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Fourier Expansion of Holomorphic Siegel Modular Forms of Genus n *along the Minimal Parabolic Subgroup*

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Abstract. The aim of this paper is to establish a theory of Fourier expansion of holomorphic Siegel modular forms of genus n along the minimal parabolic subgroup. There are two known Fourier expansions of holomorphic Siegel modular forms, i.e. classical Fourier expansion and Fourier-Jacobi expansion (cf. §6). We also give a comparison of our expansion with them.

0. Introduction

In this paper, we study a Fourier expansion of scalar-valued holomorphic Siegel modular forms of arbitrary genus with respect to the minimal parabolic subgroup. In the theory of automorphic forms, the investigation of their Fourier expansions along various parabolic subgroups is fundamental and gives us significant information on such theory. For example, it gives a starting point for the construction of automorphic L-functions (cf. A.N.Andrianov [1], W.Kohnen and N.P.Skoruppa [8]).

The Fourier expansions of holomorpic Siegel modular forms along the maximal parabolic subgroups have already been studied. Up to conjugation, there are n maximal parabolic subgroups of the real symplectic group $Sp(n;\mathbb{R})$ of degree n. We have the unique maximal parabolic subgroup whose unipotent radical is abelian. This should be called the Siegel parabolic subgroup. The Fourier expansion along this parabolic subgroup is the most classical one. Its detailed investigation was initiated by C.L.Siegel (cf. [14]) in his theory of quadratic forms. The Fourier expansions along the other maximal parabolic subgroups are called Fourier-Jacobi expansions (cf. I.I.Piatetskii-Shapiro [12]). There are some detailed results [4], [17] etc. in the literature. In this paper, we are interested in the Fourier expansion along the *minimal* parabolic subgroup.

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Let us formulate our problem for a real semi-simple Lie group G of Hermitian type, constructed from $\mathbb{R}$-rational points of a simple $\mathbb{Q}$-algebraic group. The group $Sp(n;\mathbb{R})$ gives an example of such a group. Additionally, let K denote a maximal compact subgroup of G. The group G has unitary representations called holomorphic discrete series and we can define holomorphic automorphic forms on G. Given some irreducible finite dimensional representation (τ, V_τ) of the complexification $K_{\mathbb{C}}$ of K, let f be a V_τ-valued holomorphic automorphic form on G with respect to an arithmetic subgroup Γ. Let N be the unipotent radical of the minimal parabolic subgroup of G and $N_\Gamma := N \cap \Gamma$. We regard $f(xg)$ as a function in $x \in N$ with a fixed $g \in G$. From the Γ-invariance of f, we deduce $f(xg) \in L^2(N_\Gamma\backslash N) \otimes V_\tau$, where $L^2(N_\Gamma\backslash N)$ denotes the space of square-integrable functions on the quotient $N_\Gamma\backslash N$. Since $N_\Gamma\backslash N$ is compact, the space $L^2(N_\Gamma\backslash N)$ decomposes discretely into

$$L^2(N_\Gamma\backslash N) \simeq \tilde{\bigoplus}_{(\eta,H_\eta)\in\hat{N}} \mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N)) \otimes H_\eta,$$

where $\tilde{\bigoplus}$ denotes the Hilbert space direct sum, $\hat{N}$ the unitary dual of N and note $\dim_{\mathbb{C}} \mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N)) < \infty$. According to this decomposition, we have

$$f(*g) = \sum_Q \sum_{\eta\in\hat{N}} \sum_{1\le m\le m(\eta)} (\Theta_\eta^m \otimes W_{\eta,Q}^m(g)) \otimes v_Q,$$

where Q runs through an index set for a basis $\{v_Q\}$ of V_τ, $\{\Theta_\eta^m\}_{1\le m\le m(\eta)}$ is a basis of $\mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N))$ and $W_{\eta,Q}^m(g) \in H_\eta$ the (η, m, Q)-component of the decomposition. Via the evaluation map at $x \in N$, $\Theta_\eta^m \otimes W_{\eta,Q}^m(g)$ is identified with an element $\Theta_\eta^m(W_{\eta,Q}^m(g))(x)$ of $L^2(N_\Gamma\backslash N)$. Hence the decomposition above can be rewritten as

$$f(xg) = \sum_Q \sum_{\eta\in\hat{N}} \sum_{1\le m\le m(\eta)} \Theta_\eta^m(W_{\eta,Q}^m(g))(x) \cdot v_Q.$$

We call this decomposition *the Fourier expansion of the form f along the minimal parabolic subgroup.* In this paper, we consider the case where $G = Sp(n;\mathbb{R})$ and $\dim_{\mathbb{C}} V_\tau = 1$.

To investigate such an expansion, the following questions are fundamental;

(1) Determine $W_{\eta,Q}^m$ explicitly;

(2) Describe the multiplicity $m(\eta)$ concretely and find a basis of $\mathrm{Hom}_N(\eta, L^2(N_\Gamma \backslash N))$ for each $\eta \in \hat{N}$.
The function $W_\eta^m = \sum_Q W_{\eta,Q}^m \cdot v_Q$ is found to be a *generalized Whittaker function for holomorphic discrete series with K-type τ* (for a definition, see Definition 4.1, which treats the case of the one-dimensional K-type). An explicit formula of W_η^m is obtained by solving the differential equations arising from the "*Cauchy-Riemann condition*" (for a definition, see the end of §4.1), which characterizes the minimal K-type of holomorphic discrete series. The results on the generalized Whittaker functions are given as Theorem 4.5, Theorem 4.7, Theorem 4.12 and Theorem 4.13. These are solutions for the problem (1). Here we state our explicit formula of generalized Whittaker funcrions.

Theorem 0.1 (Theorem 4.7). *Let π_κ be a holomorphic discrete series with the one-dimensional minimal K-type $\tau_\kappa \simeq \det^\kappa$ (cf. §2). If there exists a non-zero gereralized Whittaker functions for π_κ attached to $\eta_l \in \hat{N}$ parametrized by $l \in \mathfrak{n}^*$ (for a detail on η_l, see §3), its explicit formula is given as*

$$W_{\kappa,l}(x_L a) = C(a_1 a_2 \cdots a_n)^\kappa \exp(-2\pi \operatorname{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n))),$$

where $\mathfrak{n}^*$ denotes the dual space of $\mathfrak{n} := \mathrm{Lie}(N)$, $(x_L, a) = \left(\begin{pmatrix} X_L & \\ & {}^t X_L^{-1} \end{pmatrix}, \begin{pmatrix} A_n & \\ & A_n^{-1} \end{pmatrix} \right) \in N_L \times A$ with N_L (resp. A) denoting the subgroup of N, canonically isomorphic to the standard maximal unipotent subgroup U_n of $GL_n(\mathbb{R})$ (resp. split component of an Iwasawa decomposition of G), $Y_n(l)$ is a certain symmetric matrix of degree n attached to l (cf. §3) and C denotes an arbitrary constant.

By a result of L.Corwin and F.P.Greenleaf [2], we can describe the multiplicity $m(\eta)$ in terms of the integral coadjoint orbit of a character inducing $\eta \in \hat{N}$, and find a basis of the space of intertwining operators in the problem (2) by using a notion of theta series on $N_\Gamma \backslash N$.

These solutions for the two problems (1) and (2) give our Fourier expansion, stated as Theorem 5.8 or equivalently as Theorem 5.10.

Theorem 0.2 (Theorem 5.8). *Let $(x, a) \in N \times A$ and let f be a holomorphic Siegel modular form on G of weight κ with respect to $Sp(n; \mathbb{Z})$.*

Then a Fourier expansion of f along the minimal parabolic subgroup is written as

$$f(xa) = \sum_{l \in \tilde{L}} \sum_{l' \in \mathfrak{M}(l)} C^l_{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x),$$

where $\Theta_{l'}(W_{\kappa,l'}(*a))(x) :=$

$$\sum_{l'' \in \mathrm{Ad}^*_S N_L(\mathbb{Z}) \cdot l'} \chi_{l''}(x_S)(a_1 a_2 \cdots a_n)^{\kappa} \exp(-2\pi \operatorname{Tr}({}^t(X_L A_n)\, Y_n(l'')(X_L A_n))).$$

THEOREM 0.3 (Theorem 5.10). *Let F_f be the holomorphic Siegel modular form on the Siegel upper half space $\mathfrak{H}_n$ of degree n, constructed from the form f on G (cf. §5). The Fourier expansion in Theorem 5.8 is rewritten as*

$$F_f(Z) = \sum_{S \in \tilde{\Omega}_{n,\mathbb{Z}}} \sum_{T \in \mathfrak{M}_n(S)} C^S_T \Theta_T(Z),$$

where $\Theta_T(Z) = \sum_{R \in \Omega_n(T)} \exp 2\pi\sqrt{-1} \operatorname{Tr}(RZ)$.

Here we explain the notations in the two theorems above. The set $\tilde{\Omega}_{n,\mathbb{Z}}$ denotes the $U_n(\mathbb{Q})$-equivalence classes of the set $\Omega_{n,\mathbb{Z}}$ of symmetric positive semi-definite semi-integral matrices of degree n, with $U_n(\mathbb{Q}) := U_n \cap GL_n(\mathbb{Q})$. The sets $\tilde{L}$ is the quotient of L by co-adjoint $N_L(\mathbb{Q})$-action (denoted by Ad^*_S), where L is a lattice in a certain subspace $\mathfrak{n}^*_S$ (cf. §5) of $\mathfrak{n}^*$, canonically bijective with $\Omega_{n,\mathbb{Z}}$ and where $N_L(\mathbb{Q}) := N_L \cap Sp(n;\mathbb{Q})$. These two sets $\tilde{\Omega}_{n,\mathbb{Z}}$ and $\tilde{L}$ parametrize irreducible unitary representations of N contributing to our Fourier expansion. For an $S \in \Omega_{n,\mathbb{Z}}$ (resp. $l \in L$), $\mathfrak{M}_n(S)$ (resp. $\mathfrak{M}(l)$) denotes the quotient of the $U_n(\mathbb{Q})$ (resp. $N_L(\mathbb{Q})$)-equivalence class of S (resp. l) in $\tilde{\Omega}_{n,\mathbb{Z}}$ (resp. $\tilde{L}$) by $U_n(\mathbb{Z})$ (resp. $N_L(\mathbb{Z})$)-equivalence, where $U_n(\mathbb{Z}) := U_n \cap GL_n(\mathbb{Z})$ and $N_L(\mathbb{Z}) := N_L \cap Sp(n;\mathbb{Z})$. The cardinalities of two sets $\mathfrak{M}(l)$ and $\mathfrak{M}_n(S)$ are equal to the multiplicity $m(\eta_l)$ when S corresponds to l via $\Omega_{n,\mathbb{Z}} \simeq L$. For a $T \in \Omega_{n,\mathbb{Z}}$, $\Omega_n(T) := \{{}^t u T u \mid u \in U_n(\mathbb{Z})\}$. The theta series $\Theta_{l'}(W_{\kappa,l'}(*a))(x)$ and $\Theta_T(Z)$ correspond to theta series $\Theta^m_{\eta}(W^m_{\eta,Q}(g))(x)$ in the formulation of our Fourier expansion. The constants $C^l_{l'}$ and C^S_T denote the Fourier coefficients.

Here we give some remarks on our results. Generalized Whittaker functions for admissible representations are of great interest in terms of representation theory (cf. B.Kostant [9],H.Yamashita [16]). It is known that holomorphic discrete series on semi-simple Lie groups of Hermitian type do not admit any Whittaker models attached to non-singular characters (for a definition of "non-singular characters", see [9],§2.3). For holomorphic discrete series of $Sp(n;\mathbb{R})$ with one-dimensional K-type, our results, Theorem 4.5, Theorem 4.7, Theorem 4.12 and Theorem 4.13 completely describe Whittaker functions attached to *all* irreducible unitary representations of N. With regard to our theory of Fourier expansion, we have already obtained such a Fourier expansion for vector valued holomorphic Siegel modular forms of genus 2 and those of genus 3 (cf. [10],[11]). The results in these previous papers are prototypes of our present study here.

Now we explain the contents of this paper. In §1, we introduce some basic notations for the real symplectic group, its standard subgroups, the associated Lie algebras, and the root systems for them. In §2, we give a parametrization of holomorphic discrete series using Harish-Chandra's theory on discrete series of semi-simple Lie groups. In §3, we recall a classification of irreducible unitary representations of N, using the "orbit method" for nilpotent Lie groups, established by A.A.Kirillov. We also give a formula for the infinitesimal actions of the representations of N. In §4, we obtain an explicit formula for the generalized Whittaker function. To be more precise, we first define the generalized Whittaker function in §4.1. In §4.2, we deduce the differential equations characterizing it from the Cauchy-Riemann condition. In §4.3, we get an explicit formula for the generalized Whittaker function by solving the differential equations. In §5, we express our Fourier expansion using the generalized Whittaker functions obtained above and the results of Corwin and Greenleaf [2]. In fact, this is accomplished by constructing theta series from the generalized Whittaker functions. In §6, we compare our expansion with the other two known Fourier expansions, i.e. classical Fourier expansion and Fourier-Jacobi expansion. In §6,1, we obtain a relation between Fourier coefficients of the classical expansion and those of our expansion. The result is

THEOREM 0.4 (Theorem 6.1). *Let* $T \in \Omega_{n,\mathbb{Z}}$ *belong to* $\mathfrak{M}_n(S)$ *with some* $S \in \Omega_{n,\mathbb{Z}}$ *and* C_T^S *(resp.* C_T*) denote the Fourier coefficient of our Fourier expansion in Theorem* 5.10 *(resp. classical Fourier expansion). Then we have*

$$C_T^S = C_T$$

and, for every $u \in U_n(\mathbb{Z})$,

$$C_{{}^t u T u} = C_T^S.$$

In §6.2, we study a relation between Fourier-Jacobi coefficients and theta series appearing in our Fourier expansion. Our result is stated as

Theorem 0.5 (Theorem 6.4, Corollary 6.5). (1) *Let* ϕ_{R_1} *be the Fourier-Jacobi coefficients of Fourier-Jacobi expansion of* f *indexed by* $R_1 \in \Omega_{j,\mathbb{Z}}$ *with* $1 \leq j \leq n-1$. *Then one has*

$$\sum_{T_1 \in \mathfrak{M}_j(S_1)} \sum_{R_1 \in \Omega_j(T_1)} \phi_{R_1}(Z_2, Z_3) \exp(2\pi\sqrt{-1}\,\mathrm{Tr}\, R_1 Z_1) = \sum_{S \in \tilde{\Omega_{S_1}}} \sum_{T \in \mathfrak{M}_n(S)} C_T^S \Theta_T(Z),$$

where $\Omega_{S_1} := \left\{ S = \begin{pmatrix} S_1 & S_2 \\ {}^t S_2 & S_3 \end{pmatrix} \in \Omega_{n,\mathbb{Z}} \,\middle|\, S_2 \in M_{j,n-j}(\mathbb{Q}),\ S_3 \in M_{j-n}(\mathbb{Q}) \right\}$ *and* $\tilde{\Omega_{S_1}}$ *denotes the* $U_n(\mathbb{Q})$*-equivalence classes of* Ω_{S_1}.
(2) *When* $j = 1$, *this formula becomes*

$$\phi_{S_1}(Z_2, Z_3) \exp 2\pi\sqrt{-1}\,\mathrm{Tr}\, S_1 Z_1 = \sum_{T \in \tilde{\Omega_{S_1}}} \sum_{S \in \mathfrak{M}_n(T)} C_T^S \Theta_S(Z).$$

Theorem 0.4 and Theorem 0.5 tell us how the known two Fourier expansions and our Fourier expansion are related to each other. We hope that these two theorems provide us some new information on the two known expansions in terms of our theory of Fourier expansion.

