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Blow-up Solutions of the Constantin-Lax-Majda Equation with a Generalized Viscosity Term

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Abstract. A generalized one-dimensional model for the three-dimensional vorticity equation of incompressible and viscous fluid is considered. Its viscosity term is given by an arbitrary order of derivative of the vorticity. A formal solution of the equation is given explicitly by using the spectral method. We investigate convergence of the solution and show that the solution blows up in finite time for sufficiently small viscosity coefficient regardless of the order of derivative of the viscosity term.

1. Introduction

The vorticity equation of three-dimensional inviscid and incompressible fluids is given by

$$\frac{Dw}{Dt} = w \cdot \nabla v, \qquad x \in \mathbf{R^3},\ t > 0, \tag{1}$$

where $\frac{D}{Dt}$ is the convective derivative, w is the vorticity and v is the velocity. Since $w = \nabla \times v$ and div $v = 0$, the velocity is recovered from the vorticity by the Biot-Savart integral;

$$v(x,t) = -\frac{1}{4\pi} \int \frac{(x-y)}{|x-y|^3} \times w(y,t) dy. \tag{2}$$

Constantin, Lax and Majda[1] rewrote the quadratic term $w \cdot \nabla v$ as an operator form $\mathrm{D}(w)w$ by using (2), where $\mathrm{D}(w)$ is a strongly singular integral operator acting on w. By considering some properties of the operator, they proposed a one-dimensional model for the vorticity equation (1), which is called Constantin-Lax-Majda (CLM) equation:

$$\frac{\partial \omega}{\partial t} = \omega H(\omega), \qquad x \in \mathbf{R},\ t > 0.$$

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The operator H is the Hilbert transform, which is a scalar one-dimensional analogue of the operator $\mathrm{D}(w)$. The model described the 3-D Euler equations successfully. See [1] for the details.

Now, we consider the CLM equation with a generalized viscosity term:

$$\frac{\partial \omega}{\partial t} = \omega \mathrm{H}(\omega) - \nu(-\Delta)^{\frac{\alpha}{2}}\omega, \quad x \in \mathbf{R},\ t > 0, \tag{3}$$

where a positive real constant ν represents the viscosity coefficient and $\alpha \in \mathbf{R}$ is order of derivative. This is a simple model of the 3-D Navier-Stokes equations.

Schochet[4] gave an explicit solution of the equation when $\alpha = 2$ by using pole expansion method. He showed that the solution blowed up in finite time and the velocity recovered from the solution also blowed up at the same time. He concluded that the model with second-order viscosity term was less successful than the CLM equation, since the vorticity and the velocity never blowed up simultaneously in the 3-D Navier-Stokes equations.

Murthey[2] and Wegert & Murthy[5] proposed a non-local viscosity term, $\nu H(\omega_x)$ instead of the diffusion. They obtained an explicit solution that blowed up after the solution of the CLM equation blowed up. Thus, they improved one of the drawbacks of Schochet's model.

However, it is still unknown what kind of the viscosity term is appropriate for the model of the 3-D Navier-Stokes equations. This is why we consider the generalized equation. What we prove in the article is the following interesting property: For an arbitrary α, there exists a small viscosity coefficient ν_0 such that solutions of the equation blow up in finite time for $0 < \nu < \nu_0$. This means the equation has a strong nonlinearity, since there exists a blow-up solution whatever order of viscous diffusion it may has.

The paper consists of five sections. We reformulate the CLM equation (3) by the spectral method and give a formal solution in §2. In §3, we prove mathematically blow-up of the solution for initial data whose Fourier coefficients are all non-negative. In §4, we give some numerical results that show blow-up of the solution for other initial data. Final section is devoted to conclusion.

2. The Spectral Method

In the present article, we consider the periodic boundary condition in x, i.e. $\omega(x + 2\pi, t) = \omega(x, t)$. Hence, we can represent solutions of the CLM

equation by

$$\omega(x,t) = \sum_{n=-\infty}^{\infty} \omega_n(t)\mathrm{e}^{\mathrm{i}nx}, \quad \omega_n(t) \in \mathbf{C}. \tag{4}$$

Then, the Hilbert transform of (4) is expressed as follows[3].

$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{\infty} \mathrm{i}\,\mathrm{sgn}(n)\omega_n(t)\mathrm{e}^{\mathrm{i}nx},$$

where the function $\mathrm{sgn}(n)$ is

$$\mathrm{sgn}(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n < 0. \end{cases}$$

Hence, we represent the nonlinear term $\omega\mathrm{H}(\omega)$ by

(5)

$$\begin{aligned} \omega\mathrm{H}(\omega) &= \left(\omega_0 + \sum_{n=1}^{\infty}\left(\omega_n\mathrm{e}^{\mathrm{i}nx} + \omega_{-n}\mathrm{e}^{-\mathrm{i}nx}\right)\right) \times \sum_{n=1}^{\infty}\mathrm{i}\left(\omega_n\mathrm{e}^{\mathrm{i}nx} - \omega_{-n}\mathrm{e}^{-\mathrm{i}nx}\right) \\ &= \sum_{n=1}^{\infty}\mathrm{i}\left\{\left(\sum_{k=0}^{n-1}\omega_k\omega_{n-k} - \sum_{k=1}^{\infty}\omega_{n+k}\omega_{-k} + \sum_{k=1}^{\infty}\omega_{-k}\omega_{n+k}\right)\mathrm{e}^{\mathrm{i}nx}\right. \\ &\quad \left. + \left(-\sum_{k=0}^{n-1}\omega_{-k}\omega_{-n+k} - \sum_{k=1}^{\infty}\omega_k\omega_{-n-k} + \sum_{k=1}^{\infty}\omega_{-n-k}\omega_k\right)\mathrm{e}^{-\mathrm{i}nx}\right\} \\ &= \sum_{n=1}^{\infty}\left\{\left(\mathrm{i}\sum_{k=0}^{n-1}\omega_k\omega_{n-k}\right)\mathrm{e}^{\mathrm{i}nx} - \left(\mathrm{i}\sum_{k=0}^{n-1}\omega_{-k}\omega_{-n+k}\right)\mathrm{e}^{-\mathrm{i}nx}\right\}. \end{aligned}$$

