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Malliavin Calculus Revisited

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Abstract. The author considers the regularity on diffusion semigroups, and shows a precise estimate under a certain assumption which is much weaker than hypoellipticity assumptions.

1. Introduction and Main Results

Let $W_0 = \{w \in C([0,\infty);\mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and P be the Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1,\dots,d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t),\dots,B^d(t)); t \in [0,\infty)\}$ is a d-dimensional Brownian motion under $P(dw)$. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $V_0, V_1, \dots, V_d \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$. Here $C_b^\infty(\mathbf{R}^N;\mathbf{R}^n)$ denotes the space of $\mathbf{R}^n$-valued smooth functions defined in $\mathbf{R}^N$ whose derivatives of any order are bounded. We regard elements in $C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ as vector fields on $\mathbf{R}^N$. For simplicity, we sometimes denote (i) by i, $i = 0,1,\dots,d$, and $C_b^\infty(\mathbf{R}^N;\mathbf{R})$ by $C_b^\infty(\mathbf{R}^N)$.

Now let $X(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be the solution to the Stratonovich stochastic integral equation

$$(1) \qquad X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s).$$

Then there is a unique solution to this equation. Moreover we may assume that $X(t,x)$ is continuous in t and smooth in x and $X(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{k=1}^{\infty} \{0,1,\dots,d\}^k$. Then $\mathcal{A}$ becomes a semigroup with a product $*$ defined by $\alpha * \beta = (\alpha^1,\dots,\alpha^k,\beta^1,\dots,\beta^\ell)$ for $\alpha = (\alpha^1,\dots,\alpha^k) \in \mathcal{A}$ and $\beta = (\beta^1,\dots,\beta^\ell) \in \mathcal{A}$. For $\alpha \in \mathcal{A}$, let $|\alpha| = 0$ if $\alpha = \emptyset$, let $|\alpha| = k$ if $\alpha = (\alpha^1,\dots,\alpha^k) \in \{0,1,\dots,d\}^k$, and let $\| \alpha \| = |\alpha| + \mathrm{card}\{1 \le i \le |\alpha|;\ \alpha^i = 0\}$. Let $\mathcal{A}_0$ and $\mathcal{A}_1$ denote $\mathcal{A} \setminus \{\emptyset\}$ and $\mathcal{A} \setminus \{\emptyset, 0\}$, respectively. Also, for each $m \ge 1$, $\mathcal{A}(m)$, $\mathcal{A}_0(m)$ and $\mathcal{A}_1(m)$ denote $\{\alpha \in \mathcal{A};\ \| \alpha \| \le m\}$, $\{\alpha \in \mathcal{A}_0;\ \| \alpha \| \le m\}$ and $\{\alpha \in \mathcal{A}_1;\ \| \alpha \| \le m\}$ respectively.

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We define vector fields $V_{[\alpha]}$, $\alpha \in \mathcal{A}$, inductively by

$$V_{[\emptyset]} = 0, \qquad V_{[i]} = V_i, \quad i = 0, 1, \ldots, d,$$

$$V_{[\alpha * i]} = [V_{[\alpha]}, V_i], \qquad i = 0, 1, \ldots, d.$$

Definition 1. We say that a system $\{V_i; i = 0, 1, \ldots, d\}$ of vector fields satisfies the condition (UFG), if there are an integer ℓ and $\varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N)$, $\alpha \in \mathcal{A}_1$, $\beta \in \mathcal{A}_1(\ell)$, satisfying the following.

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(\ell)} \varphi_{\alpha,\beta} V_{[\beta]}, \qquad \alpha \in \mathcal{A}_1.$$

Let $c \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$ and let us define a semigroup of linear operators $\{P_t^c\}_{t \in [0,\infty)}$ by

$$(P_t^c f)(x) = E[\exp(\int_0^t c(X(s,x))ds) f(X(t,x))], \qquad t \in [0,\infty),\ f \in C_b(\mathbf{R}^N).$$

Our main result is the following.

Theorem 2. *Suppose that* $\{V_i; i = 0, 1, \ldots, d\}$ *satisfies the* (UFG) *condition. Then for any* $k, m \geq 0$ *and* $\alpha_1, \ldots, \alpha_{k+m} \in \mathcal{A}_1$, *there is a constant* $C > 0$ *such that*

$$\| V_{[\alpha_1]} \cdots V_{[\alpha_k]} P_t^c V_{[\alpha_{k+1}]} \cdots V_{[\alpha_{k+m}]} f \|_{L^p(dx)} \\ \leq C t^{-(\|\alpha_1\| + \cdots + \|\alpha_{k+m}\|)/2} \| f \|_{L^p(dx)}$$

for any $f \in C_0(\mathbf{R}^N)$, $t \in (0,1]$ *and* $p \in [1,\infty]$.

Definition 3. We say that a system $\{V_i; i = 0, 1, \ldots, d\}$ satisfies the condition (UH), if there are an integer ℓ such that

$$\inf\{ \sum_{\alpha \in \mathcal{A}_1(\ell)} (V_{[\alpha]}(x), \xi)^2;\ x, \xi \in \mathbf{R}^N,\ |\xi| = 1\} > 0.$$

Remark 4. (1) If a system $\{V_i; i = 0, 1, \ldots, d\}$ of vector fields satisfies the condition (UH), then it satisfies the condition (UFG).

(2) Theorem 2 is proved in Kusuoka-Stroock [4] under the assumption that $\{V_i; i = 0, 1, \dots, d\}$ satisfies the condition (UH).

REMARK 5. Sussman [5] introduced a local version of the condition (UFG). By his argument, we see that if V_i, $i = 0, 1, \dots, d$, are real analytic and periodic with the same period, then the system $\{V_i; i = 0, 1, \dots, d\}$ satisfies the condition (UFG).

