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Birational Symmetries, Hirota Bilinear Forms and Special Solutions of the Garnier Systems in 2-variables

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Abstract. Hirota bilinear forms of the Garnier system in 2-variables, $G(1,1,1,1,1)$, are given. By using Hirota bilinear forms we construct new birational symmetries of $G(1,1,1,1,1)$. We obtain special solutions of the Garnier system in n-variables, which are described in terms of solutions of the Garnier system in $(n-1)$-variables. We investigate also algebraic solutions for $n = 2$.

Introduction

The Painlevé equations P_J $(J = I, \cdots, VI)$ are derived from the theory of monodromy preserving deformation of the linear differential equation of second order:

$$(0.1) \qquad (L_J) \quad \frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0.$$

R. Fuchs ([1]) had obtained the sixth Painlevé equation P_{VI} by considering monodromy preserving deformation of (0.1). In fact, P_{VI} is deduced from complete integrability conditions for an extended system of (0.1). For each of the other Painlevé equations P_J $(J = I, \cdots, V)$, such construction from integrability conditions was established firstly by R. Garnier ([2]) without any mention about monodromy property and later more precise consideration has been done by M. Jimbo, T. Miwa, K. Ueno ([4, 16]) and by K. Okamoto ([11]). In this paper we do not enter into details of the theory of monodromy preserving deformation; we give below the list of singularities

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of linear equations L_J $(J = II, \cdots, VI)$.

(0.2)

	Singularities of L_J
P_{VI}	$(1,1,1,1)$
P_V	$(1,1,2)$
P_{IV}	$(1,3)$
P_{III}	$(2,2)$
P_{II}	(4)

In this table, $(r_1, r_2, \cdots, r_m)$ means that L_J has m singular points with the Poincaré ranks, $r_1 - 1, r_2 - 1, \cdots, r_m - 1$, respectively. We can thus regard each Painlevé equation P_J $(J = II, \cdots, VI)$ as corresponding to a partition of 4 through the monodromy preserving deformation. Note that the first Painlevé equation P_I contains no constant parameter and there is no correspondence to any partition of 4.

A generalization of the sixth Painlevé equation P_{VI} was also obtained by R. Garnier ([2]) from the viewpoint of the theory of monodromy preserving deformation. He considered the monodromy preserving deformation of the linear differential equation of second order of the form (0.1), with $n+3$ regular singularities and n apparent singularities $x = \lambda_j$, whose Riemannian scheme is given by

$$\left(\begin{array}{ccccc} x=0 & x=1 & x=\infty & x=t_i & x=\lambda_j \\ 0 & 0 & \alpha & 0 & 0 \\ \kappa_0 & \kappa_1 & \alpha+\kappa_\infty & \theta_i & 2 \end{array} \right), \quad i,j = 1, \cdots, n, \tag{0.3}$$

where

$$\alpha = -\frac{1}{2}\left(\kappa_0 + \kappa_1 + \kappa_\infty + \sum_i \theta_i - 1\right).$$

Then he obtained the system of nonlinear partial differential equations for $\lambda_j = \lambda_j(t)$, called the Garnier system in n-variables.

It is known ([3, 7, 12]) that through a certain change of variables the Garnier system in n-variables is equivalent to the following Hamiltonian system:

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i,j = 1, \cdots, n), \tag{0.4}$$

with Hamiltonians:

$$H_i = \frac{1}{s_i(s_i-1)} \left(\sum_{j,k} E^i_{jk}(s,q) p_j p_k - \sum_j F^i_j(s,q) p_j + \kappa q_i \right). \tag{0.5}$$

Here $E^i_{jk} = E^i_{kj}, F^i_j \in \mathbb{C}(s)[q]$ are given by

$$E^i_{jk} = \begin{cases} q_i q_j q_k, & \text{if} \quad i \neq j \neq k \neq i, \\ q_i q_j (q_j - R_{ji}), & \text{if} \quad i \neq j = k, \\ q_i q_k (q_i - R_{ik}), & \text{if} \quad i = j \neq k, \\ q_i(q_i - 1)(q_i - s_i) - \sum_{l(\neq i)} S_{il} q_i q_l, & \text{if} \quad i = j = k, \end{cases} \tag{0.6}$$

$$F^i_j = \begin{cases} A q_i q_j - \theta_i R_{ij} q_j - \theta_j R_{ji} q_i, & \text{if} \quad i \neq j, \\ \\ (\kappa_0 - 1) q_i (q_i - 1) + \kappa_1 q_i (q_i - s_i) & \\ \quad + \theta_i (q_i - 1)(q_i - s_i) & \\ \quad + \sum_{k(\neq i)} (\theta_k q_i (q_i - R_{ik}) - \theta_i S_{ik} q_k), & \text{if} \quad i = j, \end{cases} \tag{0.7}$$

with

$$R_{ij} = \frac{s_i(s_j - 1)}{s_j - s_i}, \quad S_{ij} = \frac{s_i(s_i - 1)}{s_i - s_j}, \tag{0.8}$$

$$A = \kappa_0 + \kappa_1 + \sum_l \theta_l - 1, \quad \kappa = \frac{1}{4}(A^2 - \kappa_\infty^2). \tag{0.9}$$

In the same way as the Painlevé equations, we can regard the Garnier system in n-variables as corresponding to the partition $(1, 1, \cdots, 1)$ of $n + 3$.

