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Borel Summability of Divergent Solutions for Singular First Order Linear Partial Differential Equations with Polynomial Coefficients

By Masaki HIBINO

Abstract. This paper is concerned with the study of the Borel summability of divergent power series solutions for singular first order linear partial differential equations of nilpotent type. The conditions under which formal solutions are Borel summable are given by analytic continuation properties and growth (or decreasing) estimates for coefficients.

1. Introduction and Main Result

In this paper, we are concerned with the Borel summability of the formal solution for the following singular first order linear partial differential equation of nilpotent type:

$$(1.1)\qquad \begin{gathered} Pu(x,y) = f(x,y), \\ P = 1 + \{a + bx + c(x,y)\}yD_x + \{d + e(x,y)\}y^2 D_y, \end{gathered}$$

where $x,\ y \in \mathbf{C}$, $D_x = \partial/\partial x$, $D_y = \partial/\partial y$, and a, b and d are complex constants, and $f(x,y)$ is holomorphic at the origin. $c(x,y)$ and $e(x,y)$ are polynomials of at least degree 1 with respect to y, that is, $c(x,y)$ and $e(x,y)$ have following forms:

$$(1.2)\qquad c(x,y) = \sum_{p=p_0}^{p_1} c_p(x) y^p,$$

$$(1.3)\qquad e(x,y) = \sum_{q=q_0}^{q_1} e_q(x) y^q,$$

where $1 \le p_0 \le p_1$ and $1 \le q_0 \le q_1$. $c_p(x)$ $(p = p_0,\ p_0+1,\ \ldots,\ p_1)$ and $e_q(x)$ $(q = q_0,\ q_0+1,\ \ldots,\ q_1)$ are holomorphic at the origin. Throughout this paper, we always assume that

$$(1.4)\qquad a \neq 0,$$

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$$c_{p_0}(x) \not\equiv 0 \quad \text{unless} \quad c(x,y) \equiv 0, \tag{1.5}$$

and

$$e_{q_0}(x) \not\equiv 0 \quad \text{unless} \quad e(x,y) \equiv 0. \tag{1.6}$$

1.1. Motivation

In the paper Hibino[5], we studied the following more general equation:

$$\begin{gathered} P_0 u(x,y) = f(x,y), \\ P_0 = 1 + \{a(x,y)y + b(x,y)xy + c(x,y)y^2\}D_x + d(x,y)y^2 D_y, \end{gathered} \tag{1.7}$$

where the coefficients a, b, c, d and f are holomorphic at the origin with $a(0,0) \neq 0$.

First, in order to state the problem and the result in [5], let us introduce some notations.

Definition 1.1. (1) $\mathcal{O}[R]$ denotes the ring of holomorphic functions on the closed ball $B(R) := \{x \in \mathbf{C};\ |x| \leq R\}$.

(2) The ring of formal power series in y ($\in \mathbf{C}$) over the ring $\mathcal{O}[R]$ is denoted as $\mathcal{O}[R][[y]]$: $\mathcal{O}[R][[y]] = \{u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n;\ u_n(x) \in \mathcal{O}[R]\}$.

(3) We say that $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]$ *belongs to* $\mathcal{O}[R][[y]]_2$ if there exist some positive constants C and K such that $\max_{|x| \leq R} |u_n(x)| \leq CK^n n!$ for all $n \in \mathbf{N}$.

(4) For $\theta \in \mathbf{R}$, $\alpha > 0$ and $0 < \rho \leq +\infty$, the sector $S(\theta, \alpha, \rho)$ in the universal covering space of $\mathbf{C} \setminus \{0\}$ is defined by

$$S(\theta, \alpha, \rho) = \left\{ y;\ |\arg(y) - \theta| < \frac{\alpha}{2},\ 0 < |y| < \rho \right\}. \tag{1.8}$$

(5) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x)y^n \in \mathcal{O}[R][[y]]_2$ and let $U(x,y)$ be a holomorphic function on $X := B(R) \times S(\theta, \alpha, \rho)$. Then we say that $U(x,y)$ *has* $u(x,y)$ *as an asymptotic expansion of Gevrey order* 2 *in* X if the following asymptotic extimates hold: For any α' and ρ' ($0 < \alpha' < \alpha$, $0 < \rho' < \rho$), there exist some positive constants C and K such that

$$\max_{|x| \leq R} \left| U(x,y) - \sum_{n=0}^{N-1} u_n(x)y^n \right| \leq CK^N N! |y|^N, \quad y \in S(\theta, \alpha', \rho'); \qquad N = 1, 2, \ldots. \tag{1.9}$$

Then we write this as

$$U(x,y) \cong_2 u(x,y) \text{ in } X.$$

Now we already know that *the equation* (1.7) *has a unique formal power series solution in* $\mathcal{O}[R][[y]]_2$ *for some* $R > 0$ (see Hibino[4]). The main problem in [5] was the existence of holomorphic solutions $U(x,y)$ on some $X = B(R) \times S(\theta, \alpha, \rho)$ such that $U(x,y) \cong_2 u(x,y)$ in X. [5] proved that this problem was solved positively if $\alpha < \pi$:

Theorem 1.1 (Hibino[5]). *Let* $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ *be the formal solution of the equation* (1.7)*, and let* θ *be an arbitrary real number. Let us assume that* $0 < \alpha < \pi$*. Then there exist some positive constants* r_α*,* ρ_α *and a holomorphic solution* $U(x,y)$ *of* (1.7) *on* $X_\alpha = B(r_\alpha) \times S(\theta, \alpha, \rho_\alpha)$ *such that* $U(x,y) \cong_2 u(x,y)$ *in* X_α*. Such asymptotic solutions* $U(x,y)$ *exist infinitely many.*

Let us remark that we do not require any additional condition for coefficients.

Theorem 1.1 does not necessarily hold when $\alpha > \pi$. When we consider an open disk (see Definition 1.2) instead of a sector also, we can not unconditionally expect the existence of asymptotic solutions as the above. However, in these cases, if there exists such an asymptotic solution, then we see that it is unique by a general theory of Gevrey asymptotic expansions (cf. Balser[1][2], Lutz-Miyake-Schäfke[8] and Malgrange[9]). Such a solution is called the Borel sum of the formal solution. In the paper Hibino[6], we studied the condition under which the formal solution is Borel summable in the case where the region is an open disk. Since it is difficult to obtain such a condition for the general equation (1.7), [6] restricted that the coefficients a, b, c and d are constants. In this paper, we give one of the generalizations of [6].

1.2. Main Result

Now let us return to the equation (1.1) and let us consider the condition under which the formal solution is Borel summable. First of all, let us give the precise definition of the Borel summability.

Definition 1.2. (1) For $\theta \in \mathbf{R}$ and $T > 0$, we define the region $O(\theta, T)$ by

$$O(\theta, T) = \{y;\ |y - T\mathrm{e}^{i\theta}| < T\}, \tag{1.10}$$

where $\mathrm{e}^t = \exp t$.

(2) Let $u(x,y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$. We say that $u(x,y)$ *is Borel summable in* θ if there exists a holomorphic function $U(x,y)$ on $B(r) \times O(\theta, T)$ for some $0 < r \le R$ and $T > 0$ which satisfies the following asymptotic estimates: There exist some positive constants C and K such that

$$\max_{|x|\le r} \left| U(x,y) - \sum_{n=0}^{N-1} u_n(x) y^n \right| \le C K^N N! |y|^N, \quad y \in O(\theta, T); \quad N = 1, 2, \ldots. \tag{1.11}$$

As mentioned in §1.1, when $u(x,y)$ is Borel summable in θ, the above holomorphic function $U(x,y)$ is unique. So we call this $U(x,y)$ the *Borel sum of* $u(x,y)$ *in* θ.

We divide the problem into the following four cases:

Case (1): $b = d = 0$,

Case (2): $b = 0$, $d \neq 0$,

Case (3): $b \neq 0$, $d = 0$,

Case (4): b, $d \neq 0$.