2. Basic Relations

Form now on, we assume the assumption (UFG) throughout this paper. We define $\hat{B}^{\circ\alpha}(t)$, $t \in [0, \infty)$, $\alpha \in \mathcal{A}$, inductively by

$$\hat{B}^{\circ\emptyset}(t) = 1,$$

$$\hat{B}^{\circ(i)}(t) = B^i(t), \ i = 0, 1, \dots, d$$

and

$$\hat{B}^{\circ(i*\alpha)}(t) = \int_0^t \hat{B}^{\circ\alpha}(s) \circ dB^i(s) \qquad i = 0, 1, \dots, d.$$

Let $J_i^j(t, x) = \frac{\partial}{\partial x^i} X^j(t, x)$. Then for any C_b^∞ vector field W on $\mathbf{R}^N$, we see that

$$(X(t)_* W)^i(X(t, x)) = \sum_{j=1}^N J_j^i(t, x) W^j(x),$$

where $X(t)_*$ is a push-forward operator with respect to the diffeomorphism $X(t, \cdot) : \mathbf{R}^N \to \mathbf{R}^N$. Therefore we see that

$$d(X(t)_*^{-1} W)(x) = -\sum_{i=0}^d (X(t)_*^{-1}[W, V_i])(x) \circ dB^i(t)$$

for any C_b^∞ vector field W on $\mathbf{R}^N$ (cf. [3]). So we have for $\alpha \in \mathcal{A}_1(\ell)$,

$$d(X(t)_*^{-1} V_{[\alpha]})(x) = \sum_{i=0}^d \sum_{\beta \in \mathcal{A}_1(\ell)} c_{\alpha,i}^\beta(X(t, x))(X(t)_*^{-1} V_{[\beta]})(x) \circ dB^i(t),$$

where

$$c_{\alpha,i}^\beta(x) = \begin{cases} -1, & \text{if } \alpha * i \in \mathcal{A}_1(\ell) \text{ and } \beta = \alpha * i \\ 0, & \text{if } \alpha * i \in \mathcal{A}_1(\ell) \text{ and } \beta \neq \alpha * i \\ -\varphi_{\alpha*i,\beta}(x), & \text{otherwise.} \end{cases}$$

Note that $c^{\beta}_{\alpha,i} \in C^{\infty}_b(\mathbf{R}^N)$. Let $a^{\beta}_{\alpha}(t,x)$, $\alpha,\beta \in \mathcal{A}_1(\ell)$, be the solution to the following SDE

$$da^{\beta}_{\alpha}(t,x)$$

$$= \sum_{i=0}^{d} \sum_{\gamma\in\mathcal{A}_1(\ell)} c^{\gamma}_{\alpha,i}(X(t,x))a^{\beta}_{\gamma}(t,x)dB^i(t) + \frac{1}{2}\sum_{i=1}^{d} \sum_{\gamma\in\mathcal{A}_1(\ell)} (V_i c^{\gamma}_{\alpha,i})(X(t,x))a^{\beta}_{\gamma}dt$$

$$+\frac{1}{2}\sum_{i=1}^{d} \sum_{\gamma\xi\in\mathcal{A}_1(\ell)} (c^{\xi}_{\alpha,i}c^{\gamma}_{\xi,i})(X(t,x))a^{\beta}_{\gamma}(t,x)dt,$$

$$a^{\beta}_{\alpha}(0,x) = \delta^{\beta}_{\alpha}.$$

Here δ^{β}_{α} is Kronecker's delta.

Such a solution exists uniquely, and moreover, we may assume that $a^{\beta}_{\alpha}(t,x)$ is smooth in x with probability one and that

$$\sup_{x\in\mathbf{R}^N} E^P[\sup_{t\in[0,T]} |\frac{\partial^{|\gamma|}}{\partial x^{\gamma}} a^{\beta}_{\alpha}(t,x)|^p] < \infty, \qquad p \in [1,\infty),\ T > 0$$

for any multi-index γ. One can easily see that

$$da^{\beta}_{\alpha}(t,x) = \sum_{i=0}^{d} \sum_{\gamma\in\mathcal{A}_1(\ell)} (c^{\gamma}_{\alpha,i}(X(t,x))a^{\beta}_{\gamma}(t,x)) \circ dB^i(t). \tag{2}$$

Then the uniqueness of SDE implies

$$(X(t)^{-1}_* V_{[\alpha]})(x) = \sum_{\beta\in\mathcal{A}_1(\ell)} a^{\beta}_{\alpha}(t,x)V_{[\beta]}(x),\ \alpha \in \mathcal{A}_1(\ell).$$

Similarly we see that there exists a unique solution $b^{\beta}_{\alpha}(t,x)$, $\alpha,\beta \in \mathcal{A}_1(\ell)$, to the SDE

$$b^{\beta}_{\alpha}(t,x) = \delta^{\beta}_{\alpha} - \sum_{i=0}^{d} \sum_{\gamma\in\mathcal{A}_1(\ell)} \int_0^t b^{\gamma}_{\alpha}(s,x)c^{\beta}_{\gamma,i}(X(s,x)) \circ dB^i(s). \tag{3}$$

and we see that $b^{\beta}_{\alpha}(t,x)$ is smooth in x with probability one ,

$$\sup_{x\in\mathbf{R}^N} E^P[\sup_{t\in[0,T]} |\frac{\partial^{|\gamma|}}{\partial x^{\gamma}} b^{\beta}_{\alpha}(t,x)|^p] < \infty, \qquad p \in [1,\infty),\ T > 0$$

for any multi-index γ, and that

$$V_{[\alpha]}(x) = \sum_{\beta \in \mathcal{A}_1(\ell)} b_\alpha^\beta(t,x)(X(t)_*^{-1} V_{[\beta]})(x), \ \alpha \in \mathcal{A}_1(\ell).$$