It is well known that the confluence of singularities of L_J causes the step-by-step degeneration of the Painlevé equations P_J ([12]). In a way similar to the Painlevé equations, the degeneration of the Garnier system can be considered. Many degenerate Garnier systems are studied by several authors ([5, 6, 8, 9, 15]). Each of degenerate Garnier systems corresponds to a certain partition of natural number through the theory of monodromy preserving deformation. In this paper we denote by $G(\#)$ the Painlevé equation or the (degenerate) Garnier system corresponding to the partition $(\#)$. For example, we refer by $G(1, 1, 1, 1, 1)$ the Garnier system in 2-variables, by $G(1, 3)$ the fourth Painlevé equation P_{IV} and so on; see Figure 1.

Painlevé equations

$$\begin{tikzcd}
 & & (2,2) \arrow[dr] & \\
(1,1,1,1) \arrow[r] & (1,1,2) \arrow[ur] \arrow[r] & (1,3) \arrow[r] & (4)
\end{tikzcd}$$

Garnier systems in 2-variables

$$\begin{tikzcd}
 & & (1,2,2) \arrow[r] \arrow[dr] & (2,3) \arrow[dr] & \\
(1,1,1,1,1) \arrow[r] & (1,1,1,2) \arrow[ur] \arrow[r] & (1,1,3) \arrow[ur] \arrow[r] & (1,4) \arrow[r] & (5)
\end{tikzcd}$$

Fig. 1. degeneration scheme

In this paper we study special solutions, Hirota bilinear forms, and birational symmetries of the Garnier system in 2-variables $G(1,1,1,1,1)$. Particular solutions of the system which are described in terms of hypergeometric functions in 2-variables, are known ([14]), and we discuss in the present article other types of particular solutions; we consider special solutions of $G(1,1,1,1,1)$, given in terms of solutions of the sixth Painlevé equation P_{VI}; see Theorem 2.1. Moreover, we will see that for $(\#) = (1,1,\cdots,1)$, the Garnier system $G(1,\#)$ has particular solutions given in terms of solutions of $G(\#)$. It is natural to make the following conjecture.

Conjecture. *For any partition $(\#)$ of an integer $n(\geq 4)$, $G(1,\#)$ has a particular solution written in terms of solutions of $G(\#)$.*

If the statement of the conjecture is true, we denote this fact simply by

$$G(1,\#) \supset G(\#). \tag{0.10}$$

For example, we obtain (0.10) for $(\#) = (1,1,\cdots,1)$; see Theorem 6.1. Moreover we can verify (0.10) for $(1,1,2)$, $(1,3)$, (4), (5); details will be discussed in forthcoming papers.

The second subject of the investigation concerns Hirota bilinear forms of the Garnier system $G(1,1,1,1,1)$, which plays very important roles in this paper. In fact, we study special solutions and birational symmetries by means of them.

Let us consider, for example, P_{II}, which is equivalent to the following Hamiltonian system $\mathcal{H}(\alpha)$:

$$\frac{\mathrm{d}q}{\mathrm{d}s} = \frac{\partial H}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}s} = -\frac{\partial H}{\partial q}, \tag{0.11}$$

with the Hamiltonian $H = H(\alpha)$:

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{s}{2}\right)p - \left(\alpha + \frac{1}{2}\right)q. \tag{0.12}$$

By defining $\mathrm{d}\log\tau(\alpha) = H(\alpha)\mathrm{d}s$, Hirota bilinear forms of P_{II} are described as

$$\left(\mathcal{D}^2 + \frac{s}{2}\right) g \cdot f = 0, \tag{0.13}$$

$$\left(\mathcal{D}^3 + \frac{s}{2}\mathcal{D} - \alpha\right) g \cdot f = 0, \tag{0.14}$$

where $f = \tau(\alpha), g = \tau(\alpha - 1)$ and $\mathcal{D}$ is the Hirota derivative with respect to $\mathrm{d}/\mathrm{d}s$. If we put $f = 1$ and $\alpha = -1/2$, above bilinear forms reduce to the linear differential equation for g:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2} + \frac{s}{2}\right) g = 0,$$

which is the Airy differential equation. This gives a classical solution of P_{II}; see [15].

Return to bilinear forms (0.13)-(0.14), it is easy to see that these are invariant under the action $w : (f, g; \alpha) \mapsto (g, f; -\alpha)$. This trivial symmetry can be lifted to a birational canonical transformation of $\mathcal{H}(\alpha)$. And the fixed solution with respect to w, $(f, g; \alpha) = (\exp(-s^3/24), \exp(-s^3/24); 0)$, gives a rational solution of P_{II}, $(q, p; \alpha) = (0, s/2; 0)$.

