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Elementary Moduli Space of Triangles and Iterative Processes

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Abstract. We regard the unit disk of the complex plane as the moduli space of the similarity classes of triangles. A certain plane geometric operation on triangles may then be interpreted as a rotation of the moduli disk. We examine conditions under which the operation has a finite period, and discuss examples and related topics.

§1. Introduction

Iterative processes on triangles are studied in several contemporary articles (e.g. Shapiro [1], Chang-Davis [2], Davis [3,4]), where standard approach seems to have been making use of circulant matrices.

In this article, we propose another approach to such sorts of problems — the *moduli approach*. In fact, the similarity classes of triangles are naturally parametrized by the points of the unit disk $\mathcal{D} = \{w \in \mathbb{C}; |w| < 1\}$ (§2) and a certain type of plane geometric operator acts on $\mathcal{D}$ as its anticlockwise rotation around the origin through a certain angle (§3). We will investigate the conditions under which our operator has a finite period (§3, §4, §5). Finally we consider cases where a "reversely similar" triangle appears after iterative actions of this operator (§6).

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§2. Moduli Space $\mathcal{D}$

A triangle is a subset $\Delta = \{a, b, c\}$ of $\mathbb{C}$ with its cardinality 3 which satisfies the condition : $(a-b)/(c-b) \notin \mathbb{R}$. The transformation group $G = \{f : \mathbb{C} \to \mathbb{C} \mid f(z) = \alpha z + \beta (\alpha, \beta \in \mathbb{C}, \alpha \neq 0)\}$ acts naturally on the set T of all triangles. We call each orbit ($\in S := T/G$) a(n ordinary) similarity class of triangles. The similarity class to which a triangle Δ belongs will be denoted by $[\Delta]$.

Notation 1.

$$\begin{aligned}
\rho &:= \exp(2\pi i/6),\ \omega := \exp(2\pi i/3),\\
\mathcal{H} &:= \{z \in \mathbb{C}; \mathrm{Im} z > 0\} : \text{the upper half plane in variable } z.\\
\mathcal{B} &:= \{Z \in \mathbb{C}; |Z| < 1\} : \text{the unit disk in variable } Z.\\
\mathcal{D} &:= \{w \in \mathbb{C}; |w| < 1\} : \text{the unit disk in variable } w.
\end{aligned}$$

Definition 2. We define the maps $\lambda : \mathcal{H} \to \mathcal{H}$, $\mu : \mathcal{B} \to \mathcal{B}$, $f : \mathcal{H} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{D}$ by

$$\lambda(z) = \frac{1}{1-z}\ ,\ \mu(Z) = \omega Z\ ,\ f(z) = \frac{\omega - \rho z}{z + \omega}\ ,\ g(Z) = Z^3.$$

The bijectivity of f, the surjectivity of g and the commutativity of Diagram 1 below follow from simple calculations.

$$\begin{array}{ccccc}
\mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D} \\
\lambda\downarrow & & \mu\downarrow & & \| \\
\mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D}
\end{array} \qquad \text{Diagram 1}$$

The operator λ (resp. μ) generates an automorphism group of $\mathcal{H}$ (resp. $\mathcal{B}$) of order 3, whose orbits will be called λ-orbits (resp. μ-orbits).

Now we define the surjection $\phi : \mathcal{H} \to S$ by

$$\phi(z) = [\{0, 1, z\}].$$

It is easy to see that for any triangle $\Delta \in T$ the fiber $\phi^{-1}([\Delta])$ coincides with exactly one λ-orbit. It is also clear that for any $w \in \mathcal{D}$ the fiber $g^{-1}(w)$ forms a single μ-orbit; hence there arises a bijection $\psi : \mathcal{D} \to S$ which makes Diagram 2 commute.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "f", "\sim"'] \arrow[d, "\phi"'] & \mathcal{B} \arrow[d, "g"] \\ S & \mathcal{D} \arrow[l, "\psi"] \end{tikzcd} \qquad \text{Diagram 2}$$

Definition 3. Through the above bijection ψ, we call $\mathcal{D}$ the moduli space of the similarity classes of triangles.