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1. Basic Notations

Let $G = Sp(n;\mathbb{R})$ be the real symplectic group of degree n, defined by

$$\{g \in GL_{2n}(\mathbb{R}) \mid {}^t g\, J_n g = J_n\}$$

with $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We often use the block notation $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C and $D \in M_n(\mathbb{R})$. Let θ denote the Cartan involution defined by $G \ni g \mapsto {}^t g^{-1}$ and $K := \{g \in G \mid \theta(g) = g\}$, which coincides with $\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \right\}$. Then the group K is a maximal compact subgroup of G, which is isomorphic to the unitary group $U(n)$ of degree n under the map

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B.$$

Let $\mathfrak{g} = \mathfrak{sp}(n; \mathbb{R})$ be the Lie algebra of G, which is given as

$$\{X \in M_{2n}(\mathbb{R}) \mid {}^t X \, J_n + J_n X = 0_{2n}\}.$$

We denote also by θ the Cartan involution on $\mathfrak{g}$ given by

$$X \mapsto -\, {}^t X \,.$$

Let $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$. Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g}$ admits a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Throughout the subsequent argument, E_{ij} denotes the matrix unit $(\delta_{ip}\delta_{jq})_{1 \le p,q \le N}$ in the matrix algebra M_N (either $N = n$ or $N = 2n$). In order to formulate the Iwasawa decomposition of $\mathfrak{g}$, we prepare the restricted root system of it. Let $\mathfrak{a} = \sum_{k=1}^{n} \mathbb{R} H_k$ with $H_k = E_{kk} - E_{k+n,k+n}$, which is a maximal abelian subalgebra of $\mathfrak{p}$. We write A for

$$\exp(\mathfrak{a}) = \left\{ a = \begin{pmatrix} A_n & \\ & A_n^{-1} \end{pmatrix} \;\middle|\; A_n = \operatorname{diag}(a_1, a_2, \dots, a_n), \; a_i \in \mathbb{R}_+ \right\}.$$

The root system $\Delta(\mathfrak{a}, \mathfrak{g})$ of $(\mathfrak{g}, \mathfrak{a})$ is of type C_n and given by

$$\{\pm e_i \pm e_j, \; \pm 2e_k \mid 1 \le i < j \le n, \; 1 \le k \le n\},$$

where e_i denotes the linear functional on $\mathfrak{a}$ defined by $e_i(H_j) = \delta_{ij}$. We denote by E_α the root vector corresponding to a root $\alpha \in \Delta(\mathfrak{a}, \mathfrak{g})$, explicitly given as

$$E_{e_i+e_j} = E_{i,j+n} + E_{j,i+n}, \; E_{e_i-e_j} = E_{ij} - E_{j+n,i+n},$$
$$E_{2e_k} = E_{k,k+n}, \; E_{-\alpha} = {}^t E_\alpha \,.$$

Let $\Delta^+(\mathfrak{a},\mathfrak{g}) = \{e_i \pm e_j,\ 2e_k \mid 1 \le i < j \le n,\ 1 \le k \le n\}$ be the standard set of positive roots. Furthermore set $\mathfrak{n} = \sum_{\alpha\in\Delta^+(\mathfrak{a},\mathfrak{g})} \mathbb{R}E_\alpha$, which is the nilradical of the minimal parabolic subalgebra. Then we have an Iwasawa decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

With $N := \exp(\mathfrak{n})$, the group G also has such decomposition

$$G = NAK.$$

Additionally, we introduce a compact Cartan subalgebra $\mathfrak{t}$ given by

$$\mathfrak{t} = \oplus_{1\le k\le n}\mathbb{R}T_k,$$

with $T_k = E_{k,k+n} - E_{k+n,k}$. Consider the root system $\Delta(\mathfrak{t}_\mathbb{C},\mathfrak{g}_\mathbb{C})$ of $(\mathfrak{g}_\mathbb{C},\mathfrak{t}_\mathbb{C})$, where $\mathfrak{g}_\mathbb{C}$ and $\mathfrak{t}_\mathbb{C}$ denote the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$ respectively. This root system is also of type C_n and given by $\{\pm f_i \pm f_j,\ \pm 2f_k \mid 1 \le i < j \le n,\ 1 \le k \le n\}$, where f_i denotes the linear functional on $\mathfrak{t}_\mathbb{C}$ defined by $f_i(T_j) = \sqrt{-1}\delta_{ij}$. We denote by F_β the root vector for $\beta \in \Delta(\mathfrak{t}_\mathbb{C},\mathfrak{g}_\mathbb{C})$, explicitly given as

$$\begin{aligned}
F_{f_i+f_j} &= E_{ij} + E_{ji} - E_{i+n,j+n} - E_{j+n,i+n}\\
&\quad + \sqrt{-1}(E_{i,j+n} + E_{j,i+n} + E_{i+n,j} + E_{j+n,i}),\\
F_{2f_k} &= E_{kk} - E_{k+n,k+n} + \sqrt{-1}(E_{k,k+n} + E_{k+n,k}),\\
F_{f_i-f_j} &= E_{ij} - E_{ji} + E_{i+n,j+n} - E_{j+n,i+n}\\
&\quad - \sqrt{-1}(E_{i+n,j} + E_{j+n,i} - E_{i,j+n} - E_{j,i+n}),\\
F_{-\beta} &= \bar{F}_\beta.
\end{aligned}$$

The set $\Delta^+ = \{f_i \pm f_j,\ 2f_k \mid 1 \le i < j \le n,\ 1 \le k \le n\}$ forms the standard positive root system and $\Delta_n^+ = \{f_i + f_j,\ 2f_k \mid 1 \le i < j \le n,\ 1 \le k \le n\}$ the set of non-compact positive roots. Put

$$\mathfrak{p}^+ = \oplus_{\beta\in\Delta_n^+}\mathbb{C}F_\beta,\ \mathfrak{p}^- = \oplus_{\beta\in\Delta_n^+}\mathbb{C}F_{-\beta} = \overline{\mathfrak{p}^+}.$$

Then, in the complexification $\mathfrak{p}_\mathbb{C}$ of $\mathfrak{p}$, these two spaces $\mathfrak{p}^+$ and $\mathfrak{p}^-$ form its holomorphic part and anti-holomorphic part respectively, and we have a decomposition of $\mathfrak{g}_\mathbb{C}$:

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-.$$

In §4, we will consider the infinitesimal actions of the generators of $\mathfrak{p}^-$. For that purpose we introduce the Iwasawa decomposition of $F_{-\beta}$ for $\beta \in \Delta_n^+$, which is settled by direct computation.

LEMMA 1.1. *Let* $\mathrm{Ad}\, a$ *denote the adjoint action of* $a \in A$ *on* $\mathfrak{n}$. *Then we have the following decompositions:*

$$\begin{aligned} F_{-f_i-f_j} &= 2a_i a_j^{-1}\,\mathrm{Ad}(a^{-1})E_{e_i-e_j} - 2a_i a_j\sqrt{-1}\,\mathrm{Ad}(a^{-1})E_{e_i+e_j} - F_{-f_i+f_j},\\ F_{-2f_k} &= -2a_k^2\sqrt{-1}\,\mathrm{Ad}(a^{-1})E_{2e_k} + H_k + \sqrt{-1}T_k. \end{aligned}$$

2. Holomorphic Discrete Series of $Sp(n;\mathbb{R})$

We recall a notion of holomorphic discrete series representations of $Sp(n;\mathbb{R})$ in terms of Harish-Chandra's parametrization of the discrete series representations of a semisimple Lie group (cf. [6],Chap.IX,§7,Theorem 9.20, Chap.XII,§5,Theorem 12.21). Consider an arbitrary continuous character on the compact Cartan subgroup $T := \exp(\mathfrak{t})$, which is of the form:

$$T \ni \exp(\sum_{1\le i\le n} \theta_i T_i) \mapsto \exp\sqrt{-1}(\sum_{1\le i\le n} \Lambda_i\theta_i) \in U(1)\ (\theta_i \in \mathbb{R}),$$

where $(\Lambda_1, \Lambda_2, \dots, \Lambda_n) \in \mathbb{Z}^{\oplus n}$ and $U(1)$ is the set of complex numbers of absolute value 1. The vector $(\Lambda_1, \Lambda_2, \dots, \Lambda_n)$ is identified with a differential of the above character and with a linear functional $\Lambda = \Lambda_1 f_1 + \Lambda_2 f_2 + \cdots + \Lambda_n f_n$ on $\mathfrak{t}_\mathbb{C}$. Such Λ is called an analytically integral weight (cf. [6],Chap.IV,§5, Proposition 4.13). The subset $\{f_i - f_j \mid 1 \le i < j \le n\}$ of $\Delta(\mathfrak{t}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ forms a set of compact positive roots. We denote by ρ and ρ_c the half-sum of positive roots and that of compact positive roots, respectively. Due to the Harish-Chandra's parametrization of discrete series, holomorphic discrete series representations of $Sp(n;\mathbb{R})$ can be parametrized by the following set of analytically integral weights:

$$\begin{aligned} &\{\Lambda \mid \rho + \Lambda \text{ is analytically integral and}\\ &\quad \Lambda \text{ is regular dominant with respect to } \Delta^+\} \simeq\\ &\{\Lambda = \Lambda_1 f_1 + \Lambda_2 f_2 + \cdots + \Lambda_n f_n \mid (\Lambda_1, \Lambda_2, \dots, \Lambda_n) \in \mathbb{Z}^{\oplus n},\\ &\quad \Lambda_1 > \Lambda_2 > \ldots > \Lambda_n > 0\} \end{aligned}$$

(cf. [6],Chap.VI,§4,Theorem 6.6, Chap.IX,§7,Theorem 9.20,Remarks (1)). Such Λ's are called *Harish-Chandra parameters* for holomorphic discrete series. We denote by π_Λ the holomorphic discrete series with Harish-Chandra parameter Λ. The hightest weight of the minimal K-type of π_Λ is given by the special weight $\lambda = \Lambda+\rho-2\rho_c = (\Lambda_1+1)f_1+(\Lambda_2+2)f_2+\cdots+(\Lambda_n+n)f_n$, which is called the *Blattner parameter*.

Let the minimal K-type τ_λ of π_Λ be one-dimensional. Then the Harish-Chandra parameter Λ (resp. Blattner parameter λ) is given as $(\kappa-1)f_1 + (\kappa-2)f_2+\cdots+(\kappa-n)f_n$ (resp. $\kappa f_1+\kappa f_2+\cdots \kappa f_n$) with $\kappa > n$. The minimal K-type τ_λ can be expressed as $\det^\kappa$ on $U(n)$ via the isomorphism $K \simeq U(n)$ in §1. We denote this τ_λ by τ_κ and π_κ by the holomorphic discrete series with the minimal K-type τ_κ.

3. Classification of Unitary Representation of N

The group $N = \exp(\mathfrak{n})$ is the standard maximal unipotent subgroup of G. We want to describe the unitary dual $\hat{N}$ of N using Kirillov's construction of irreducible unitary representations of nilpotent Lie groups.

First we introduce some notations. The group N can be written as $N_S \rtimes N_L$ with

$$N_S := \left\{ x_S = \begin{pmatrix} 1_n & X_S \\ 0_n & 1_n \end{pmatrix} \;\middle|\; X_S \in M_n(\mathbb{R}),\ {}^tX_S = X_S \right\},$$

$$N_L := \left\{ x_L = \begin{pmatrix} X_L & 0_n \\ 0_n & {}^tX_L^{-1} \end{pmatrix} \;\middle|\; X_L \in U_n \right\},$$

where U_n denotes the standard maximal unipotent subgroup of $GL_n(\mathbb{R})$. We denote the (i,j)-component of X_S (resp. X_L) by x_{ij} (resp. x'_{ij}). Let $\mathfrak{n}_S$ (resp. $\mathfrak{n}_L$) be the Lie algebra of N_S (resp. N_L). Then these two Lie algebras are given as

$$\mathfrak{n}_S = \oplus_{1\le i\le j\le n}\mathbb{R}E_{e_i+e_j},\ \mathfrak{n}_L = \oplus_{1\le i<j\le n}\mathbb{R}E_{e_i-e_j}$$

and we have

$$\mathfrak{n} = \mathfrak{n}_S \oplus \mathfrak{n}_L.$$

Let $\mathfrak{n}^*$ denote the dual space of $\mathfrak{n}$. We denote by $\{l_{ij}, l_{kk}, l'_{ij}\}_{\{1\le i<j\le n,\ 1\le k\le n\}}$ the basis of $\mathfrak{n}^*$ dual to $\{E_{e_i+e_j}, E_{2e_k}, E_{e_i-e_j}\}_{\{1\le i<j\le n, 1\le k\le n\}}$. We write $l \in \mathfrak{n}^*$ as $l = \sum_{1\le i\le j\le n}\xi_{ij}l_{ij} + \sum_{1\le i<j\le n}\xi'_{ij}l'_{ij}$ with ξ_{ij} and $\xi'_{ij} \in \mathbb{R}$.

For $l \in \mathfrak{n}^*$, let $\mathfrak{m}$ denote a polarization subalgebra with respect to an inner product $l([*,*])$ on $\mathfrak{n}$ (for a definition, see [3],p27-p28), where $[*,*]$ denotes the bracket product on $\mathfrak{n}$. Furthermore, set $M := \exp(\mathfrak{m})$ and let $\chi_l : M \to U(1)$ be a character defined as $\chi_l(m) := \exp(2\pi\sqrt{-1}l(\log(m)))$ for $m \in M$. Using these notations, we state the following theorem established by A.A.Kirillov (cf. [3],Theorems 2.2.1-2.2.4).

PROPOSITION 3.1. (1) *Every $\eta \in \hat{N}$ is unitarily equivalent to a representation of the form*

$$\eta_l := \mathrm{L}^2\text{-}\operatorname{Ind}_M^N \chi_l$$

with some $l \in \mathfrak{n}^*$. Up to unitary equivalence, η_l does not depend on the choice of M. Additionally, we remark that, if η_l is not a character, a representation space H_{η_l} of η_l is given as

$$\left\{ h : \textit{measurable function on } N \;\middle|\; \begin{array}{r} h(mx) = \chi_l(m)h(x) \\ \textit{for } (m,x) \in M \times N \\ \|h\|_l^2 := \int_{M\backslash N} h(x)\overline{h(x)}d\dot{x} < \infty \end{array} \right\},$$

where $d\dot{x}$ denotes an invariant measure on the quotient $M\backslash N$.
(2) *Two representations η_l and $\eta_{l'}$ with l and $l' \in \mathfrak{n}^*$ are unitarily equivalent if and only if $l = \operatorname{Ad}^* x \cdot l'$ with some $x \in N$. That is, we have a bijection*

$$\hat{N} \simeq \mathfrak{n}^* / \operatorname{Ad}^* N.$$

In order to find convenient choices of polarization subalgebras for our argument, we state

LEMMA 3.2. (1) *This time, let $\mathfrak{n}$ be a general m-dimensional nilpotent Lie algebra with a chain of ideals*

$$\{0\} \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \ldots \subset \mathfrak{n}_m = \mathfrak{n},$$

where $\dim \mathfrak{n}_i = i$. *For $l \in \mathfrak{n}^*$, we set $l_i := l|_{\mathfrak{n}_i}$ and*

$$\mathfrak{r}_{\mathfrak{n}_i}(l_i) := \{X \in \mathfrak{n}_i \mid l_i([X,Y]) = 0 \ \forall Y \in \mathfrak{n}_i\}.$$

Then $\sum_{i=1}^{m} \mathfrak{r}_{\mathfrak{n}_i}(l_i)$ *forms a polarization subalgebra for* l.
(2) *Let* $\mathfrak{n}$ *be our Lie algebra. It has a filtration of ideals satisfying the condition in* (1), *given by*

$$\begin{aligned}\{0\} &\subset \mathfrak{n}_{11} \subset \mathfrak{n}_{12} \subset \ldots\ldots \subset \mathfrak{n}_{1n} \subset \mathfrak{n}_{22} \subset \ldots\ldots \subset \mathfrak{n}_{2n} \subset \ldots \\ &\subset \mathfrak{n}_{ii} \subset \ldots\ldots\ldots \quad \subset \mathfrak{n}_{in} \subset \ldots\ldots\ldots\ldots \subset \mathfrak{n}_{nn} = \mathfrak{n}_S \\ &\subset \mathfrak{n}'_{1n} \subset \mathfrak{n}'_{1,n-1} \subset \ldots \subset \mathfrak{n}'_{12} \subset \mathfrak{n}'_{2n} \subset \ldots\ldots \subset \mathfrak{n}'_{23} \subset \ldots \\ &\subset \mathfrak{n}'_{in} \subset \ldots\ldots\ldots\ldots \subset \mathfrak{n}'_{i,i+1} \subset \ldots\ldots\ldots \subset \mathfrak{n}'_{n-1,n} = \mathfrak{n}.\end{aligned}$$

Here

$$\mathfrak{n}_{ij} := \bigoplus_{\substack{u,v \\ s.t.\ u<i,\ v\geq u}} \mathbb{R}E_{e_u+e_v} \oplus \bigoplus_{i\leq v\leq j} \mathbb{R}E_{e_i+e_v} \subset \mathfrak{n}_S \quad \textit{for } 1 \leq i \leq j \leq n,$$

and

$$\mathfrak{n}'_{ij} := \mathfrak{n}_S \oplus \bigoplus_{\substack{u,v \\ s.t.\ u<i\ v>u}} \mathbb{R}E_{e_u-e_v} \oplus \bigoplus_{v\geq j} \mathbb{R}E_{e_i-e_v} \quad \textit{for } 1 \leq i < j \leq n.$$

PROOF. For a proof of (1), see [3], Theorem 1.3.5. Regarding (2), we can check that each subspace in the filtration above forms an ideal of $\mathfrak{n}$ by direct computation. □

For each l, we can take a polarization subalgebra $\mathfrak{m}$ so that it contains $\mathfrak{n}_S$. In fact, since $\mathfrak{r}_{\mathfrak{n}_{nn}}(l|_{\mathfrak{n}_{nn}}) = \mathfrak{n}_S$, Lemma 3.2 implies that there exists such a polarization subalgebra. In terms of Proposition 3.1 (1), there is no loss of generality if we impose the following condition on $\mathfrak{m}$:

ASSUMPTION 1. *From now on, we assume that the polarization subalgebra* $\mathfrak{m}$ *for an* l *contains* $\mathfrak{n}_S$.