In the present article we study mainly the Garnier system in 2-variables $G(1, 1, 1, 1, 1)$. We will give particular solutions which are described in terms of solutions of P_{VI} (Theorem 2.1), Hirota bilinear forms (Theorem 3.2), birational symmetries (Theorem 4.1, 4.3) and consider algebraic solutions. Finally we consider the Garnier system in n-variables, where we denote it by $\mathcal{G}_n$. We obtain particular solutions of $\mathcal{G}_n$ given in terms of solutions of $\mathcal{G}_{n-1}$ (Theorem 6.1).

1. Hamiltonian System of G(1,1,1,1,1)

The Garnier system $G(1,1,1,1,1)$ is equivalent to the Hamiltonian system

$$\frac{\partial q_i}{\partial s_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{\partial p_i}{\partial s_j} = -\frac{\partial H_j}{\partial q_i}, \quad (i, j = 1, 2),$$

with Hamiltonians:

$$\begin{aligned}
(1.1)\quad s_1(s_1-1)H_1 = & \left(q_1(q_1-1)(q_1-s_1) - \frac{s_1(s_1-1)}{s_1-s_2}q_1q_2\right)p_1^2 \\
& + 2q_1q_2\left(q_1 - \frac{s_1(s_2-1)}{s_2-s_1}\right)p_1p_2 \\
& + q_1q_2\left(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\right)p_2^2 \\
& -\Bigg\{(\kappa_0-1)q_1(q_1-1) + \kappa_1 q_1(q_1-s_1) \\
& \quad +\theta_1(q_1-1)(q_1-s_1) \\
& \quad +\theta_2 q_1\left(q_1 - \frac{s_1(s_2-1)}{s_2-s_1}\right) - \theta_1\frac{s_1(s_1-1)}{s_1-s_2}q_2\Bigg\}p_1 \\
& -\left\{\theta q_1q_2 - \theta_2 q_1\frac{s_2(s_1-1)}{s_1-s_2} - \theta_1 q_2\frac{s_1(s_2-1)}{s_2-s_1}\right\}p_2 \\
& + \kappa q_1,
\end{aligned}$$

and H_2 is of the form obtained by the replacement

$$\{q_1 \leftrightarrow q_2, p_1 \leftrightarrow p_2, s_1 \leftrightarrow s_2, \theta_1 \leftrightarrow \theta_2\},$$

in H_1. Here we consider $\vec{\kappa} = (\kappa_0, \kappa_1, \kappa_\infty, \theta_1, \theta_2) \in \mathbb{C}^5$ as parameters and put $\kappa = (\theta - \kappa_\infty)(\theta + \kappa_\infty)/4$, $\theta = \kappa_0 + \kappa_1 + \theta_1 + \theta_2 - 1$.

2. Particular Solutions

It is known ([14]) that $G(1,1,1,1,1)$ with certain special values of parameters admits a particular solution expressed in terms of Appell's hypergeometric function $F_1(\alpha, \beta, \beta', \gamma; x, y)$. In this section we show that $G(1,1,1,1,1)$ admits a particular solution expressed in terms of the sixth Painlevé transcendent; in fact we have the

THEOREM 2.1. *If $\theta_2 = 0$, then $G(1,1,1,1,1)$ has a particular solution of the form:*

$$q_2 = 0, \quad \frac{\partial q_1}{\partial s_2} = \frac{\partial p_1}{\partial s_2} = 0.$$

Moreover (q_1, p_1) satisfy

$$\begin{aligned} s_1(s_1-1)\frac{\partial q_1}{\partial s_1} &= 2q_1(q_1-1)(q_1-s_1)p_1 \\ &\quad -\Big\{(\kappa_0-1)q_1(q_1-1)+\kappa_1 q_1(q_1-s_1) \\ &\qquad +\theta_1(q_1-1)(q_1-s_1)\Big\}, \\ s_1(s_1-1)\frac{\partial p_1}{\partial s_1} &= -\Big\{3q_1^2-2(s_1+1)q_1+s_1\Big\}p_1^2 \\ &\quad +\Big\{(\kappa_0-1)(2q_1-1)+\kappa_1(2q_1-s_1) \\ &\qquad +\theta_1(2q_1-s_1-1)\Big\}p_1-\kappa, \end{aligned}$$

which is equivalent to the sixth Painlevé equation P_{VI}. And p_2 satisfies Riccati type equations whose coefficients are polynomials in (q_1, p_1).

PROOF. Consider the case $\theta_2 = 0$. Take $q_2 = 0$ then

$$\begin{aligned} s_1(s_1-1)H_1 &= q_1(q_1-1)(q_1-s_1)p_1^2 \\ &\quad -((\kappa_0-1)q_1(q_1-1)+\kappa_1 q_1(q_1-s_1) \\ &\qquad +\theta_1(q_1-1)(q_1-s_1))p_1+\kappa q_1. \end{aligned}$$

This is nothing but the Hamiltonian of P_{VI}. And it can be verified by computations,

$$\frac{\partial q_1}{\partial s_2} = \frac{\partial H_2}{\partial p_1} = 0, \quad \frac{\partial p_1}{\partial s_2} = -\frac{\partial H_2}{\partial q_1} = 0.$$