In order to state the main result, we introduce some notations. Let us define the function $\Phi(x, \eta)$ by

$$\Phi(x,\eta) = \begin{cases} x - a\eta & \text{(Case (1))} \\ x - \dfrac{a}{d}\log(1 + d\eta) & \text{(Case (2))} \\ \left(\dfrac{a}{b} + x\right)\mathrm{e}^{-b\eta} - \dfrac{a}{b} & \text{(Case (3))} \\ \left(\dfrac{a}{b} + x\right)(1 + d\eta)^{-b/d} - \dfrac{a}{b} & \text{(Case (4))}, \end{cases} \tag{1.12}$$

and let us define the region $\Omega_{r,\theta,\kappa}$ ($\kappa > 0$) by

$$\Omega_{r,\theta,\kappa} = \{\Phi(x,\eta);\ |x| \le r,\ \eta \in E_+(\theta, \kappa)\}. \tag{1.13}$$

Here $E_+(\theta,\kappa)$ is a region defined by

$$E_+(\theta,\kappa)=\{\eta;\ \text{dist}\,(\eta,\mathbf{R}_+\mathrm{e}^{i\theta})\le\kappa\}, \tag{1.14}$$

where $\mathbf{R}_+=[0,+\infty)$. In order to ensure the well-definedness of $\Omega_{r,\theta,\kappa}$, we always assume the following:

$$\theta\neq\arg\left(-\frac{1}{d}\right)\ \text{in Case (2) and Case (4).} \tag{1.15}$$

In Case (3) and Case (4), $\Omega_{r,\theta,\kappa}$ is usually a spiral region turning around $-a/b$, and analytic functions on $\Omega_{r,\theta,\kappa}$ have a branch point at $-a/b$ in general. Therefore in such cases we regard $\Omega_{r,\theta,\kappa}$ as a region in the Riemann surface of $\log\left(x+\dfrac{a}{b}\right)$.

Under these notations, let us give the assumptions for the equation (1.1).

(A1) In each case, $f(x,y)$ can be continued analytically to $\Omega_{r,\theta,\kappa}\times\{y\in\mathbf{C};\ |y|\le r'\}$ for some r, κ and r'.

(A2) $f(x,y)$ has the following growth estimate on $\Omega_{r,\theta,\kappa}\times\{y\in\mathbf{C};\ |y|\le r'\}$: There exist some positive constants C and δ such that:

Case (1):

$$\max_{|y|\le r'}|f(x,y)|\le C\mathrm{e}^{\delta|x|},\quad x\in\Omega_{r,\theta,\kappa}; \tag{1.16}$$

Case (2):

$$\max_{|y|\le r'}|f(x,y)|\le C\exp\left\{\delta\left|\exp\left(-\frac{d}{a}x\right)\right|\right\},\quad x\in\Omega_{r,\theta,\kappa}; \tag{1.17}$$

Case (3):

$$\max_{|y|\le r'}|f(x,y)|\le C\exp\left\{\delta\left|\log\left(x+\frac{a}{b}\right)\right|\right\},\quad x\in\Omega_{r,\theta,\kappa}; \tag{1.18}$$

Case (4):

$$\max_{|y|\le r'}|f(x,y)|\le C\exp\left\{\delta\left|\left(x+\frac{a}{b}\right)^{-d/b}\right|\right\},\quad x\in\Omega_{r,\theta,\kappa}. \tag{1.19}$$

(A3) In each case, $c_p(x)$ and $e_q(x)$ ($p=p_0,\ p_0+1,\ \ldots,\ p_1$; $q=q_0$, $q_0+1,\ \ldots,\ q_1$) can be continued analytically to $\Omega_{r,\theta,\kappa}$.

(A4) In each case $c_p(x)$ has the following estimate on $\Omega_{r,\theta,\kappa}$:
Case (1): $c_p(x)$ is bounded:

$$M_p := \sup_{x\in\Omega_{r,\theta,\kappa}} |c_p(x)| < \infty; \tag{1.20}$$

Case (2): For some positive constants $M_p > 0$ and $\beta_p < p$,

$$|c_p(x)| \le M_p \left|\exp\left(-\frac{d}{a}x\right)\right|^{\beta_p}, \quad x \in \Omega_{r,\theta,\kappa}; \tag{1.21}$$

Case (3): For some positive constant $M_p > 0$,

$$|c_p(x)| \le M_p \left|x + \frac{a}{b}\right|, \quad x \in \Omega_{r,\theta,\kappa}; \tag{1.22}$$

Case (4): For some positive constants $M_p > 0$ and $\beta_p < p$,

$$|c_p(x)| \le M_p \left|\left(x + \frac{a}{b}\right)^{-d/b}\right|^{\beta_p} \left|x + \frac{a}{b}\right|, \quad x \in \Omega_{r,\theta,\kappa}. \tag{1.23}$$

(A5) In each case $e_q(x)$ has the following estimate on $\Omega_{r,\theta,\kappa}$:
Case (1): For some positive constants $N_q > 0$ and $\alpha_q > 1$,

$$|e_q(x)| \le \frac{N_q}{(1+|x|)^{\alpha_q}}, \quad x \in \Omega_{r,\theta,\kappa}; \tag{1.24}$$

Case (2): For some positive constants $N_q > 0$ and $\beta_q{}' < q$,

$$|e_q(x)| \le N_q \left|\exp\left(-\frac{d}{a}x\right)\right|^{\beta_q{}'}, \quad x \in \Omega_{r,\theta,\kappa}; \tag{1.25}$$

Case (3): For some positive constants $N_q > 0$ and $\alpha_q > 1$,

$$|e_q(x)| \le \frac{N_q}{\left\{1 + \left|\log\left(x + \frac{a}{b}\right)\right|\right\}^{\alpha_q}}, \quad x \in \Omega_{r,\theta,\kappa}; \tag{1.26}$$

Case (4): For some positive constants $N_q > 0$ and $\beta_q{}' < q$,

$$|e_q(x)| \le N_q \left|\left(x + \frac{a}{b}\right)^{-d/b}\right|^{\beta_q{}'}, \quad x \in \Omega_{r,\theta,\kappa}. \tag{1.27}$$

The main result in this paper is stated as follows:

MAIN THEOREM *Under the assumptions* (A1)*,* (A2)*,* (A3)*,* (A4) *and* (A5)*, the formal solution* $u(x, y)$ *of the equation* (1.1) *is Borel summable in* θ*, and its Borel sum is a holomorphic solution of* (1.1)*.*

REMARK 1.1. In Case (3) and Case (4), it depends on the values of $\Re(b e^{i\theta})$ and $\Re(b/d)$, respectively, whether the condition (A4) is a growth or decreasing condition. For example, let us consider the equation

$$\{1 + (a + bx + cy)yD_x + dy^2 D_y\}u(x, y) = f(x, y). \tag{1.28}$$

This is the case where $c(x, y) = cy$ ($c_1(x) \equiv c$) and $e(x, y) \equiv 0$. In Case (3), if $\Re(b e^{i\theta}) > 0$, our condition (1.22) is the decreasing condition. Therefore a constant c which satisfies such a decreasing condition is exactly zero. Similarly in Case (4), if $\Re(b/d) \geq 1$, we see that a constant which satisfies the condition (1.23) for $p = 1$ is exactly zero. Let us remark that in Case (3) (resp. Case (4)) if $\Re(b e^{i\theta}) \leq 0$ (resp. $\Re(b/d) < 1$), the condition (A4) is always satisfied for (1.28). Therefore for the equation (1.28), our condition (A4) can be written as follows:

Case (3):

$$c = 0 \quad \text{or} \quad \Re(b e^{i\theta}) \leq 0;$$

Case (4):

$$c = 0 \quad \text{or} \quad \Re\left(\frac{b}{d}\right) < 1.$$

In [6], the equation (1.28) is studied in detail, and we see there that the above conditions are not technical but essential.

Finally, we give some remarks on the precedent results.

On the problem of the existence of Gevrey asymptotic solutions for divergent formal solutions, there are many results in the theory of ordinary differential equations, which can be seen in Balser[1][2]. On the other hand, in the theory of partial differential equations, such studies started recently, and there are not so many articles. On the existence of Gevrey asymptotic solutions in small sectors, such as in Theorem 1.1, we can find some interesting results in Ōuchi[11], where very general nonlinear partial differential equations are dealt with. In Ōuchi's results and our Theorem 1.1, in order to prove the existence of Gevrey asymptotic solutions, the Gevrey version of Borel-Ritt's theorem (cf. Balser[1][2]) plays a fundamental role (on the original version of Borel-Ritt's theorem, see Wasow[12]). Moreover, we remark that the existence of Gevrey asymptotic solutions in small sectors is assured without any global condition for coefficients. Unfortunately, the theorem of Borel-Ritt is not useful in the argument of Borel summability. Being different from small sector cases, in order to assure the Borel summability of formal solutions the global conditions, such as in our main theorem, are demanded despite that the domain $O(\theta, T)$ of the Borel sum is local. Lutz-Miyake-Schäfke[8] and Miyake[10] gave the necessary and sufficient conditions for the Borel summability of formal solutions for non-Kowalevskian equations with constant coefficients like heat equations. Balser-Miyake[3] dealt with more general equations with constant coefficients and gave the sufficient conditions for the Borel summability. The equation which is studied in this paper is the one with variable coefficients and is a different type of equation from theirs.

2. Formal Borel Transform of Equations

Before proving the main theorem, we give some preliminaries. First, we remark that if the formal solution $u(x, y)$ of (1.1) is Borel summable, then it is easily proved that its Borel sum $U(x, y)$ is a holomorphic solution of (1.1) as follows: Since $U(x, y)$ is the Borel sum of $u(x, y)$, it follows that $PU(x, y)$ is the Borel sum of $Pu(x, y)$. On the other hand, it is clear that $f(x, y)$ is the Borel sum of $Pu(x, y)$. Therefore it follows from the uniqueness of the Borel sum that $PU(x, y) = f(x, y)$.