Substituting (4) and (5) into (3) and equating coefficients of $\mathrm{e}^{\mathrm{i}nx}$, we obtain ordinary differential equations for $\{\omega_n(t)\}_{n\geq 1}$, $\{\omega_{-n}(t)\}_{n\geq 1}$ and $\omega_0(t)$:

$$\begin{cases} \dfrac{\mathrm{d}\omega_n}{\mathrm{d}t} &= -\nu n^{\alpha}\omega_n + \mathrm{i}\displaystyle\sum_{k=0}^{n-1}\omega_k\omega_{n-k}, \\ \dfrac{\mathrm{d}\omega_0}{\mathrm{d}t} &= -\nu\omega_0, \\ \dfrac{\mathrm{d}\omega_{-n}}{\mathrm{d}t} &= -\nu n^{\alpha}\omega_{-n} - \mathrm{i}\displaystyle\sum_{k=0}^{n-1}\omega_{-k}\omega_{-n+k}. \end{cases} \tag{6}$$

Here, we assume that initial condition of ω is given by

$$\omega(x,0) = \sum_{n=1}^{\infty} A_n \sin nx. \tag{7}$$

Then, it follows from the equations (6) that solutions are symmetric with respect to the origin for all the time. That is to say,

$$\omega_n(t) + \omega_{-n}(t) = 0, \qquad \text{for } t > 0. \tag{8}$$

Hence, we have $\omega_0(t) = 0$ for all t. Furthermore, the equations (6) are reduced to equations only for $\{\omega_n(t)\}_{n\geq 1}$,

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}t} = -\nu n^\alpha \omega_n + \mathrm{i}\sum_{k=1}^{n-1} \omega_k \omega_{n-k}, \quad n = 1, 2, \cdots. \tag{9}$$

Due to (7), initial conditions for $\omega_n(t)$ are given by

$$\omega_n(0) = \frac{A_n}{2\mathrm{i}}. \tag{10}$$

It is possible to solve the equations (9) and (10) recursively:

$$\omega_1(t) = \frac{A_1}{2\mathrm{i}} \mathrm{e}^{-\nu t}, \tag{11}$$

$$\omega_n(t) = \frac{A_n}{2\mathrm{i}} \mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{n^\alpha \nu s} \mathrm{i} \sum_{k=1}^{n-1} \omega_k(s) \omega_{n-k}(s) ds. \tag{12}$$

Note that the solution of the equation (3) for the initial condition (7) is represented formally by

$$\omega(x,t) = \sum_{n=1}^{\infty} 2\mathrm{i}\omega_n(t) \sin nx. \tag{13}$$

For the rest of this section, we consider a solution of the equation without viscosity (i.e. $\nu = 0$) for $A_1 \neq 0$ and $A_n = 0$, $(n \geq 2)$.

PROPOSITION 1. *When $\nu = 0$, the solutions (11) and (12) for $A_1 \neq 0$ and $A_n = 0, (n \geq 2)$ are expressed by*

$$\mathrm{i}\omega_n(t) = \left(\frac{A_1}{2}\right)^n t^{n-1}. \tag{14}$$

PROOF. We prove it by the mathematical induction. It follows from (11) that $\mathrm{i}\omega_1(t)$ is equivalent to (14). If the solutions $\mathrm{i}\omega_k(t)$ for $k = 1, \cdots, n-1$ are represented by (14), then $\mathrm{i}\omega_n(t)$ becomes

$$\begin{aligned}\mathrm{i}\omega_n(t) &= \int_0^t \sum_{k=1}^{n-1} \left\{ \left(\frac{A_1}{2}\right)^k s^{k-1} \right\} \left\{ \left(\frac{A_1}{2}\right)^{n-k} s^{n-k-1} \right\} \\ &= \left(\frac{A_1}{2}\right)^n (n-1) \int_0^t s^{n-2} ds \\ &= \left(\frac{A_1}{2}\right)^n t^{n-1}. \ \square\end{aligned}$$

Since the solution $\omega(x,t)$ is given by

$$\omega(x,t) = \sum_{n=1}^{\infty} A_1 \left(\frac{A_1}{2}t\right)^{n-1} \sin nx = \frac{4A_1 \sin x}{4 - 4A_1 t \cos x + A_1^2 t^2}$$

for $\left|\frac{A_1}{2}t\right| < 1$, it blows up like $\frac{1}{x}$ as $t \to \frac{2}{A_1}$.

REMARK. In the paper[1], Constantin, Lax and Majda gave an example of a blow-up solution whose initial condition was $\omega(x,0) = \sin x$. They showed that the solution blowed up at $t = 2$. Since their initial condition corresponds to $A_1 = 1$ in the present analysis, the blow-up time agrees with their result.

3. Blow-up of Solutions for Non-negative Initial Coefficients

In this section, we assume that all the Fourier coefficients of the initial data (7) are non-negative, i.e. $A_n \geq 0$. First of all, we consider the equation (9) with the following special initial condition:

$$\tilde{\omega}_n(0) = \begin{cases} \dfrac{A_1}{2\mathrm{i}}, & n = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Let $\tilde{\omega}_n(t)$ be the solutions of the equation (9), which are given by

$$\tilde{\omega}_1(t) = \frac{A_1}{2\mathrm{i}} \mathrm{e}^{-\nu t}, \tag{16}$$

$$\tilde{\omega}_n(t) = \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{n^\alpha \nu s} \mathrm{i} \sum_{k=1}^{n-1} \tilde{\omega}_k(s) \tilde{\omega}_{n-k}(s) ds, \qquad (n \geq 2). \tag{17}$$

Then, we try to find a lower comparison function of $\mathrm{i}\tilde{\omega}_n(t)$. Finally, we show blow-up of the solution (13) for general initial conditions with non-negative Fourier coefficients by using the following comparison.

PROPOSITION 2. *For $n \geq 1$ and $t \geq 0$, $\mathrm{i}\omega_n(t) \geq \mathrm{i}\tilde{\omega}_n(t) \geq 0$.*

PROOF. This is easily proved by the mathematical induction. For $n = 1$, $\mathrm{i}\omega_1(t) = \mathrm{i}\tilde{\omega}_1(t) > 0$. If $\mathrm{i}\omega_k(t) \geq \mathrm{i}\tilde{\omega}_k(t) \geq 0$ holds for $k = 1, \cdots, n-1$, then we have

$$\begin{aligned}
\mathrm{i}\omega_n(t) &= \frac{A_n}{2} \mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{n^\alpha \nu s} \sum_{k=1}^{n-1} \mathrm{i}\omega_k(s) \mathrm{i}\omega_{n-k}(s) ds \\
&\geq \frac{A_n}{2} \mathrm{e}^{-n^\alpha \nu t} + \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{n^\alpha \nu s} \sum_{k=1}^{n-1} \mathrm{i}\tilde{\omega}_k(s) \mathrm{i}\tilde{\omega}_{n-k}(s) ds \\
&= \frac{A_n}{2} \mathrm{e}^{-n^\alpha \nu t} + \mathrm{i}\tilde{\omega}_n(t) \geq 0.
\end{aligned}$$

Since $A_n \geq 0$, we obtain $\mathrm{i}\omega_n(t) \geq \mathrm{i}\tilde{\omega}_n(t) \geq 0$. □

3.1. First order viscosity term

We consider the equation (3) when the viscosity term is given by the first derivative of ω, i.e. $\alpha = 1$. Then, it is possible to express the solutions $\mathrm{i}\tilde{\omega}_n(t)$ explicitly.