For $l \in \mathfrak{n}^*$, we set

$$Y_n(l) := \begin{pmatrix} \xi_{11} & \xi_{12}/2 & \cdots & \xi_{1n}/2 \\ \xi_{12}/2 & \xi_{22} & \cdots & \xi_{2n}/2 \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{1n}/2 & \xi_{2n}/2 & \cdots & \xi_{nn} \end{pmatrix}.$$

Consider the total set of indices $[1,n]=\{1,2,\dots,n\}$. For $l\in\mathfrak{n}^*$, we define a subset $I(l)$ of indices by

$$I(l):=\begin{cases}\{i\in[1,n]\mid \text{there exists } j\in[1,n] \\ \qquad\qquad \text{such that } \xi_{ij}\neq 0 \text{ or } \xi_{ji}\neq 0\} & \text{if } Y_n(l)\neq 0_n; \\ \{n\} & \text{if } Y_n(l)=0_n.\end{cases}$$

We set $r=\sharp I(l)$ and write $I(l)$ as $\{n_1,n_2,\dots,n_r\}$ with $n_1<n_2<\dots<n_r$.

Proposition 3.3. *Let $l\in\mathfrak{n}^*$ satisfy $I(l)\neq\{n\}$.*
(1) *A polarization subalgebra $\mathfrak{m}$ for l satisfies*

$$E_{e_{n_p}-e_j}\notin\mathfrak{m} \text{ for any } (n_p,j) \text{ with } n_p\in I(l) \text{ and } j>n_p.$$

(2) *We define an $r\times r$ symmetric matrix $Y(l)$ by*

$$\begin{pmatrix}\xi_{n_1n_1} & \xi_{n_1n_2}/2 & \cdots & \xi_{n_1n_r}/2\\ \xi_{n_1n_2}/2 & \xi_{n_2n_2} & \cdots & \xi_{n_2n_r}/2\\ \vdots & \vdots & \ddots & \vdots\\ \xi_{n_1n_r}/2 & \xi_{n_2n_r}/2 & \cdots & \xi_{n_rn_r}\end{pmatrix},$$

and its minor matrices $Y(l)_s$ by

$$\begin{pmatrix}\xi_{n_1n_1} & \xi_{n_1n_2}/2 & \cdots & \xi_{n_1n_s}/2\\ \xi_{n_1n_2}/2 & \xi_{n_2n_2} & \cdots & \xi_{n_2n_s}/2\\ \vdots & \vdots & \ddots & \vdots\\ \xi_{n_1n_s}/2 & \xi_{n_2n_s}/2 & \cdots & \xi_{n_sn_s}\end{pmatrix}$$

for $1\le s\le r$. Moreover, we set

$$\mathfrak{n}_l:=\begin{cases}\bigoplus_{\substack{i,j\\ s.t.\ i\notin I(l),\ j>i}}\mathbb{R}E_{e_i-e_j} & (I(l)\neq[1,n]);\\ \{0\} & (I(l)=[1,n]).\end{cases}$$

Assume that $\det Y(l)_s\neq 0$ for $1\le s\le r$ (resp. $1\le s\le r-1$) if $n_r\neq n$ (resp. $n_r=n$) in $I(l)$. Then a polarization subalgebra $\mathfrak{m}$ satisfies

$$\mathfrak{m}\subset\mathfrak{n}_S\oplus\mathfrak{n}_l.$$

In particular, if l additionally satisfies $l(\mathfrak{n}_l) = \{0\}$, then $\mathfrak{m} = \mathfrak{n}_S \oplus \mathfrak{n}_l$.

PROOF. (1) Let $\mathfrak{m}$ contain an $E_{e_{n_p}-e_j}$ for some (n_p, j) with $n_p \in I(l)$ and $j > n_p$. Then $l([E_{e_{n_p}-e_j}, \mathfrak{n}_S]) = \{0\}$ has to hold since we assume that $\mathfrak{m} \supset \mathfrak{n}_S$. But there is a non-zero $\xi_{n_p n_q}$ or $\xi_{n_q n_p}$ with some $n_q \in I(l)$, so that $l([E_{e_{n_p}-e_j}, E_{e_{n_q}+e_j}]) \neq 0$ or $l([E_{e_{n_p}-e_j}, E_{e_j+e_{n_q}}]) \neq 0$. This is a contradiction. Hence $E_{e_{n_p}-e_j} \notin \mathfrak{m}$.
(2) Let $X := \sum_{\substack{n_p \in I(l), j \\ \text{s.t. } j > n_p}} a_{n_p j} E_{e_{n_p}-e_j}$ with $a_{n_p j} \in \mathbb{R}$ belong to a subspace $\bigoplus_{\substack{n_p \in I(l), j \\ \text{s.t. } j > n_p}} \mathbb{R} E_{e_{n_p}-e_j}$, complementary to $\mathfrak{n}_S \oplus \mathfrak{n}_l$ in $\mathfrak{n}$. In order to prove (2), it suffices to show the following claim:

$$l([X, \mathfrak{n}_S]) = \{0\} \text{ means } X = 0.$$

If this is serttled, we obtain $\mathfrak{m} \subset \mathfrak{n}_S \oplus \mathfrak{n}_l$. In fact, decompose $Y \in \mathfrak{n}$ into $Y = Y_{S,l} + Y'$ with $Y_{S,l} \in \mathfrak{n}_S \oplus \mathfrak{n}_l$ and Y' in the complementary subspace. In order that Y belongs to $\mathfrak{m}$, $l([Y, \mathfrak{n}_S]) = \{0\}$ has to hold. Since $l([\mathfrak{n}_S \oplus \mathfrak{n}_l, \mathfrak{n}_S]) = \{0\}$, $l([Y', \mathfrak{n}_S]) = \{0\}$ is satisfied. Therefore we get $Y' = 0$ if the claim is proved. Hence $\mathfrak{m} \subset \mathfrak{n}_S \oplus \mathfrak{n}_l$. Moreover if $l(\mathfrak{n}_l) = \{0\}$, $\mathfrak{m} = \mathfrak{n}_S \oplus \mathfrak{n}_l$ holds since $l([\mathfrak{n}_S \oplus \mathfrak{n}_l, \mathfrak{n}_S \oplus \mathfrak{n}_l]) = \{0\}$.

We start proving the claim. Let $l([X, \mathfrak{n}_S]) = \{0\}$. By direct computation, we see that

$$l([X, E_{e_{n_p}+e_j}]) = \begin{cases} \sum_{1 \le s \le p-1} \xi_{n_s n_p} a_{n_s j} + 2\xi_{n_p n_p} a_{n_p j} \\ \quad + \sum_{p+1 \le s \le k(j)} \xi_{n_p n_s} a_{n_s j} = 0 & (j \notin I(l),\ j > n_p); \\ \sum_{1 \le s \le p-1} \xi_{n_s n_p} a_{n_s j} + 2\xi_{n_p n_p} a_{n_p j} \\ \quad + \sum_{p+1 \le s \le k(j)} \xi_{n_p n_s} a_{n_s j} \\ \quad + \sum_{1 \le s \le p-1} \xi_{n_s j} a_{n_s n_p} = 0 & (j \in I(l),\ j > n_p), \end{cases}$$

where we set $k(j) := \max\{s \mid n_s < j,\ n_s \in I(l)\}$. From these formulas, we have

$$Y(l)_{k(j)} \begin{pmatrix} a_{n_1 j} \\ a_{n_2 j} \\ \vdots \\ a_{n_{k(j)} j} \end{pmatrix}$$

$$= \begin{cases} \mathbf{0}_{k(j)} & \text{for } j \notin I(l) \text{ with } j > n_1; \\ \begin{pmatrix} 0 \\ -\frac{\xi_{n_1 j}}{2} a_{n_1 n_2} \\ \vdots \\ -\sum_{1 \le t \le k(j)-1} \frac{\xi_{n_t j}}{2} a_{n_t n_{k(j)}} \end{pmatrix} & \text{for } j \in I(l) \text{ with } j > n_1, \end{cases}$$

where $\mathbf{0}_{k(j)}$ denotes the zero vector in $\mathbb{R}^{k(j)}$. From the assumption on the minor matrices $Y(l)_{k(j)}$, we see that $a_{n_p j} = 0$ for any (n_p, j) with $n_p \in I(l)$ and $j > n_p$. Therefore we obtain $X = 0$ and complete the proof. $\square$

Let $H_{\eta_l}^\infty$ denote the space of C^∞-vectors in H_{η_l}. We calculate the infinitesimal actions of generators of $\mathfrak{n}$ on $H_{\eta_l}^\infty$ via the differential $d\eta_l$ of η_l. For that purpose, we denote by $\xi_{ij}({}^t X_L\, Y_n(l) X_L)$ the coefficient of $\frac{1}{2}(E_{ij} + E_{ji})$ in ${}^t X_L\, Y_n(l) X_L$ for $1 \le i \le j \le n$.

Proposition 3.4. (1) *$d\eta_l(E_{e_i+e_j}) = 2\pi\sqrt{-1}\xi_{ij}({}^t X_L\, Y_n(l) X_L)$ for $1 \le i \le j \le n$.*
(2) *$d\eta_l(E_{e_i-e_j}) = \frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}}$ for $1 \le i < j \le n$.*

Proof. Let $h \in H_\eta^\infty$. Then $y = y_S y_L \in N$ with $y_S \in N_S$ and $y_L \in N_L$ acts on H_η^∞ by right translation R_N:

$$\begin{aligned} R_N(y)h(x) &= h(x_S x_L y_S y_L) = h(x_S(x_L y_S x_L^{-1}) x_L y_L) \\ &= \chi_l(x_S(x_L y_S x_L^{-1}))h(x_L y_L). \end{aligned}$$

We compute $d\eta_l(X) = dR_N(X)$ for each generator $X \in \mathfrak{n}$. To show the formula (1), the following obvious equality is convenient.

Lemma 3.5. *The matrix $Y_n(l)$ is characterized by the following property:*

$$l(\log(x_S)) = \mathrm{Tr}(Y_n(l) X_S) \quad \textit{with } x_S = \begin{pmatrix} 1_n & X_S \\ & 1_n \end{pmatrix} \in N_S.$$

Noting this and a well-known formula $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ for two matrices X and Y with the same degree, we have

$$\begin{aligned} d\eta_l(E_{e_i+e_j}) &= d\chi_l(x_L E_{e_i+e_j} x_L^{-1}) = 2\pi\sqrt{-1}l(x_L E_{e_i+e_j} x_L^{-1}) \\ &= 2\pi\sqrt{-1}\,\mathrm{Tr}(Y_n(l)X_L(E_{ij}+E_{ji})\,{}^tX_L) \\ &= 2\pi\sqrt{-1}\,\mathrm{Tr}(({}^tX_L\,Y_n(l)X_L)(E_{ij}+E_{ji})) \\ &= 2\pi\sqrt{-1}\xi_{ij}({}^tX_L\,Y_n(l)X_L) \end{aligned}$$

for (i,j) with $i<j$. We can compute $d\eta_l(E_{2e_k})$ similarly.

In order to obtain the formula (2), we note the equality

$$x_L \exp(tE_{e_i-e_j}) = 1_{2n} + (x'_{ij}+t)E_{e_i-e_j} + \sum_{1\le u<i} (x'_{uj} + x'_{ui}t)E_{e_u-e_j} + \sum_{\substack{(u,v)\notin\{(w,j)\mid 1\le w\le i\} \\ \text{s.t } u<v}} x'_{uv}E_{e_u-e_v},$$

which can be checked by direct computation. Using this, compute the differential $\frac{d}{dt}\big|_{t=0} h(x_L \exp(tE_{e_i-e_j}))$. $\square$

4. Generalized Whittaker Functions on G for Holomorphic Discrete Series

4.1. Definition

Recall that π_κ denotes the holomorphic discrete series representation of G with the minimal K-type τ_κ (cf. §2).

Let $\iota : \tau_\kappa \to \pi_{\kappa K}$ be the inclusion map of τ_κ into the space $\pi_{\kappa K}$ of K-finite vectors in π_κ. We simply denote $\pi_{\kappa K}$ by π_κ. Before giving the definition of generalized Whittaker functions, we introduce the following spaces:

$$\begin{aligned} C^\infty_{\eta_l}(N\backslash G) &:= \{F : H^\infty_{\eta_l}\text{-valued } C^\infty\text{-function on } G \mid \\ &\qquad F(xg) = \eta_l(x)F(g)\ \}; \\ C^\infty_{\eta_l}(N\backslash G)_K &:= \{F \in C^\infty_{\eta_l}(N\backslash G) \mid F \text{ is } K\text{-finite}\}; \\ C^\infty_{\eta_l,\tau^*_\kappa}(N\backslash G/K) &:= \{W : H^\infty_{\eta_l}\text{-valued } C^\infty\text{-function on } G \mid \\ &\qquad W(xgk) = \eta_l(x)\tau^*_\kappa(k)^{-1}W(g) = \eta_l(x)\tau_\kappa(k)W(g)\}, \end{aligned}$$

where $(x, g, k) \in N \times G \times K$, τ_κ^* denotes the contragredient representation of τ_κ. The two spaces π_κ and $C_{\eta_l}^\infty(N\backslash G)_K$ form $(\mathfrak{g}_\mathbb{C}, K)$-modules respectively (for a definition of a $(\mathfrak{g}_\mathbb{C}, K)$-module, see [15], Chap.0, §3, Definition 0.3.8).

Definition 4.1. Let ι^* be a map defined as

$$\iota^* : \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi_\kappa, C_{\eta_l}^\infty(N\backslash G)_K) \ni F \mapsto F \cdot \iota \in \mathrm{Hom}_K(\tau_\kappa, C_{\eta_l}^\infty(N\backslash G)_K).$$

An element of $\mathrm{Im}\,\iota^*$ is called a *generalized Whittaker function on G for the representation π_κ with K-type τ_κ.*

We have a canonical identification

$$\mathrm{Hom}_K(\tau_\kappa, C_{\eta_l}^\infty(N\backslash G)_K) \simeq C_{\eta_l,\tau_\kappa^*}^\infty(N\backslash G/K).$$

Now we introduce

$$\begin{aligned} &S_{\eta_l,\tau_\kappa^*}(N\backslash G/K) \\ &:= \left\{ W : \mathbb{C}\text{-valued } C^\infty \text{function on } G \;\middle|\; \begin{array}{c} W(gk) = \tau_\kappa(k)W(g) \\ W(xg) \in H_{\eta_l}^\infty \end{array} \right\}, \end{aligned}$$

where $(g, k) \in G \times K$ and we regard $W(xg)$ as a function in $x \in N$ with a fixed $g \in G$. Here note that $\tau_\kappa^*(k) = \tau_\kappa(k)^{-1}$. Then we have a bijection

$$C_{\eta_l,\tau_\kappa^*}^\infty(N\backslash G/K) \simeq S_{\eta_l,\tau_\kappa^*}(N\backslash G/K)$$

via the evaluation map $W(g)(*) \mapsto W(g)(1)$ at $1 \in N$, since $W(g)(xx_0) = \eta_l(x_0)W(g)(x) = W(x_0g)(x)$ for $x_0, x \in N$ and $g \in G$.