Remark 4. For any real number $x \in \mathcal{D}$, $\psi(x)$ is a similarity class of isosceles triangles. In particular, $\psi(0)$ is the similarity class of regular triangles. The similarity classes of rectangular triangles are parameterized by

$$\left\{2\cos(\frac{\theta-\pi}{3}) - \sqrt{4\cos^2(\frac{\theta-\pi}{3}) - 1}\right\}^3 e^{i\theta} \qquad (0 \le \theta < 2\pi).$$

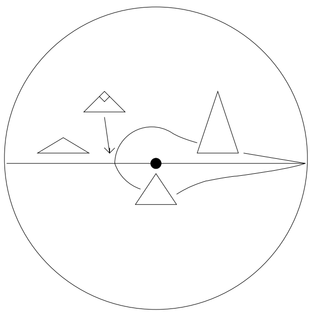

Figure 1a

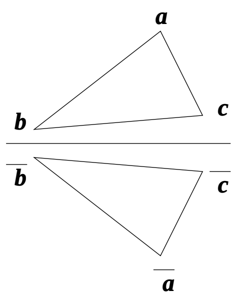

Figure 1b

Two "complex conjugate" triangles $\{a, b, c\}$ and $\{\bar{a}, \bar{b}, \bar{c}\}$ are generally not similar due to our definition of (ordinary) similarity class of triangles. For each similarity class $[\Delta] = [\{a, b, c\}]$, the conjugate similarity class $\overline{[\Delta]} := [\{\bar{a}, \bar{b}, \bar{c}\}]$ is clearly well defined. If $\Delta_1 \in [\Delta]$ and $\Delta_2 \in \overline{[\Delta]}$ for some Δ, then Δ_1 and Δ_2 are called reversely similar (Fig.1b). The reverse similarity will be studied later in §6.

§3. Operator $T_{p,q}$

Let $\Delta = \{a, b, c\}$ be a triangle and let p, q be real numbers which satisfy $(p, q) \neq (1/2, 1/2)$ and $pq \neq 1$. We put

$$u_1 := pc + (1-p)b,\ u_2 := pa + (1-p)c,\ u_3 := pb + (1-p)a,$$
$$v_1 := qb + (1-q)c,\ v_2 := qc + (1-q)a,\ v_3 := qa + (1-q)b,$$

and

$$\begin{aligned} a' &:= \text{ the intersection point of the lines } \overline{bu_2} \text{ and } \overline{av_1}, \\ b' &:= \text{ the intersection point of the lines } \overline{cu_3} \text{ and } \overline{bv_2}, \\ c' &:= \text{ the intersection point of the lines } \overline{au_1} \text{ and } \overline{cv_3}. \end{aligned}$$

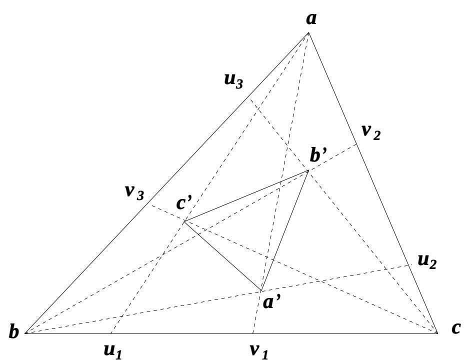

Figure 2

Let us show that $\{a', b', c'\}$ becomes a triangle. Since the process described above belongs to the similarity geometry, it suffices to consider the case where $a = z,\ b = 0,\ c = 1$ $(z \in \mathcal{H})$. By using the well known theorem of Menelaos, we get the equations

$$(\#) : \begin{cases} a' := (1-q)(pz + (1-p))/(1-pq), \\ b' := (1-p)((1-q)z + q)/(1-pq), \\ c' := \{(1-p)qz + (1-q)p\}/(1-pq). \end{cases}$$

Our assumption on the pair (p, q) insures that the cardinality of $\{a', b', c'\}$ is equal to 3. In fact, $(\#)$ implies

$$\begin{aligned} a' - b' &= \{(1-q)(2p-1)z + (1-p)(1-2q)\}/(1-pq), \\ c' - b' &= \{(1-p)(2q-1)z + (p-q)\}/(1-pq). \end{aligned}$$