Note that

$$a_\alpha^\beta(t,x)$$
$$= \delta_\alpha^\beta + \sum_{i=0}^d \sum_{\gamma \in \mathcal{A}_1(\ell)} \int_0^t (c_{\alpha,i}^\gamma(X(s,x)) a_\gamma^\beta(s,x)) \circ dB^i(s).$$

So if $\| \alpha \| \le \ell - 2$,

$$a_\alpha^\beta(t,x) = \delta_\alpha^\beta + \sum_{i=0}^d \int_0^t (-1) a_{\alpha * i}^\beta(s,x) \circ dB^i(s),$$

and if $\| \alpha \| = \ell - 1$,

$$a_\alpha^\beta(t,x)$$
$$= \delta_\alpha^\beta + \sum_{i=1}^d \int_0^t (-1) a_{\alpha * i}^\beta(s,x) \circ dB^i(s)$$
$$+ \sum_{\gamma \in \mathcal{A}_1(\ell)} \int_0^t c_{\alpha,0}^\gamma(X(s,x)) a_\gamma^\beta(s,x) ds.$$

So we have for any $\alpha, \beta \in \mathcal{A}_1(\ell)$ with $\| \alpha \| \le \| \beta \|$,

$$a_\alpha^\beta(t,x) = a_\alpha^{0,\beta}(t,x) + r_\alpha^\beta(t,x), \tag{4}$$

where

$$a_\alpha^{0,\beta}(t,x) = \begin{cases} (-1)^{|\gamma|} \hat{B}^{\circ\gamma}(t), & \text{if } \beta = \alpha * \gamma \text{ for some } \gamma \in \mathcal{A}, \\ 0, & \text{otherwise }, \end{cases} \tag{5}$$

and

$$r_\alpha^\beta(t,x)$$
$$= {\sum_{\gamma,j}}' \sum_{\delta \in \mathcal{A}_1(\ell)} \int_0^t \circ dB^{\gamma_1}(s_1) (\int_0^{s_1} \circ dB^{\gamma_2}(s_2) \dots (\int_0^{s_{k-1}} \circ dB^{\gamma_k}(s_k)$$
$$(\int_0^{s_k} \circ dB^j(s_{k+1}) (-1)^{|\gamma|} (c_{\alpha*\gamma,j}^{0,\delta}(X(s_{k+1},x)) a_\delta^\beta(X(s_{k+1},x)))) \dots),$$

where $\sum'_{\gamma,j}$ is the summation taken for $\gamma \in \mathcal{A}$ and $j = 0, 1, \ldots, d$ such that $\| \gamma \| \leq \ell - \| \alpha \|$ and $\| \gamma * j \| \geq \ell + 1 - \| \alpha \|$. Therefore we have

$$\sup_{x \in \mathbf{R}^N} E[(\sup_{t \in (0,1]} t^{-(\ell+1-\|\alpha\|)/2+1/4} |r_\alpha^\beta(t,x)|)^p] < \infty \tag{6}$$

for any $p \in (1, \infty)$, $\alpha, \beta \in \mathcal{A}_1(\ell)$ with $\| \alpha \| \leq \| \beta \|$.

3. Integration by Parts Formula

In this section, we use Malliavin calculus to analyze the operator P_t^c. We use the notation in [1] and [2]. Let $k^\alpha : [0, \infty) \times \mathbf{R}^N \times W_0 \to H$, $\alpha \in \mathcal{A}_1(\ell)$, be given by

$$k^\alpha(t,x) = (\int_0^{t\wedge\cdot} a_i^\alpha(s,x) ds)_{i=1,\ldots,d}, \qquad (t,x) \in [0,\infty) \times \mathbf{R}^N.$$

Then we have by

$$X(t)_*^{-1} DX(t,x) = (\int_0^{t\wedge\cdot} (X(s)_*^{-1} V_i)(x) ds)_{i=1,\ldots,d} = \sum_{\alpha \in \mathcal{A}_1(\ell)} k^\alpha(t,x) V_{[\alpha]}(x)$$

for $(t,x) \in [0,\infty) \times \mathbf{R}^N$ (c.f.[3]). Then we have

$$\begin{aligned} D(f(X(t,x)) &= {}_{T_x^*}\langle (X(t)^* df)(x), X(t)_*^{-1} DX(t,x) \rangle_{T_x} \\ &= \sum_{\beta \in \mathcal{A}_1(\ell)} {}_{T_x^*}\langle (X(t)^* df)(x), V_{[\beta]}(x) \rangle_{T_x} k^\beta(t,x) \\ &= \sum_{\beta \in \mathcal{A}_1(\ell)} (V_{[\beta]} f)(X(t,x)) k^\beta(t,x). \end{aligned} \tag{7}$$

Let $M^{\alpha,\beta}(t,x)$, $(t,x) \in [0,\infty) \times \mathbf{R}^N$, $\alpha, \beta \in \mathcal{A}_1(\ell)$, be given by

$$\begin{aligned} M^{\alpha,\beta}(t,x) &= t^{-(\|\alpha\|+\|\beta\|)/2} (k^\alpha(t,x), k^\beta(t,x))_H \\ &= t^{-(\|\alpha\|+\|\beta\|)/2} \sum_{i=1}^d \int_0^t a_i^\alpha(s,x) a_i^\beta(s,x) ds \end{aligned} \tag{8}$$

The following will be shown in the next section.