The holomorphic discrete series π_κ forms a highest weight module with highest weight $\kappa(f_1 + f_2 + \cdots + f_n)$ (cf. [16],Proposition 7.4). Due to this and [16],Proposition 12.2, we obtain a bijection

$$\mathrm{Im}\,\iota^* \simeq \{W \in S_{\eta_l,\tau_\kappa^*}(N\backslash G/K) \mid dR_X W = 0 \ \forall X \in \mathfrak{p}^-\}$$

by the method of highest weight module, where dR denotes the differential of right regular representation R of G on the space of C^∞-functions on G. The condition

$$dR_X W = 0 \quad \forall X \in \mathfrak{p}^-$$

is called the *Cauchy-Riemann condition.*

4.2. Explicit formulas for differential equations

From the Cauchy-Riemann condition, we obtain the differential equations characterizing the generalized Whittaker functions. Let $W_{\kappa,l}$ denote a generalized Whittaker function attached to π_κ and η_l. It is determined by its restriction to NA because of its K-equivariance. Furthermore recall that, for each $l \in \mathfrak{n}^*$, a polarization subalgebra $\mathfrak{m}$ is assumed to be taken so that $\mathfrak{m} \supset \mathfrak{n}_S$ (cf. §3, Assumption 1). Therefore we see that $W_{\kappa,l}(x_S x_L a) = \chi_l(x_S) W_{\kappa,l}(x_L a)$ with $(x_S, x_L, a) \in N_S \times N_L \times A$. Hence it suffices to consider the differential equations for the restriction of $W_{\kappa,l}$ to $N_L A$. In order to simplify the equations, we introduce the Euler operators $\partial_k := a_k \frac{\partial}{\partial a_k}$ for $1 \le k \le n$. Using the infinitesimal actions ∂_k and $d\eta_l$, the Cauchy-Riemann condition can be rewritten as the following differential equations:

PROPOSITION 4.2. (1) *The conditions* $dR_{F_{-e_i-e_j}} W_{\kappa,l} = 0$ *with* $1 \le i < j \le n$ *are equivalent to*

$$a_i a_j^{-1} d\eta_l(E_{e_i-e_j}) W_{\kappa,l} - \sqrt{-1} a_i a_j d\eta_l(E_{e_i+e_j}) W_{\kappa,l} = 0 \quad (1 \le i < j \le n).$$

(2) *The conditions* $dR_{F_{-2e_k}} W_{\kappa,l} = 0$ *with* $1 \le k \le n$ *are equivalent to*

$$\partial_k W_{\kappa,l} - 2\sqrt{-1} a_k^2 d\eta_l(E_{2e_k}) W_{\kappa,l} - \kappa W_{\kappa,l} = 0 \quad (1 \le k \le n).$$

PROOF. Note that the infinitesimal actions of $\mathfrak{k}$ via the differential $d\tau_\kappa$ of τ_κ are given as follows;

$$d\tau_\kappa(F_{\pm(f_i-f_j)}) = 0 \text{ for } 1 \le i < j \le n \quad d\tau_\kappa(T_k) = \sqrt{-1}\kappa \text{ for } 1 \le k \le n.$$

The formulas in the assertion follow from the Iwasawa decompositions of $F_{-e_i-e_j}$ and F_{-2e_k} in Lemma 1.1 and from the above formula of $d\tau_\kappa$. □

Inserting the formulas in Proposition 3.4 into Proposition 4.2, we get more explicit forms of the differential equations for $W_{\kappa,l}$.

PROPOSITION 4.3. (1) *The differential equations in Proposition* 4.2 (1) *are rewritten as*

$$\left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}} \right) W_{\kappa,l}(x_L a) + 2\pi a_j^2 \xi_{ij}({}^t X_L Y_n(l) X_L) W_{\kappa,l}(x_L a) = 0$$

for $1 \le i < j \le n$.
(2) *The differential equations in Proposition* 4.2 (2) *are rewritten as*

$$\partial_k W_{\kappa,l}(x_L a) + 4\pi a_k^2 \xi_{kk}({}^t X_L\, Y_n(l) X_L) W_{\kappa,l}(x_L a) - \kappa W_{\kappa,l}(x_L a) = 0$$

for $1 \le k \le n$.

4.3. Explicit formula of generalized Whittaker functions

In this subsection, we solve the differential equations in Proposition 4.3 and obtain explicit formulas of generalized Whittaker functions. To simplify the equations, we need

LEMMA 4.4. (1) *For* $1 \le i < j \le n$, *we have*

$$\left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}}\right) (\mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n))) \\ = a_j^2 \xi_{ij}({}^t X_L\, Y_n(l) X_L).$$

(2) *For* $1 \le k \le n$, *we have*

$$\partial_k \,\mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)) = 2a_k^2 \xi_{kk}({}^t X_L\, Y_n(l) X_L).$$

For notations X_L *and* A_n, *see the definition of* N_L *in* §3 *and the definition of* A *in* §1 *respectively.*

PROOF. (1) The proof of Proposition 3.4 (2) implies that a C^∞-function f on N_L satisfies

$$\left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}}\right) f(x_L) = \left.\frac{d}{dt}\right|_{t=0} f(x_L \exp(tE_{e_i - e_j})).$$

Apply this to $f(x_L) := \mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n))$ with a fixed A_n. Then we have

$$\left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}}\right) \mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)) \\ = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Tr}({}^t(X_L(1_n + tE_{ij})A_n)\, Y_n(l)(X_L(1_n + tE_{ij})A_n))$$

$$
\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Tr}({}^t X_L\, Y_n(l) X_L((1_n + tE_{ij})A_n)\, {}^t((1_n + tE_{ij})A_n)) \\
&= \lim_{t\to 0}\left(\frac{(a_i^2 + a_j^2 t^2) - a_i^2}{t}\xi_{ii}({}^t X_L\, Y_n(l) X_L) + \frac{a_j^2 t}{t}\xi_{ij}({}^t X_L\, Y_n(l) X_L)\right) \\
&= a_j^2 \xi_{ij}({}^t X_L\, Y_n(l) X_L).
\end{aligned}
$$

(2) This is settled by a calculation as follows;

$$
\begin{aligned}
\partial_k \,\mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)) &= a_k \frac{\partial}{\partial a_k}\mathrm{Tr}({}^t X_L\, Y_n(l) X_L A_n^2) \\
&= 2a_k^2 \xi_{kk}({}^t X_L\, Y_n(l) X_L). \; \square
\end{aligned}
$$

We first consider the generalized Whittaker functions attached to a representation η_l in the following case:

Case 1. $I(l) \neq [1, n]$, and for some (i, j) with $i \notin I(l)$ and $j > i$, $\xi'_{ij} \neq 0$ and $E_{e_i - e_j} \in \mathfrak{m}$ holds, i.e. $d\chi_l(E_{e_i - e_j}) = 2\pi\sqrt{-1}\xi'_{ij} \neq 0$.

THEOREM 4.5. *Let $W_{\kappa,l}$ be a generalized Whittaker functions attached to η_l in Case 1. Then we have $W_{\kappa,l} \equiv 0$.*

PROOF. We set $i(l) := \min\{i \notin I(l) \mid \xi'_{ij} \neq 0 \text{ and } E_{e_i - e_j} \in \mathfrak{m} \text{ for some } j\}$ and $j(l) := \max\{j \mid \xi'_{i(l)j} \neq 0 \text{ and } E_{e_{i(l)} - e_j} \in \mathfrak{m}\}$ for $l \in \mathfrak{n}^*$. Furthermore, we give the following order for the set $I := \{(i, j) \in [1, n] \times [1, n] \mid 1 \leq i < j \leq n\}$:

$$
\begin{aligned}
&(i, j) > (i', j') \quad \text{for any } (j, j') \text{ if } i > i' \\
&(i, j) > (i, j') \quad \text{if } j' > j,
\end{aligned}
$$

and define a subset $I_{i(l)j(l)}$ of I by

$$
I_{i(l)j(l)} := \{(i, j) \in I \mid 1 \leq i < i(l),\ j > i\} \cup \{(i(l), j) \in I \mid j \geq j(l)\}.
$$

We set

$$
W'_{\kappa,l}(x_L a) := \exp(2\pi\,\mathrm{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)))W_{\kappa,l}(x_L a).
$$

Here we need

LEMMA 4.6. (1) *For $l \in \mathfrak{n}^*$, we have*

$$Y_n(\mathrm{Ad}^* x_L^{-1} \cdot l) = {}^t X_L \, Y_n(l) X_L$$

for $x_L = \begin{pmatrix} X_L & \\ & {}^t X_L{}^{-1} \end{pmatrix} \in N_L$.
(2) *The function $W'_{\kappa,l}(x_L a)$ satisfies $W'_{\kappa,l}(m_L x_L a) = \chi_l(m_L) W'_{\kappa,l}(x_L a)$ for $m_L \in M \cap N_L$.*

PROOF. (1) By direct computation, one obtains

$$\begin{aligned} &\mathrm{Ad}^* x_L^{-1} \cdot l \left(\begin{pmatrix} 0_n & X_S \\ 0_n & 0_n \end{pmatrix} \right) = l \left(\begin{pmatrix} 0_n & X_L X_S \, {}^t X_L \\ 0_n & 0_n \end{pmatrix} \right) \\ &= \mathrm{Tr}(Y_n(l) X_L X_S \, {}^t X_L) = \mathrm{Tr}({}^t X_L \, Y_n(l) X_L X_S). \end{aligned}$$

Lemma 3.5 means $Y_n(\mathrm{Ad}^* x_L^{-1} \cdot l) = {}^t X_L \, Y_n(l) X_L$.
(2) Since $\chi_l = \chi_{\mathrm{Ad}^* m^{-1} \cdot l}$ and $\chi_l|_{N_S} = \chi_{\mathrm{Ad}^* m^{-1} \cdot l}|_{N_S}$ for $m \in M$, we have $Y_n(l) = Y_n(\mathrm{Ad}^* m^{-1} \cdot l)$. This and the assertion (1) means that $\exp(2\pi \, \mathrm{Tr}({}^t(X_L A_n) \, Y_n(l)(X_L A_n)))$ is left $M \cap N_L$-invariant. Since $W_{\kappa,l}(m_L x_L a) = \chi_l(m_L) W_{\kappa,l}(x_L a)$ for $m_L \in M \cap N_L$, the assertion (2) holds. □

Inserting $W_{\kappa,l}(x_L a) = \exp(-2\pi \, \mathrm{Tr}({}^t(X_L A_n) \, Y_n(l)(X_L A_n))) W'_{\kappa,l}(x_L a)$ into the differerntial equations in Proposition 4.3 (1) and noting Lemma 4.4 (1), we get

$$\left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx_{x'_{uj}}} \right) W'_{\lambda,l}(x_L a) = 0 \quad (1 \le i < j \le n).$$

From these differential equations for $(i,j) \in I_{i(l)j(l)} \setminus \{(i(l), j(l))\}$, we observe that

$$\frac{d}{dx'_{ij}} W'_{\lambda,l} = 0 \quad \text{for any } (i,j) \in I_{i(l)j(l)} \setminus \{(i(l), j(l))\},$$

by induction on the order of $(i,j) \in I$. The validity of this is assured by the condition in Case 1, Lemma 4.6 (2) and Proposition 3.3 (1). In particular, Proposition 3.3 (1) implies that $\frac{d}{dx'_{ij}}$ for (i,j) with $i \in I(l)$ and $j > i$ is the non-trivial derivation in a direction $E_{e_i - e_j}$, transversal to $\mathfrak{m}$. Noting the formulas just above and Lemma 4.6 (2), we observe that the differential equation

$$\left(\frac{d}{dx'_{i(l)j(l)}} + \sum_{1 \le u < i(l)} x'_{ui(l)} \frac{d}{dx_{x'_{uj(l)}}}\right) W'_{\lambda,l}(x_L a) = 0$$

is equivalent to

$$\frac{d}{dx'_{i(l)j(l)}} W'_{\lambda,l}(x_L a) = 2\pi\sqrt{-1}\xi'_{i(l)j(l)} W'_{\lambda,l} = 0.$$

This implies $W_{\kappa,l} \equiv 0$. □

Now we consider the generalized Whittaker function attached to a representation η_l not in Case 1. Namely we assume that η_l is in the following case:

Case 2. η_l satisfies one of the following conditions:
(1) $I(l) = [1,n]$;
(2) $I(l) \neq [1,n]$, and for *any* (i,j) with $i \notin I(l)$ and $j > i$, $\xi'_{ij} = 0$ or $E_{e_i - e_j} \notin \mathfrak{m}$ holds.

THEOREM 4.7. *If η_l is in Case 2, we obtain the unique solution*

$$W_{\kappa,l}(x_L a) = C(a_1 a_2 \cdots a_n)^{\kappa} \exp(-2\pi \operatorname{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)))$$

of the differential equations in Proposition 4.3, up to an arbitrary constant C.

PROOF. Let $W'_{\kappa,l}(x_L a)$ and the set I be as in the proof of Theorem 4.5. Inserting $W_{\kappa,l}(x_L a) = \exp(-2\pi \operatorname{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n))) W'_{\kappa,l}(x_L a)$ into the differential equations (1) and (2) in Proposition 4.3 and noting Lemma 4.4, we obtain

$$(1)\ \left(\frac{d}{dx'_{ij}} + \sum_{1 \le u < i} x'_{ui} \frac{d}{dx'_{uj}}\right) W'_{\kappa,l}(x_L a) = 0 \quad (1 \le i < j \le n),$$

$$(2)\ \partial_k W'_{\kappa,l}(x_L a) - \kappa W'_{\kappa,l}(x_L a) = 0 \quad (1 \le k \le n).$$

From (1) and the two conditions in Case 2, we see that $\frac{d}{dx'_{ij}} W'_{\kappa,l}(x_L a) = 0$ for any $(i,j) \in I$ by induction on (i,j) with respect to the order of I given in the proof of Theorem 4.5. That is, $W'_{\kappa,l}(x_L a)$ does not depend on x_L. From (2), we see that $W'_{\kappa,l}(x_L a) = C(a_1 a_2 \cdots a_n)^\kappa$ with an arbitrary constant C. Eventually, we get the solution in the assertion. □

We consider a necessary and sufficient condition for the above solution to give the non-zero generalized Whittaker function. For that purpose, we state some lemmas:

LEMMA 4.8. (1) *Let $W_{\kappa,l}(*a)$ with a fixed $a \in A$ denote a function on N defined by $N \ni x \mapsto W_{\kappa,l}(xa)$. Consider the restriction $W_{\kappa,l}|_{N_L}(*a)$ of $W_{\kappa,l}(*a)$ to N_L, explicitly given in Theorem* 4.5, 4.7. *It satisfies*

$$W_{\kappa,l}|_{N_L}(m_L x_L a) = W_{\kappa,l}|_{N_L}(x_L a)$$

for any $m_L \in M \cap N_L$, i.e. $W_{\kappa,l}|_{N_L}(*a)$ defines a well-defined function on $M \cap N_L \backslash N_L$.
(2) *If there is a non-zero generalized Whittaker function $W_{\kappa,l}$ for η_l, the character χ_l inducing η_l has to satisfy $\chi_l(M \cap N_L) = \{1\}$.*

PROOF. The first assertion follows from the left $M \cap N_L$-invariance of $\exp(2\pi \operatorname{Tr}({}^t(X_L A_n)\, Y_n(l)(X_L A_n)))$, stated in the proof for Lemma 4.6 (2). The second assertion is an immediate consequence of the first. □

LEMMA 4.9. *Every $l \in \mathfrak{n}^*$ with the non-zero $Y_n(l)$ is* $\mathrm{Ad}^* N_L$*-equivalent to a linear form l' such that* $\det Y(l') \neq 0$.