Thus in order to prove the main theorem, it is sufficient to prove that the formal solution $u(x, y)$ is Borel summable under the conditions in the theorem.

When we want to check the Borel summability of the formal power series $u(x, y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$, the following theorem plays a fundamental role in general.

Theorem 2.1 (Lutz-Miyake-Schäfke[8]). *In order that a formal power series* $u(x, y) = \sum_{n=0}^{\infty} u_n(x) y^n \in \mathcal{O}[R][[y]]_2$ *is Borel summable in* θ*, the following condition* (BS) *is necessary and sufficient: Let* $\mathcal{B}(u)(x, \eta)$

be the formal Borel transform of $u(x,y)$ *defined by*

$$\mathcal{B}(u)(x,\eta)=\sum_{n=0}^{\infty}u_n(x)\frac{\eta^n}{n!}, \tag{2.1}$$

which is holomorphic in a neighborhood of the origin. Then the condition (BS) *is stated as follows*:

(BS) $\mathcal{B}(u)(x,\eta)$ *can be continued analytically to* $B(r_0)\times E_+(\theta,\kappa_0)$ *for some* $r_0>0$ *and* $\kappa_0>0$, *and has the following exponential growth estimate for some positive constants* C *and* δ:

$$\max_{|x|\le r_0}|\mathcal{B}(u)(x,\eta)|\le C\mathrm{e}^{\delta|\eta|},\quad \eta\in E_+(\theta,\kappa_0). \tag{2.2}$$

When (BS) *is satisfied, the Borel sum* $U(x,y)$ *of* $u(x,y)$ *in* θ *is given by*

$$U(x,y)=\frac{1}{y}\int_{\mathbf{R}_+\mathrm{e}^{i\theta}}\mathrm{e}^{-\eta/y}\mathcal{B}(u)(x,\eta)\mathrm{d}\eta. \tag{2.3}$$

It is thus sufficient to prove that the formal Borel transform $\mathcal{B}(u)(x,\eta)$ of the formal solution $u(x,y)$ satisfies the above condition (BS) under the conditions in the theorem. In order to do that, firstly let us write down the equation which $\mathcal{B}(u)(x,\eta)$ should satisfy. By the formal Borel transform, the operators y and D_y are transformed to the operators ${D_\eta}^{-1}=\int_0^\eta \mathrm{d}\eta$ and $D_\eta\eta D_\eta$, respectively. They are easily seen from the following commutative diagrams:

$$\begin{array}{ccc} y^n & \xrightarrow{\text{Borel tr.}} & \dfrac{\eta^n}{n!} \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle {D_\eta}^{-1}} \\ y^{n+1} & \xrightarrow[\text{Borel tr.}]{} & \dfrac{\eta^{n+1}}{(n+1)!}, \end{array} \qquad \begin{array}{ccc} y^n & \xrightarrow{\text{Borel tr.}} & \dfrac{\eta^n}{n!} \\ {\scriptstyle D_y}\downarrow & & \downarrow{\scriptstyle D_\eta\eta D_\eta} \\ ny^{n-1} & \xrightarrow[\text{Borel tr.}]{} & n\dfrac{\eta^{n-1}}{(n-1)!}. \end{array} \tag{2.4}$$

Therefore we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following equation:

$$\begin{aligned} &\Big\{1+(a+bx){D_\eta}^{-1}D_x+{D_\eta}^{-1}\sum_{p=p_0}^{p_1}c_p(x){D_\eta}^{-p}D_x \\ &\qquad\qquad + d{D_\eta}^{-1}\eta D_\eta+{D_\eta}^{-1}\sum_{q=q_0}^{q_1}e_q(x){D_\eta}^{-q}\eta D_\eta\Big\}v(x,\eta) \\ &= \mathcal{B}(f)(x,\eta), \end{aligned} \tag{2.5}$$

where $\mathcal{B}(f)(x,\eta)$ is the formal Borel transform of $f(x,y)=\sum_{n=0}^{\infty} f_n(x)y^n$, that is,

$$\mathcal{B}(f)(x,\eta)=\sum_{n=0}^{\infty} f_n(x)\frac{\eta^n}{n!}.$$

Furthermore by operating D_η to the equation (2.5) from the left, we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the initial value problem of the following integro-differential equation:

$$(2.6)\quad \begin{aligned} Lv(x,\eta) &= -\left\{\sum_{p=p_0}^{p_1} c_p(x){D_\eta}^{-p}D_x+\sum_{q=q_0}^{q_1} e_q(x){D_\eta}^{-q}\eta D_\eta\right\}v(x,\eta)\\ &\quad + g(x,\eta),\\ v(x,0) &= f(x,0), \end{aligned}$$

where

$$(2.7)\qquad L=(1+d\eta)D_\eta+(a+bx)D_x,$$

and $g(x,\eta)=D_\eta\mathcal{B}(f)(x,\eta)$.

Here it follows from an integration by parts that

$$\begin{aligned} {D_\eta}^{-q}\eta D_\eta v(x,\eta) &= \int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{q-1}} s_q\cdot v_\eta(x,s_q)\mathrm{d}s_q\cdots\mathrm{d}s_2\mathrm{d}s_1\\ &= \int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{q-1}} s_q\cdot\frac{\mathrm{d}}{\mathrm{d}s_q}v(x,s_q)\mathrm{d}s_q\cdots\mathrm{d}s_2\mathrm{d}s_1\\ &= \int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{q-2}} s_{q-1}\cdot v(x,s_{q-1})\mathrm{d}s_{q-1}\cdots\mathrm{d}s_2\mathrm{d}s_1\\ &\quad -\int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{q-1}} v(x,s_q)\mathrm{d}s_q\cdots\mathrm{d}s_2\mathrm{d}s_1. \end{aligned}$$

Therefore we see that $\mathcal{B}(u)(x,\eta)$ is a solution of the following initial value problem:

$$(2.8)\qquad \begin{aligned} Lv(x,\eta) &= \sum_{i=1}^{3} I_i v(x,\eta)+g(x,\eta),\\ v(x,0) &= f(x,0), \end{aligned}$$

where each operator I_i is given by

$$I_1 v(x,\eta) = -\sum_{p=p_0}^{p_1} c_p(x)\int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{q-1}} v_x(x,s_p)\mathrm{d}s_p\cdots\mathrm{d}s_2\mathrm{d}s_1,$$

$$
\begin{aligned}
I_2 v(x,\eta) &= -\sum_{q=q_0}^{q_1} e_q(x) \\
&\quad \times \int_0^{\eta}\int_0^{s_1}\cdots\int_0^{s_{q-2}} s_{q-1}\cdot v(x,s_{q-1})\mathrm{d}s_{q-1}\cdots \mathrm{d}s_2\mathrm{d}s_1, \\
I_3 v(x,\eta) &= \sum_{q=q_0}^{q_1} e_q(x)\int_0^{\eta}\int_0^{s_1}\cdots\int_0^{s_{q-1}} v(x,s_q)\mathrm{d}s_q\cdots \mathrm{d}s_2\mathrm{d}s_1.
\end{aligned}
$$

The main theorem will be proved by showing that the solution $v(x,\eta)$ of the equation (2.8) satisfies the condition (BS).

We will prove this fact in the next section. If $c(x,y)=\sum_{p=p_0}^{p_1} c_p(x)y^p \equiv 0$ or $e(x,y)=\sum_{q=q_0}^{q_1} e_q(x)y^q \equiv 0$, the proof becomes easier. Therefore in the following we consider only the case $c(x,y)\not\equiv 0$ and $e(x,y)\not\equiv 0$. In this case, we remark that $c_{p_0}(x)\not\equiv 0$ and $e_{q_0}(x)\not\equiv 0$ by the assumptions (1.5) and (1.6).

The proof will be done only in Case (4). Since the proof for other cases can be seen in Chapter 5 of author's doctoral thesis Hibino[7], here we only remark the following: Case (1) and Case (3) are proved by an essentially same method. On the other hand, Case (2) and Case (4) are proved by an essentially same method different from the one used in the proofs of Case (1) and Case (3). For details, refer to [7].