Thus $q_1(s_1), p_1(s_1)$ are solved by the solutions of P_{VI}. Also it can be seen easily that $p_2(s_1, s_2)$ satisfies Riccati type equations:

$$\begin{aligned} s_1(s_1-1)\frac{\partial p_2}{\partial s_1} &= \frac{s_2(s_1-1)}{s_1-s_2}q_1 p_2^2 \\ &\quad -\left\{(2q_1p_1-\theta)q_1+\frac{s_1(s_2-1)}{s_2-s_1}(2q_1p_1-\theta_1)\right\}p_2 \end{aligned}$$

$$
\begin{aligned}
& +\frac{s_1(s_1-1)}{s_1-s_2}(q_1p_1-\theta_1)p_1, \\
s_2(s_2-1)\frac{\partial p_2}{\partial s_2} \quad = \quad & \left(\frac{s_2(s_2-1)}{s_2-s_1}q_1-s_2\right)p_2^2 \\
& +\left\{\frac{s_2(s_1-1)}{s_1-s_2}(2q_1p_1-\theta_1)+1-\kappa_0-\kappa_1 s_2\right\}p_2 \\
& -(q_1p_1-\theta)q_1p_1+\frac{s_1(s_2-1)}{s_2-s_1}(q_1p_1-\theta_1)p_1-\kappa. \ \square
\end{aligned}
$$

As is shown by Theorem 2.1, we have $G(1,1,1,1,1) \supset G(1,1,1,1)$; the sixth Painlevé equation P_{VI} is contained in the two dimensional Garnier system, $G(1,1,1,1,1)$.

3. τ-functions and Hirota Bilinear Forms

We can verify that, for the Hamiltonians of the Garnier system,

$$
\begin{aligned}
\frac{\partial H_i}{\partial s_j} &= \sum_{k=1,2}\left(\frac{\partial H_i}{\partial q_k}\frac{\partial q_k}{\partial s_j}+\frac{\partial H_i}{\partial p_k}\frac{\partial p_k}{\partial s_j}\right)+\left(\frac{\partial}{\partial s_j}\right)H_i \\
&= \sum_{k=1,2}\left(\frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k}-\frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k}\right)+\left(\frac{\partial}{\partial s_j}\right)H_i \\
&= \left(\frac{\partial}{\partial s_j}\right)H_i,
\end{aligned}
$$

where $(\partial/\partial s_j)$ denotes differentiation with respect to s_j such that (q,p) are viewed to be independent of s. By the use of (1.1), we have

$$
\frac{\partial H_1}{\partial s_2}=\frac{\partial H_2}{\partial s_1}=\frac{A(q,p,s)}{(s_1-s_2)^2}, \tag{3.1}
$$

$$
\begin{aligned}
A(q,p,s) \quad = \quad & q_1q_2p_1^2-2q_1q_2p_1p_2+q_1q_2p_2^2 \\
& +(\theta_2q_1-\theta_1q_2)p_1+(\theta_1q_2-\theta_2q_1)p_2.
\end{aligned} \tag{3.2}
$$

It is not difficult to show the

PROPOSITION 3.1 ([7]). *The 1-form $\omega \equiv H_1 \mathrm{d}s_1 + H_2 \mathrm{d}s_2$ is closed.*

Then we can define, up to multiplicative constants, the τ-function, $\tau = \tau(\vec{\kappa})$, related to $G(1,1,1,1,1)$ as follows:

$$\omega = \mathrm{d}\log\tau. \tag{3.3}$$

Now we set a pair of τ-functions (f, g) as

$$\begin{aligned} \mathrm{d}\log f &= H_1\mathrm{d}s_1 + H_2\mathrm{d}s_2, && (3.4)\\ \mathrm{d}\log g &= \overline{H}_1\mathrm{d}s_1 + \overline{H}_2\mathrm{d}s_2, && (3.5)\end{aligned}$$

where

$$s_i\overline{H}_i = s_iH_i + x_i, \quad x_i = -q_ip_i, \quad (i = 1, 2). \tag{3.6}$$

REMARK. (i) Existence of the function $g = g(s_1, s_2)$ satisfying (3.5) is assured by means of the equation:

$$s_2\frac{\partial x_1}{\partial s_2} = s_1\frac{\partial x_2}{\partial s_1}.$$

(ii) If we write as $f = \tau(\vec{\kappa})$, then we will see later that $g = \tau(\rho(\vec{\kappa})) = \tau(R_\tau(\vec{\kappa}))$, where $\rho(\vec{\kappa}) = (\kappa_0 + 1, \kappa_1 - 1, \kappa_\infty, \theta_1, \theta_2)$ and $R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2)$.