PROOF. Let $X(i,j;c) := 1_n + cE_{ij}$ for $1 \le i < j \le n$ and $c \in \mathbb{R}$. With a suitable choice of a product X_0 of some $X(i,j;c)$'s, we can delete all non-zero linearly dependent column vectors and row vectors in $Y_n(l)$ by considering ${}^t X_0\, Y_n(l) X_0$. By Lemma 4.6 (1),

$${}^t X_0\, Y_n(l) X_0 = Y_n(\mathrm{Ad}^*\, x_0^{-1} \cdot l)$$

with $x_0 = \begin{pmatrix} X_0 & \\ & {}^t X_0{}^{-1} \end{pmatrix} \in N_L$. Therefore $\det Y(\mathrm{Ad}^*\, x_0^{-1} \cdot l) \neq 0$. We can take $\mathrm{Ad}^*\, x_0^{-1} \cdot l$ as l' in the assertion. □

LEMMA 4.10. *For all the assertions, assume that l has a polarization subalgebra $\mathfrak{m}$ such that $l(\mathfrak{m} \cap \mathfrak{n}_L) = \{0\}$. For the assertions* (1) *and* (2), *additionally assume that $l \in \mathfrak{n}^*$ satisfies $I(l) \neq \{n\}$.*
(1) *Let $l \in \mathfrak{n}^*$ with $n_r \neq n$ (resp. $n_r = n$) in $I(l)$ and the positive definite $Y(l)$ (resp. $Y(l)_{r-1}$). Then $\mathfrak{m}$ is equal to $\mathfrak{n}_S \oplus \mathfrak{n}_l$.*
(2) *Let $l \in \mathfrak{n}^*$ be $\mathrm{Ad}^* N_L$-equivalent to $l' \in \mathfrak{n}^*$ with $n_r \neq n$ (resp. $n_r = n$) in $I(l')$ and the positive definite $Y(l')$ (resp. $Y(l')_{r-1}$). Then, for any $x'_L \in N_L$ such that $\mathrm{Ad}^* x'_L \cdot l = l'$, we have $\mathfrak{m} = \mathrm{Ad}\, {x'_L}^{-1} \cdot (\mathfrak{n}_S \oplus \mathfrak{n}_{l'})$.*
(3) *The condition $I(l) = \{n\}$ holds for l if and only if $l = \xi_{nn} l_{nn}$. For such an l, $\mathfrak{m}$ is equal to $\mathfrak{n}$.*

PROOF. (1) We prove $l(\mathfrak{n}_l) = \{0\}$. Then $\mathfrak{m} = \mathfrak{n}_S \oplus \mathfrak{n}_l$ holds by Proposition 3.3 (2). When $I(l) = [1, n]$, there is nothing to prove since $\mathfrak{n}_l = \{0\}$. We assume $I(l) \neq [1, n]$. We prove $\xi'_{ij} = 0$ for any (i, j) with $i \notin I(l)$ and $j > i$, which means $l(\mathfrak{n}_l) = \{0\}$. Let $\xi'_{ij} \neq 0$ for some (i, j) with $i \notin I(l)$ and $j > i$. Set $i(l)' := \min\{i \notin I(l) \mid \xi'_{ij} \neq 0 \text{ for some } j\}$ and $j(l)' := \max\{j \mid \xi'_{i(l)'j} \neq 0\}$. Since $l(\mathfrak{m} \cap \mathfrak{n}_L) = \{0\}$ by the assumption, $E_{e_{i(l)'} - e_{j(l)'}} \notin \mathfrak{m}$. We can check that $l([E_{e_{i(l)'} - e_{j(l)'}}, \mathfrak{n}_S \oplus \mathfrak{n}_l]) = \{0\}$ by direct computation, and Proposition 3.3 (2) says that $\mathfrak{m} \subset \mathfrak{n}_S \oplus \mathfrak{n}_l$. Hence $\mathfrak{m} \oplus \mathbb{R} E_{e_{i(l)'} - e_{j(l)'}}$ forms an isotropic subspace with respect to an inner product $l([*, *])$. But this contradicts the maximality of $\mathfrak{m}$ as an isotropic subspace. Therefore $\xi'_{ij} = 0$ for any (i, j) with $i \notin I(l)$ and $j > i$.
(2) Let $x'_L \in N_L$ be in the assertion (2) and a pair $(l, \mathfrak{m})$ satisfy the condition in the assertion (2). Then $(\mathrm{Ad}^* {x'}_L \cdot l, \mathrm{Ad}\, x'_L \cdot (\mathfrak{m}))$ forms a pair with the condition in the assertion (1). Hence we have

$$\mathrm{Ad}\, x'_L \cdot (\mathfrak{m}) = \mathfrak{n}_S \oplus \mathfrak{n}_{\mathrm{Ad}^* x'_L \cdot l}$$

that is,

$$\mathfrak{m} = \mathrm{Ad}\, {x'_L}^{-1} \cdot (\mathfrak{n}_S \oplus \mathfrak{n}_{l'}).$$

(3) If $l = \xi_{nn} l_{nn}$, clearly $I(l) = \{n\}$ holds. Conversely, assume that l satisfies $I(l) = \{n\}$. We prove $\xi'_{ij} = 0$ for $1 \leq i < j \leq n$. Then we get $l = \xi_{nn} l_{nn}$. Let some $\xi'_{ij} \neq 0$. Set $i(l)'' := \min\{i \in [1, n] \mid \xi'_{ij} \neq 0 \text{ for some } j > i\}$ and $j(l)'' := \max\{j \in [1, n] \mid \xi'_{i(l)''j} \neq 0\}$. The assumption $l(\mathfrak{m} \cap \mathfrak{n}_L) = \{0\}$ means $E_{e_{i(l)''} - e_{j(l)''}} \notin \mathfrak{m}$. We check $l([E_{e_{i(l)''} - e_{j(l)''}}, \mathfrak{n}_L]) = 0$ by direct calculation, and $l([E_{e_{i(l)''} - e_{j(l)''}}, \mathfrak{n}_S]) = \{0\}$ by the assumption $I(l) = \{n\}$. Therefore $\mathfrak{m} \oplus \mathbb{R} E_{e_{i(l)''} - e_{j(l)''}}$ forms an isotropic subspace with respect to an inner

product $l([*,*])$. But this contradicts the maximality of $\mathfrak{m}$ as an isotropic subspace. Hence any $\xi'_{ij}=0$. For a linear form $l=\xi_{nn}l_{nn}$, we check that $l([\mathfrak{n},\mathfrak{n}])=\{0\}$ by direct computation. Hence $\mathfrak{m}=\mathfrak{n}$. □

LEMMA 4.11. *Let $(l,\mathfrak{m})$ be a pair of a linear form l and a polarization subalgebra $\mathfrak{m}$, satisfying $I(l)\neq\{n\}$ and $l(\mathfrak{m}\cap\mathfrak{n}_L)=\{0\}$. For $n_1<i\le n$, we define $k(i):=\max\{p\mid n_p\in I(l),\ n_p<i\}$ and introduce a coordinate $\mathbf{x}_i:=(x'_{n_1,i},x'_{n_2,i},\dots,x'_{n_{k(i)},i})$ of $\mathbb{R}^{k(i)}$.*
(1) Let $X(\mathbf{x}_i):=x'_{n_1i}E_{n_1i}+x'_{n_2i}E_{n_2i}+\cdots+x'_{n_{k(i)}i}E_{n_{k(i)}i}$ with $\mathbf{x}_i\in\mathbb{R}^{k(i)}$. If l satisfies $n_r\neq n$ (resp. $n_r=n$) in $I(l)$ and $\det Y(l)_s\neq 0$ for $1\le s\le r$ (resp. $1\le s\le r-1$), the set

$$\left\{\begin{pmatrix}1_n+\sum_{n_1<i\le n}X(\mathbf{x}_i) & \\ & {}^t(1_n+\sum_{n_1<i\le n}X(\mathbf{x}_i))^{-1}\end{pmatrix}\in N_L \;\middle|\; \mathbf{x}_i\in\mathbb{R}^{k(i)}\right\}$$

is bijective with the quotient $M\cap N_L\backslash N_L\simeq M\backslash N$. Hence an invariant measure $d\dot{x}$ on $M\backslash N$ can be written as $\prod_{\substack{n_p\in I(l),j\\ s.t. j>n_p}}dx'_{n_pj}$ up to constant multiple. In particular, this assertion holds if l satisfies $n_r\neq n$ (resp. $n_r=n$) in $I(l)$ and the positive-definiteness $Y(l)$ (resp. $Y(l)_{r-1}$).
(2) Each diagonal entry of ${}^t(X_LA_n)\,Y_n(l)(X_LA_n)$ is given by

$$\begin{cases}0 & (i<n_1);\\ a_{n_1}^2\xi_{n_1n_1} & (i=n_1);\\ a_{n_p}^2(\mathbf{x}_{n_p},1)Y(l)_p\,{}^t(\mathbf{x}_{n_p},1) & \\ =a_{n_p}^2(\mathbf{x}_{n_p}Y(l)_{p-1}\,{}^t\mathbf{x}_{n_p}+2\mathbf{y}_{p-1}\,{}^t\mathbf{x}_{n_p}+\xi_{n_pn_p}) & (i=n_p\in I(l)\setminus\{n_1\});\\ a_i^2\mathbf{x}_iY(l)_{k(i)}\,{}^t\mathbf{x}_i & (i\notin I(l),\ i>n_1),\end{cases}$$

where we write $Y(l)_p=\begin{pmatrix}Y(l)_{p-1} & {}^t\mathbf{y}_{p-1}\\ \mathbf{y}_{p-1} & \xi_{n_pn_p}\end{pmatrix}$ with $\mathbf{y}_{p-1}=(\xi_{n_1n_p}/2,\xi_{n_2n_p}/2,\dots,\xi_{n_{p-1}n_p}/2)$. The explicit formula of $W_{\kappa,l}(x_La)$ can be written as

$$\begin{aligned}&(a_1a_2\cdots a_n)^\kappa\exp(-2\pi a_{n_1}^2\xi_{n_1n_1})\\ &\quad\times\prod_{n_p\in I(l)\setminus\{n_1\}}\exp(-2\pi a_{n_p}^2(\mathbf{x}_{n_p},1)Y(l)_p\,{}^t(\mathbf{x}_{n_p},1))\\ &\quad\times\prod_{\substack{i\notin I(l)\\ s.t. i>n_1}}\exp(-2\pi a_i^2\mathbf{x}_iY(l)_{k(i)}\,{}^t\mathbf{x}_i),\end{aligned}$$

up to constant multiple.
(3) *If $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable on $M \cap N_L \backslash N_L$, $Y(l)_{k(i)}$ has to be positive semi-definite for $n_1 < i \leq n$.*

Proof. The assertion (1) follows from Proposition 3.3 (2), and the assertion (2) is obtained by direct computation. We prove the assertion (3). The square-integrability of $W_{\kappa,l}|_{N_L}(*a)$ means that, for any $n_1 < i \leq n$, $\mathbf{x}_i Y(l)_{k(i)} {}^t\mathbf{x}_i$ has to be non-negative, i.e. $Y(l)_{k(i)}$ is positive semi-definite. In fact, otherwise, there exists a non-zero $\mathbf{x}_i^0 \in \mathbb{R}^{k(i)}$ such that $\mathbf{x}_i^0 Y(l)_{k(i)} {}^t\mathbf{x}_i^0$ is negative for some $n_1 < i \leq n$. Noting the formula of the diagonal entries of ${}^t(X_L A_n) Y_n(l)(X_L A_n)$ and the formula of $W_{\kappa,l}(x_L a)$ in the assertion (2), we see that $W_{\kappa,l}|_{N_L}(*a)$ is neither trivial nor square-integrable on

$$\left\{ \begin{pmatrix} 1_n + X(\mathbf{x}_i) & \\ & {}^t(1_n + X(\mathbf{x}_i))^{-1} \end{pmatrix} \middle| \; \mathbf{x}_i \in \mathbb{R} \cdot \mathbf{x}_i^0 \right\} \subset N_L.$$

Therefore we obtain the assertion (3). □

Theorem 4.5 and 4.7 tell us that $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\kappa, C^\infty_{\eta_l}(N\backslash G)_K) \leq 1$. To be more precise, we have

Theorem 4.12. $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\kappa, C^\infty_{\eta_l}(N\backslash G)_K) = 1$ *holds if and only if η_l satisfies one of the following conditions:*
(a) *$\eta_l = \chi_l$ and $\chi_l(M \cap N_L) = \{1\}$, i.e. $l = \xi_{nn} l_{nn}$.*
(b) *$\eta_l \neq \chi_l$, $\chi_l(M \cap N_L) = \{1\}$ and l is $\mathrm{Ad}^* N_L$-equivalent to $l' \in \mathfrak{n}^*$ such that $n_r \neq n$ in $I(l')$ and $Y(l')$ is positive definite, or to $l' \in \mathfrak{n}^*$ such that $n_r = n$ in $I(l')$ and $Y(l')_{r-1}$ is positive definite.*

For the condition (a), remark that $\eta_l = \chi_l$ means $\mathfrak{m} = \mathfrak{n}$ and that $\eta_l = \chi_l$ if and only if non-zero is at least one of the parameters ξ_{nn} and $\xi'_{i,i+1}$ of l with $1 \leq i < n$, which correspond to the simple roots of the restricted root system $\Delta(\mathfrak{a}, \mathfrak{g})$. We see that $\eta_l = \chi_l$ and $\chi_l(N_L \cap M) = \{1\}$ if and only if $l = \xi_{nn} l_{nn}$.

Proof. By virtue of Lemma 4.8 (2), it suffices to consider only the case where η_l satisfies $\chi_l(M \cap N_L) = \{1\}$. Therefore we assume this throughout our proof. Here remark that $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, C^\infty_{\eta_l}(N\backslash G)_K) = 1$ if and only if $W_{\kappa,l}(*a) \in H^\infty_{\eta_l}$ for any fixed $a \in A$. Hence, under the condition $\chi_l(M \cap N_L) = \{1\}$, it suffices to prove

The condition (a) or (b) on η_l in the assertion holds if and only if $W_{\kappa,l}(*a) \in H^\infty_{\eta_l}$ for any fixed $a \in A$.

Here recall that $W_{\kappa,l}(*a)$ denotes a function on N defined by $N \ni x \mapsto W_{\kappa,l}(xa)$ (cf. Lemma 4.8 (1)).

For any $\eta_l = \chi_l$ with $\chi_l(M \cap N_L) = \{1\}$, $W_{\kappa,l}(*a)$ with any fixed $a \in A$ belongs to $H^\infty_{\eta_l} \simeq \mathbb{C}$ as its explicit formula

$$C(a_1 a_2 \ldots a_n)^\kappa \exp(-2\pi a_n^2 \xi_{nn})$$

of $W_{\kappa,l}|_{N_L}(*a)$ in Theorem 4.7 indicates. Therefore $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\Lambda, C^\infty_{\eta_l}(N\backslash G)) = 1$ for such $\eta_l = \chi_l$.

Let $\eta_l \neq \chi_l$. This means $I(l) \neq \{n\}$ under our assumption $\chi_l(M \cap N_L) = \{1\}$. In fact, $\eta_l = \chi_l$ and $\chi_l(N_L \cap M) = \{1\}$ if and only if $l = \xi_{nn} l_{nn}$ as is remarked in the assertion, and we see from this and Lemma 4.10 (3) that $\eta_l = \chi_l$ if and only if $I(l) = \{n\}$ under the assumption $\chi_l(N_L \cap M) = \{1\}$. For any fixed $a \in A$, $W_{\kappa,l}(*a) \in H^\infty_{\eta_l}$ holds if and only if $W_{\kappa,l}|_{N_L}(*a)$ is a square-integrable function on the quotient $M \cap N_L \backslash N_L$. We prove that such square-integrability condition on $W_{\kappa,l}|_{N_L}(*a)$ is equivalent to the condition (b) in the assertion.