3. Proof of Main Theorem (Case (4))

Let us prove that the solution $v(x,\eta)$ of the equation (2.8) for b, $d\neq 0$, satisfies the condition (BS) in Theorem 2.1. Firstly we remark that in general the solution $w(x,\eta)$ of the initial value problem of the following first order linear partial differential equation

$$
\text{(3.1)}\qquad \begin{aligned}
&\{(1+d\eta)D_\eta + (a+bx)D_x\}w(x,\eta) = k(x,\eta), \\
&w(x,0) = l(x)
\end{aligned}
$$

is given by

$$
\text{(3.2)}\qquad \begin{aligned}
w(x,\eta) &= l\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d} - \frac{a}{b}\right) \\
&\quad + \int_0^{\eta} k\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d} - \frac{a}{b}, t\right)\frac{1}{1+dt}\mathrm{d}t.
\end{aligned}
$$

Proof of Main Theorem (Case (4)). First, let us transform (2.8) into the integral equation. It follows from (3.2) that the equation (2.8) is equivalent to the following equation:

$$\begin{aligned} v(x,\eta) &= f\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d}-\frac{a}{b},0\right) \\ &+ \int_0^\eta g\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t \\ &+ \sum_{i=1}^{3}\int_0^\eta (I_i v)\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t. \end{aligned}$$

Let us transform $\int_0^\eta (I_1 v)\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t$. This is given by

$$\begin{aligned} &\int_0^\eta (I_1 v)\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t \\ = & -\sum_{p=p_0}^{p_1}\int_0^\eta c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}\right) \\ &\times \int_0^t\int_0^{s_1}\cdots\int_0^{s_{p-1}} v_x\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},s_p\right) \\ &\qquad\qquad \times \mathrm{d}s_p\cdots\mathrm{d}s_2\mathrm{d}s_1\frac{1}{1+dt}\mathrm{d}t \\ = & -\sum_{p=p_0}^{p_1}\int_0^\eta\int_0^{s_1}\cdots\int_0^{s_{p-1}}\int_{s_1}^\eta c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}\right) \\ &\times v_x\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},s_p\right)\frac{1}{1+dt}\mathrm{d}t\mathrm{d}s_p\cdots\mathrm{d}s_2\mathrm{d}s_1. \end{aligned}$$

(by Fubini's Theorem)

Here we remark that

$$\begin{aligned} &\int_{s_1}^\eta c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}\right) \\ &\quad\times v_x\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},s_p\right)\frac{1}{1+dt}\mathrm{d}t \end{aligned}$$

$$= \frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}(1+d\eta)^{b/d}\int_{s_1}^{\eta} c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}\right) \times (1+dt)^{-b/d}\frac{\mathrm{d}}{\mathrm{d}t}\left\{v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}, s_p\right)\right\}\mathrm{d}t.$$

Therefore by an integration by parts and Fubini's Theorem again, we see that (2.8) is equivalent to the following integral equation:

$$v(x,\eta) = f\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d}-\frac{a}{b},0\right) + \int_0^{\eta} g\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t + \sum_{i=1}^{6} J_i v(x,\eta), \tag{3.3}$$

where each integral operator J_i is given by

$$J_1 v(x,\eta) = \frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\sum_{p=p_0}^{p_1}(1+d\eta)^{b/d} \times \int_0^{\eta}(1+ds_1)^{-b/d}c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}\right) \times \int_0^{s_1}\cdots\int_0^{s_{p-1}} v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}, s_p\right)\mathrm{d}s_p\cdots\mathrm{d}s_2\mathrm{d}s_1,$$

$$J_2 v(x,\eta) = -\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\sum_{p=p_0}^{p_1} c_p(x)\int_0^{\eta}\int_0^{s_1}\cdots\int_0^{s_{p-1}} v(x,s_p)\mathrm{d}s_p\cdots\mathrm{d}s_2\mathrm{d}s_1,$$

$$J_3 v(x,\eta) = -\left(\frac{a}{b}+x\right)^{-1}\sum_{p=p_0}^{p_1}(1+d\eta)^{b/d} \times \int_0^{\eta}(1+ds_1)^{-b/d}c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}\right) \times \int_0^{s_1}\cdots\int_0^{s_p} v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}, s_{p+1}\right)$$

$$\times\ \mathrm{d}s_{p+1}\cdots \mathrm{d}s_2 \frac{1}{1+ds_1}\mathrm{d}s_1,$$

$$\begin{aligned}
&J_4 v(x,\eta)\\
=\ &\sum_{p=p_0}^{p_1}\int_0^{\eta} c_p{}'\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}\right)\\
&\times \int_0^{s_1}\cdots\int_0^{s_p} v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}, s_{p+1}\right)\\
&\qquad\qquad\qquad\qquad \times\ \mathrm{d}s_{p+1}\cdots \mathrm{d}s_2 \frac{1}{1+ds_1}\mathrm{d}s_1,\\
&J_5 v(x,\eta)\\
=\ &\int_0^{\eta}(I_2 v)\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}, t\right)\frac{1}{1+dt}\mathrm{d}t\\
=\ &-\sum_{q=q_0}^{q_1}\int_0^{\eta} e_q\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}\right)\\
&\times \int_0^{s_1}\cdots\int_0^{s_{q-1}} s_q\cdot v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}, s_q\right)\\
&\qquad\qquad\qquad\qquad \times\ \mathrm{d}s_q\cdots \mathrm{d}s_2 \frac{1}{1+ds_1}\mathrm{d}s_1,\\
&J_6 v(x,\eta)\\
=\ &\int_0^{\eta}(I_3 v)\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b}, t\right)\frac{1}{1+dt}\mathrm{d}t\\
=\ &\sum_{q=q_0}^{q_1}\int_0^{\eta} e_q\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}\right)\\
&\times \int_0^{s_1}\cdots\int_0^{s_q} v\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d}-\frac{a}{b}, s_{q+1}\right)\\
&\qquad\qquad\qquad\qquad \times\ \mathrm{d}s_{q+1}\cdots \mathrm{d}s_2 \frac{1}{1+ds_1}\mathrm{d}s_1,
\end{aligned}$$

where $c_p{}'(x)=\dfrac{\mathrm{d}c_p}{\mathrm{d}x}(x)$.

In order to prove that the solution $v(x,\eta)$ of (3.3) satisfies the condition (BS), we employ the iteration method. Let us define $\{v_n(x,\eta)\}_{n=0}^{\infty}$ as follows:

$$\begin{aligned} v_0(x,\eta) &= f\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d}-\frac{a}{b},0\right) \\ &\quad + \int_0^\eta g\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right)\frac{1}{1+dt}\mathrm{d}t. \end{aligned}$$

For $n \geq 0$,

$$v_{n+1}(x,\eta) := v_0(x,\eta) + \sum_{i=1}^{6} J_i v_n(x,\eta). \tag{3.4}$$

Next, we define $\{w_n(x,\eta)\}_{n=0}^{\infty}$ by $w_0(x,\eta) := v_0(x,\eta)$ and $w_n(x,\eta) = v_n(x,\eta) - v_{n-1}(x,\eta)$ $(n \geq 1)$, and define $\{W_n(x,\eta,t)\}_{n=0}^{\infty}$ by

$$W_n(x,\eta,t) := w_n\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+dt}\right)^{-b/d}-\frac{a}{b},t\right). \tag{3.5}$$

Definition 3.1. (1) For $\lambda \geq 0$ and $\varepsilon > 0$, $U_\varepsilon[0,\lambda]$ denotes the ε-neighborhood of $[0,\lambda]$ in $\mathbf{C}$.

(2) For $\eta \in \mathbf{C}$, we define the function $\mathcal{G}_\eta(\tau)$ by

$$\mathcal{G}_\eta(\tau) = \frac{\tau e^{i\arg(\eta)}}{1+d(|\eta|-\tau)e^{i\arg(\eta)}}, \quad \tau \in \mathbf{C},$$

and define $\mathcal{G}_\eta$ and $\mathcal{G}_\eta^\varepsilon$ as follows:

$$\begin{aligned} \mathcal{G}_\eta &:= \{\mathcal{G}_\eta(R) \in \mathbf{C};\ 0 \leq R \leq |\eta|\}, \\ \mathcal{G}_\eta^\varepsilon &:= \{\mathcal{G}_\eta(\tau) \in \mathbf{C};\ \tau \in U_\varepsilon[0,|\eta|]\}. \end{aligned}$$

Now let us define $\mathcal{C}_p(x,\eta)$ by

$$\mathcal{C}_p(x,\eta) := c_p\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d}-\frac{a}{b}\right).$$

It follows from the assumptions (A3) and (A4) that $\mathcal{C}_p(x,\eta)$ is holomorphic on $B(r) \times E_+(\theta,\kappa)$ with the estimate

$$\max_{|x|\leq r} |(1+d\eta)^{b/d}\mathcal{C}_p(x,\eta)| \leq M_p{}'(1+|\eta|)^{\beta_p}, \quad \eta \in E_+(\theta,\kappa), \tag{3.6}$$

for some positive constant ${M_p}'$. Therefore by Cauchy's integral formula, we see that for $\kappa' := \kappa/2$ there exists a positive constant ${M_p}''$ such that

$$\max_{|x|\le r} |(1+d\eta)^{b/d}\mathcal{C}_{p,\eta}(x,\eta)| \le {M_p}''(1+|\eta|)^{\beta_p}, \quad \eta \in E_+(\theta,\kappa'), \tag{3.7}$$

where $\mathcal{C}_{p,\eta}(x,\eta) = \dfrac{\partial \mathcal{C}_p}{\partial \eta}(x,\eta)$.