LEMMA 6. *For any $p \in (1, \infty)$,*

$$\sup_{t \in (0,1], x \in \mathbf{R}^N} E^\mu[\det(M^{\alpha,\beta}(t,x))_{\alpha,\beta \in \mathcal{A}_1(\ell)}^{-p}] < \infty.$$

Let E be a separable real Hilbert space and $r \in \mathbf{R}$. Let $\mathcal{K}_r(E)$ denote the set of $f : (0,1] \times \mathbf{R}^N \to \mathbf{D}^{\infty}_{\infty-}(E)$ satisfying the following two conditions.
(1) $f(t,x)$ is smooth in x and $\frac{\partial^\nu f}{\partial x^\nu}(t,x)$ is continuous in $(t,x) \in (0,1] \times \mathbf{R}^N$ with probability one for any multi-index ν.
(2) $\sup_{t \in (0,1], x \in \mathbf{R}^N} t^{-r/2} \| \frac{\partial^\nu f}{\partial^\nu x}(t,x) \|_{\mathbf{D}^s_p(E)} < \infty$ for any $s \in \mathbf{R}$ and $p \in (1,\infty)$.
We denote $\mathcal{K}_r(\mathbf{R})$ by $\mathcal{K}_r$.

Then we have the following.

Lemma 7. (1) *Suppose that* $f \in \mathcal{K}_r(E)$, $r \geq 0$, *and let* $g_i(t,x) = \int_0^t f(s,x)dB^i(s)$, $i = 0,1,\dots,d$, $t \in (0,1]$, $x \in \mathbf{R}^N$. *Then* $g_0 \in \mathcal{K}_{r+2}(E)$ *and* $g_i \in \mathcal{K}_{r+1}(E)$, $i = 1,\dots,d$.
(2) a^β_α, $b^\beta_\alpha \in \mathcal{K}_{(\|\beta\|-\|\alpha\|)\vee 0}$ *for* $\alpha,\beta \in \mathcal{A}_1(\ell)$.
(3) $k^\alpha \in \mathcal{K}_{\|\alpha\|}(H)$, $\alpha \in \mathcal{A}_1(\ell)$.
(4) *Let* $\{M^{-1}_{\alpha,\beta}(t,x)\}_{\alpha,\beta \in \mathcal{A}_1(\ell)}$ *be the inverse matrix of* $\{M^{\alpha,\beta}(t,x)\}_{\alpha,\beta \in \mathcal{A}_1(\ell)}$. *Then* $M^{-1}_{\alpha,\beta} \in \mathcal{K}_0$, $\alpha,\beta \in \mathcal{A}_1(\ell)$.

Proof. Note that

$$Dg_0(t,x)(h) = \int_0^t Df(s,x)(h)dB_0(t), \text{ and}$$

$$Dg_i(t,x)(h) = \int_0^t Df(s,x)(h)dB_i(t) + \int_0^t f(s,x)h_i(s)ds, \quad i = 1,\dots,d,$$

for any $h \in H$. This implies our assertion (1)(c.f.[4]).

We see that $f(X(t,x))$, $a^\beta_\alpha(t,x)$, $b^\beta_\alpha(t,x) \in \mathcal{K}_0(\mathbf{R})$ for any $f \in C^\infty_b(\mathbf{R}^N,\mathbf{R})$ and $\alpha,\beta \in \mathcal{A}_1(\ell)$, since they are solutions to good stochastic differential equations (c.f. [3]). Then the assertion (2) follows from the assertion (1) and Equations (2), (3) and (4).

The assertion (3) follows from the definition of $k^\alpha(t,x)$ and the assertions (1) and (2). Then we see that $M^{\alpha,\beta}(t,x) \in \mathcal{K}_0(\mathbf{R})$, $\alpha,\beta \in \mathcal{A}_1(\ell)$, by the assertion (3). This fact and Lemma 6 imply the assertion (4). This completes the proof. □

For each $\alpha \in \mathcal{A}_1(\ell)$, and $u \in \mathcal{K}_r(\mathbf{R})$, $r \in \mathbf{R}$, let

$$(D^{(\alpha)}u)(t,x) = (Du(t,x), k^\alpha(t,x))_H.$$

Then we have the following.

Lemma 8. (1) *For any* $\alpha \in \mathcal{A}_1(\ell)$, *and* $u \in \mathcal{K}_r$, $r \in \mathbf{R}$, $D^{(\alpha)}u \in \mathcal{K}_{r+\|\alpha\|}$.
(2) *For any* $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, $t \in (0,1]$, *and* $x \in \mathbf{R}^N$, *we have*

$$(V_{[\alpha]}f)(X(t,x))$$
$$= t^{-\|\alpha\|/2} \sum_{\beta \in \mathcal{A}_1(\ell)} t^{-\|\beta\|/2} M^{-1}_{\alpha\beta}(t,x) D^{(\beta)}(f(X(t,x))).$$

(3) *For any* $r \in \mathbf{R}$, $\Phi \in \mathcal{K}_r$, *and* $\alpha \in \mathcal{A}_1(\ell)$,

$$E^\mu[\Phi(t,x)(V_{[\alpha]}f)(X(t,x))] = t^{-\|\alpha\|/2} E^\mu[\Phi_\alpha(t,x) f(X(t,x))],$$

where

$$\begin{aligned} &\Phi_\alpha(t,x) \\ = &\sum_{\beta \in \mathcal{A}_1(\ell)} t^{-\|\beta\|/2} \{ -D^{(\beta)}\Phi(t,x) M^{-1}_{\alpha\beta}(t,x) \\ &-(\sum_{\gamma_1,\gamma_2 \in \mathcal{A}_1(\ell)} \Phi(t,x) M^{-1}_{\alpha\gamma_1}(t,x) D^{(\beta)} M^{\gamma_1\gamma_2}(t,x) M^{-1}_{\gamma_2\beta}(t,x)) \\ &\qquad +\Phi(t,x) M^{-1}_{\alpha\beta}(t,x) D^* k^\beta(t,x) \}, \qquad t > 0,\ x \in \mathbf{R}^N. \end{aligned}$$

In particular, $\Phi_\alpha \in \mathcal{K}_r$.