Now we recall the definition of Hirota derivatives (in 2-variables):

$$P(\mathcal{D}_1, \mathcal{D}_2)g\cdot f = P(D_1, D_2)(g(s+t)g(s-t))\big|_{t_1=t_2=0}\,, \tag{3.7}$$

where $P(\mathcal{D}_1, \mathcal{D}_2)$ is a polynomial in $(\mathcal{D}_1, \mathcal{D}_2)$ and D_i is a derivation. In this paper we deal with

$$D_i = s_i\frac{\partial}{\partial s_i}. \tag{3.8}$$

By definition we have

$$\begin{aligned} \mathcal{D}_ig\cdot f &= (D_ig)f - g(D_if), && (3.9)\\ \mathcal{D}_i\mathcal{D}_jg\cdot f &= (D_iD_jg)f - (D_ig)(D_jf) - (D_jg)(D_if) && (3.10)\\ &\quad + g(D_iD_jf), \end{aligned}$$

for $i, j = 1, 2$. It is easy to verify the following identities:

$$D_i \log \frac{g}{f} = \frac{\mathcal{D}_i g \cdot f}{g \cdot f}, \tag{3.11}$$

$$D_i D_j \log gf = \frac{\mathcal{D}_i \mathcal{D}_j g \cdot f}{g \cdot f} - \frac{\mathcal{D}_i g \cdot f}{g \cdot f} \frac{\mathcal{D}_j g \cdot f}{g \cdot f}, \tag{3.12}$$

$$\begin{aligned} D_i^2 D_j \log \frac{g}{f} &= \frac{\mathcal{D}_i^2 \mathcal{D}_j g \cdot f}{g \cdot f} - \frac{\mathcal{D}_i^2 g \cdot f}{g \cdot f} \frac{\mathcal{D}_j g \cdot f}{g \cdot f} \\ &\quad -2 \frac{\mathcal{D}_i \mathcal{D}_j g \cdot f}{g \cdot f} \frac{\mathcal{D}_i g \cdot f}{g \cdot f} + 2 \left(\frac{\mathcal{D}_i g \cdot f}{g \cdot f} \right)^2 \frac{\mathcal{D}_j g \cdot f}{g \cdot f}, \end{aligned} \tag{3.13}$$

for $i, j = 1, 2$.

For the pair of τ-functions (f, g), we have the

THEOREM 3.2. *The pair of τ-functions (f, g) satisfies bilinear equations of the forms:*

$$B_1(g, f; \vec{\kappa}) + s_1(s_2 - 1) B_2(g, f; \vec{\kappa}) = 0, \tag{3.14}$$

$$\begin{aligned} &\frac{s_1 - 1}{s_1} B_3(g, f; \vec{\kappa}) + \frac{(s_1 - 1)^2}{s_1} B_4(g, f; \vec{\kappa}) + \frac{s_1^2 - s_2}{s_1 s_2} B_5(g, f; \vec{\kappa}) \\ &\quad +2(\kappa_0 - \kappa_1)(s_1 - s_2) B_2(g, f; \vec{\kappa}) + B_6(g, f; \vec{\kappa}) = 0, \end{aligned} \tag{3.15}$$

and satisfies also the equations obtained by the replacement $\{s_1 \leftrightarrow s_2, \theta_1 \leftrightarrow \theta_2\}$ in (3.14), (3.15). *Here $\mathcal{D}_i$ is the Hirota derivative and $B_i(g, f; \vec{\kappa})$ are given by:*

$$\begin{aligned} B_1(g, f; \vec{\kappa}) &= (s_1 - 1) \mathcal{D}_1^2 g \cdot f \\ &\quad + \{(\kappa_1 + \theta_1) s_1 - (\kappa_0 + \theta_1 - 1)\} \mathcal{D}_1 g \cdot f \\ &\quad + (s_1 + 1) g \cdot D_1 f, \end{aligned} \tag{3.16}$$

$$B_2(g, f; \vec{\kappa}) = \frac{1}{s_1 + s_2} \Big(2 \mathcal{D}_1 \mathcal{D}_2 g \cdot f + \theta_2 \mathcal{D}_1 g \cdot f + \theta_1 \mathcal{D}_2 g \cdot f \Big), \tag{3.17}$$

$$\begin{aligned} B_3(g, f; \vec{\kappa}) &= (s_1 - 1) \mathcal{D}_1^3 g \cdot f \\ &\quad + \{(\kappa_1 + \theta_1) s_1 - (\kappa_0 + \theta_1 - 1)\} \mathcal{D}_1^2 g \cdot f \\ &\quad + (s_1 + 1) \mathcal{D}_1 g \cdot D_1 f \\ &\quad + \frac{s_1}{s_1 - 1} \Big(\{(\kappa_0 - \kappa_1 - \kappa_\infty)(\kappa_0 - \kappa_1 + \kappa_\infty) \\ &\qquad + \theta_1 (2\kappa_0 + 2\kappa_1 + \theta_1 - 2)\} \mathcal{D}_1 g \cdot f \end{aligned} \tag{3.18}$$

$$+(\kappa_1-\kappa_0)(g\cdot D_1f+D_1g\cdot f)+2\theta_1\kappa g\cdot f\Big),$$

$$\begin{aligned}
(3.19)\quad B_4(g,f;\vec{\kappa}) &= 2\mathcal{D}_1^2\mathcal{D}_2g\cdot f+\theta_2\mathcal{D}_1^2g\cdot f+\theta_1\mathcal{D}_1\mathcal{D}_2g\cdot f,\\
(3.20)\quad B_5(g,f;\vec{\kappa}) &= (s_2-1)\mathcal{D}_1\mathcal{D}_2^2g\cdot f\\
&\quad +\{(\kappa_1+\theta_2)s_2-(\kappa_0+\theta_2-1)\}\mathcal{D}_1\mathcal{D}_2g\cdot f\\
&\quad +(s_2+1)\mathcal{D}_1g\cdot D_2f,\\
(3.21)\quad B_6(g,f;\vec{\kappa}) &= \theta_2(2\theta_1+\theta_2)\mathcal{D}_1g\cdot f+2\theta_1(\kappa_0+\kappa_1-1)\mathcal{D}_2g\cdot f.
\end{aligned}$$