Similarly, let us define $\mathcal{E}_q(x,\eta)$ by

$$\mathcal{E}_q(x,\eta) := e_q\left(\left(\frac{a}{b}+x\right)(1+d\eta)^{-b/d} - \frac{a}{b}\right).$$

Then it follows from the assumptions (A3) and (A5) that $\mathcal{E}_q(x,\eta)$ is holomorphic on $B(r)\times E_+(\theta,\kappa)$ with the estimate

$$\max_{|x|\le r} |\mathcal{E}_q(x,\eta)| \le {N_q}'(1+|\eta|)^{{\beta_q}'}, \quad \eta \in E_+(\theta,\kappa), \tag{3.8}$$

for some positive constant ${N_q}'$.

Next, we take a positive constant K so that

$$K^{-1}\frac{1}{1+|\eta|} \le \left|\frac{1}{1+d\eta}\right| \le K\frac{1}{1+|\eta|} \ (\le K), \quad \eta \in E_+(\theta,\kappa), \tag{3.9}$$

and we define $\beta > 0$ by

$$\beta = \min\{p-\beta_p,\ q-{\beta_q}';\ p = p_0, p_0+1,\dots,p_1 \text{ and } q = q_0, q_0+1,\dots,q_1\}. \tag{3.10}$$

Finally let us define $L > 0$ by

$$L = \max\left\{\frac{3}{\beta},\ 1+\frac{2}{\beta}\right\}. \tag{3.11}$$

Under these preparations, let us take a positive constant A so that

$$\begin{aligned} &{M_p}'K^{p+2}\cdot \max_{|x|\le r}\left|\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\right| \le A, \quad {M_p}'K^{p}\cdot \max_{|x|\le r}\left|\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\right| \le A, \\ &{M_p}'K^{p+1}\cdot \max_{|x|\le r}\left|\left(\frac{a}{b}+x\right)^{-1}\right| \le A, \quad {M_p}''K^{p}\cdot \max_{|x|\le r}\left|\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\right| \le A, \\ &\qquad (p = p_0, p_0+1,\dots,p_1), \end{aligned}$$

$$N_q{}'K^{q+1} \leq A, \quad (q = q_0, q_0+1, \ldots, q_1),$$

and let us take a monotonically decreasing positive sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ satisfying

$$\widetilde{\kappa} := \kappa' - \sum_{n=0}^{\infty} \varepsilon_n > 0. \tag{3.12}$$

Then we obtain the following lemma:

Lemma 3.1. *$W_n(x,\eta,t)$ is continued analytically to $\{(x,\eta,t);\ |x| \leq r,\ \eta \in E_+(\theta,\kappa' - \sum_{j=0}^{n}\varepsilon_j),\ t \in \mathcal{G}_\eta^{\varepsilon_n}\}$. Furthermore on $\{(x,\eta,t);\ |x| \leq r,\ \eta \in E_+(\theta,\kappa' - \sum_{j=0}^{n}\varepsilon_j),\ t \in \mathcal{G}_\eta\}$ we have the following estimate: For some positive constants $C_1 > 0$ and $\delta_1 > 1$,*

$$\begin{aligned} &|W_n(x,\eta,\mathcal{G}_\eta(R))| \\ &\quad\leq C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \frac{1}{(1+|\eta|-R)^{n\beta}} \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^k}{k!}, \end{aligned} \tag{3.13}$$

where $\xi = \min\{p_0, q_0\}$ and $\omega = \max\{p_1, q_1\}$.

If we admit Lemma 3.1, the theorem is proved as follows: It follows from Lemma 3.1 that $w_n(x,\eta)$ $(= W_n(x,\eta,\eta))$ is continued analytically to $B(r) \times E_+(\theta,\kappa' - \sum_{j=0}^{n}\varepsilon_j)\}$ with the estimate

$$\begin{aligned} |w_n(x,\eta)| &= |W_n(x,\eta,\mathcal{G}_\eta(|\eta|))| \\ &\leq C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^k}{k!}. \end{aligned}$$

Hence on $B(r) \times E_+(\theta,\widetilde{\kappa})$ we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} |w_n(x,\eta)| &\leq C_1 \mathrm{e}^{\delta_1|\eta|} \sum_{n=0}^{\infty} (AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^k}{k!} \\ &= C_1 \mathrm{e}^{\delta_1|\eta|} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} (AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^k}{k!}. \end{aligned}$$

Since $\xi \geq 1$, we may take δ_1 so large that

$$A' := 2AL \sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}} < 1,$$

which implies that

$$\sum_{n=k}^{\infty}(AL)^n\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^n\binom{n}{k}\le\sum_{n=k}^{\infty}(A')^n=\frac{(A')^k}{1-A'}.$$

Therefore on $B(r)\times E_+(\theta,\widetilde{\kappa})$ it holds that

$$\begin{aligned}\sum_{n=0}^{\infty}|w_n(x,\eta)| &\le \frac{C_1}{1-A'}\mathrm{e}^{\delta_1|\eta|}\sum_{k=0}^{\infty}\frac{\{A'\delta_1(1+|\eta|)\}^k}{k!}\\ &= \frac{C_1\mathrm{e}^{A'\delta_1}}{1-A'}\exp[(\delta_1+A'\delta_1)|\eta|].\end{aligned}$$

This shows that $v_n(x,\eta)$ $(=\sum_{k=0}^{n}w_k(x,\eta))$ converges to the solution $V(x,\eta)$ of (3.3) uniformly on $B(r)\times E_+(\theta,\widetilde{\kappa})$. It is clear that $V(x,\eta)$ is an analytic continuation of $v(x,\eta)$ and that

$$\max_{|x|\le r}|V(x,\eta)|\le\frac{C_1\mathrm{e}^{A'\delta_1}}{1-A'}\exp[(\delta_1+A'\delta_1)|\eta|],\quad \eta\in E_+(\theta,\widetilde{\kappa}).$$

It follows from the above argument that $v(x,\eta)$ satisfies the condition (BS). The theorem is proved. □

Therefore it is sufficient to prove Lemma 3.1.

Proof of Lemma 3.1. It is proved by the induction. Since the case $n=0$ have been already proved in [6], we assume that the claim of the lemma is proved up to n and prove it for $n+1$.

By (3.4) and (3.5), we have the following relation between W_n and W_{n+1}:

$$W_{n+1}(x,\eta,t)=\sum_{i=1}^{6}\mathcal{J}_i(x,\eta,t), \tag{3.14}$$

where

$$\begin{aligned}&\mathcal{J}_1(x,\eta,t)\\ =\ &\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\sum_{p=p_0}^{p_1}(1+d\eta)^{b/d}\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^t (1+ds_1)^{-b/d} c_p \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times \int_0^{s_1} \cdots \int_0^{s_{p-1}} W_n(x, \zeta(\eta, s_1, s_p), s_p) \mathrm{d}s_p \cdots \mathrm{d}s_2 \mathrm{d}s_1, \\
&\mathcal{J}_2(x,\eta,t) \\
= \ & -\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1} \sum_{p=p_0}^{p_1} (1+d\eta)^{b/d} c_p \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+dt}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times (1+dt)^{-b/d} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{p-1}} W_n(x, \zeta(\eta, t, s_p), s_p) \mathrm{d}s_p \cdots \mathrm{d}s_2 \mathrm{d}s_1, \\
&\mathcal{J}_3(x,\eta,t) \\
= \ & -\left(\frac{a}{b}+x\right)^{-1} \sum_{p=p_0}^{p_1} (1+d\eta)^{b/d} \\
&\times \int_0^t (1+ds_1)^{-b/d} c_p \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times \int_0^{s_1} \cdots \int_0^{s_p} W_n(x, \zeta(\eta, s_1, s_{p+1}), s_{p+1}) \mathrm{d}s_{p+1} \cdots \mathrm{d}s_2 \frac{1}{1+ds_1} \mathrm{d}s_1, \\
&\mathcal{J}_4(x,\eta,t) \\
= \ & \sum_{p=p_0}^{p_1} \int_0^t {c_p}' \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times \int_0^{s_1} \cdots \int_0^{s_p} W_n(x, \zeta(\eta, s_1, s_{p+1}), s_{p+1}) \mathrm{d}s_{p+1} \cdots \mathrm{d}s_2 \frac{1}{1+ds_1} \mathrm{d}s_1, \\
&\mathcal{J}_5(x,\eta,t) \\
= \ & -\sum_{q=q_0}^{q_1} \int_0^t e_q \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times \int_0^{s_1} \cdots \int_0^{s_{q-1}} s_q \cdot W_n(x, \zeta(\eta, s_1, s_q), s_q) \mathrm{d}s_q \cdots \mathrm{d}s_2 \frac{1}{1+ds_1} \mathrm{d}s_1, \\
&\mathcal{J}_6(x,\eta,t) \\
= \ & \sum_{q=q_0}^{q_1} \int_0^t e_q \left(\left(\frac{a}{b}+x\right) \left(\frac{1+d\eta}{1+ds_1}\right)^{-b/d} - \frac{a}{b} \right) \\
&\times \int_0^{s_1} \cdots \int_0^{s_q} W_n(x, \zeta(\eta, s_1, s_{q+1}), s_{q+1}) \mathrm{d}s_{q+1} \cdots \mathrm{d}s_2 \frac{1}{1+ds_1} \mathrm{d}s_1,
\end{aligned}
$$

