(L_n-\nu_0)(dy)\right)\right) \\
& \leq 2C_0,
\end{aligned}
$$

which implies that

$$nR(\nu_0, L_n - \nu_0) \to 0$$

in law under $Q_{0,x}^{n-1,y}$ as $n \to \infty$.

Therefore, by the definition of $\widetilde{\Phi}$, we get that

$$
n\widetilde{\Phi}(L_n) \to \frac{1}{2}\left(: D^2\Phi(\nu_0)(W,W) : + \int_E \overline{G}_x \Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}\nu_0(du)\right) \tag{4.6}
$$

in distribution under $Q_{0,x}^{n-1,y}$ as $n \to \infty$.

This together with Lemma 4.2 imply that for any $x \in E$,

$$
\begin{aligned}
& \lim_{n\to\infty} E^{Q_x}\left[\exp\left(n\widetilde{\Phi}(L_n)\right), A_\varepsilon \,\middle|\, X_{n-1} = y\right] \\
& = E\left[\exp\left(\frac{1}{2} : D^2\Phi(\nu_0)(W,W) : + \frac{1}{2}\int_E \overline{G}_x \Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}\nu_0(du)\right)\right] \\
& = E\left[\exp(\frac{1}{2} : D^2\Phi(\nu_0)(W,W) :)\right] \\
& \quad \times \exp\left(\frac{1}{2}\int_E \overline{G}_x \Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}\nu_0(du)\right) \\
& = det_2(I_H - D^2\Phi(\nu_0))^{-1/2} \times \exp\left(\frac{1}{2}\int_E \overline{G}_x \Phi^{(2)}(\nu_0,\cdot,\cdot)|_{(u,u)}\nu_0(du)\right),
\end{aligned}
$$

which is just what we need. □

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