Remark. (i) If we put $\theta_2=0$, $\dfrac{\partial g}{\partial s_2}=\dfrac{\partial f}{\partial s_2}=0$, then above bilinear forms (3.14)-(3.15) reduce to the following:

$$\begin{aligned}
(3.22)\quad & (s_1-1)\mathcal{D}_1^2g\cdot f+\{(\kappa_1+\theta_1)s_1-(\kappa_0+\theta_1-1)\}\mathcal{D}_1g\cdot f\\
&+(s_1+1)g\cdot D_1f=0,\\
(3.23)\quad & (s_1-1)\mathcal{D}_1^3g\cdot f+\{(\kappa_1+\theta_1)s_1-(\kappa_0+\theta_1-1)\}\mathcal{D}_1^2g\cdot f\\
&+(s_1+1)\mathcal{D}_1g\cdot D_1f\\
&+\frac{s_1}{s_1-1}\Big(\{(\kappa_0-\kappa_1-\kappa_\infty)(\kappa_0-\kappa_1+\kappa_\infty)\\
&\quad +\theta_1(2\kappa_0+2\kappa_1+\theta_1-2)\}\mathcal{D}_1g\cdot f\\
&\quad +(\kappa_1-\kappa_0)(g\cdot D_1f+D_1g\cdot f)+2\theta_1\kappa g\cdot f\Big)=0.
\end{aligned}$$

These are equivalent to the bilinear forms of P_{VI} ([15]).
(ii) If we put $\kappa=0$, $f=1$, then the bilinear forms of $G(1,1,1,1,1)$ reduce to the system of linear partial differential equations for g, which is equivalent to Appell's hypergeometric differential equation.

Proof of Theorem 3.2. Recall the definitions of τ-functions (f,g):

$$(3.24)\qquad s_iH_i=D_i\log f,\quad s_i\overline{H}_i=D_i\log g,$$

where

$$(3.25)\qquad s_i\overline{H}_i=s_iH_i+x_i,\quad x_i=-q_ip_i,$$

for $i=1,2$. Using the formulae (3.11)-(3.13), we have expressions of Hirota derivatives of (f,g), in terms of x_i and H_i, as follows:

$$(3.26)\qquad \frac{\mathcal{D}_ig\cdot f}{g\cdot f}=x_i,$$

$$\frac{\mathcal{D}_i\mathcal{D}_j g\cdot f}{g\cdot f} = 2D_j(s_iH_i)+D_jx_i+x_ix_j, \tag{3.27}$$

$$\begin{aligned}\frac{\mathcal{D}_i^2\mathcal{D}_j g\cdot f}{g\cdot f} &= D_i^2x_j-(2D_i(s_iH_i)+D_ix_i+x_i^2)x_j \\ &\quad -2(2D_j(s_iH_i)+D_jx_i+x_ix_j)x_i+2x_i^2x_j.\end{aligned} \tag{3.28}$$

Put these into $B_i(g,f;\vec{\kappa})$, we can verify the bilinear relations (3.14) and (3.15) by computations. □

4. Birational Symmetries

In this section we consider birational symmetries of $G(1,1,1,1,1)$. The Hamiltonians H_i $(i=1,2)$ are invariant under the action: $\kappa_\infty \mapsto -\kappa_\infty$. This trivial symmetry can be lifted to a birational canonical transformation of $G(1,1,1,1,1)$.

On the other hand, from the viewpoint of monodromy preserving deformations, H. Kimura constructed birational symmetries of $G(1,1,1,1,1)$ which act on the parameters as permutations; see [3].

Then combining the above results, we obtain the following theorem.

THEOREM 4.1. *There exist birational canonical transformations*

$$\mathcal{H}(\vec{\kappa}) \to \mathcal{H}(R_\Delta(\vec{\kappa}))$$

of $G(1,1,1,1,1)$*, where* $\mathcal{H}(\vec{\kappa})=(q(\vec{\kappa}),p(\vec{\kappa}),H(\vec{\kappa}),s)$*. Here the transformations* $R_\Delta:(q,p)\mapsto(Q,P)$ *are given as follows:*

(4.1)

R_Δ	action on $\vec{\kappa}$	$Q_i \quad (i=1,2)$	$P_i \quad (i=1,2)$
R_{κ_∞}	$\kappa_\infty \mapsto -\kappa_\infty$	$Q_i=q_i$	$P_i=p_i$
R_{κ_1}	$\kappa_1 \mapsto -\kappa_1$	$Q_i=q_i$	$P_i=p_i-\dfrac{\kappa_1}{q_1+q_2-1}$
R_{κ_0}	$\kappa_0 \mapsto -\kappa_0$	$Q_i=q_i$	$P_i=p_i-\dfrac{\kappa_0}{s_i(q_1/s_1+q_2/s_2-1)}$
R_{θ_1}	$\theta_1 \mapsto -\theta_1$	$Q_i=q_i$	$P_1=p_1-\theta_1/q_1, \quad P_2=p_2$
R_{θ_2}	$\theta_2 \mapsto -\theta_2$	$Q_i=q_i$	$P_1=p_1, \quad P_2=p_2-\theta_2/q_2$

Furthermore, another birational symmetry can be derived from the Hirota bilinear forms.