First we assume that $\det Y(l) \neq 0$. We prove that the square-integrability condition on $W_{\kappa,l}|_{N_L}(*a)$ holds if and only if

$$\begin{cases} Y(l) \text{ is positive definite} & \text{when } n_r \neq n \text{ in } I(l), \\ Y(l)_{r-1} \text{ is positive definite} & \text{when } n_r = n \text{ in } I(l). \end{cases}$$

Let $n_r \neq n$ in $I(l)$ and $\det Y(l) \neq 0$. If $Y(l)$ is positive definite, we see from Lemma 4.11 (1), (2) that $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable on $M \cap N_L \backslash N_L$. Conversely, if such square-integrability condition on $W_{\kappa,l}|_{N_L}(*a)$ holds, Lemms 4.11 (3) and the assumption $\det Y(l) \neq 0$ means that $Y(l)$ is positive definite.

Let $n_r = n$ in $I(l)$ and $\det Y(l) \neq 0$. Assume that $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable on $M \cap N_L \backslash N_L$. Here recall the formula of the n-th diagonal entry of ${}^t(X_L A)\, Y_n(l)(X_L A_n)$ in Lemma 4.11 (2):

$$a_{n_r}^2(\mathbf{x}_{n_r} Y(l)_{r-1}\, {}^t\mathbf{x}_{n_r} + 2\mathbf{y}_{r-1}\, {}^t\mathbf{x}_{n_r} + \xi_{n_r n_r}).$$

By this formula, we check that $Y(l)_{r-1}$ is positive definite. In fact, the square-integrability condition on $W_{\kappa,l}|_{N_L}(*a)$ and Lemma 4.11 (3) means that $Y(l)_{r-1}$ is positive semi-definite. Let $\det Y(l)_{r-1} = 0$. Then there exists a non-zero $\mathbf{x}^0_{n_r} \in \mathbb{R}^{r-1}$ such that $\mathbf{x}^0_{n_r} Y(l)_{r-1} = (0, \ldots, 0)$. If $\mathbf{y}_{r-1} {}^t\mathbf{x}^0_{n_r} = 0$, this contradicts the assumption $\det Y(l) \neq 0$. If $\mathbf{y}_{r-1} {}^t\mathbf{x}^0_{n_r} \neq 0$, we see that $W_{\kappa,l}|_{N_L}(*a)$ is neither trivial nor square-integrable on

$$\left\{ \begin{pmatrix} 1_n + X(\mathbf{x}_{n_r}) & \\ & {}^t(1_n + X(\mathbf{x}_{n_r}))^{-1} \end{pmatrix} \middle| \ \mathbf{x}_{n_r} \in \mathbb{R} \cdot \mathbf{x}^0_{n_r} \right\} \subset N_L,$$

by noting the above formula of the n-th diagonal entry and the formula of $W_{\kappa,l}(x_L a)$ in Lemma 4.11 (2). Therefore $\det Y(l)_{r-1} \neq 0$, hence $Y(l)_{r-1}$ is positive definite.

Conversely, let $Y(l)_{r-1}$ be positive definite. Then Lemma 4.11 (1) is valid. By the formula of $W_{\kappa,l}(x_L a)$ in Lemma 4.11 (2), we check that $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable with respect to $\mathbf{x}_i$ for $n_1 < i < n$. For $\mathbf{x}_n = \mathbf{x}_{n_r}$, note that the n-th diagonal entry is written as

$$a^2_{n_r}((\mathbf{x}_{n_r} + \mathbf{y}'_{r-1})Y(l)_{r-1} {}^t(\mathbf{x}_{n_r} + \mathbf{y}'_{r-1}) + (-\mathbf{y}'_{r-1}, 1)Y(l)_{r-1} {}^t(-\mathbf{y}'_{r-1}, 1)),$$

where $\mathbf{y}'_{r-1} := \mathbf{y}_{r-1} Y(l)^{-1}_{r-1}$. This is checked by direct calculation. From the positive-definiteness of $Y(l)_{r-1}$, we see that $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable with respect to $\mathbf{x}_{n_r}$, which means that it is square-integrable with respect to all $\mathbf{x}_i$. Therefore $W_{\kappa,l}|_{N_L}(*a)$ is square-integrable on $M \cap N_L \backslash N_L$.

Next assume that $\eta_l \neq \chi_l$ and $\det Y(l) = 0$. Remark that $Y_n(l) \neq 0_n$ since $\eta_l \neq \chi_l$ means $I(l) \neq \{n\}$ as is noted above. By Lemma 4.9, we can take an $x'_L \in N_L$ so that $\det Y(\mathrm{Ad}^* x'_L \cdot l) \neq 0$. Set $l' := \mathrm{Ad}^* x'_L \cdot l$. Since $W_{\kappa,l}(x_L a) = W_{\kappa,l'}(x'_L x_L a)$, the argument for the case $\det Y(l) \neq 0$ means that, when $n_r \neq n$ (resp. $n_r = n$) in $I(l')$, $W_{\kappa,l}$ is a generalized Whittaker function if and only if $Y(l')$ (resp. $Y(l')_{r-1}$) is positive definite. This implies the assertion for this case. As a result, we complete the proof. □

Recall that $\| * \|_l$ denote the norm on H_{η_l} (cf. Proposition 3.1 (1)), and let $\| * \|$ denote the norm on the matrix algebra $M_{2n}(\mathbb{R})$ defined by $\|Y\| := \mathrm{Tr}\, {}^tY\, Y$. Moreover, let $U(\mathfrak{g}_{\mathbb{C}})$ be the universal envelopping algebra of $\mathfrak{g}_{\mathbb{C}}$. Now we consider the space $\mathcal{A}_{\eta_l}(N\backslash G)$ of all $F \in C^\infty_{\eta_l}(N\backslash G)_K$ satisfying the *moderate growth condition*, i.e.

$$\|X \cdot F(g)\|_l < C\|g\|^m \quad \text{for any } g \in G \text{ and any } X \in U(\mathfrak{g}_{\mathbb{C}}),$$

where a constant C and an integer m depend only on F and X. This forms a $(\mathfrak{g}_{\mathbb{C}}, K)$-submodule of $C^\infty(N\backslash G)_K$.

THEOREM 4.13. $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\kappa, \mathcal{A}_{\eta_l}(N\backslash G)) = 1$ *if and only if* η_l *satisfies the condition that* $Y_n(l)$ *is positive semi-definite and that* $\chi_l(M \cap N_L) = \{1\}$.

PROOF. As we remarked in the beginning of the proof of Theorem 4.12, it suffices to consider the case where η_l satisfies $\chi_l(M \cap N_L) = \{1\}$. Note that, under this condition, $\dim_{\mathbb{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbb{C}},K)}(\pi_\kappa, \mathcal{A}_{\eta_l}(N\backslash G)) = 1$ is equivalent to the moderate growth condition of $W_{\kappa,l}$ in the following sense:

$$X \cdot W_{\kappa,l}(*a) \in H^\infty_{\eta_l}, \quad \|X \cdot W_{\kappa,l}(*a)\|_l < C\|a\|^m$$

for any $a \in A$ and any $X \in U(\mathfrak{g}_{\mathbb{C}})$, where a constant C and an integer m depend only on $W_{\kappa,l}$ and X. Assuming the condition $\chi_l(M \cap N_L) = \{1\}$ on η_l, we prove that this moderate growth condition of $W_{\kappa,l}(*a)$ holds if and only if η_l satisfies the condition that $Y_n(l)$ is positive semi-definite.

Let $\eta_l = \chi_l$. Then the explicit formula of $W_{\kappa,l}|_{N_L}(*a)$ implies that the moderate growth condition holds if and only if $\xi_{nn} \geq 0$, i.e. $Y_n(l)$ is positive semi-definite.

Let $\eta_l \neq \chi_l$. Recall that this implies $I(l) \neq \{n\}$ as we remarked in the 3-rd paragraph of the proof of Theorem 4.12. First assuming $\det Y(l) \neq 0$, we prove that the moderate growth condition of $W_{\kappa,l}$ holds if and only if $Y(l)$ is positive definite. If $Y(l)$ is positive definite, we can express $W_{\kappa,l}(x_L a)$ as

$$\begin{aligned}
&(a_1 a_2 \dots a_n)^\kappa \exp(-2\pi a_{n_1}^2 \xi_{n_1 n_1}) \\
&\times \prod_{n_p \in I(l)\backslash\{n_1\}} \exp(-2\pi a_{n_p}^2 (-\mathbf{y}'_{p-1}, 1) Y(l)_p \, {}^t(-\mathbf{y}'_{p-1}, 1)) \\
&\times \prod_{n_p \in I(l)\backslash\{n_1\}} \exp(-2\pi a_{n_p}^2 (\mathbf{x}_{n_p} + \mathbf{y}'_{p-1}) Y(l)_{p-1} \, {}^t(\mathbf{x}_{n_p} + \mathbf{y}'_{p-1})) \\
&\times \prod_{\substack{i \notin I(l) \\ \text{s.t. } i > n_1}} \exp(-2\pi a_i^2 \mathbf{x}_i Y(l)_{k(i)} \, {}^t\mathbf{x}_i)
\end{aligned}$$

up to constant multiple. Here see Lemma 4.11 for the notation $k(i)$ and we set $\mathbf{y}'_{p-1} := \mathbf{y}_{p-1} Y(l)_{p-1}^{-1}$. In fact, this also holds for l such that $n_r = n$ in $I(l)$ and $Y(l)_{r-1}$ is positive definite. From the positive-definiteness of $Y(l)$, we see that Lemma 4.11 (1) holds and that the exponential part of $W_{\kappa,l}(x_L a)$ is constant on $\{(a_1, a_2, \dots, a_{n_1-1}) \in \mathbb{R}_+^{n_1-1}\}$ but defines a Schwartz function on $(M \cap N_L \backslash N_L) \times \{(a_{n_1}, a_{n_1+1}, \dots, a_n) \in \mathbb{R}_+^{n-n_1+1}\}$. We can check that $W_{\kappa,l}$ satisfies the moderate growth condition.

Conversely, if the moderate growth condition is satisfied, $W_{\kappa,l}(*a) \in H_{\eta_l}^\infty$ and $\|W_{\kappa,l}(*a)\|_l < C\|a\|^m$ hold with a constant C and an integer m depending only on $W_{\kappa,l}$. Theorem 4.12 and $W_{\kappa,l}(*a) \in H_{\eta_l}^\infty$ means that $Y(l)$ (resp. $Y(l)_{r-1}$) is positive definite if l satisfies $n_r \neq n$ (resp. $n_r = n$) in $I(l)$. Hence it suffices to consider the case where l satisfies $n_r = n$ in $I(l)$ and the positive-definiteness of $Y(l)_{r-1}$. For such an l, Lemma 4.11 (1) and the expression of $W_{\kappa,l}(x_L a)$ in the preceding paragraph are valid. Here note the formula

$$a_{n_r}^2((\mathbf{x}_{n_r} + \mathbf{y}'_{r-1})Y(l)_{r-1}\,{}^t(\mathbf{x}_{n_r} + \mathbf{y}'_{r-1}) + (-\mathbf{y}'_{r-1}, 1)Y(l)_{r-1}\,{}^t(-\mathbf{y}'_{r-1}, 1))$$

of the n-th diagonal entry of ${}^t(X_L A_n)\, Y_n(l)(X_L A_n)$, which is also stated in the 7-th paragraph of the proof of Theorem 4.12. The positive-definiteness of $Y(l)_{r-1}$ means the non-negativity of $(\mathbf{x}_{n_r} + \mathbf{y}'_{r-1})Y(l)_{r-1}\,{}^t(\mathbf{x}_{n_r} + \mathbf{y}'_{r-1})$, and the condition $\|W_{\kappa,l}(*a)\|_l < C\|a\|^m$ the non-negativity of $(-\mathbf{y}'_{r-1}, 1)Y(l)_{r-1}\,{}^t(-\mathbf{y}'_{r-1}, 1)$. Hence the n-th diagonal entry has to be non-negative. It tells us that ${}^t\mathbf{x}\, Y(l)\mathbf{x} \geq 0$ for a column vector $\mathbf{x}$ with r entries and with non-zero r-th component. By this condition and the positive definiteness of $Y(l)_{r-1}$, we check that $Y(l)$ is positive definite.

For $\eta_l \neq \chi_l$ with $\det Y(l) = 0$, we see that the problem is reduced to the previous case by the same reasoning as in the last paragraph of the proof of Theorem 4.12. Indeed, $l \in \mathfrak{n}^*$ has the non-zero positive semi-definite $Y_n(l)$ if and only if l is $\mathrm{Ad}_S^* N_L$-equivalent to $l' \in \mathfrak{n}^*$ with the positive definite $Y(l')$. Hence we obtain the result. □

Remark 4.14. Theorem 4.13 corresponds to the "Koecher principle" for holomorphic Siegel modular forms (cf. [7], Satz 1, Satz 2).

By reviewing the proof of Theorem 4.13, we have

Corollary 4.15. *Assume* $\chi_l(M \cap N_L) = \{1\}$. *Then* η_l *satisfies the positive-semi-definiteness of* $Y_n(l)$ *if and only if*

$$W_{\kappa,l}(*a) \in H_{\eta_l}^\infty, \ \|W_{\kappa,l}(*a)\|_l < C\|a\|^m$$

with the constant C *and the integer* m *depending only on* $W_{\kappa,l}$.

5. Construction of the Fourier Expansion

From now on, let $\Gamma := Sp(n;\mathbb{Z})$, $N_{\mathbb{Z}} := N \cap \Gamma$, $N_S(\mathbb{Z}) := N_S \cap \Gamma$ and $N_L(\mathbb{Z}) := N_L \cap \Gamma$. Furthermore let $N_{\mathbb{Q}}$ and $N_L(\mathbb{Q})$ denote the groups of $\mathbb{Q}$-rational points in N and in N_L, respectively. We recall a definition of $\mathbb{C}$-valued holomorphic Siegel modular form with respect to Γ.

Definition 5.1. Let $\kappa > n$ be an integer. A C^∞-function $f : G \to \mathbb{C}$ is called a holomorphic Siegel modular form of weight κ with respect to Γ if it satisfies
(1) $f(\gamma g k) = \det(A + \sqrt{-1}B)^\kappa f(g)$ for any $(\gamma, g, k) \in \Gamma \times G \times K$, where we denote k by $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.
(2) f satisfies the Cauchy-Riemann condition, i.e.

$$dR_X f = 0 \quad \text{for any } X \in \mathfrak{p}^-,$$

where dR denotes the differential of right regular representation R of G on the space of C^∞-functions on G.

Here remark that the weight κ satisfies $\kappa n \equiv 0 \mod 2$ since $-1_{2n} \in \Gamma \cap K$ and that, if $n = 1$, we have to add the moderate growth condition (for a definition, see the remark just before Lemma 5.7) of f to the definition above.

We formulate the Fourier expansion of a modular form f along the minimal parabolic subgroup. For a fixed g, we can regard $f(xg)$ as a function in $x \in N$. Note $f(xg) \in L^2(N_{\mathbb{Z}}\backslash N)$, where $L^2(N_{\mathbb{Z}}\backslash N)$ denotes the space of square-integrable functions on the quotient $N_{\mathbb{Z}}\backslash N$. Since $N_{\mathbb{Z}}\backslash N$ is compact, we have

Proposition 5.2. *The space $L^2(N_{\mathbb{Z}}\backslash N)$ decomposes discretely into*

$$L^2(N_{\mathbb{Z}}\backslash N) \simeq \tilde{\bigoplus}_{(\eta,H_\eta)\in\hat{N}} m(\eta) H_\eta \simeq \tilde{\bigoplus}_{(\eta,H_\eta)\in\hat{N}} \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N)) \otimes H_\eta,$$

where H_η denotes a representation space of η, $\tilde{\bigoplus}$ the Hilbert space direct sum and $m(\eta) = \dim_{\mathbb{C}} \mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N)) < \infty$

For a proof, see [5],Chap I,§2.3.