LEMMA 3. *When $\alpha = 1$, the solutions (16) and (17) are represented by*

$$\mathrm{i}\tilde{\omega}_n(t) = \frac{1}{t} \left(\frac{A_1 t}{2} \mathrm{e}^{-\nu t} \right)^n. \tag{18}$$

PROOF. For $n = 1$, (18) is nothing but (16). If we assume that the functions $\mathrm{i}\tilde{\omega}_k(t)$ for $k = 1, \cdots, n-1$ are represented by (18), then it follows

from (17) that $\mathrm{i}\tilde{\omega}_n(t)$ becomes

$$\begin{aligned}\mathrm{i}\tilde{\omega}_n(t) &= \mathrm{e}^{-n\nu t}\int_0^t \mathrm{e}^{n\nu s}\sum_{k=1}^{n-1}\left\{\frac{1}{s}\left(\frac{A_1 s}{2}\mathrm{e}^{-\nu s}\right)^k\right\}\left\{\frac{1}{s}\left(\frac{A_1 s}{2}\mathrm{e}^{-\nu s}\right)^{n-k}\right\}ds \\ &= \left(\frac{A_1}{2}\mathrm{e}^{-\nu t}\right)^n (n-1)\int_0^t s^{n-2}ds = \frac{1}{t}\left(\frac{A_1 t}{2}\mathrm{e}^{-\nu t}\right)^n. \ \square\end{aligned}$$

It follows from Parseval's equality and Proposition 2 that

$$\begin{aligned}||\omega(x,t)||^2_{L^2[-\pi,\pi)} &= \pi\sum_{n=1}^{\infty}\{2\mathrm{i}\omega_n(t)\}^2 \\ &\geq \pi\sum_{n=1}^{\infty}\{2\mathrm{i}\tilde{\omega}_n(t)\}^2 = \pi\sum_{n=1}^{\infty}\left\{A_1\mathrm{e}^{-\nu t}\left(\frac{A_1 t}{2}\mathrm{e}^{-\nu t}\right)^{n-1}\right\}^2.\end{aligned}$$

Now, we define a function $r(t)$ by $\frac{A_1 t}{2}\mathrm{e}^{-\nu t}$. When $|r(t)|<1$, the last infinite summation is represented by $\pi\frac{A_1^2\mathrm{e}^{-2\nu t}}{1-r^2(t)}$. Since the function $r(t)$ has the maximum value $\frac{A_1}{2\nu\mathrm{e}}$ at $t=\frac{1}{\nu}$, there exists a finite time $T_1^*(\nu)>0$ such that $r\left(T_1^*(\nu)\right)=1$, if $0<\nu\leq\frac{A_1}{2\mathrm{e}}$. Thus we prove the following theorem.

THEOREM 4. *When $\alpha=1$, for $0<\nu\leq\frac{A_1}{2\mathrm{e}}$, there exists a finite time $T_1^*(\nu)$ such that $||\omega(x,t)||_{L^2}\to\infty$ as $t\to T_1^*(\nu)$.*

REMARK. All the theorems given in the article are stated in terms of blow-up of L^2 norm of the solution. However, in the present problem, not only the L^2 norm but also the maximum of the solution, $M(t)=\sup_{x\in[-\pi,\pi)}|\omega(x,t)|$, blows up at the same time, since we have

$$||\omega(x,t)||^2_{L^2}=\int_{-\pi}^{\pi}|\omega(x,t)|^2dx<2\pi M^2(t),$$

due to the periodic boundary condition.

3.2. Higher order viscosity term

We consider the equation (3) and (7) when $\alpha>1$, which contains a model for the Navier-Stokes equations. What we are going to prove is the following theorem.

THEOREM 5. *When $\alpha > 1$, for $0 < \nu \leq \frac{A_1}{2}\mathrm{e}^{-3\alpha-1}$, there exists a finite time $T_2^*(\nu)$ such that $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_2^*(\nu)$.*

The first step to prove the theorem is to define a sequence $\left\{p_n^{(\alpha)}\right\}_{n\geq 1}$ recursively by

$$p_1^{(\alpha)} = 1, \qquad p_n^{(\alpha)} = \frac{1}{n^\alpha}\sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} \quad \text{for } n = 2, 3, \cdots.$$

We give a lower bound for $p_n^{(\alpha)}$.

LEMMA 6. *For $\alpha > 1$, let D_α be a constant such that $\alpha 2^{3\alpha} < D_\alpha \leq \mathrm{e}^{3\alpha}$. Then, the sequence $p_n^{(\alpha)}$ is bounded as follows:*

$$\mathrm{e}^{-3\alpha n} < D_\alpha n^{\alpha-1}\mathrm{e}^{-3\alpha n} < p_n^{(\alpha)}. \tag{19}$$

PROOF. For $n = 1$, since $D_\alpha \leq \mathrm{e}^{3\alpha}$, $D_\alpha \mathrm{e}^{-3\alpha} \leq 1 = p_1^{(\alpha)}$. To obtain the estimate for $p_n^{(\alpha)}$ $(n \geq 2)$, we show the following inequality.