where ζ is defined by

$$\frac{(1+d\eta)(1+d\nu)}{1+d\mu} = 1 + d\zeta(\eta,\mu,\nu). \tag{3.15}$$

Let us prove that each $\mathcal{J}_i(x,\eta,t)$ is well-defined on $\{(x,\eta,t);\ |x| \le r,\ \eta \in E_+(\theta,\kappa' - \sum_{j=0}^{n+1}\varepsilon_j),\ t \in \mathcal{G}_\eta^{\varepsilon_{n+1}}\}$ by taking suitable integral paths. Let us write $t \in \mathcal{G}_\eta^{\varepsilon_{n+1}}$ as $t = \mathcal{G}_\eta(\tau)$ $(\tau \in U_{\varepsilon_{n+1}}[0,|\eta|])$.

On $\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(\tau))$: Let us take integral paths as

$$s_1(\sigma_1) = \frac{\sigma_1 \mathrm{e}^{i\arg(\eta)}}{1 + d(|\eta| - \sigma_1)\mathrm{e}^{i\arg(\eta)}} \quad (\sigma_1 \in [0,\tau]), \tag{3.16}$$

and

$$s_j(\sigma_j) = \frac{\sigma_j \mathrm{e}^{i\arg(\eta)}}{1 + d(|\eta| - \sigma_1)\mathrm{e}^{i\arg(\eta)}} \quad (\sigma_j \in [0,\sigma_{j-1}]) \quad (j = 2,\dots,p). \tag{3.17}$$

Then we have $\zeta(\eta, s_1(\sigma_1), s_p(\sigma_p)) \in E_+(\theta,\kappa' - \sum_{j=0}^{n}\varepsilon_j)$ and $s_p(\sigma_p) \in \mathcal{G}^{\varepsilon_n}_{\zeta(\eta,s_1(\sigma_1),s_p(\sigma_p))}$. Hence $W_n(x,\zeta(\eta,s_1(\sigma_1),s_p(\sigma_p)),s_p(\sigma_p))$ is well-defined. It is clear that

$$\begin{aligned} & c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+ds_1(\sigma_1)}\right)^{-b/d} - \frac{a}{b}\right) \\ = \ & c_p\left(\left(\frac{a}{b}+x\right)\{1 + d(|\eta|-\sigma_1)\mathrm{e}^{i\arg(\eta)}\}^{-b/d} - \frac{a}{b}\right) \end{aligned}$$

is well-defined. Therefore $\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(\tau))$ is well-defined.

On $\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(\tau))$: Let us take integral paths as

$$s_j(\sigma_j) = \frac{\sigma_j \mathrm{e}^{i\arg(\eta)}}{1 + d(|\eta| - \tau)\mathrm{e}^{i\arg(\eta)}} \quad (\sigma_j \in [0,\sigma_{j-1}]) \quad (j = 1,\dots,p), \tag{3.18}$$

where $\sigma_0 := \tau$. Then we have $\zeta(\eta,\mathcal{G}_\eta(\tau),s_p(\sigma_p)) \in E_+(\theta,\kappa' - \sum_{j=0}^{n}\varepsilon_j)$ and $s_p(\sigma_p) \in \mathcal{G}^{\varepsilon_n}_{\zeta(\eta,\mathcal{G}_\eta(\tau),s_p(\sigma_p))}$. Hence $W_n(x,\zeta(\eta,\mathcal{G}_\eta(\tau),s_p(\sigma_p)),s_p(\sigma_p))$ is well-defined. It is clear that

$$\begin{aligned} & c_p\left(\left(\frac{a}{b}+x\right)\left(\frac{1+d\eta}{1+d\mathcal{G}_\eta(\tau)}\right)^{-b/d} - \frac{a}{b}\right) \\ = \ & c_p\left(\left(\frac{a}{b}+x\right)\{1 + d(|\eta|-\tau)\mathrm{e}^{i\arg(\eta)}\}^{-b/d} - \frac{a}{b}\right) \end{aligned}$$

is well-defined. Therefore $\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(\tau))$ is well-defined.

On $\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(\tau))$, $\mathcal{J}_4(x,\eta,\mathcal{G}_\eta(\tau))$, $\mathcal{J}_5(x,\eta,\mathcal{G}_\eta(\tau))$ and $\mathcal{J}_6(x,\eta,\mathcal{G}_\eta(\tau))$, we only state the integral paths. The suitable integral paths are as follows:

On $\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(\tau))$ and $\mathcal{J}_4(x,\eta,\mathcal{G}_\eta(\tau))$: (3.16) and

$$s_j(\sigma_j)=\frac{\sigma_j \mathrm{e}^{i\arg(\eta)}}{1+d(|\eta|-\sigma_1)\mathrm{e}^{i\arg(\eta)}}\quad(\sigma_j\in[0,\sigma_{j-1}])\quad(j=2,\dots,p+1). \tag{3.19}$$

On $\mathcal{J}_5(x,\eta,\mathcal{G}_\eta(\tau))$: (3.16) and

$$s_j(\sigma_j)=\frac{\sigma_j \mathrm{e}^{i\arg(\eta)}}{1+d(|\eta|-\sigma_1)\mathrm{e}^{i\arg(\eta)}}\quad(\sigma_j\in[0,\sigma_{j-1}])\quad(j=2,\dots,q). \tag{3.20}$$

On $\mathcal{J}_6(x,\eta,\mathcal{G}_\eta(\tau))$: (3.16) and

$$s_j(\sigma_j)=\frac{\sigma_j \mathrm{e}^{i\arg(\eta)}}{1+d(|\eta|-\sigma_1)\mathrm{e}^{i\arg(\eta)}}\quad(\sigma_j\in[0,\sigma_{j-1}])\quad(j=2,\dots,q+1). \tag{3.21}$$

By taking the above integral paths, we see that each $\mathcal{J}_i(x,\eta,t)$ (therefore $W_{n+1}(x,\eta,t)$) is well-defined on $\{(x,\eta,t);\ |x|\le r,\ \eta\in E_+(\theta,\kappa'-\sum_{j=0}^{n+1}\varepsilon_j),\ t\in\mathcal{G}_\eta^{\varepsilon_{n+1}}\}$. Moreover on $\{(x,\eta,t);\ |x|\le r,\ \eta\in E_+(\theta,\kappa'-\sum_{j=0}^{n+1}\varepsilon_j),\ t\in\mathcal{G}_\eta\}$ we have the following representations:

$$\begin{aligned}
&\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(R))\\
=\ &\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\int_0^R\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{b/d}\mathcal{C}_p(x,(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)})\\
&\times\int_0^{R_1}\cdots\int_0^{R_{p-1}}\mathcal{W}_n(x,\eta,R_1,R_p)\mathrm{d}R_p\cdots\mathrm{d}R_2\\
&\qquad\qquad\times\frac{(1+d\eta)\{\mathrm{e}^{i\arg(\eta)}\}^p}{\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{p+1}}\mathrm{d}R_1,\\
&\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(R))\\
=\ &-\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1}\sum_{p=p_0}^{p_1}\{1+d(|\eta|-R)\mathrm{e}^{i\arg(\eta)}\}^{b/d}\mathcal{C}_p(x,(|\eta|-R)\mathrm{e}^{i\arg(\eta)})\\
&\times\frac{\{\mathrm{e}^{i\arg(\eta)}\}^p}{\{1+d(|\eta|-R)\mathrm{e}^{i\arg(\eta)}\}^p}
\end{aligned}$$