PROPOSITION 4.2. *Hirota bilinear forms of* $G(1,1,1,1,1)$ *are invariant under the action*

$$R_\tau : (f, g; \vec{\kappa}) \mapsto (g, f; R_\tau(\vec{\kappa}))$$

where $R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2)$.

PROOF. If $P(\mathcal{D})$ is a monomial, then we have:

$$\begin{aligned} P(\mathcal{D}) g \cdot f &= -P(\mathcal{D}) f \cdot g \quad (P : odd), \\ P(\mathcal{D}) g \cdot f &= P(\mathcal{D}) f \cdot g \quad (P : even), \end{aligned}$$

and it is easy to verify that

$$\begin{aligned} g \cdot D_1 f &= \mathcal{D}_1 f \cdot g + f \cdot D_1 g, \\ \mathcal{D}_1 g \cdot D_1 f &= -\mathcal{D}_1^2 f \cdot g - \mathcal{D}_1 f \cdot D_1 g, \\ \mathcal{D}_1 g \cdot D_2 f &= -\mathcal{D}_1 \mathcal{D}_2 f \cdot g - \mathcal{D}_1 f \cdot D_2 g, \end{aligned}$$

which we use in the following.

Consider the exchange of τ-functions $f \leftrightarrow g$ in Hirota bilinear forms of $G(1,1,1,1,1)$, (3.14)-(3.15), then we obtain again Hirota bilinear forms of $G(1,1,1,1,1)$ with parameters $R_\tau(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, -\kappa_\infty, -\theta_1, -\theta_2)$. For example we compute:

$$\begin{aligned} B_1(g, f; \vec{\kappa}) &= (s_1 - 1)\mathcal{D}_1^2 g \cdot f \\ &\quad + \{(\kappa_1 + \theta_1)s_1 - (\kappa_0 + \theta_1 - 1)\}\mathcal{D}_1 g \cdot f \\ &\quad + (s_1 + 1) g \cdot D_1 f \\ &= (s_1 - 1)\mathcal{D}_1^2 f \cdot g \\ &\quad + \{(-\kappa_1 - \theta_1 + 1)s_1 - (-\kappa_0 - \theta_1)\}\mathcal{D}_1 f \cdot g \\ &\quad + (s_1 + 1) f \cdot D_1 g \\ &= B_1(f, g; R_\tau(\vec{\kappa})), \end{aligned} \tag{4.2}$$

similarly we have $B_2(g, f; \vec{\kappa}) = B_2(f, g; R_\tau(\vec{\kappa}))$ and $B_i(g, f; \vec{\kappa}) = -B_i(f, g; R_\tau(\vec{\kappa}))$ for $i = 3, 4, 5, 6$. We obtain thus Hirota bilinear forms of $G(1,1,1,1,1)$ with parameters $R_\tau(\vec{\kappa})$. $\square$

This symmetry of τ-functions can be lifted to a birational canonical transformation of $G(1,1,1,1,1)$.

THEOREM 4.3. *There exists a birational canonical transformation*

$$R_\tau : \mathcal{H}(q_i, p_i; \vec{\kappa}) \to \mathcal{H}(Q_i, P_i; R_\tau(\vec{\kappa}))$$

of $G(1,1,1,1,1)$ *where* $R_\tau(\vec{\kappa}) = (-\kappa_0+1, -\kappa_1+1, -\kappa_\infty, -\theta_1, -\theta_2)$ *described as*

$$Q_i = \frac{s_i p_i(q_i p_i - \theta_i)}{(\alpha + q_1 p_1 + q_2 p_2)(\alpha + \kappa_\infty + q_1 p_1 + q_2 p_2)}, \tag{4.3}$$

$$Q_i P_i = -q_i p_i, \tag{4.4}$$

for $i = 1, 2$ *with* $\alpha = -(\theta + \kappa_\infty)/2$.