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{\alpha-1}\left(1-\frac{k}{n}\right)^{\alpha-1} &> 2\int_0^{\frac{n-1}{2n}} x^{\alpha-1}(1-x)^{\alpha-1}dx \\ &\geq 2\int_0^{\frac{1}{4}} x^{\alpha-1}(1-x)^{\alpha-1}dx \quad (n \geq 2) \\ &> 2\int_0^{\frac{1}{4}} \left(\frac{3}{4}x\right)^{\alpha-1} dx \\ &= \frac{2}{\alpha}\frac{3^{\alpha-1}}{4^{2\alpha-1}}. \end{aligned} \tag{20}$$

See Figure 1 for the first inequality. The third inequality comes from $x(1-x) \geq \frac{3}{4}x$ for $0 \leq x \leq \frac{1}{4}$. Since $D_\alpha > \alpha 2^{3\alpha}$, we obtain

$$\frac{D_\alpha}{n}\sum_{k=1}^{n-1}\left(\frac{k}{n}\right)^{\alpha-1}\left(1-\frac{k}{n}\right)^{\alpha-1} > 2\frac{2^{3\alpha}3^{\alpha-1}}{4^{2\alpha-1}} = 4\left(\frac{3}{2}\right)^{\alpha-1} > 1.$$

Hence, for $n \geq 2$ and $\alpha > 1$, we obtain

$$\frac{D_\alpha}{n^\alpha}\sum_{k=1}^{n-1} k^{\alpha-1}(n-k)^{\alpha-1} > n^{\alpha-1}. \tag{21}$$

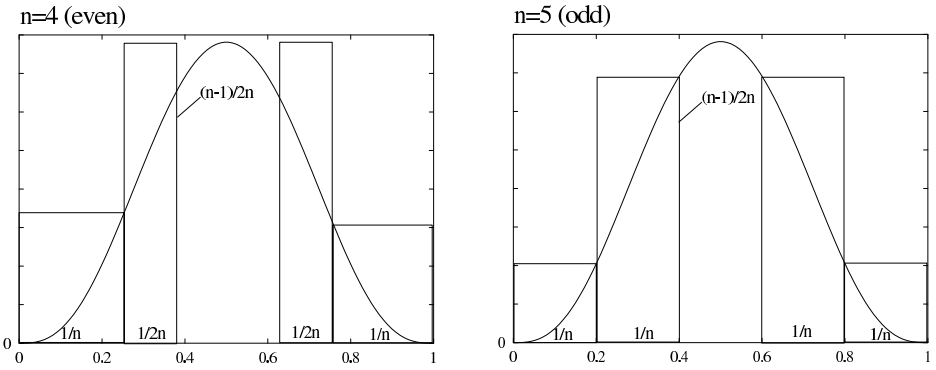

Fig. 1. Schematic figures to show how the inequality (20) holds true. The summation in the left hand side of (20) equals to area of boxes, which is greater than double of the integral of function $x^\alpha(1-x)^\alpha$ from 0 to $\frac{n-1}{2n}$.

Suppose that the estimate (19) holds for $k = 1, \cdots, n-1$. Then multiplying both sides of the inequality (21) by $D_\alpha \mathrm{e}^{-3\alpha n}$, we obtain

$$\begin{aligned} D_\alpha n^{\alpha-1}\mathrm{e}^{-3\alpha n} &< \frac{1}{n^\alpha}\sum_{k=1}^{n-1}\left(D_\alpha k^{\alpha-1}\mathrm{e}^{-3\alpha k}\right)\left(D_\alpha (n-k)^{\alpha-1}\mathrm{e}^{-3\alpha(n-k)}\right) \\ &< \frac{1}{n^\alpha}\sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} \\ &= p_n^{(\alpha)}. \ \square \end{aligned}$$

Next lemma gives a lower estimate for $\mathrm{i}\tilde{\omega}_n(t)$.

LEMMA 7. *Let functions $f_n^{(\alpha)}(t)$ be defined by*

$$\mathrm{i}\tilde{\omega}_n(t) = \nu p_n^{(\alpha)} \left(\frac{A_1}{2\nu}\mathrm{e}^{-\nu t}\right)^n f_n^{(\alpha)}(t). \tag{22}$$

Then, for $0 \le \nu t \le 1$, the functions $f_n^{(\alpha)}(t)$ satisfy

$$f_n^{(\alpha)}(t) \ \ge \ (\nu t)^{n-1}. \tag{23}$$

PROOF. Since $f_1^{(\alpha)} \equiv 1$, the statement (23) holds true for $n = 1$. Substituting (22) into (17), we obtain a recursive definition of the functions $f_n^{(\alpha)}$ for $n \geq 2$;

$$p_n^{(\alpha)} f_n^{(\alpha)}(t) = \nu \mathrm{e}^{-(n^\alpha - n)\nu t} \sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} f_k^{(\alpha)}(s) f_{n-k}^{(\alpha)}(s) ds.$$

Suppose that the functions $f_k^{(\alpha)}(t)$ have the lower estimate (23) for $k = 1, \cdots, n-1$ and $0 \leq \nu t \leq 1$, then we have

$$\begin{aligned} f_n^{(\alpha)}(t) &\geq \nu \frac{1}{p_n^{(\alpha)}} \sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} \, \mathrm{e}^{-(n^\alpha - n)\nu t} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} (\nu s)^{n-2} ds \\ &= \nu^{n-1} n^\alpha \mathrm{e}^{-(n^\alpha - n)\nu t} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} s^{n-2} ds. \end{aligned}$$

Now, let functions $F_1(t)$ and $F_2(t)$ be defined by

$$F_1(t) = n^\alpha \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} s^{n-2} ds \quad \text{and} \quad F_2(t) = t^{n-1} \mathrm{e}^{(n^\alpha - n)\nu t}.$$

Then, for $n \geq 2$, $\alpha > 1$ and $0 \leq \nu t \leq 1$, we obtain $F_1(t) \geq F_2(t)$, since $F_1(0) = F_2(0)$ and

$$F_1'(t) - F_2'(t) = t^{n-2} \mathrm{e}^{(n^\alpha - n)\nu t} \left(n^\alpha - n + 1 - (n^\alpha - n)\nu t\right) > 0.$$

Hence, $f_n^{(\alpha)}(t) \geq (\nu t)^{n-1}$ for $0 \leq \nu t \leq 1$. □

We finally prove Theorem 5 by using these lemmas.

PROOF OF THEOREM 5. From Parseval's equality and Proposition 2, we have

$$\begin{aligned} ||\omega(x,t)||^2_{L^2[-\pi,\pi)} &= \pi \sum_{n=1}^{\infty} \left\{2\mathrm{i}\omega_n(t)\right\}^2 \\ &\geq \pi \sum_{n=1}^{\infty} \left\{2\mathrm{i}\tilde{\omega}_n(t)\right\}^2 \\ &= \pi \sum_{n=1}^{\infty} \left\{2\nu p_n^{(\alpha)} \left(\frac{A_1}{2\nu} \mathrm{e}^{-\nu t}\right)^n f_n^{(\alpha)}(t)\right\}^2. \end{aligned}$$