Proof. By Equation (7), we have

$$D^{(\alpha)}(f(X(t,x)) = \sum_{\beta \in \mathcal{A}_1(\ell)} t^{(\|\alpha\|+\|\beta\|)/2} M^{\alpha\beta}(t,x)(V_{[\beta]}f)(X(t,x)).$$

This implies our assertion. □

For any $\Phi \in \mathcal{K}_r(\mathbf{R})$, $r \in \mathbf{R}$, let us define an operator $T_\Phi(t)$, $t \in (0,1]$ in $C_b^\infty(\mathbf{R}^N; \mathbf{R})$ by

$$(T_\Phi(t)f)(x) = E^\mu[\Phi(t,x) f(X(t,x))], \qquad x \in \mathbf{R}^N.$$

Corollary 9. *Let $r \in \mathbf{R}$, and $\Phi \in \mathcal{K}_r$.*
(1) *There is a constant $C < \infty$ such that*

$$\| T_\Phi(t) f \|_\infty \leq \frac{C}{t^{r/2}} \| f \|_\infty$$

for any $t \in (0,1]$ and $f \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$.
(2) *For any $\alpha \in \mathcal{A}_1(\ell)$ there are $\Phi_{\alpha,i} \in \mathcal{K}_{r-\|\alpha\|}$, $i = 0,1$, such that*

$$T_\Phi(t) V_{[\alpha]} = T_{\Phi_{\alpha,0}}(t) \qquad \textit{and} \qquad V_{[\alpha]} T_\Phi(t) = T_{\Phi_{\alpha,1}}(t).$$

Proof. The assetion (1) is obvious. So we will prove the assertion (2). The existence of $\Phi_{\alpha,0}(t,x)$ follows from Lemma 8(3). Let

$$\Psi'_\alpha(t,x) = \sum_{i=1}^N V_{[\alpha]}^i(x) \frac{\partial}{\partial x^i} \Phi(t,x)$$

and

$$\Psi_{\alpha,\beta}(t,x) = \Phi(t,x) b_\alpha^\beta(t,x), \qquad \beta \in \mathcal{A}_1(\ell).$$

Then we see that $\Psi' \in \mathcal{K}_r$ and $\Psi_\beta \in \mathcal{K}_{r+\|\beta\|-\|\alpha\|}$. Also, we see that

$$V_{[\alpha]} T_\Phi(t) = T_{\Psi'_\alpha}(t) + \sum_{\beta \in \mathcal{A}_1(\ell)} T_{\Psi_{\alpha,\beta}}(t) V_{[\beta]}.$$

So we see the existence of $\Phi_{\alpha,1}(t,x)$ from Lemma 8(3). □

Now let us prove Theorem 2. Let $\Phi(t,x) = \exp(\int_0^t c(X(s,x))ds)$. Then we see that $\Phi \in \mathcal{K}_0$, and that $P_t^c = T_\Phi(t)$, $t \in (0,1]$. So Corollary 9 implies the assertion for $p = \infty$.

Now let $g_i \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$, $i = 0, \ldots, d$, be given by

$$g_i(x) = \sum_{j=1}^d (\frac{\partial}{\partial x^j} V_i^j)(x), \quad x \in \mathbf{R}^N.$$

Then we see that the formal self adjoint operator V_i^* is $-V_i - g_i$. Let $\tilde{V}_i \in C_b^\infty(\mathbf{R}^N; \mathbf{R}^N)$, $i = 0, \ldots, d$, and $\tilde{c} \in C_b^\infty(\mathbf{R}^N; \mathbf{R})$ be given by

$$\tilde{V}_0 = -V_0 + \sum_{j=1}^d g_j V_j, \qquad \tilde{V}_i = V_i, \quad i = 1, \ldots, d,$$

and

$$\tilde{c} = -g_0 + \frac{1}{2}\sum_{j=1}^{d}(g_j^2 + V_j g_j).$$

Then we see that the system of vector fields $\{\tilde{V}_i; i = 0, 1, \ldots, \}$ satisfies the condition (UFG). Let us think of the SDE

$$\tilde{X}(t,x) = x + \sum_{i=0}^{d}\int_0^t \int_0^d \tilde{V}_i(\tilde{X}(s,x)) \circ dB_i(s).$$

and let $\tilde{P}_t$, $t \geq 0$, be a linear operator in $C_b^\infty(\mathbf{R}^N)$, given by

$$\tilde{P}_t = E^\mu[\exp(\int_0^t \tilde{c}(\tilde{X}(t,x)))f(\tilde{X}(t,x))].$$

Then we see (c.f. [3]) that

$$\int_{\mathbf{R}^N}(P_t^c f)(x)g(x)dx = \int_{\mathbf{R}^N} f(x)(\tilde{P}_t g)(x)dx, \quad f, g \in C_0^\infty(\mathbf{R}^N).$$

So we see that for any $f \in C_0^\infty(\mathbf{R}^N)$

$$\| V_{[\alpha_1]} \cdots V_{[\alpha_k]} P_t^c V_{[\alpha_{k+1}]} \cdots V_{[\alpha_{k+m}]} f \|_{L^1(dx)}$$

$$\leq \sup\{| \int_{\mathbf{R}^N} f(x)(V_{[\alpha_{k+m}]}^* \cdots V_{[\alpha_{k+1}]}^* \tilde{P}_t V_{[\alpha_k]}^* \cdots V_{[\alpha_1]}^* g)(x)dx|; \\ g \in C_0^\infty(\mathbf{R}^N), \ \| g \|_\infty \leq 1\}.$$

So we have our assertion for the case $p = 1$. Then by the interpolation theory we have our assertion.