Let $\{\Theta^{\eta}_m\}_{1\le m\le m(\eta)}$ be a basis of $\mathrm{Hom}_N(\eta, L^2(N_{\mathbb{Z}}\backslash N))$. According to the decomposition above, $f(xg)$ decomposes into

$$f(xg)=\sum_{\eta\in\hat{N}}\sum_{1\le m\le m(\eta)}\Theta^{\eta}_m(W^f_{\eta,m}(g))(x),$$

where $W^f_{\eta,m}(g)\in H_\eta$ denotes the (η,m)-component of $f(xg)$ and we regard $\Theta^{\eta}_m(W^f_{\eta,m}(g))$ as a function on N. For two x_1, $x_2\in N$, we deduce

$$\Theta^{\eta}_m(W^f_{\eta,m}(g))(x_1x_2)=\Theta^{\eta}_m(\eta(x_2)W^f_{\eta,m}(g))(x_1)$$

from the N-equivariance of Θ^{η}_m, and

$$\Theta^{\eta}_m(W^f_{\eta,m}(g))(x_1x_2)=\Theta^{\eta}_m(W^f_{\eta,m}(x_2g))(x_1)$$

from the trivial formula $f(x_1x_2\cdot g)=f(x_1\cdot x_2g)$. Therefore we obtain

$$W^f_{\eta,m}(xg)=\eta(x)W^f_{\eta,m}(g)$$

for any $x\in N$. From the right-K-equivariance of f and the holomorphy of f, we find $W^f_{\eta,m}(g)$ a generalized Whittaker function for holomorphic discrete series π_κ with K-type τ_κ (for the notations π_κ and τ_κ, see §2). We have already obtained the explicit formula of $W^f_{\eta,m}$.

Our remaining work is to determine the dimension $m(\eta)$ and a basis of the space $\mathrm{Hom}_N(\eta, L^2(N_\Gamma\backslash N))$. For such determination, we recall some results established by L.Corwin and F.P.Greenleaf [2]. The paper treats a spectral decomposition of $L^2\text{-}\mathrm{Ind}_{N_\Gamma}^N\,\rho$, where this time N is a general simply connected nilpotent Lie group with some $\mathbb{Q}$-rational structure and ρ denotes a character on a uniform discrete subgroup N_Γ in N. There is another result by L.Richardson [13], which treats only the special case where ρ is trivial. We first state their result on $m(\eta)$.

PROPOSITION 5.3. *In this assertion, we do not assume the assumption* 1 (*cf.* §3) *for the polarization subalgebras* $\mathfrak{m}$ *and* $\mathfrak{m}_0$.
(1) *If* η_l *occurs in* $L^2(N_{\mathbb{Z}}\backslash N)$, *a coadjoint orbit* $\mathrm{Ad}^*N\cdot l$ *contains a* $\mathbb{Q}$-*rational* $l'\in\mathfrak{n}^*$, *i.e.* $l'(\log(N_{\mathbb{Z}}))\subset\mathbb{Q}$.
(2) *We define, by* $\mathrm{Ad}^*x\cdot(\chi_0,M_0):=(\chi_0(x^{-1}*x),xM_0x^{-1})$ *with* $x\in N$, *the action* Ad^* *of* N *on the set of pairs* (χ_0,M_0), *where* $M_0:=\exp(\mathfrak{m}_0)$

with a polarrization subalgebra $\mathfrak{m}_0$ *for some linear form in* $\mathfrak{n}^*$, *and* χ_0 *is a character on* M_0.

Let $l \in \mathfrak{n}^*$ *be* $\mathbb{Q}$*-rational and* $M := \exp(\mathfrak{m})$ *with a* $\mathbb{Q}$*-rational polarization subalgebra* $\mathfrak{m}$ *for* l, *i.e.* $\mathfrak{m} \cap \mathfrak{n}_{\mathbb{Q}}$ *forms a* $\mathbb{Q}$*-sructure of* $\mathfrak{m}$ *with* $\mathfrak{n}_{\mathbb{Q}} := \mathbb{Q}$*-span of* $\{\log(N_{\mathbb{Z}})\}$ (*the existence of such an* $\mathfrak{m}$ *is proved in* [3], *Proposition* 5.2.6).

Let η_l *be induced from* (χ_l, M) *with* l *and* M *above and* $O(\eta_l)_{\mathbb{Z}} := \{(\chi_{l'}, M') \in \operatorname{Ad}^* N_{\mathbb{Q}} \cdot (\chi_l, M) \mid \chi_{l'}(N_{\mathbb{Z}} \cap M') = \{1\}\}$. *Then the representation* η_l *occurs in* $L^2(N_{\mathbb{Z}}\backslash N)$ *if and only if* $O(\eta_l)_{\mathbb{Z}}$ *is non-empty. The multiplicity* $m(\eta_l)$ *of the representation* η_l *in* $L^2(N_{\mathbb{Z}}\backslash N)$ *is equal to the cardinality of* $\mathfrak{M}(\eta_l) := O(\eta_l)_{\mathbb{Z}}/\operatorname{Ad}^* N_{\mathbb{Z}}$.

For a proof, see Theorem 5.1 in [2].

Let $\mathfrak{n}_S^* := \{l \in \mathfrak{n}^* \mid l(\mathfrak{n}_L) = \{0\}\}$. Let Ad_S^* denote the coadjoint action of N on $\mathfrak{n}_S^*$ and also denote the action of N on the set of pairs (l, M) with $l \in \mathfrak{n}_S^*$ and an associated polarization subgroup $M := \exp(\mathfrak{m})$, defined by

$$\operatorname{Ad}_S^* x \cdot (l, M) := (\operatorname{Ad}_S^* x \cdot l, xMx^{-1})$$

with $x \in N$. Remark that both actions satisfy the triviality of $\operatorname{Ad}_S^*|_{N_S}$.

Theorem 4.12 and Proposition 5.3 implies that a representation η_l occuring in the Fourier expansion is attached to $(l, \mathfrak{m})$ in (1), (2) or (3) of Lemma 4.10 with a $\mathbb{Q}$-rational l. There is no loss of generality if we assume $l \in \mathfrak{n}_S^*$. In fact, we check that, under the assumption 1, the polarization subalgebra $\mathfrak{m}$ for l in Lemma 4.10 (1), (2) or (3) such that $l(\mathfrak{m} \cap \mathfrak{n}_L) = \{0\}$ coincides with the polarization subalgebra for $l' \in \mathfrak{n}_S^*$ with $Y_n(l') = Y_n(l)$, which implies $\eta_l = \eta_{l'}$. From now on, we assume

Assumption 2. *l is in* $\mathfrak{n}_S^*$, $\mathbb{Q}$*-rational and satisfies the condition in Lemma 4.10 (1), (2) or (3).*

For a $\mathbb{Q}$-rational $l \in \mathfrak{n}_S^*$ with the condition in Lemma 4.10 (1) (resp. Lemma 4.10 (3)), the polarization subalgebra $\mathfrak{m}$ is $\mathbb{Q}$-rational as its explicit form $\mathfrak{m} = \mathfrak{n}_S \oplus \mathfrak{n}_l$ (resp. $\mathfrak{m} = \mathfrak{n}$) indicates. For a $\mathbb{Q}$-rational $l \in \mathfrak{n}_S^*$ with the condition in Lemma 4.10 (2), the polarization subalgebra $\mathfrak{m}$ is of the form $\operatorname{Ad}_S^* {x'_L}^{-1} \cdot (\mathfrak{n}_S \oplus \mathfrak{n}_{l'})$, where $x'_L \in N_L(\mathbb{Q})$ and $l' \in \mathfrak{n}_S^*$ with the condition in Lemma 4.10 (1). Hence $\mathfrak{m}$ is $\mathbb{Q}$-rational.

PROPOSITION 5.4. *Let $l \in \mathfrak{n}_S^*$ satisfy the assumption 2. Then we have the following identifications:*

$$O(\eta_l)_{\mathbb{Z}} \simeq O(l)_{\mathbb{Z}} := \{l' \in \mathrm{Ad}_S^* N_L(\mathbb{Q}) \cdot l \mid l'(\log(N_S(\mathbb{Z}))) \subset \mathbb{Z}\};$$
$$\mathfrak{M}(\eta_l) \simeq \mathfrak{M}(l) := O(l)_{\mathbb{Z}} / \mathrm{Ad}_S^* N_L(\mathbb{Z}).$$

PROOF. Characters χ_l for $l \in \mathfrak{n}_S^*$ are determined by its restriction to N_S. The set of characters on N_S is in bijection with $\mathfrak{n}_S^*$. Therefore, in the expression of $O(\eta_l)_{\mathbb{Z}}$ in Proposition 5.3 (2), we may replace $\chi_{l'}$ and χ_l by l' and l respectively. Since the action of N on χ_l via Ad^* is identical with the action of it on $l \in \mathfrak{n}_S^*$ via Ad_S^*, we can replace Ad^* by Ad_S^*. Under such replacement, we have a bijection

$$O(\eta_l)_{\mathbb{Z}} \simeq \{(l', M') \in \mathrm{Ad}_S^* N_L(\mathbb{Q}) \cdot (l, M) \mid l'(\log(M \cap N_{\mathbb{Z}})) \subset \mathbb{Z}\}.$$

In order to deduce the bijection $O(\eta_l)_{\mathbb{Z}} \simeq O(l)_{\mathbb{Z}}$, we insert

LEMMA 5.5. *For a pair $(l, \mathfrak{m})$ with $l \in \mathfrak{n}_S^*$, $x \in N$ satisfies $\mathrm{Ad}_S^* x \cdot l = l$ if and only if $x \in M$. This also holds for any $(l, \mathfrak{m})$ with $l \in \mathfrak{n}_S^*$ not satisfying the assumption 2.*

PROOF. It suffices to prove that, for $x_L \in N_L$, $\mathrm{Ad}_S^* x_L \cdot l = l$ if and only if $x_L \in M$. Let $x_L = \exp(X)$ with $X \in \mathfrak{n}_L$. Since l is trivial on $\mathfrak{n}_L$, we see that $\mathrm{Ad}_S^* x_L \cdot l = l$ if and only if $l([X, Y_S]) = 0$ for any $Y_S \in \mathfrak{n}_S$. But the condition $l([X, \mathfrak{n}_S]) = \{0\}$ is equivalent to $X \in \mathfrak{m} \cap \mathfrak{n}_L$. In fact, $X \in \mathfrak{m} \cap \mathfrak{n}_L$ clearly satisfies $l([X, \mathfrak{n}_S]) = \{0\}$. Conversely, assume that $l([X, \mathfrak{n}_S]) = \{0\}$ holds but $X \notin \mathfrak{m} \cap \mathfrak{n}_L$. Then $\mathfrak{m} \oplus \mathbb{R}X$ forms an isotropic subspace for an inner product $l([*, *])$, but this contradicts the maximality of $\mathfrak{m}$ as an isotropic subspace. Hence $X \in \mathfrak{m} \cap \mathfrak{n}_L$. As a result, we obtain the assertion. □

We return to the proof of the proposition. The condition $l'(\log(M \cap N_{\mathbb{Z}})) \subset \mathbb{Z}$ can be replaced by the condition $l'(\log(N_S(\mathbb{Z}))) \subset \mathbb{Z}$. Since Lemma 5.5 means that for an $x_L \in N_L$, $\mathrm{Ad}_S^* x_L \cdot (l, M) = (l, M)$ if and only if $\mathrm{Ad}_S^* x_L \cdot l = l$, the bijection $O(\eta_l)_{\mathbb{Z}} \simeq O(l)_{\mathbb{Z}}$ is obtained by deleting M and M' in the set on the right hand side of the bijection just before Lemma 5.5. The bijection on $\mathfrak{M}(\eta_l)$ follows immediately from that on $O(\eta_l)_{\mathbb{Z}}$. □

We recall the construction of a basis of $\mathrm{Hom}_N(\eta_l, L^2(N_{\mathbb{Z}}\backslash N))$ stated in [2]. For each $l' \in \mathfrak{M}(l)$, we define $\Theta_{l'} \in \mathrm{Hom}_N(\eta_{l'}, L^2(N_{\mathbb{Z}}\backslash N))$ by

$$\Theta_{l'}(h)(x) := \sum_{\gamma \in N_{\mathbb{Z}} \cap M' \backslash N_{\mathbb{Z}}} h(\gamma x),$$

where $M' = \exp(\mathfrak{m}')$ with the polarization subalgebra $\mathfrak{m}'$ for $l' \in \mathfrak{M}(l)$. Here remark that $\Theta_{l'}$ may depend on the choice of $\mathfrak{m}'$ but that thanks to Lemma 4.10 $\mathfrak{m}'$ is uniquely determined by l'. We obtain

Proposition 5.6. *The space $\tilde{\bigoplus}_{l' \in \mathfrak{M}(l)} \Theta_{l'}(H_{\eta_{l'}})$ forms the η_l-isotypic component of $L^2(N_{\mathbb{Z}}\backslash N)$, where recall that $\tilde{\bigoplus}$ denotes the Hilbert space direct sum.*

For a proof, see [2],§6.

By virtue of Proposition 5.6, the η_l-component of our Fourier expansion is given as

$$\sum_{l' \in \mathfrak{M}(l)} C^l_{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x) = \sum_{l' \in \mathfrak{M}(l)} C^l_{l'} \sum_{\gamma \in M' \cap N_{\mathbb{Z}} \backslash N_{\mathbb{Z}}} W_{\kappa,l'}(\gamma x a),$$

where $C^l_{l'}$ denotes the constant factor of the Whittaker function $W_{\kappa,l'}$ with the boundary condition $W_{\kappa,l'}(1) = 1$. An element $\gamma \in N_{\mathbb{Z}}$ can be decomposed into $\gamma = \gamma_S \gamma_L$ with $\gamma_S \in N_S(\mathbb{Z})$ and $\gamma_L \in N_L(\mathbb{Z})$ since $N_{\mathbb{Z}} = N_S(\mathbb{Z}) \rtimes N_L(\mathbb{Z})$. Noting this, the Whittaker function $W_{\kappa,l'}(\gamma x a)$ twisted by $\gamma \in N_{\mathbb{Z}}$ can be written as

$$\begin{aligned} W_{\kappa,l'}(\gamma x a) &= W_{\kappa,l'}(\gamma_S \gamma_L x_S x_L a) = \chi_{\mathrm{Ad}^*_S \gamma_L^{-1} \cdot l'}(x_S) W_{\kappa,l'}(\gamma_L x_L a) \\ &= \chi_{\mathrm{Ad}^*_S \gamma_L^{-1} \cdot l'}(x_S) W_{\kappa,\mathrm{Ad}^*_S \gamma_L^{-1} \cdot l'}(x_L a) = W_{\kappa,\mathrm{Ad}^*_S \gamma_L^{-1} \cdot l'}(xa), \end{aligned}$$

where we use Lemma 4.6 (1) in order to deduce the third equation. Lemma 5.5 yields a bijection:

$$M' \cap N_{\mathbb{Z}} \backslash N_{\mathbb{Z}} \simeq M' \cap N_L(\mathbb{Z}) \backslash N_L(\mathbb{Z}) \simeq \mathrm{Ad}^*_S N_L(\mathbb{Z}) \cdot l'.$$

Therefore the η_l-component of the Fourier expansion can be written as

$$\sum_{l' \in \mathfrak{M}(l)} C^l_{l'} \Theta_{l'}(W_{\kappa,l'}(*a))(x) = \sum_{l' \in \mathfrak{M}(l)} C^l_{l'} \sum_{l'' \in \mathrm{Ad}^*_S N_L(\mathbb{Z}) \cdot l'} W_{\kappa,l''}(xa),$$

Due to the Koecher principle (cf. [7], Satz 1, Satz 2), a holomorphic Siegel modular form f satisfies the moderate growth condition as follows:

$$|f(g)| < C_f \|g\|^{m_f} \quad \text{for any } g \in G,$$

with some constant C_f and integer m_f, where $|*|$ denotes the norm on $\mathbb{C}$ defined as $|z| := \bar{z} \cdot z$. In fact, we can check this by observing the relation between modular forms on G and those on the Siegel upper half space, which will be referred to just after Definition 5.9. Here we insert

LEMMA 5.7. (1) *A theta series $\Theta_l(W_{\kappa,l}(*a))$ contributing to the Fourier expansion of a holomorphic Siegel modular form f satisfies the moderate growth condition in the following sense:*

$$\|\Theta_l(W_{\kappa,l}(*a))(x)\|_{L_2} < C' \|a\|^{m'},$$

where $\| * \|_{L_2}$ denotes the L^2-norm on $L^2(N_{\mathbb{Z}} \backslash N)$, and C' and m' are a constant and an integer not dependent on $a \in A$ respectively.
(2) *The moderate growth condition for $\Theta_l(W_{\kappa,l}(*a))$ stated in* (1) *holds if and only if $Y_n(l)$ is positive semi-definite.*

PROOF. (1) For a fixed $g \in G$, we regard $f(xg)$ as a function in $x \in N$. It belongs to $L^2(N_{\mathbb{Z}} \backslash N)$ as we remarked in the formulation of the Fourier expansion.