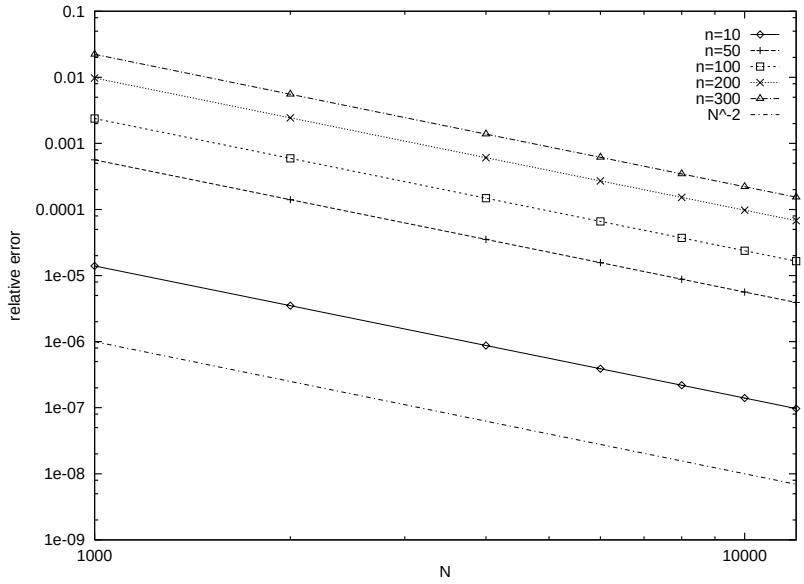

Fig. 2. Relative approximation errors between numerical coefficients $\mathrm{i}\omega_n^N$ and exact coefficients $\mathrm{i}\omega_n(t_N)$, that are given in Theorem 4, for $n = 10$, 50, 100, 200 and 300. Computational parameters are $\alpha = 1$ and $\nu = 0.25$. These errors decreases at $O(N^{-2})$.

It follows from Lemma 6 and Lemma 7 that for $0 \leq \nu t \leq 1$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ 2\nu p_n^{(\alpha)} \left(\frac{A_1}{2\nu} \mathrm{e}^{-\nu t} \right)^n f_n^{(\alpha)}(t) \right\}^2 \\
&\quad > \sum_{n=1}^{\infty} \left\{ A_1 \mathrm{e}^{-\nu t} \mathrm{e}^{-3\alpha n} \left(\frac{A_1 t}{2} \mathrm{e}^{-\nu t} \right)^{n-1} \right\}^2 \\
&\quad = A_1^2 \mathrm{e}^{-2\nu t} \mathrm{e}^{-6\alpha} \sum_{n=1}^{\infty} \left(\frac{A_1 t}{2\mathrm{e}^{3\alpha}} \mathrm{e}^{-\nu t} \right)^{2(n-1)}.
\end{aligned}$$

Therefore, if $R(t) = \frac{A_1 t}{2\mathrm{e}^{3\alpha}} \mathrm{e}^{-\nu t}$ is less than one, we obtain

$$\begin{aligned}
(24) \qquad ||\omega(x,t)||_{L^2}^2 &> \pi A_1^2 \mathrm{e}^{-2\nu t} \mathrm{e}^{-6\alpha} \sum_{n=1}^{\infty} \left\{ R^2(t) \right\}^{n-1} \\
&= \pi \frac{A_1^2 \mathrm{e}^{-2\nu t} \mathrm{e}^{-6\alpha}}{1 - R^2(t)}.
\end{aligned}$$

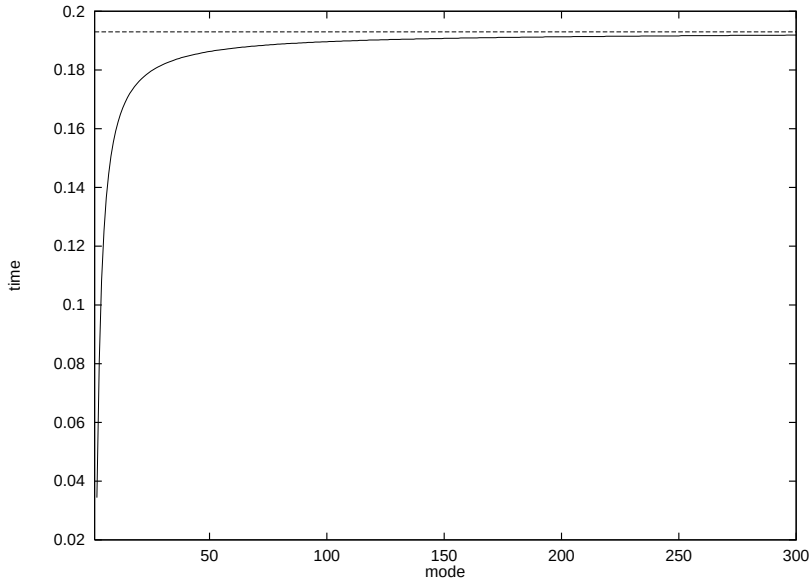

Fig. 3. Estimated time $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $n = 1, \cdots, 300$. It approaches asymptotically to some finite value, which is an estimated blow-up time of the solution.

Since $R(t)$ is monotonically increasing for $0 \leq \nu t \leq 1$, there exists a time $T_2^*(\nu) \in [0, \frac{1}{\nu}]$ such that $R\left(T_2^*(\nu)\right) = 1$, if the viscosity coefficient satisfies

$$R\left(\frac{1}{\nu}\right) = \frac{A_1}{2\nu \mathrm{e}^{3\alpha+1}} \geq 1.$$

Hence, it follows from (24) that if $0 < \nu \leq \frac{A_1}{2}\mathrm{e}^{-3\alpha-1}$, $||\omega(x,t)||_{L^2}$ blows up as $t \to T_2^*(\nu)$. □

3.3. Lower order viscosity term

We consider blow-up of the solution (13) when the order of viscosity term is less than one ($\alpha < 1$). We prove the following lemma.