Therefore, to show the non-degeneracy of $\{a', b', c'\}$, it suffices to see the non-vanishing of

$$\begin{aligned} D : &= \{(1-q)(2p-1)\}\{(p-q)\} - \{(1-p)(1-2q)\}\{(1-p)(2q-1)\} \\ &= 4p^2q^2 - 6p^2q - 6pq^2 + 3p^2 + 3q^2 + 7pq - 3p - 3q + 1. \end{aligned}$$

If we introduce $u := p + q$, $v := pq$, then we see $D = 3u^2 + 4v^2 - 6uv + v - 3u + 1$. This quadratic form of elliptic type can easily be estimated under the condition (*) : $u^2 - 4v \geq 0$. In fact, the solution of the equations $\frac{\partial D}{\partial u} = \frac{\partial D}{\partial v} = 0$ is $(u, v) = (3/2, 1)$, which does not satisfy (*). Hence, D takes the minimal value on the boundary set $\{(u, v); u^2 - 4v = 0\}$. Put $v = u^2/4$, then $D = (u-1)^2(u-2)^2/4$. From this, it follows that D is always positive, because the cases of $p = q = 1/2$ and of $pq = 1$ are excluded by our assumption on (p, q). Thus, $(a' - b')/(c' - b')$ belongs to $\mathcal{H}$, and especially $\Delta' := \{a', b', c'\}$ becomes triangle.

So we shall define the map $S_{p,q} : S \to S$ and $T_{p,q} : \mathcal{D} \to \mathcal{D}$ by

$$S_{p,q}([\Delta]) := [\Delta'],\ T_{p,q} := \psi^{-1} \circ S_{p,q} \circ \psi,$$

where "$\circ$" denotes the composition of maps.

$$\begin{array}{ccc} S & \xrightarrow{S_{p,q}} & S \\ \psi\big\uparrow & & \big\uparrow\psi \\ \mathcal{D} & \xrightarrow{T_{p,q}} & \mathcal{D} \end{array}$$

Diagram 3

THEOREM 1. *For any* $w \in \mathcal{D}$,

$$T_{p,q}(w) = \frac{t(p,q)}{|t(p,q)|} w,$$

where $t(p,q) = \{(p-1)(2q-1)\rho - (q-1)(2p-1)\}^6$.

PROOF. Let a, b, c be $z, 0, 1$ respectively ($z \in \mathcal{H}$) and Δ be the triangle $\{a, b, c\}$. Suppose that $\Delta' = \{a', b', c'\}$ is given by (#). It is clear that $[\Delta']$ is equal to $[\{0, 1, (a' - b')/(c' - b')\}]$, where

$$\frac{a' - b'}{c' - b'} = \frac{(1-q)(2p-1)z + (1-p)(1-2q)}{(1-p)(2q-1)z + (p-q)}.$$

This is already known to belong to $\mathcal{H}$. So we define $k(z)$ to be the right hand side of this equation. Then,

$$f \circ k \circ f^{-1}(Z) = \frac{(-4pq + 3p + 3q - 2)\rho + 2pq - 3p + 1}{(2pq - 3q + 1)\rho + 2pq - 3p + 1} Z.$$

Multiplying $-(1+\rho)/3$ to both the numerator and the denominator above, we may also write

$$f \circ k \circ f^{-1}(Z) = \frac{(p-1)(2q-1)\rho - (q-1)(2p-1)}{(p-1)(2q-1)\rho^{-1} - (q-1)(2p-1)} Z.$$

Since ρ^{-1} and ρ are complex conjugate to each other, the coefficient $A_{p,q}$ of Z in $f \circ k \circ f^{-1}(Z)$ has absolute value 1. This together with the right commutativity of Diagram 4 concludes the statement of the theorem. □

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D} \\ {\scriptstyle k}\downarrow & & {\scriptstyle \cdot A_{p,q}}\downarrow & & \downarrow{\scriptstyle \cdot A_{p,q}^3} \\ \mathcal{H} & \xrightarrow{f} & \mathcal{B} & \xrightarrow{g} & \mathcal{D} \end{array} \qquad \text{Diagram 4}$$

By Theorem 1, the operator $T_{p,q}$ acts on $\mathcal{D}$ as the anticlockwise rotation around the origin through the angle $\arg(t(p,q))$.