$$
\begin{aligned}
& \times \int_0^R \int_0^{R_1} \cdots \int_0^{R_{p-1}} \mathcal{W}_n(x,\eta,R_1,R_p)\mathrm{d}R_p \cdots \mathrm{d}R_2 \mathrm{d}R_1, \\
& \mathcal{J}_3(x,\eta,\mathcal{G}_\eta(R)) \\
= & -\left(\frac{a}{b}+x\right)^{-1} \sum_{p=p_0}^{p_1} \int_0^R \{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{b/d} \\
& \times \mathcal{C}_p(x,(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}) \\
& \times \int_0^{R_1} \cdots \int_0^{R_p} \mathcal{W}_n(x,\eta,R_1,R_{p+1})\mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \\
& \qquad\qquad \times \frac{\{\mathrm{e}^{i\arg(\eta)}\}^{p+1}}{\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{p+1}}\mathrm{d}R_1, \\
& \mathcal{J}_4(x,\eta,\mathcal{G}_\eta(R)) \\
= & -\frac{1}{b}\left(\frac{a}{b}+x\right)^{-1} \sum_{p=p_0}^{p_1} \int_0^R \{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{b/d} \\
& \times \mathcal{C}_{p,\eta}(x,(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}) \\
& \times \int_0^{R_1} \cdots \int_0^{R_p} \mathcal{W}_n(x,\eta,R_1,R_{p+1})\mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \\
& \qquad\qquad \times \frac{\{\mathrm{e}^{i\arg(\eta)}\}^{p+1}}{\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{p}}\mathrm{d}R_1, \\
& \mathcal{J}_5(x,\eta,\mathcal{G}_\eta(R)) \\
= & -\sum_{q=q_0}^{q_1} \int_0^R \mathcal{E}_q(x,(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}) \\
& \times \int_0^{R_1} \cdots \int_0^{R_{q-1}} \mathcal{W}_n(x,\eta,R_1,R_q)R_q\mathrm{d}R_q \cdots \mathrm{d}R_2 \\
& \qquad\qquad \times \frac{\{\mathrm{e}^{i\arg(\eta)}\}^{q+1}}{\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{q+1}}\mathrm{d}R_1, \\
& \mathcal{J}_6(x,\eta,\mathcal{G}_\eta(R)) \\
= & \sum_{q=q_0}^{q_1} \int_0^R \mathcal{E}_q(x,(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}) \\
& \times \int_0^{R_1} \cdots \int_0^{R_q} \mathcal{W}_n(x,\eta,R_1,R_{q+1})\mathrm{d}R_{q+1} \cdots \mathrm{d}R_2 \\
& \qquad\qquad \times \frac{\{\mathrm{e}^{i\arg(\eta)}\}^{q+1}}{\{1+d(|\eta|-R_1)\mathrm{e}^{i\arg(\eta)}\}^{q+1}}\mathrm{d}R_1,
\end{aligned}
$$

where $\mathcal{W}_n$ is defined by

$$\mathcal{W}_n(x,\eta,\mu,\nu) = W_n(x,(|\eta|-\mu+\nu)\mathrm{e}^{i\arg(\eta)},\mathcal{G}_{(|\eta|-\mu+\nu)\mathrm{e}^{i\arg(\eta)}}(\nu)). \tag{3.22}$$

Let us estimate each $\mathcal{J}_i(x,\eta,\mathcal{G}_\eta(R))$.

First, we prepare the following lemma. We omit the proof.

LEMMA 3.2. *For $\delta > 0$, it holds that*

$$\int_0^R \int_0^{R_1} \cdots \int_0^{R_{p-1}} \mathrm{e}^{\delta R_p} \mathrm{d}R_p \cdots \mathrm{d}R_2 \mathrm{d}R_1 \le \frac{1}{\delta^p}\mathrm{e}^{\delta R} \quad (R \ge 0).$$

On $\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(R))$: It follows from the assumption of the induction that

$$\begin{aligned} (3.23)\quad & |\mathcal{W}_n(x,\eta,R_1,R_p)| \\ \le\ & C_1 \mathrm{e}^{\delta_1|\eta|}\mathrm{e}^{-\delta_1 R_1}\mathrm{e}^{\delta_1 R_p}(AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \frac{1}{(1+|\eta|-R_1+R_p-R_p)^{n\beta}} \\ & \times \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|-R_1+R_p)^k}{k!} \\ \le\ & C_1 \mathrm{e}^{\delta_1|\eta|}\mathrm{e}^{-\delta_1 R_1}\mathrm{e}^{\delta_1 R_p}(AL)^n \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \frac{1}{(1+|\eta|-R_1)^{n\beta}} \\ & \times \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^k}{k!}, \end{aligned}$$

which implies that

$$\begin{aligned} & |\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(R))| \\ \le\ & C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n A \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k(1+|\eta|)^{k+1}}{k!} \\ & \times \sum_{p=p_0}^{p_1} \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p+1}} \\ & \times \int_0^{R_1} \cdots \int_0^{R_{p-1}} \mathrm{e}^{\delta_1 R_p} \mathrm{d}R_p \cdots \mathrm{d}R_2 \mathrm{d}R_1. \end{aligned}$$

Here it follows from

$$\frac{1}{(1+|\eta|-R_1)^{n\beta+p-\beta_p+1}} \leq \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta+1}}$$

and Lemma 3.2 that

$$\begin{aligned}
&\int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p+1}} \int_0^{R_1} \cdots \int_0^{R_{p-1}} \mathrm{e}^{\delta_1 R_p} \mathrm{d}R_p \cdots \mathrm{d}R_2 \mathrm{d}R_1 \\
\leq\ & \frac{1}{{\delta_1}^{p-1}} \int_0^R \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta+1}} \mathrm{d}R_1 \\
\leq\ & \frac{1}{{\delta_1}^{p-1}} \frac{1}{(n+1)\beta} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\
\leq\ & \frac{1}{\beta} \frac{1}{{\delta_1}^{p-1}} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \frac{1}{k+1} \quad (k=0,1,2,\ldots,n).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&|\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(R))| \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n \frac{A}{\beta} \\
&\times \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \cdot \left[\sum_{p=p_0}^{p_1} \frac{1}{{\delta_1}^{p}}\right] \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\
&\times \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^{k+1}(1+|\eta|)^{k+1}}{(k+1)!} \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n \frac{A}{\beta} \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^{n+1} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\
&\times \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{{\delta_1}^{k}(1+|\eta|)^{k}}{k!}.
\end{aligned}$$

On $\mathcal{J}_4(x,\eta,\mathcal{G}_\eta(R))$: Since a similar estimate to (3.23) holds for $|\mathcal{W}_n(x,\eta,R_1,R_{p+1})|$ (let us consider R_{p+1} instead of R_p), we have

$$\begin{aligned}
&|\mathcal{J}_4(x,\eta,\mathcal{G}_\eta(R))| \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n A \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^{k}(1+|\eta|)^{k}}{k!}
\end{aligned}$$

$$\times \sum_{p=p_0}^{p_1} \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p}} \times \int_0^{R_1} \cdots \int_0^{R_p} \mathrm{e}^{\delta_1 R_{p+1}} \mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \mathrm{d}R_1.$$

Here it follows from

$$\begin{aligned} \frac{1}{(1+|\eta|-R_1)^{n\beta+p-\beta_p}} &\leq \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta}} \\ &\leq \frac{1+|\eta|}{(1+|\eta|-R_1)^{(n+1)\beta+1}}, \quad \delta_1 > 1 \end{aligned}$$

and Lemma 3.2 that

$$\begin{aligned} &\int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p}} \int_0^{R_1} \cdots \int_0^{R_p} \mathrm{e}^{\delta_1 R_{p+1}} \mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \mathrm{d}R_1 \\ \leq\ & \frac{1+|\eta|}{{\delta_1}^p} \int_0^R \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta+1}} \mathrm{d}R_1 \\ \leq\ & \frac{1+|\eta|}{{\delta_1}^{p-1}} \frac{1}{(n+1)\beta} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\ \leq\ & \frac{1}{\beta} \frac{1+|\eta|}{{\delta_1}^{p-1}} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \frac{1}{k+1} \quad (k=0,1,2,\ldots,n). \end{aligned}$$

Hence we obtain

$$\begin{aligned} &|\mathcal{J}_4(x,\eta,\mathcal{G}_\eta(R))| \\ \leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n \frac{A}{\beta} \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}} \right]^{n+1} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\ & \times \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{{\delta_1}^k (1+|\eta|)^k}{k!}. \end{aligned}$$

On $\mathcal{J}_5(x,\eta,\mathcal{G}_\eta(R))$: Let us consider R_q instead of R_p in (3.23). Then we have

$$\begin{aligned} &|\mathcal{J}_5(x,\eta,\mathcal{G}_\eta(R))| \\ \leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n A \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}} \right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k (1+|\eta|)^k}{k!} \end{aligned}$$

$$\times \sum_{q=q_0}^{q_1} \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+q-\beta_q'+1}}$$
$$\times \int_0^{R_1} \cdots \int_0^{R_{q-1}} \mathrm{e}^{\delta_1 R_q} R_q \mathrm{d}R_q \cdots \mathrm{d}R_2 \mathrm{d}R_1.$$