LEMMA 8. *For $\alpha < 1$, $n \geq 1$ and $t \geq 0$,*

$$\mathrm{i}\tilde{\omega}_n(t) \geq \left(\frac{A_1}{2}\right)^n t^{n-1}\mathrm{e}^{-n\nu t}. \tag{25}$$

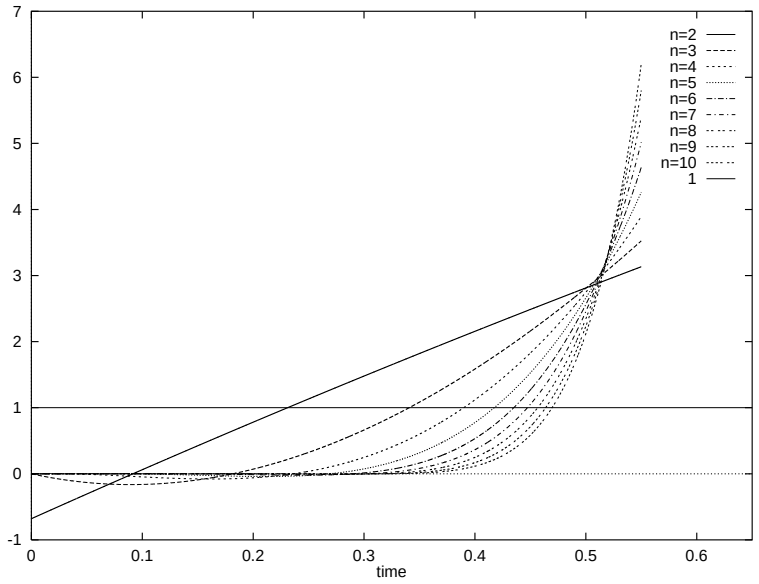

Fig. 4. Numerical solutions $\mathrm{i}\omega_n^j$ for $n = 1, \cdots, 10$ and $j = 0, \cdots, 5500$. The initial conditions are $A_1 = 2e$, $A_2 = -\frac{A_1}{4}$ and $\nu = 0.05$. The target time T is 0.55.

PROOF. It follows from (16) that (25) holds for $n = 1$. If $\mathrm{i}\tilde{\omega}_k(t)$ for $k = 1, \cdots, n-1$ are bounded by (25), then for $\mathrm{i}\tilde{\omega}_n(t)$ we obtain

$$\begin{aligned} \mathrm{i}\tilde{\omega}_n(t) &\geq \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{n^\alpha \nu s} \sum_{k=1}^{n-1} \left(\frac{A_1}{2}\right)^n s^{n-2} \mathrm{e}^{-n\nu s} ds \\ &= \left(\frac{A_1}{2}\right)^n \mathrm{e}^{-n^\alpha \nu t} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} (n-1) s^{n-2} ds. \end{aligned}$$

Suppose that functions $G_1(t)$ and $G_2(t)$ are defined by

$$G_1(t) = t^{n-1} \qquad \text{and} \qquad G_2(t) = \mathrm{e}^{-(n^\alpha - n)\nu t} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} (n-1) s^{n-2} ds,$$

then for $\alpha < 1$, $n \geq 2$ and $t \geq 0$, we have $G_1(t) \leq G_2(t)$. This is because $G_1(0) = G_2(0)$ and

$$G_2'(t) - G_1'(t) = (n - n^\alpha)(n-1)\nu \mathrm{e}^{-(n^\alpha - n)\nu t} \int_0^t \mathrm{e}^{(n^\alpha - n)\nu s} s^{n-2} ds \geq 0.$$

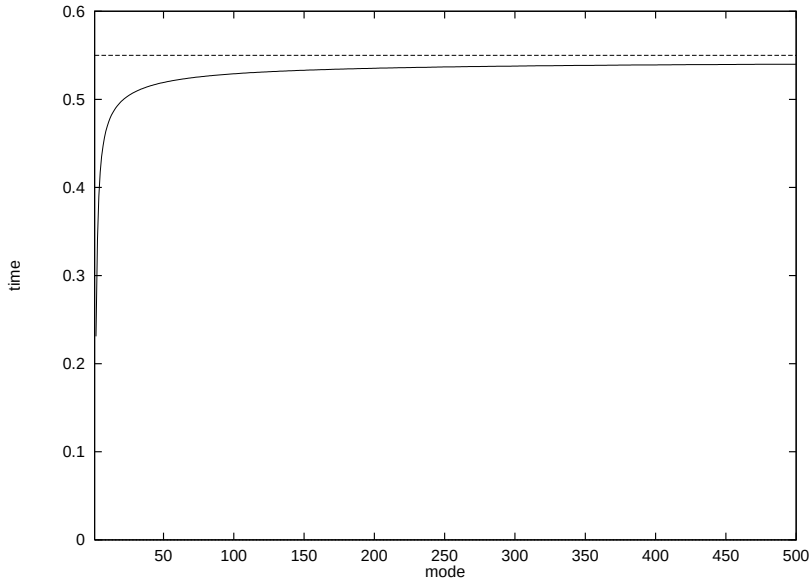

Fig. 5. Estimated time $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $n = 1, \cdots, 500$. Initial data are $A_1 = 2e$, $A_2 = -\frac{A_1}{4}$, $\nu = 0.05$ and $\alpha = 2$. The time approaches asymptotically to some finite value near 0.55. This is blow-up time of the solution.

Hence, we obtain

$$\mathrm{i}\tilde{\omega}_n(t) \geq \left(\frac{A_1}{2}\right)^n t^{n-1}\mathrm{e}^{-n\nu t}. \ \square$$

Proof of blow-up is now easy.

THEOREM 9. *When $\alpha < 1$, for $0 < \nu \leq \frac{A_1}{2e}$, there exists a finite time $T_1^*(\nu)$ such that $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_1^*(\nu)$.*

PROOF. In the same way as the proof of Theorem 5,

$$\begin{aligned} ||\omega(x,t)||_{L^2}^2 &> \pi \sum_{n=1}^{\infty} \{2\mathrm{i}\tilde{\omega}_n(t)\}^2 \\ &> \pi \sum_{n=1}^{\infty} \left\{2\left(\frac{A_1}{2}\right)^n t^{n-1}\mathrm{e}^{-n\nu t}\right\}^2 \end{aligned}$$

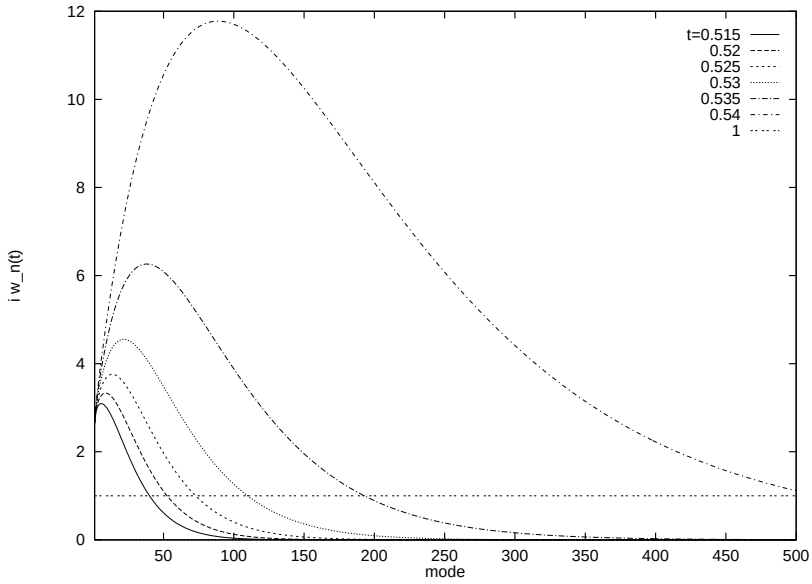

Fig. 6. Numerical solutions $\mathrm{i}\omega_n(t)$ vs. n from $t = 0.515$ to 0.54. Range of modes which exceed 1 expands rapidly as time approaches 0.54, at which all the coefficients are greater than 1.