Corollary 1. *$T_{p,q}$ and $T_{r,s}$ are commutative :*

$$T_{p,q} \circ T_{r,s} = T_{r,s} \circ T_{p,q}.$$

Proof. This is clear from the above description. □

Corollary 2.

(1) $T_{q,p} = {T_{p,q}}^{-1}$;

(2) $T_{\frac{3}{2}-p,\frac{3}{2}-q} = {T_{p,q}}^{-1}$.

Proof. Put $s(p,q) = (p-1)(2q-1)\rho - (q-1)(2p-1)$ so that $s(p,q)^6 = t(p,q)$. Then, it is easy to see that

$$s(p,q) = s((3/2)-q, (3/2)-p) = -\rho \cdot \overline{s(q,p)}.$$

The corollary follows immediately from this. □

In our formulation of the iterative process $S_{p,q}$ on triangles described above, we are mainly interested in the conditions for (p,q) under which the cyclic group $\{{T_{p,q}}^n | n \in \mathbb{Z}\}$ acting on $\mathcal{D}$ becomes finite.

Definition 5. We define the period of $T_{p,q}$ to be the order of the cyclic group $\{{T_{p,q}}^n | n \in \mathbb{Z}\} \subset Aut(\mathcal{D})$. If it is not finite, the period is set to be ∞.

Lemma 6. *Let $K(\subset \mathbb{C})$ be an algebraic number field which is stable under the complex conjugation. For $x \in K$, the following two conditions are equivalent.*

(a) *There exists an integer n such that $(x/|x|)^n = 1$.*

(b) *$(x/|x|)^2$ is a root of unity in K .*

Proof. Clear from the fact that $|x|^2 = x\bar{x} \in K$. □

Theorem 2. *Let (p,q) be any pair of real numbers satisfying $(p,q) \neq (1/2, 1/2)$ and $pq \neq 1$. Then,*

(1) *The period of $T_{p,q}$ is equal to 1 if and only if at least one of the following equations holds: $2p = 1, 2q = 1, p = 1, q = 1, p = q$.*

(2) *The period of $T_{p,q}$ is equal to 2 if and only if at least one of the following equations holds: $2pq - 3q + 1 = 0, 2pq - 3p + 1 = 0$, $4pq - 3p - 3q + 2 = 0$.*

(3) *If p, q are rational numbers satisfying neither (1) nor (2), then the period of $T_{p,q}$ is ∞ .*

Proof. Put $A := (p-1)(2q-1), B := -(q-1)(2p-1)$. Then $T_{p,q} : \mathcal{D} \to \mathcal{D}$ is the rotation around 0 through the angle $6\arg(A\rho + B)$.

(1) The period of $T_{p,q}$ is 1 if and only if $\arg(A\rho + B) = \pi n/3$ $(n = 0, 1, 2, \cdots)$. This is equivalent to say that $A : B = 1 : 0$ or $0 : 1$ or $1 : (-1)$. (See Figure 3 below).

(2) The period of $T_{p,q}$ is 2 if and only if $\arg(A\rho + B) = \pi(2n + 1)/6$ $(n = 0, 1, 2, \cdots)$. This is equivalent to say $A : B = 1 : 1$, $2 : (-1)$ or $1 : (-2)$.

(3) For rational numbers p, q, the number $A\rho + B$ belongs to the quadratic field $\mathbb{Q}(\rho)$. Since the roots of unity in $\mathbb{Q}(\rho)$ are ρ^m $(m = 0, 1, 2, \cdots, 5)$,