This completes the proof of Theorem 2.

4. Proof of Lemma 6

First note the following theorem due to [3].

THEOREM 10. *For any $m \geq 0$ and $p \in (1, \infty)$, we have*

$$E[\inf\{\int_0^1 (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha \hat{B}^\alpha(t))^2 dt; \sum_{\alpha \in \mathcal{A}(m)} a_\alpha^2 = 1\}^{-p}] = C_{m,p} < \infty.$$

Note that for any $m \geq 0$, $p \in (1, \infty)$ and $T, s > 0$,

$$E[\inf\{\int_0^T (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha \hat{B}^\alpha(t))^2 dt; \sum_{\alpha \in \mathcal{A}(m)} T^{\|\alpha\|+1} a_\alpha^2 \geq s\}^{-p}] = C_{m,p} s^{-p}.$$

Then we have the following.

LEMMA 11. *Let $m \geq 0$, and $f_\alpha : [0,1] \times \Omega \to \mathbf{R}$, $\alpha \in \mathcal{A}(m)$, be continuous processes. If*

$$A_p = \sup_{T \in (0,1]} E[(T^{-(m/2+3/4)}(\sum_{\alpha \in \mathcal{A}(m)} \int_0^T f_\alpha(t)^2 dt)^{1/2})^p] < \infty, \qquad p \in [1, \infty),$$

then

$$P(\inf\{(\int_0^T (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha (\hat{B}^\alpha(t) + f_\alpha(t)))^2 dt)^{1/2}; \sum_{\alpha \in \mathcal{A}(m)} T^{\|\alpha\|+1} a_\alpha^2 = 1\} \leq z^{-1})$$
$$\leq (4^p C_{m,p} + A_{2p}) z^{-p\gamma}$$

for any $T \in (0,1]$ and $z \geq 1$. Here $\gamma = (4m+5)^{-1}$.

PROOF. Note that for $T \in (0,1]$, and $y \geq 1$

$$(\int_0^T (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha (\hat{B}^\alpha(t) + f_\alpha(t)))^2 dt)^{1/2}$$
$$\geq (\int_0^{T/y} (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha (\hat{B}^\alpha(t) + f_\alpha(t)))^2 dt)^{1/2}$$
$$\geq (\int_0^{T/y} (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha \hat{B}^\alpha(t))^2 dt)^{1/2}$$
$$-(\sum_{\alpha \in \mathcal{A}(m)} T^{\|\alpha\|+1} a_\alpha^2)^{1/2} (\sum_{\alpha \in \mathcal{A}(m)} T^{-(m+1)} \int_0^{T/y} f_\alpha(t)^2 dt)^{1/2}.$$

Then we have for any $T \in (0,1]$, $y \geq 1$, and $z = y^{m/2+5/8}$, we have

$$P(\inf\{(\int_0^T (\sum_{\alpha \in \mathcal{A}(m)} a_\alpha (\hat{B}^\alpha(t) + f_\alpha(t)))^2 dt)^{1/2}; \sum_{\alpha \in \mathcal{A}(m)} T^{\|\alpha\|+1} a_\alpha^2 = 1\} \leq z^{-1})$$

$$\leq P(\inf\{(\int_0^{T/y}(\sum_{\alpha\in\mathcal{A}(m)} a_\alpha \hat{B}^\alpha(t))^2 dt)^{1/2}; \sum_{\alpha\in\mathcal{A}(m)} T^{\|\alpha\|+1}a_\alpha^2 = 1\} \leq 2z^{-1})$$

$$+P(T^{-(m+1)/2}(\sum_{\alpha\in\mathcal{A}(m)}\int_0^{T/y} f_\alpha(t)^2 dt)^{1/2} \geq z^{-1})$$

$$\leq P(\inf\{\int_0^{T/y}(\sum_{\alpha\in\mathcal{A}(m)} a_\alpha \hat{B}^\alpha(t))^2 dt;$$

$$\sum_{\alpha\in\mathcal{A}(m)} (T/y)^{\|\alpha\|+1} a_\alpha^2 \geq y^{-(m+1)}\} \leq 4z^{-2})$$

$$+P((T/y)^{-(m/2+3/4)}\sum_{\alpha\in\mathcal{A}(m)}(\int_0^{T/y} f_\alpha(t)^2 dt)^{1/2} \geq y^{m/2+3/4}z^{-1})$$

$$\leq (4z^{-2}y^{(m+1)})^p C_{m,p} + (y^{-(m/2+3/4)}z)^{2p} A_{2p} \leq (4^p C_{m,p} + A_{2p})y^{-p/8}.$$

Thus we have our assertion. □

Applying Lemma 11 for $m = \ell - 1$, we have the following from Equations (4),(5) and (6).

Corollary 12. *For any $p \in (1,\infty)$, there is a constant $C > 0$ such that*

$$P(\inf\{\sum_{\alpha,\beta\in\mathcal{A}_1(\ell)} \xi_\alpha\xi_\beta M^{\alpha,\beta}(t,x);\ \xi\in\mathbf{R}^{\mathcal{A}_1(\ell)}, \sum_{\alpha\in\mathcal{A}_1(\ell)}|\xi_\alpha|^2 = 1\} \leq \frac{1}{n}) \leq Cn^{-p}$$

for any $n \geq 1$, $t \in (0,1]$, and $x \in \mathbf{R}^N$.