$$= \pi A_1^2 \mathrm{e}^{-2\nu t} \sum_{n=1}^{\infty} \left\{ r^2(t) \right\}^{n-1},$$

where the function $r(t)$ is the same function defined in the proof of Lemma 3. If $|r(t)| < 1$, then we obtain

$$||\omega(x,t)||_{L^2}^2 > \pi \frac{A_1^2 \mathrm{e}^{-2\nu t}}{1 - r^2(t)}.$$

If $0 < \nu \leq \frac{A_1}{2e}$, there exists the time $T_1^*(\nu) > 0$ such that $r\left(T_1^*(\nu)\right) = 1$. Accordingly, $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_1^*(\nu)$. □

4. Numerical Examples for Other Initial Data

It is not easy to prove blow-up of solutions when some of their initial Fourier coefficients are negative, since the comparison like Proposition 2 never holds. In the subsequent subsections, we give some numerical examples that indicate blow-up could also occur for other initial conditions.

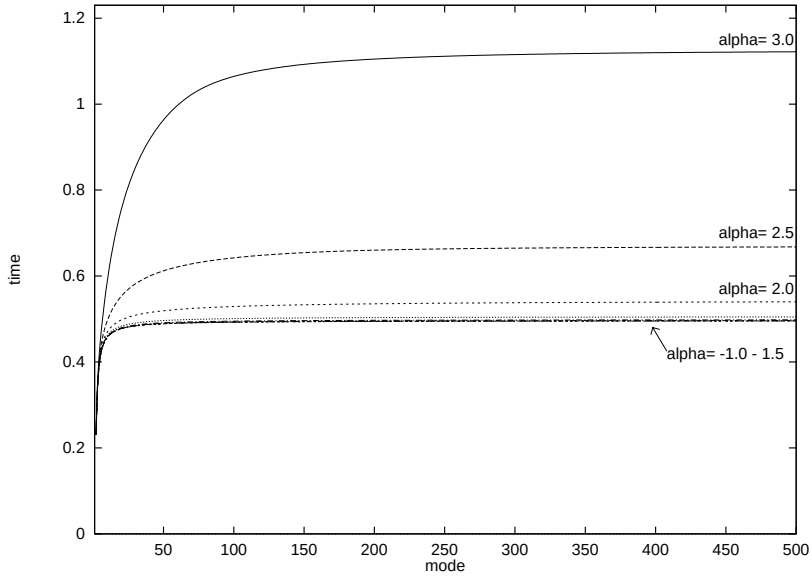

Fig. 7. Estimated times $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. Initial data are $A_1 = 2e$, $A_2 = -\frac{A_1}{4}$ and $\nu = 0.05$.

4.1. Numerical method

Since the equation (3) has already been reformulated with the system of ordinary differential equations (9), we have only to solve them by a conventional numerical scheme. We integrate the formula (11) and (12) recursively by the trapezoidal rule. That is to say, for a given time interval $[0, T]$ and $t_j = \frac{j}{N}T$, we approximate the Fourier coefficients $\mathrm{i}\omega_n(t_j)$ by

$$\begin{aligned} \mathrm{i}\omega_1(t_j) = \mathrm{i}\omega_1^j &= \frac{A_1}{2}\mathrm{e}^{-\nu t_j}, \\ \mathrm{i}\omega_n(t_j) \approx \mathrm{i}\omega_n^j &= \frac{A_n}{2}\mathrm{e}^{-n^\alpha \nu t_j} + \frac{1}{N}\sum_{m=0}^{j} \mathrm{e}^{n^\alpha \nu (t_j - t_m)} \sum_{k=1}^{n-1} \mathrm{i}\omega_k^m \mathrm{i}\omega_{n-k}^m. \end{aligned}$$

According to the proofs in the previous section, $||\omega(x,t)||_{L^2}$ blows up, if almost all the Fourier coefficients are greater than one at some time t^*. Hence, we search a time t^* when $\mathrm{i}\omega_n(t^*) \geq 1$ in the numerical computation.

We compare a numerical solution with the exact solution given in Lemma 3 in order to show the numerical method works well. The proof

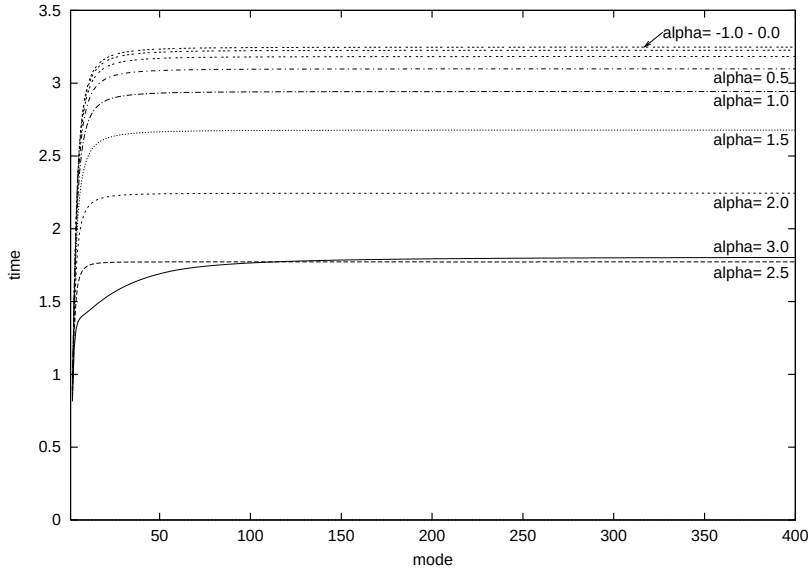

Fig. 8. Estimated times $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. Initial data are $A_1 = 2e$, $A_2 = -2A_1$ and $\nu = 0.05$.

of Theorem 4 tells us that $||\omega(x,t)||_{L^2}$ blows up in finite time for $A_1 = 4e$ and $\nu = 0.25$. We compute the coefficients from $n = 1$ to 300 up to the target time $T = 0.25$ for various N and show relative maximum errors between the numerical solutions and the exact solutions for $n = 10$, 50, 100, 200 and 300 at $T = 0.25$ in Figure 2. As $N \to \infty$, the errors tend to zero at $O(N^{-2})$. Next, we plot estimated times $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ in Figure 3. The time $t^{(n)}$ approaches asymptotically to some finite time as $n \to \infty$. Hence, we expect blow-up of the solution at the time $\sup_n t^{(n)}$ near 0.193.