PROOF. The transposition of τ-functions R_τ: $f \leftrightarrow g$ yields the transposition of Hamiltonians $H_i \leftrightarrow \overline{H}_i$; we have (4.4) from (3.6). On the other hand, since $g = \tau(R_\tau(\vec{\kappa}))$, the following relation holds:

$$H_i(Q, P, s, R_\tau(\vec{\kappa})) = \overline{H}_i = H_i(q, p, s, \vec{\kappa}) - \frac{q_i p_i}{s_i}. \tag{4.5}$$

Using this and (4.4), we obtain (4.3). □

REMARK. We can construct a birational canonical transformation for $G(1,1,1,1,1)$, called a contiguity relation, which realizes the action on the space of parameters as translation. Put

$$\rho = R_{\kappa_1} \circ R_\tau \circ R_{\theta_1} \circ R_{\theta_2} \circ R_{\kappa_\infty} \circ R_{\kappa_0}, \tag{4.6}$$

then we have a birational canonical transformation

$$\rho : \mathcal{H}(\vec{\kappa}) \to \mathcal{H}(\rho(\vec{\kappa})),$$

where $\rho(\vec{\kappa}) = (\kappa_0 + 1, \kappa_1 - 1, \kappa_\infty, \theta_1, \theta_2)$. The relation between the Hamiltonians is

$$\rho(H_i) = H_i - \frac{q_i p_i}{s_i}, \quad (i = 1, 2), \tag{4.7}$$

hence we have

$$\overline{H}_i = \rho(H_i) = R_\tau(H_i), \quad (i = 1, 2), \tag{4.8}$$

i.e., $g = \tau(\rho(\vec{\kappa})) = \tau(R_\tau(\vec{\kappa}))$.

5. Algebraic Solutions

Now consider the birational canonical transformation, $w = R_\tau \circ R_{\theta_1} \circ R_{\theta_2} \circ R_{\kappa_\infty}$:

$$w : \mathcal{H}(q_i, p_i; \vec{\kappa}) \to \mathcal{H}(Q_i, P_i; w(\vec{\kappa}));$$

we have $w(\vec{\kappa}) = (-\kappa_0 + 1, -\kappa_1 + 1, \kappa_\infty, \theta_1, \theta_2)$ and

$$Q_i = \frac{s_i p_i(q_i p_i - \theta_i)}{(\alpha + q_1 p_1 + q_2 p_2)(\alpha + \kappa_\infty + q_1 p_1 + q_2 p_2)}, \tag{5.1}$$

$$Q_i P_i = -q_i p_i + \theta_i, \tag{5.2}$$

for $i = 1, 2$.

Put $\kappa_0 = \kappa_1 = 1/2$, then there is a fixed point with respect to the action w:

$$(q_i, p_i) = \pm \left(\frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \frac{\kappa_\infty}{2\sqrt{s_i}} \right) \quad i = 1, 2. \tag{5.3}$$

This gives an algebraic solution of $G(1,1,1,1,1)$. By using birational symmetries, we can construct many other algebraic solutions. For example, when $\kappa_0 = 1/2, \kappa_1 = -1/2$, we have

$$q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \tag{5.4}$$

$$p_i = \frac{\kappa_\infty}{2\sqrt{s_i}} \cdot \frac{\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty - \sqrt{s_i}}{\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty}. \tag{5.5}$$

For $\kappa_0 = 1/2, \kappa_1 = 3/2$, we have

$$q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty} \cdot \frac{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty)^2 - s_i}{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty)^2 - 1}, \tag{5.6}$$

$$p_i = \frac{\kappa_\infty}{2\sqrt{s_i}} \cdot \frac{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty)^2 - 1}{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty)(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty - \sqrt{s_i})}. \tag{5.7}$$

For $\kappa_0 = \kappa_1 = -1/2$, we have

$$q_i = \frac{\theta_i \sqrt{s_i}}{\kappa_\infty}, \tag{5.8}$$

$$p_i = \frac{\kappa_\infty}{2\sqrt{s_i}} \cdot \frac{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty - \sqrt{s_i})(\frac{\theta_1}{\sqrt{s_1}} + \frac{\theta_2}{\sqrt{s_2}} - \kappa_\infty - \frac{1}{\sqrt{s_i}}) - 1}{(\theta_1\sqrt{s_1} + \theta_2\sqrt{s_2} - \kappa_\infty)(\frac{\theta_1}{\sqrt{s_1}} + \frac{\theta_2}{\sqrt{s_2}} - \kappa_\infty)}. \tag{5.9}$$

6. Particular Solutions of the Garnier System in n-variables

THEOREM 6.1. *For special values of parameters, the Garnier system in n-variables $\mathcal{G}_n$ admits a particular solution expressed in terms of solutions of $\mathcal{G}_{n-1}$.*

PROOF. If $\theta_n = 0$, $\mathcal{G}_n$ admits a particular solution as $q_n = 0$. Take $q_n = 0$, we obtain

$$\frac{\partial q_i}{\partial s_n} = \frac{\partial H_n}{\partial p_i} = 0, \quad \frac{\partial p_i}{\partial s_n} = -\frac{\partial H_n}{\partial q_i} = 0,$$

for $1 \leq i \leq n-1$, *i.e.*, (q_i, p_i) do not depend on s_n. Put $\theta_n = 0$, $q_n = 0$ into the Hamiltonians H_i (0.5) for $1 \leq i \leq n-1$, then we obtain the Hamiltonians for $\mathcal{G}_{n-1}$. We do not enter into detail of computation. □

REMARK. In her paper [10], M. Mazzocco obtains the same type of particular solutions, by considering the monodromy preserving deformation of a linear differential equations such that some monodromy matrices are reduced to $\pm I$.

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