Now Lemma 6 is an easy consequence of Corollary 12.
This completes the proof of Lemma 6.

5. Hypoelliptic Part

In this section, we assume that the system $\{V_i; i = 0,1,\ldots,d\}$ satisfies the condition (UFG) and let ℓ be as in Definition 1. Let $A \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N\otimes\mathbf{R}^N)$ be given by

$$A(x) = \sum_{\alpha\in\mathcal{A}_1(\ell)} V_{[\alpha]}(x)\otimes V_{[\alpha]}(x), \qquad x\in\mathbf{R}^N,$$

and $\lambda_0 : \mathbf{R}^N \to [0,\infty)$ be a continuous function given by

$$\lambda_0(x) = \inf\{(A(x)\xi,\xi);\ \xi \in \mathbf{R}^N,\ |\xi| = 1\}, \qquad x \in \mathbf{R}^N.$$

Then we have the following.

Proposition 13. *For any $p \in (1,\infty)$ there are constants C_0, C_1 such that*

$$C_0\lambda_0(x)^{-1} \le E[\lambda_0(X(t,x))^{-p}]^{1/p} \le C_1\lambda_0(x)^{-1}, \qquad x \in \mathbf{R}^N.$$

Proof. Let $J(t,x) = \{\frac{\partial}{\partial x^j} X^i(t,x)\}_{i,j=1}^N$. Then we have

$$(A(X(t,x))\xi,\xi) = \sum_{\alpha\in\mathcal{A}_1(\ell)} (V_{[\alpha]}(X(t,x)),\xi)^2$$

$$= \sum_{\alpha\in\mathcal{A}_1(\ell)} (\sum_{\beta\in\mathcal{A}_1(\ell)} a_\alpha^\beta(t,x)(V_{[\beta]}(x), J(t,x)^*\xi))^2$$

$$\le (\sum_{\alpha\in\mathcal{A}_1(\ell)} \sum_{\beta\in\mathcal{A}_1(\ell)} a_\alpha^\beta(t,x)^2)(\sum_{\beta\in\mathcal{A}_1(\ell)} (V_{[\beta]}(x), J(t,x)^*\xi))^2)$$

So we have

$$\lambda_0(X(t,x)) \le (\sum_{\alpha\in\mathcal{A}_1(\ell)} \sum_{\beta\in\mathcal{A}_1(\ell)} a_\alpha^\beta(t,x)^2) \parallel J(t,x) \parallel^2 \lambda_0(x).$$

This implies that

$$\lambda_0(X(t,x))^{-1} \le (\sum_{\alpha\in\mathcal{A}_1(\ell)} \sum_{\beta\in\mathcal{A}_1(\ell)} a_\alpha^\beta(t,x)^2) \parallel J(t,x) \parallel^2 \lambda_0(x)^{-1}.$$

Similarly we have

$$\lambda_0(x)^{-1} \le (\sum_{\alpha\in\mathcal{A}_1(\ell)} \sum_{\beta\in\mathcal{A}_1(\ell)} b_\alpha^\beta(t,x)^2) \parallel J(t,x)^{-1} \parallel^2 \lambda_0(X(t,x))^{-1}.$$

These imply our assertion. $\square$

Also, let $\lambda : \mathbf{R}^N \to [0,\infty)$ be given by

$$\lambda(x) = \begin{cases} (trace A(x)^{-1})^{-1}, & \text{if } \lambda_0(x) > 0, \\ 0, & \text{if } \lambda_0(x) = 0. \end{cases}$$

Then we can easily see that

$$N^{-1}\lambda_0(x) \le \lambda(x) \le \lambda_0(x), \qquad x \in \mathbf{R}^N,$$

and so we see that λ is continuous.

Let $G_0 = \{x \in \mathbf{R}^N;\ \lambda_0(x) > 0\}$, and $e_i = \{\delta_{ji}\}_{j=1}^N \in \mathbf{R}^N$, $i = 1, \ldots, N$, and let $c_{\alpha,i} : G_0 \to \mathbf{R}$, $\alpha \in \mathcal{A}_1(\ell)$, $i = 1, \ldots, N$, be given by

$$c_{\alpha,i}(x) = (e_i, A(x)^{-1}V_{[\alpha]}(x)), \qquad x \in G_0.$$

Then we see that

$$\frac{\partial}{\partial x^i} = \sum_{\alpha \in \mathcal{A}_1(\ell)} c_{\alpha,i}V_{[\alpha]}, \qquad \text{on } G_0.$$

Since we have

$$\frac{\partial}{\partial x^i}(A(x)^{-1}) = -A(x)^{-1}(\frac{\partial}{\partial x^i}A(x))A(x)^{-1},$$

we see that for any $n \ge 1$ and $i_1, \ldots, i_n \in \{1, \ldots, N\}$, there is a $C > 0$ such that

$$|\frac{\partial^n}{\partial x^{i_1} \cdots \partial x^{i_n}} trace(A(x)^{-1})| \le C\lambda(x)^{-(n+1)}, \qquad x \in \mathbf{R}^N,$$

$$|\frac{\partial^n}{\partial x^{i_1} \cdots \partial x^{i_n}} \lambda(x)| \le C\lambda(x)^{-(n-1)}, \qquad x \in \mathbf{R}^N,$$

and

$$|\frac{\partial^n}{\partial x^{i_1} \cdots \partial x^{i_n}} c_{\alpha,i}(x)| \le C\lambda(x)^{-(n+1)}, \qquad x \in \mathbf{R}^N$$

for all $\alpha \in \mathcal{A}_1(\ell)$, $i = 1, \ldots, N$.

Combining these facts and Theorem 2, we have the following.