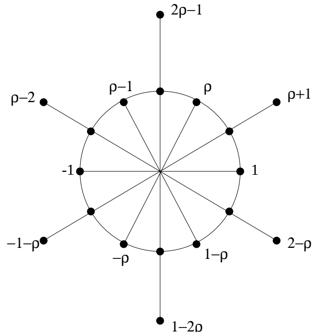

Figure 3

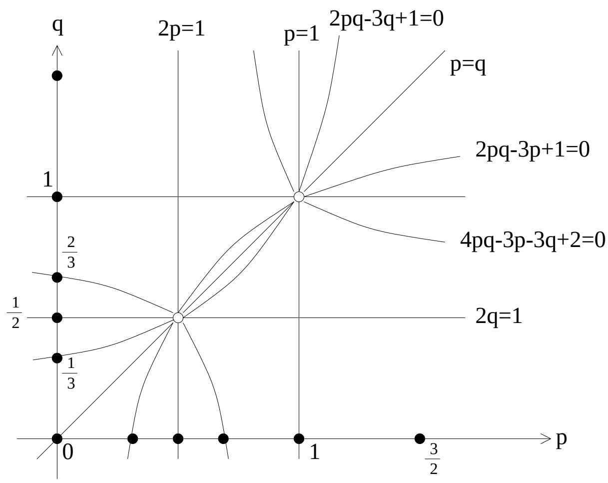

Figure 4

the above Lemma implies that, in order for the period of $T_{p,q}$ to be finite, $\arg(A\rho + B)$ should be $\pi n/6$ $(n = 0, 1, 2, \cdots)$ i.e. only in the cases of (1),(2). This verifies the statement. □

§4. **Special Case** $(p = 0)$

Let $\Delta = \{a, b, c\}$ be a triangle. If we set $p = 0$ at the beginning stage of §3, then a', b', c' are the division points of the sides $\overline{bc}, \overline{ca}, \overline{ab}$ respectively with the ratio $(1 - q) : q$. This case has been studied by many authors and some results are generalized for "p-affine n-gons" in Shapiro[1]. In this section, applying the result of §3, we give a necessary and sufficient condition for $T_{0,q}$ to have a finite period.

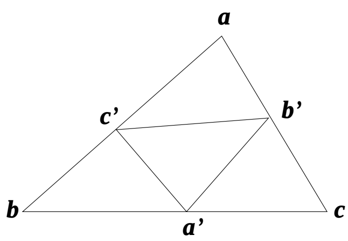

Figure 5

Note that we already know by Theorem 2 that a rational number q for which $T_{0,q}$ has a finite period is one of $0, 1/3, 1/2, 2/3, 1$.

THEOREM 3. *For any given real number q, let $\theta \in (-1, 1)$ be the unique number such that $q = \frac{1}{2}(1 + \frac{\sqrt{3}}{3}\tan\frac{\pi\theta}{2})$. Then $T_{0,q}$ is the anticlockwise rotation of $\mathcal{D}$ around the origin through the angle $3\pi\theta$. Hence $T_{0,q}$ has a finite period if and only if $\theta \in \mathbb{Q}$. Furthermore, when θ is n/m $(m \neq 0$, $(m, n) = 1)$ the period of $T_{0,q}$ equals $|\frac{2m}{(3,m)(2,n)}|$, where $(\ ,\ )$ denotes the greatest common divisor.*

PROOF. Put $p = 0$ in Theorem 1, then $T_{0,q}$ becomes the rotation of $\mathcal{D}$ through an angle of $6\arg(\omega - q\sqrt{3}i)$. Let θ satisfy:

$$-1 < \theta < 1 \text{ and } \arg(\omega - q\sqrt{3}i) = \pi + \frac{\pi\theta}{2}.$$

Then $q = \frac{1}{2}(1 + \frac{\sqrt{3}}{3}\tan\frac{\pi\theta}{2})$.

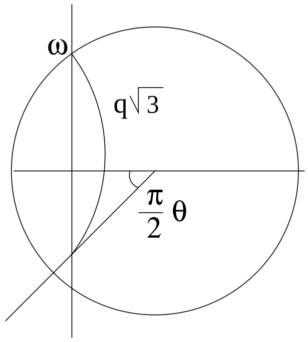

Figure 6

Obviously, there exists a natural number k with $k(\pi + \pi\theta/2) \equiv 0 \pmod{2\pi}$ iff θ belongs to $\mathbb{Q}$. If $\theta = n/m$ (irreducible fraction), then

$$6\arg(\omega - q\sqrt{3}i) \equiv 3\pi n/m \qquad (\mathrm{mod}\ 2\pi).$$

Hence the last statement also follows. □

Corollary 3. *For any natural number N there exists a real number q such that the period of $T_{0,q}$ exactly equals N.*

Proof. Using the notations of Theorem 3, we argue case by case as follows.