Then, for any $a \in A$, we have

$$\|C_l \Theta_l(W_{\kappa,l}(*a))\|_{L_2} < \|f(*a)\|_{L^2}$$

for a theta series $\Theta_l(W_{\kappa,l}(*a))$ contributing to the Fourier expansion of f, where C_l denotes the coefficient of $\Theta_l(W_{\kappa,l}(*a))$ in f.

There exists an $x_{\max} \in N$ such that $|f(x_{\max} a)|$ is the maximal value of $|f(*a)|$ since $x_{\max}$ is determined modulo $N_{\mathbb{Z}}$ and $N_{\mathbb{Z}} \backslash N$ is compact. Then we obtain

$$\|f(*a)\|_{L^2}^2 \leq \mathrm{vol}(N_{\mathbb{Z}} \backslash N) |f(x_{\max} a)|^2,$$

where $\mathrm{vol}(N_{\mathbb{Z}} \backslash N)$ denotes the volume of $N_{\mathbb{Z}} \backslash N$. Note that $x_{\max}$ may depend on $a \in A$. But the moderate growth condition of f implies

$$|f(x_{\max} a)| < C'_f \|a\|^{m'_f}$$

with a constant C'_f and an integer m'_f depending only on f. Hence $\Theta_l(W_{\kappa,l}(*a))$ satisfies the moderate growth condition in the assertion.
(2) Theorem 4.12 and Corollary 4.15 mean that, under the assumption 2, $Y_n(l)$ is positive semi-definite if and only if

$$\|W_{\kappa,l}(*a)\|_l < C\|a\|^m$$

for any $a \in A$, with a constant C and an integer m depending only on $W_{\kappa,l}$. In fact, Theorem 4.12 means that $W_{\kappa,l}(*a) \in H^\infty_{\eta_l}$ automatically holds under the assumption 2.

By the definition of Θ_l given before Proposition 5.6, we have

$$\|\Theta_l(W_{\kappa,l}(*a))\|_{L^2} = \mathrm{vol}(N_{\mathbb{Z}} \cap M\backslash M)\|W_{\kappa,l}(*a)\|_l,$$

where M denotes the polarization subgroup for l and $\mathrm{vol}(N_{\mathbb{Z}} \cap M\backslash M)$ is the volume of $N_{\mathbb{Z}} \cap M\backslash M$. Hence $Y_n(l)$ is positive semi-definite if and only if

$$\|\Theta_l(W_{\kappa,l}(*a))\|_{L^2} < C\,\mathrm{vol}(N_{\mathbb{Z}} \cap M\backslash M)\|a\|^m$$

with C and m as above. This means the assertion (2). □

In order to express our Fourier expansion, we introduce the following sets:

$$\begin{aligned}\Omega_{n,\mathbb{Z}} &:= \{T \in M_n(\mathbb{Q}) \mid \\ &\qquad T \text{ is symmetric positive semi-definite semi-integral}\},\\ L &:= \{l \in \mathfrak{n}_S^* \mid Y_n(l) \in \Omega_{n,\mathbb{Z}}\},\ \tilde{L} := L/\operatorname{Ad}_S^* N_L(\mathbb{Q}).\end{aligned}$$

Here remark that the map $L \ni l \mapsto Y_n(l) \in \Omega_{n,\mathbb{Z}}$ gives a bijection $L \simeq \Omega_{n,\mathbb{Z}}$. By virtue of Proposition 5.3 (2), Proposition 5.4 and Lemma 5.7, we see that the totallity of elements in $\hat{N}$ occurring in our Fourier expansion is in bijection with the set $\tilde{L}$. Therefore we can write our Fourier expansion of a modular form f as follows;

THEOREM 5.8. *Let f be a holomorphic Siegel modular form of weight κ with respect to Γ. Its Fourier expansion along the minimal parabolic subgroup can be written as*

$$f(xa) = \sum_{l\in\tilde{L}} \sum_{l'\in\mathfrak{M}(l)} C^l_{l'}\Theta_{l'}(W_{\kappa,l'}(*a))(x),$$

where $\Theta_{l'}(W_{\kappa,l'}(*a))(x) :=$

$$\sum_{l''\in \mathrm{Ad}^*_S N_L(\mathbb{Z})\cdot l'} \chi_{l''}(x_S)(a_1a_2\cdots a_n)^\kappa \exp(-2\pi \operatorname{Tr}({}^t(X_LA_n)\, Y_n(l'')(X_LA_n))).$$

Definition 5.9. We call the constants $C^l_{l'}$ *Fourier coefficients* of f.

Let $\mathfrak{H}_n$ be the Siegel upper half space of degree n, defined by

$$\{Z = {}^tZ \in M_n(\mathbb{C}) \mid \operatorname{Im} Z \text{ is positive definite}\},$$

where $\operatorname{Im} Z$ denotes the imaginary part of Z. The group G acts on this via the linear fractional transformation

$$\mathfrak{H}_n \ni Z \mapsto g\cdot Z := (AZ+B)(CZ+D)^{-1} \in \mathfrak{H}_n$$

with $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. For a $Z \in \mathfrak{H}_n$, let g_Z be an element of G such that $g_Z\cdot\sqrt{-1}1_n = Z$. Since the stabilizer of $\sqrt{-1}1_n$ in G is the maximal compact subgroup K, g_Z is uniquely determined modulo K. For a holomorphic modular form f on G, we define a function on $\mathfrak{H}_n$ by

$$F_f(Z) := \det(C\sqrt{-1}1_n + D)^\kappa f(g_Z),$$

where we write $g_Z = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$. The map $f \mapsto F_f$ provides a bijection between the space of holomorphic Siegel modular forms on G and the space of holomorphic Siegel modular forms on $\mathfrak{H}_n$. We want to rewrite our Fourier expasnion for F_f. For that purpose, we introduce some symbols.

Let $\tilde{\Omega_{n,\mathbb{Z}}}$ be the quotient of $\Omega_{n,\mathbb{Z}}$ by an equivalence relation:

$$S \sim S' \Leftrightarrow \text{there exists a } u \in U_n(\mathbb{Q}) \text{ such that } {}^tu\,Su = S',$$

where $U_n(\mathbb{Q}) = U_n \cap GL_n(\mathbb{Q})$. For a $S \in \Omega_{n,\mathbb{Z}}$, we define the set $\mathfrak{M}_n(S)$ as a quotient of the set $\{T \in \Omega_{n,\mathbb{Z}} \mid {}^tu\,Tu = S\ \exists u \in U_n(\mathbb{Q})\}$ by an equivalence relation

$$T \sim T' \Leftrightarrow \text{there exists a } u \in U_n(\mathbb{Z}) \text{ such that } {}^tu\,Tu = T',$$

where $U_n(\mathbb{Z}) = U_n \cap GL_n(\mathbb{Z})$. For $S \in \Omega_{n,\mathbb{Z}}$, let $l_S \in \mathfrak{n}_S^*$ such that $Y_n(l_S) = S$. Lemma 4.6 (1) means that the sets $\tilde{\Omega_{n,\mathbb{Z}}}$ and $\mathfrak{M}_n(S)$ are bijective with the sets $\tilde{L}$ and $\mathfrak{M}(l_S)$, respectively. Furthermore, for an $T \in \Omega_{n,\mathbb{Z}}$, set $\Omega_n(T) := \{{}^t u T u \mid u \in U_n(\mathbb{Z})\}$. This is in bijection with $\mathrm{Ad}_S^* N_L(\mathbb{Z}) \cdot l_T$. The map $f \mapsto F_f$ sends the η_l-component $\sum_{l' \in \mathfrak{M}(l)} C_{l'}^l \Theta_{l'}(W_{\kappa,l'}(*a))(x)$ of the Fourier expansion of f to

$$\sum_{T \in \mathfrak{M}_n(S)} C_T^S \Theta_T(Z),$$

where $l, l' \in L$ correspond to $S, T \in \Omega_{n,\mathbb{Z}}$ respectively, and we rewrite $C_{l'}^l$ as C_T^S and set $\Theta_T(Z) := \sum_{R \in \Omega_n(T)} \exp 2\pi\sqrt{-1}\,\mathrm{Tr}(RZ)$. As a result, we obtain our Fourier expansion for F_f.

THEOREM 5.10.

$$F_f(Z) = \sum_{S \in \tilde{\Omega_{n,\mathbb{Z}}}} \sum_{T \in \mathfrak{M}_n(S)} C_T^S \Theta_T(Z).$$

REMARK 5.11. Let $\Omega := \{Y \in M_n(\mathbb{R}) \mid Y$:symmetric positive-definite$\}$. The theta series $\Theta_T(Z)$ defines a holomorphic function on $\mathfrak{H}_n$. In fact, since $\Omega \simeq N_L A$ via the linear functional transformation, the uniform convergence of the absolute value of $\Theta_T(Z)$ on any compact subset of $\mathfrak{H}_n$ is equivalent to that of $\Theta_l(W_{\kappa,l}(*a))(x)$ on any compact subset of $N_L A$, with $l \in L$ such that $Y_n(l) = T$. The latter condition is justified by Lemma 5.7 (2).

6. Comparison with the Other Two Fourier Expansions

In this section, we compare our Fourier expansion with the other two known Fourier expansions, i.e. classical Fourier expansion and Fourier-Jacobi expansion. This section consists of two subsections §6.1 and §6.2. In §6.1 (resp. §6.2), we consider the comparison with the classical expansion (resp. Fourier-Jacobi expansion).

6.1. Comparison with the classical Fourier expansion

Let F_f be as in the previous section. As is well-known, the classical Fourier expansion of F_f can be written as

$$F_f(Z) = \sum_{T \in \Omega_{n,\mathbb{Z}}} C_T \exp 2\pi\sqrt{-1}\,\mathrm{Tr}\,TZ,$$

where C_T denotes the Fourier coefficient indexed by T. Compare this classical expansion with our expansion in Theorem 5.10. Then we obtain a relation between the Fourier coefficients of the classical expansion and those of our expasnion.

THEOREM 6.1. *Let $T \in \Omega_{n,\mathbb{Z}}$ belong to $\mathfrak{M}_n(S)$ with some $S \in \Omega_{n,\mathbb{Z}}$ and C_T^S denote the Fourier coefficient of our Fourier expansion in Theorem* 5.10. *Then we have*

$$C_T^S = C_T$$

and, for every $u \in U_n(\mathbb{Z})$,

$$C_{{}^t u T u} = C_T^S.$$

REMARK 6.2. This relation of Fourier coefficients is compatible with a well-known formula

$$C_{{}^t\gamma T \gamma} = C_T$$

for any $\gamma \in SL_n(\mathbb{Z})$. Noting this relation, we can deduce our Fourier expansion in Theorem 5.10 from the classical expansion since $\sum_{R \in \Omega_n(T)} C_R \exp 2\pi\sqrt{-1}\,\mathrm{Tr}(RZ) = C_T \Theta_T(Z)$.

6.2. Comparison with the Fourier-Jacobi expansion

For a field F, $M_{m,n}(F)$ denotes the set of matrices with their size $m \times n$ and coefficients in F. If $m = n$, it is nothing but $M_n(F)$.

Let $Z = \begin{pmatrix} Z_1 & Z_2 \\ {}^t Z_2 & Z_3 \end{pmatrix} \in \mathfrak{H}_n$ with $Z_1 \in M_j(\mathbb{C})$, $Z_2 \in M_{j,n-j}(\mathbb{C})$ and $Z_3 \in M_{n-j}(\mathbb{C})$, where $1 \le j \le n-1$. The Fourier-Jacobi expansion of a holomorphic form F_f on $\mathfrak{H}_n$ is written as

$$F_f(Z) = \sum_{T_1 \in \Omega_{j,\mathbb{Z}}} \phi_{T_1}(Z_2, Z_3) \exp 2\pi\sqrt{-1}\,\mathrm{Tr}\,T_1 Z_1,$$

where

$$\phi_{T_1}(Z_2, Z_3) := \sum_{T \in \Omega_{T_1}} C_T \exp 2\pi\sqrt{-1}(\mathrm{Tr}(2\,{}^tT_2\, Z_2 + T_3 Z_3)),$$

with $\Omega_{T_1} := \left\{ T = \begin{pmatrix} T_1 & T_2 \\ {}^tT_2 & T_3 \end{pmatrix} \in \Omega_{n,\mathbb{Z}} \middle| T_2 \in M_{j,n-j}(\mathbb{Q}),\ T_3 \in M_{j-n}(\mathbb{Q}) \right\}$. As is well-known, ϕ_{T_1} is a Jacobi form of weight κ and index T_1 (for a definition, see [17], Definition 1.3).

For an $S_1 \in \Omega_{j,\mathbb{Z}}$, let $\tilde{\Omega_{S_1}}$ denote the quotient of Ω_{S_1} by an equivalence relation

$$S \sim S' \Leftrightarrow {}^tu\, Su = S'\ \exists u \in U_n(\mathbb{Q}).$$

For our purpose, we need

LEMMA 6.3.

$$\bigcup_{T_1 \in \mathfrak{M}_j(S_1)} \bigcup_{R_1 \in \Omega_j(T_1)} \Omega_{R_1} = \bigcup_{S \in \tilde{\Omega_{S_1}}} \bigcup_{T \in \mathfrak{M}_n(S)} \Omega_n(T).$$

PROOF. The unions appearing in the both sides of the equation above are all disjoint. It suffices to prove that each Ω_{R_1} (resp. $\Omega_n(T)$) is contained in the right hand side (resp. left hand side). The upper-left $j \times j$ component of each element in $\Omega_n(T)$ is in $\bigcup_{T_1 \in \mathfrak{M}_j(S_1)} \Omega_j(T_1)$. Hence $\Omega_n(T)$ forms a subset of the set on the left hand side. The set Ω_{R_1} can be written as $\Omega_{{}^tu_1\, S_1 u_1} = \begin{pmatrix} {}^tu_1 & \\ & 1_{n-j} \end{pmatrix} \Omega_{S_1} \begin{pmatrix} u_1 & \\ & 1_{n-j} \end{pmatrix}$ with some $u_1 \in U_j(\mathbb{Q})$. So we see that Ω_{R_1} is contained in the right hand side. Therefore the assertion is verified. $\square$

From this lemma, we deduce a relation between the Fourier-Jacobi coefficients and theta series Θ_T, stated as

THEOREM 6.4.

$$\sum_{T_1 \in \mathfrak{M}_j(S_1)} \sum_{R_1 \in \Omega_j(T_1)} \phi_{R_1}(Z_2, Z_3) \exp(2\pi\sqrt{-1}\,\mathrm{Tr}\, R_1 Z_1)$$
$$= \sum_{S \in \tilde{\Omega_{S_1}}} \sum_{T \in \mathfrak{M}_n(S)} C_T^S \Theta_T(Z).$$

Proof. Lemma 6.3 means that the equation above formally holds. The convergence of the infinite sums on both sides is justified since the Fourier series of f in the sence of clasical Fourier expansion, which uniformly absolutely converges on $\Gamma\backslash\mathfrak{H}_n$, is a majorant of them. □

Consider the case $j=1$. Then the formula in Lemma 6.3 is rewritten as

$$\Omega_{S_1} = \bigcup_{S\in\tilde{\Omega_{S_1}}} \bigcup_{T\in\mathfrak{M}_n(S)} \Omega_n(T)$$

since $U_1=\{1\}$. This means

Corollary 6.5. *When $j=1$, one obtains*

$$\phi_{S_1}(Z_2,Z_3)\exp 2\pi\sqrt{-1}\,\mathrm{Tr}\,S_1Z_1 = \sum_{S\in\tilde{\Omega_{S_1}}} \sum_{T\in\mathfrak{M}_n(S)} C_T^S\Theta_T(Z).$$

Remark 6.6. Our Fourier expansion along the minimal parabolic subgroup, stated as Theorem 5.8 and Theorem 5.10, is the most coarse one. In this paper, we give a comparison of the Fourier expansions along the extremal parabolic subgroups, i.e. the maximal parabolic subgroups and the minimal parabolic subgroup, in terms of Fourier coefficients and theta series appearing in the expansions. We think that such comparison seems to be also possible for Fourier expansions along arbitrary parabolic subgroups.

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