Proposition 14. *Suppose that* $\{V_i; i = 0, 1, \ldots, d\}$ *satisfies the* (UFG) *condition. Then for any* $n \ge 1$ *and* $i_1, \ldots, i_n \in \{1, \ldots, N\}$, *there is a* $C > 0$ *such that*

$$\| \lambda^n \frac{\partial^n}{\partial x^{i_1} \cdots \partial x^{i_n}} P_t^c f \|_{L^p(dx)} \le Ct^{-n\ell/2} \| f \|_{L^p(dx)}$$

for any $f \in C_0^\infty(\mathbf{R}^N)$, $t \in (0, 1]$ *and* $p \in [1, \infty]$.

Also we have the following by using dual argument as in the proof of Theorem 2.

THEOREM 15. *Suppose that* $\{V_i; i = 0, 1, \ldots, d\}$ *satisfies the* (UFG) *condition. Then for any* $n, m \geq 0$ *and* $i_1, \ldots, i_n, j_1, \ldots, j_m \in \{1, \ldots, N\}$, *there is a* $C > 0$ *such that*

$$\| \lambda^n \frac{\partial^n}{\partial x^{i_1} \cdots \partial x^{i_n}} P_t^c \frac{\partial^m}{\partial x^{j_1} \cdots \partial x^{j_m}} \lambda^m f \|_{L^p(dx)} \leq C t^{-(n+m)\ell/2} \| f \|_{L^p(dx)}$$

for any $f \in C_0^\infty(\mathbf{R}^N)$, $t \in (0, 1]$ *and* $p \in [1, \infty]$.

6. Examples

Example 1. Let $d = 1$ and $N = 2$. Let $n \geq 2$, and $V_0, V_1 \in C_b^\infty(\mathbf{R}^2; \mathbf{R}^2)$ be given by

$$V_0(x^1, x^2) = (2 + (\sin x^1)^n) \frac{\partial}{\partial x^2}, \qquad V_1(x^1, x^2) = \frac{\partial}{\partial x^1}.$$

Then the condition (UH) is satisfied for $\ell = n+2$. Let $X(t, x)$ be the solution to (1) and P_t, $t > 0$, be a linear operator in $C_b^\infty(\mathbf{R}^2; \mathbf{R})$ given by

$$P_t f(x) = E[f(X(t, x))], \qquad f \in C_b^\infty(\mathbf{R}^2; \mathbf{R}),\ x \in \mathbf{R}^2.$$

Then we have the following.

PROPOSITION 16. (1) *There is a constant* $C_1 > 0$ *such that*

$$\| V_0 P_t f \|_\infty \leq C_1 t^{-(n+2)/2} \| f \|_\infty, \qquad f \in C_b^\infty(\mathbf{R}^2; \mathbf{R}),\ t \in (0, 1].$$

(2) *There is a constant* $C_2 > 0$ *such that*

$$\sup\{\| V_0 P_t f \|_\infty;\ f \in C_b^\infty(\mathbf{R}^2; \mathbf{R}), \| f \|_\infty \leq 1\} \geq C_2 t^{-(n+2)/2}, \qquad t \in (0, 1].$$

PROOF. The assertion (1) is an easy consequence of Theorem 2. So we prove the assertion (2). We can easy to see that the solution $X(t, x) = (X(t, (x^1, x^2), X^2(t, (x^1, x^2)))$ is given by

$$X^1(t, (x^1, x^2)) = x^1 + B^1(t),$$

$$X^2(t,(x^1,x^2)) = x^2 + 2t + \int_0^t \sin(x^1 + B^1(s))^n ds.$$

Then we see that

$$\frac{\partial}{\partial x^2}(P_t f)(x) = E[\frac{\partial f}{\partial x^2}(X(t,x))], \qquad f \in C_b^\infty(\mathbf{R}^2;\mathbf{R}).$$

Note that

$$E[|\int_0^t \sin(B^1(s))^n ds|] \le E[\int_0^t |B^1(s)|^n ds] = A_n t^{(n+2)/2},$$

where $A_n = E[\int_0^1 |B^1(s)|^n ds]$. So we see that $P(|\int_0^t \sin(B^1(s))^n ds| \ge 2A_n t^{(n+2)/2}) \le 1/2$.

Let us take a $g \in C_b^\infty(\mathbf{R};\mathbf{R})$ such that $g'(z) \ge 1$, $z \in [-1,1]$ and $g'(z) \ge 0$, $z \in \mathbf{R}$. Let $f^{(t)} \in C_b^\infty(\mathbf{R}^2;\mathbf{R})$, $t > 0$, be given by

$$f^{(t)}(x^1,x^2) = g((2A_n t^{(n+2)/2})^{-1}(x^2 - 2t)), \qquad (x^1,x^2) \in \mathbf{R}^2.$$

Then we see that $\| f^{(t)} \|_\infty = \| g \|_\infty$ and that

$$\frac{\partial}{\partial x^2}(P_t f^{(t)})(0) = (2A_n t^{(n+2)/2})^{-1} E[g'((2A_n t^{(n+2)/2})^{-1}(\int_0^t \sin(B^1(s))^n ds))]$$
$$\ge (4A_n t^{(n+2)/2})^{-1}.$$

This implies our assertion (2). □

Example 2. Let $d = 1$ and $N = 2$. Let $V_0, V_1 \in C_b^\infty(\mathbf{R}^2;\mathbf{R}^2)$ be given by

$$V_0(x^1,x^2) = \sin x^1 \frac{\partial}{\partial x^2}, \qquad V_1(x^1,x^2) = \sin x^1 \frac{\partial}{\partial x^1}.$$

Then the condition (UH) is not satisfied. But (UFG) is satisfied for $\ell = 4$.

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