(a) When $3 \nmid N, 2 \nmid N$, we may put $m = N, n = 2$ and $\theta = 2/N$.

(b) When $3|N, 2 \nmid N$, we may put $m = 3N, n = 2$ and $\theta = 2/3N$.

(c) When $3 \nmid N, 2|N$, we may put $m = N/2, n = 1$ and $\theta = 2/N$.

(d) When $3|N, 2|N$, we may put $m = 3N/2, n = 1$ and $\theta = 2/3N$. □

The following proposition answers to an exercise raised in the article by Shapiro[1] (16.(2)).

Proposition 7. *For $T_{0,r} \circ T_{0,s} = id$, it is necessary and sufficient that the pair (r, s) satisfies at least one of the following equations : $r + s - 1 = 0, 3rs - r - s = 0, 3rs - 2r - 2s + 1 = 0$.*

PROOF. By Corollary 2 we may consider the case where $T_{0,r} = T_{s,0}$. Since we have : $t(0,r) = \{-(2r-1)\rho + (r-1)\}^6$ and $t(s,0) = \{-(s-1)\rho + (2s-1)\}^6$, it is necessary and sufficient for $T_{0,r} = T_{s,0}$ that

$$\arg\{(2r-1)\rho + (1-r)\} \equiv \arg\{(s-1)\rho - (2s-1)\} \pmod{\pi/3}.$$

The proposition follows from the following simple fact: For any real numbers A, B, C, D with $(A^2+B^2 \neq 0, C^2+D^2 \neq 0)$, $\arg(A\rho + B) \equiv \arg(C\rho + D) \pmod{\pi/3}$ if and only if $AD - BC = 0$ or $AD + BD + AC = 0$ or $BD + AC + BC = 0$. (See Figure 3). $\square$

§5. Special Case $(p + q = 1)$

If $p + q = 1$ at the beginning stage of §3, then u_1, u_2, u_3 coincide with v_1, v_2, v_3 respectively. So the triangle $\{a', b', c'\}$ is given as in Figure 7.

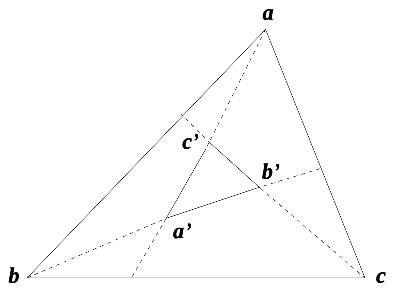

Figure 7

Apply Theorem 1 to this case, then, for any $w \in \mathcal{D}$,

$$T_{1-q,q}(w) = \frac{t(1-q,q)}{|t(1-q,q)|}w,$$

where $t(1-q,q)$ is equal to $\{(1-2q)(q\rho + (1-q))\}^6$. In this case we have

$$\lim_{q \to \frac{1}{2}} \frac{\{q\rho + (1-q)\}^6}{|q\rho + (1-q)|^6} = -1.$$

So we may define the operator $T_{\frac{1}{2},\frac{1}{2}}$ by

$$T_{\frac{1}{2},\frac{1}{2}}(w) = -w \quad (w \in \mathcal{D}).$$

THEOREM 4. *For any given real number q, take the unique θ such that $-1 < \theta < 1$ and $2q = (1 + \sqrt{3}\tan\frac{\pi\theta}{2})$. Then $T_{1-q,q}$ is the anticlockwise rotation of $\mathcal{D}$ around the origin through $\pi(1+3\theta)$. Hence it is necessary and sufficient for $T_{1-q,q}$ to have a finite period that θ belongs to $\mathbb{Q}$. Furthermore if $\theta = n/m$ $(m \neq 0, (m,n) = 1)$, the period of $T_{1-q,q}$ equals $|\frac{2m}{(3n+m,2)(m,3)}|$, where $(\ ,\ )$ is the greatest common divisor.*