Here it follows from

$$\frac{1}{(1+|\eta|-R_1)^{n\beta+q-\beta_q'+1}} \leq \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta+1}}, \quad R_q \leq 1+|\eta|,$$

and Lemma 3.2 that

$$\begin{aligned} & \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+q-\beta_q'+1}} \int_0^{R_1} \cdots \int_0^{R_{q-1}} \mathrm{e}^{\delta_1 R_q} R_q \mathrm{d}R_q \cdots \mathrm{d}R_2 \mathrm{d}R_1 \\ \leq\ & \frac{1+|\eta|}{{\delta_1}^{q-1}} \int_0^R \frac{1}{(1+|\eta|-R_1)^{(n+1)\beta+1}} \mathrm{d}R_1 \\ \leq\ & \frac{1}{\beta} \frac{1+|\eta|}{{\delta_1}^{q-1}} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \frac{1}{k+1} \quad (k=0,1,2,\ldots,n). \end{aligned}$$

Hence we obtain

$$\begin{aligned} & |\mathcal{J}_5(x,\eta,\mathcal{G}_\eta(R))| \\ \leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n \frac{A}{\beta} \\ & \times \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \cdot \left[\sum_{q=q_0}^{q_1} \frac{1}{{\delta_1}^{q}}\right] \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\ & \times \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^{k+1}(1+|\eta|)^{k+1}}{(k+1)!} \\ \leq\ & C_1 \mathrm{e}^{\delta_1|\eta|} (AL)^n \frac{A}{\beta} \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^{n+1} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\ & \times \sum_{k=1}^{n+1} \binom{n}{k-1} \frac{{\delta_1}^{k}(1+|\eta|)^{k}}{k!}. \end{aligned}$$

By the above argument, it holds that

$$\sum_{i=1,4,5} |\mathcal{J}_i(x,\eta,\mathcal{G}_\eta(R))| \tag{3.24}$$

$$\leq \quad C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^{n+1}\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}$$
$$\times \sum_{k=1}^{n+1}\binom{n}{k-1}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}.$$

On $\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(R))$: Let us consider R instead of R_1 in (3.23). Then we have

$$|\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(R))|$$
$$\leq \quad C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^{n}A\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n}\sum_{k=0}^{n}\binom{n}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}$$
$$\times \sum_{p=p_0}^{p_1}\frac{\mathrm{e}^{-\delta_1 R}}{(1+|\eta|-R)^{n\beta+p-\beta_p}}\int_0^R\int_0^{R_1}\cdots\int_0^{R_{p-1}}\mathrm{e}^{\delta_1 R_p}\mathrm{d}R_p\cdots\mathrm{d}R_2\mathrm{d}R_1.$$

Here it follows from $(1+|\eta|-R)^{n\beta+p-\beta_p}\geq(1+|\eta|-R)^{(n+1)\beta}$ and Lemma 3.2 that

$$\frac{\mathrm{e}^{-\delta_1 R}}{(1+|\eta|-R)^{n\beta+p-\beta_p}}\int_0^R\int_0^{R_1}\cdots\int_0^{R_{p-1}}\mathrm{e}^{\delta_1 R_p}\mathrm{d}R_p\cdots\mathrm{d}R_2\mathrm{d}R_1$$
$$\leq \quad \frac{1}{{\delta_1}^p}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}},$$

Hence we obtain

$$|\mathcal{J}_2(x,\eta,\mathcal{G}_\eta(R))|$$
$$\leq \quad C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^{n}A\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n}\cdot\left[\sum_{p=p_0}^{p_1}\frac{1}{{\delta_1}^{p}}\right]\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}$$
$$\times \sum_{k=0}^{n}\binom{n}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}$$
$$\leq \quad C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^{n}A\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}$$
$$\times \sum_{k=0}^{n}\binom{n}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}.$$

On $\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(R))$: Let us consider R_{p+1} instead of R_p in (3.23). Then we have

$$\begin{aligned}
&|\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(R))| \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n A \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k (1+|\eta|)^k}{k!} \\
& \times \sum_{p=p_0}^{p_1} \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p+1}} \\
& \times \int_0^{R_1} \cdots \int_0^{R_p} \mathrm{e}^{\delta_1 R_{p+1}} \mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \mathrm{d}R_1.
\end{aligned}$$

Here similarly to the calculation for $\mathcal{J}_1(x,\eta,\mathcal{G}_\eta(R))$, we have

$$\begin{aligned}
& \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+p-\beta_p+1}} \int_0^{R_1} \cdots \int_0^{R_p} \mathrm{e}^{\delta_1 R_{p+1}} \mathrm{d}R_{p+1} \cdots \mathrm{d}R_2 \mathrm{d}R_1 \\
\leq\ & \frac{1}{{\delta_1}^p} \frac{1}{(n+1)\beta} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \ \leq\ \frac{1}{\beta} \frac{1}{{\delta_1}^p} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}},
\end{aligned}$$

which implies

$$\begin{aligned}
&|\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(R))| \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n \frac{A}{\beta} \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^{n+1} \frac{1}{(1+|\eta|-R)^{(n+1)\beta}} \\
& \times \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k (1+|\eta|)^k}{k!}.
\end{aligned}$$

On $\mathcal{J}_6(x,\eta,\mathcal{G}_\eta(R))$: Let us consider R_{q+1} instead of R_q in (3.23). Then we have,

$$\begin{aligned}
&|\mathcal{J}_6(x,\eta,\mathcal{G}_\eta(R))| \\
\leq\ & C_1 \mathrm{e}^{\delta_1|\eta|}(AL)^n A \left[\sum_{p'=\xi}^{\omega} \frac{1}{{\delta_1}^{p'}}\right]^n \sum_{k=0}^{n} \binom{n}{k} \frac{{\delta_1}^k (1+|\eta|)^k}{k!} \\
& \times \sum_{q=q_0}^{q_1} \int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+q-{\beta_q}'+1}} \\
& \times \int_0^{R_1} \cdots \int_0^{R_q} \mathrm{e}^{\delta_1 R_{q+1}} \mathrm{d}R_{q+1} \cdots \mathrm{d}R_2 \mathrm{d}R_1.
\end{aligned}$$

Here similarly to the calculation for $\mathcal{J}_3(x,\eta,\mathcal{G}_\eta(R))$, we have

$$\begin{aligned}&\int_0^R \frac{\mathrm{e}^{-\delta_1 R_1}}{(1+|\eta|-R_1)^{n\beta+q-\beta_q{}'+1}}\int_0^{R_1}\cdots\int_0^{R_q}\mathrm{e}^{\delta_1 R_{q+1}}\mathrm{d}R_{q+1}\cdots\mathrm{d}R_2\mathrm{d}R_1\\ \leq\ & \frac{1}{\beta}\frac{1}{{\delta_1}^q}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}},\end{aligned}$$

which implies

$$\begin{aligned}&|\mathcal{J}_6(x,\eta,\mathcal{G}_\eta(R))|\\ \leq\ & C_1\mathrm{e}^{\delta_1|\eta|}(AL)^n\frac{A}{\beta}\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}\\ &\times\sum_{k=0}^{n}\binom{n}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}.\end{aligned}$$

By the above argument, it holds that

$$\begin{aligned}&\sum_{i=2,3,6}|\mathcal{J}_i(x,\eta,\mathcal{G}_\eta(R))|\\ \leq\ & C_1\mathrm{e}^{\delta_1|\eta|}(AL)^{n+1}\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}\\ &\times\sum_{k=0}^{n}\binom{n}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}.\end{aligned}\tag{3.25}$$

Therefore it follows from (3.24) and (3.25) that

$$\begin{aligned}&|W_{n+1}(x,\eta,\mathcal{G}_\eta(R))|\\ \leq\ & \sum_{i=1}^{6}|\mathcal{J}_i(x,\eta,\mathcal{G}_\eta(R))|\\ \leq\ & C_1\mathrm{e}^{\delta_1|\eta|}(AL)^{n+1}\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\\ &\times\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}\left\{\sum_{k=1}^{n+1}\binom{n}{k-1}+\sum_{k=0}^{n}\binom{n}{k}\right\}\frac{{\delta_1}^k(1+|\eta|)^k}{k!}\end{aligned}$$

$$= \quad C_1 e^{\delta_1|\eta|}(AL)^{n+1}\left[\sum_{p'=\xi}^{\omega}\frac{1}{{\delta_1}^{p'}}\right]^{n+1}\frac{1}{(1+|\eta|-R)^{(n+1)\beta}}$$
$$\times \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{{\delta_1}^k(1+|\eta|)^k}{k!},$$

which implies the lemma for $n+1$. The proof is completed. □

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Graduate School of Mathematics
Nagoya University
Nagoya 464-8602, Japan
E-mail: m96006y@math.nagoya-u.ac.jp

Present address
Department of Mathematics
Meijo University
Tempaku, Nagoya, Aichi 468
Japan
E-mail: hibinom@ccmfs.meijo-u.ac.jp