PROOF. Put $\arg(q\rho + 1 - q) = \frac{\pi}{6} + \frac{\pi}{2}\theta$ $(-1 < \theta < 1)$. Then, it follows that

$$q = \frac{1}{2}(1 + \sqrt{3}\tan\frac{\pi\theta}{2}).$$

(Cf. Figure 8). We see that $6\arg(q\rho + 1 - q)$ equals $6(\frac{\pi}{6} + \frac{\pi}{2}\theta) = \pi(1 + 3\theta)$ modulo 2π. If $\theta = n/m$ (irreducible fraction), then $\pi(1 + 3\theta)$ equals $(3n + m)\pi/m$, which verifies the last statement. $\square$

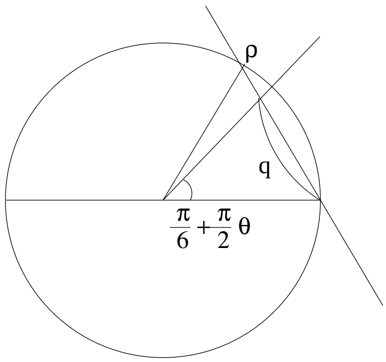

Figure 8

§6. Reverse Similarity

Let $\Delta = \Delta_0$ be a triangle and choose triangles $\Delta_k \in {S_{p,q}}^k([\Delta])$ for $k = 1, 2, 3, \cdots$. In this section we give some conditions under which there appears a triangle Δ_n $(n > 0)$ in the above sequence $\{\Delta_k; 0 < k < \infty\}$ such that Δ_n is reversely similar to Δ_0.

THEOREM 5. *Let p, q be rational numbers which satisfy the assumption of §3. Assume that there exists $n(> 0)$ such that Δ_n and Δ_0 are reversely similar. Then one of the followings holds.*

(1) *$\Delta_{2i}(i = 0, 1, 2, \cdots)$ are (ordinarily) similar, $\Delta_{2i+1}(i = 0, 1, 2, \cdots)$ are (ordinarily) similar and Δ_0 and Δ_1 are reversely similar.*

(2) *For any $m(\neq n, 0)$, Δ_m is neither ordinarily nor reversely similar to Δ_0.*

PROOF. If (2) does not hold, then there exists m $(m \neq n, 0 < m < \infty)$ such that Δ_m is ordinarily similar to Δ_0 or Δ_n. Note that we may assume that Δ is not a regular triangle. Since p and q are rational, the period of $T_{p,q}$ is 1 or 2 by Theorem 2 (3). Hence (1) holds. □

THEOREM 6. *Let p, q be real numbers which satisfy the assumption of §3 and n be any given natural number. Then there are infinitely many similarity classes $[\Delta] \in S$ such that Δ and Δ_n are reversely similar.*

PROOF. By assumption, we can take an angle θ such that

$$n \arg(t(p, q)) + \theta \equiv -\theta \pmod{2\pi}.$$

Since there are infinitely many $w \in \mathcal{D}$ such that $\arg(w) = \theta$, the statement follows. □

§7. Examples

In this section we describe a few concrete examples. For simplicity we write $T_{p,q}(\Delta) = \Delta'$ when $S_{p,q}([\Delta]) = [\Delta']$.

Example 1. The most typical examples are the cases where $(p, q) = (0, 1/2)$ and $(0, 1/3)$. Let Δ be a triangle $[A, B, C]$. The operator $T_{0,\frac{1}{2}}(\Delta)$ is the midpoint triangle of Δ which is clearly similar to Δ (Figure 9a). The operator $T_{0,\frac{1}{3}}$ is described as in Fig.9b (See Shapiro[1]). The period of $T_{0,\frac{1}{2}}$ equals 1 and that of $T_{0,\frac{1}{3}}$ is equal to 2 (cf. Theorem 2).

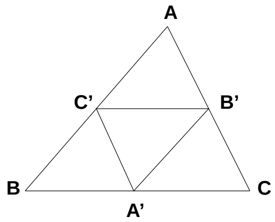

Figure 9a

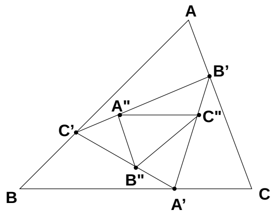

Figure 9b

Example 2. In the notation of Theorem 3, we put $\theta = -2/5$. Then $p = 0, q \fallingdotseq 0.2903$. Put $\Delta = \Delta_0 = \{0, 1, i\}$ and $\Delta_k = {T_{p,q}}^k(\Delta_0)$ $(k = 1, 2, \cdots)$. By observing the behavior of $T_{p,q}$ in our $\mathcal{D}$, we easily see the following facts (1) and (2). (Note that Δ is now isosceles):

(1) Δ_i and Δ_j are ordinarily similar iff $i \equiv j \pmod 5$.

(2) Δ_i and Δ_j are reversely similar iff $i + j \equiv 0 \pmod 5$.

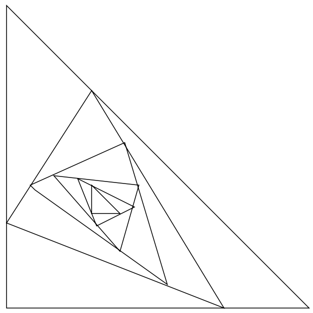

Figure 10

Example 3. The three cases of Proposition 7 in §4.

(1) $r + s = 1$ $(r = 3/7, s = 4/7)$: Figure 11a.

(2) $3rs - r - s = 0$ $(r = 3/7, s = 3/2)$: Figure 11b.

(3) $3rs - 2r - 2s + 1 = 0$ $(r = 3/7, s = 1/5)$: Figure 11c.

In these cases it is easily proven by Euclidean geometry that the corresponding sides are parallel.

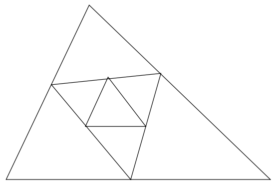

Figure 11a

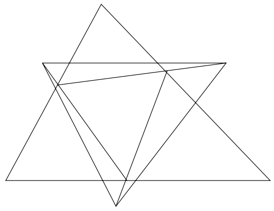

Figure 11b

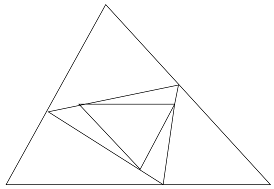

Figure 11c

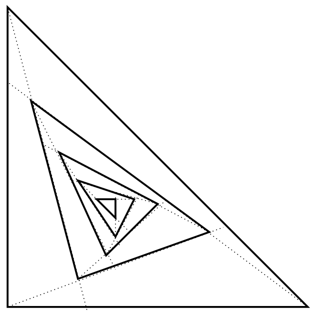

Figure 12

Example 4. In the notation of Theorem 4, we put $\theta = 1/6$. Then $q \fallingdotseq 0.7321$. The period of $T_{q,1-q}$ equals 4.

Example 5. We describe examples of (1) and (2) in Theorem 5.

(1) $\Delta_0 = \{0, 1, \sqrt{2}i\}$, $p = 0$, $q = 2/3$: $\psi^{-1}([\Delta_0])$ is on the imaginary axis in $\mathcal{D}$. Hence Δ_n $(n \geq 0)$ are divided into two similarity classes : $\{\Delta_0, \Delta_2, \cdots\}$ and $\{\Delta_1, \Delta_3, \cdots\}$. Δ_0 is reversely similar to Δ_1 (Figure 13a).

(2) $\Delta_0 = \{A, B, C\} = \{0, 1, \sqrt{3}i\}$, $p = 0$, $q = 3/4$: In this case Δ_0 and Δ_1 are reversely similar and Δ_m $(m > 1)$ is similar to neither Δ_0 nor Δ_1 (Figure 13b).

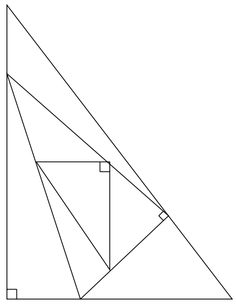

Figure 13a

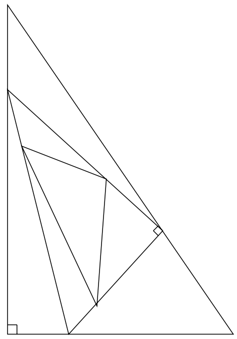

Figure 13b

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