4.2. Blow-up of the solutions when $A_1 > 0$ and $A_2 < 0$

We give numerical results when A_1 and A_2 have different signs, i.e. $A_2 = -pA_1$ $(p > 0)$. The other coefficients A_n are zero for $n \geq 3$. At first, we investigate a numerical solution when $\alpha = 2$, $\nu = 0.05$, $A_1 = 2e$ and $p = \frac{1}{4}$. The target time T is 0.55 and $N = 5500$. Figure 4 shows the coefficients $\mathrm{i}\omega_n^j$ for $n = 1, \cdots, 10$ and $j = 0, \cdots, N$. Though some of these coefficients are negative at the beginning, they become positive and

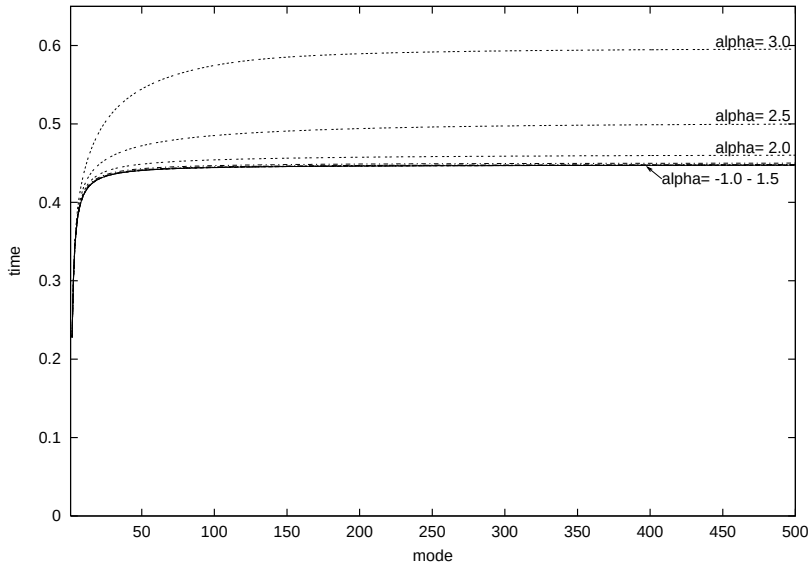

Fig. 9. Estimated times $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. Initial data are $A_n = 2e\frac{(-1)^{n+1}}{n^2}$ and $\nu = 0.01$.

eventually exceed 1. We plot estimated times $t^{(n)}$ when these coefficients exceeds 1 in Figure 5. This figure indicates that there exists $\sup_n t^{(n)} \approx 0.55$, at which all the coefficients are greater than 1 as we also see in Figure 6. These figures give evidences of blow-up of the solution (13).

Next, we compute the numerical solutions for $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. Other numerical parameters such as ν, A_1 and A_2 are unchanged. Figure 7 shows estimated times $t^{(n)}$ when the coefficient $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for each α, which indicates that there exists a blow-up time $t^* = \sup_n t^{(n)}$ for every α.

Solutions blow up as well when magnitude of the negative mode are larger than the positive mode, i.e. $|A_1| < |A_2|$. Figure 8 shows the estimated times $t^{(n)}$ at which $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ when $A_1 = 2e$, $\nu = 0.05$ and $p = 2$ for various α.

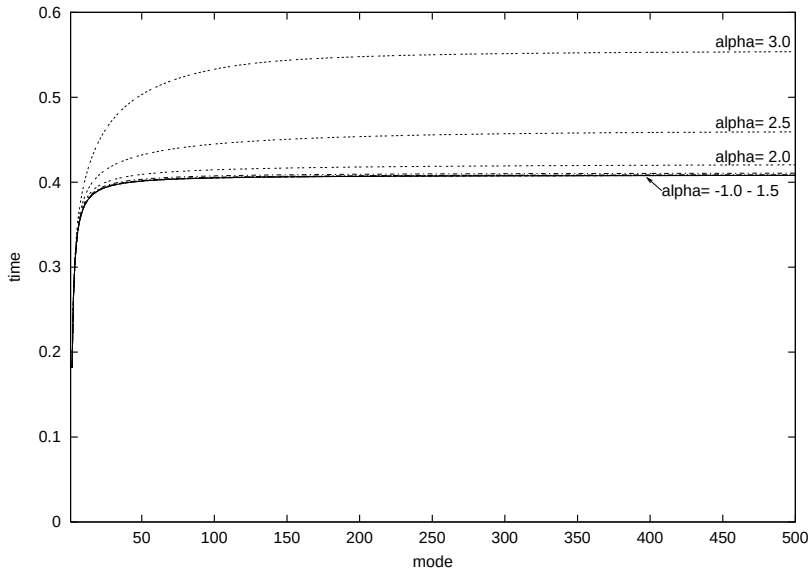

Fig. 10. Estimated times $t_{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. Initial data are $A_n = 2e\frac{(-1)^{n+1}}{n^3}$ and $\nu = 0.01$.

4.3. Blow-up of the solutions when $A_n = 2e\frac{(-1)^{n+1}}{n^p}$, $(p \geq 2)$

We show blow-up of the solutions when the initial Fourier coefficients are given by an alternative sequence, $A_n = 2e\frac{(-1)^{n+1}}{n^p}$ for $p = 2, 3$ and 4. The viscosity coefficients is $\nu = 0.01$ and the order of viscous diffusion are $\alpha = -1$, -0.5, 0, 0.5, 1, 1.5, 2, 2.5 and 3. We plot the estimated times $t^{(n)}$ such that $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $p = 2$, 3 and 4 in Figure 9, 10 and 11, respectively. These estimated times approach asymptotically to some finite times as $n \to \infty$, at which the solutions blow up.

5. Conclusion

We investigated a generalized one-dimensional model for the three-dimensional vorticity equation of incompressible and viscous flows. The viscosity term is given by an arbitrary order derivative of the vorticity. Whatever order derivative of the vorticity the viscosity term has, we proved mathematically that the solution of the equation blows up in finite time for

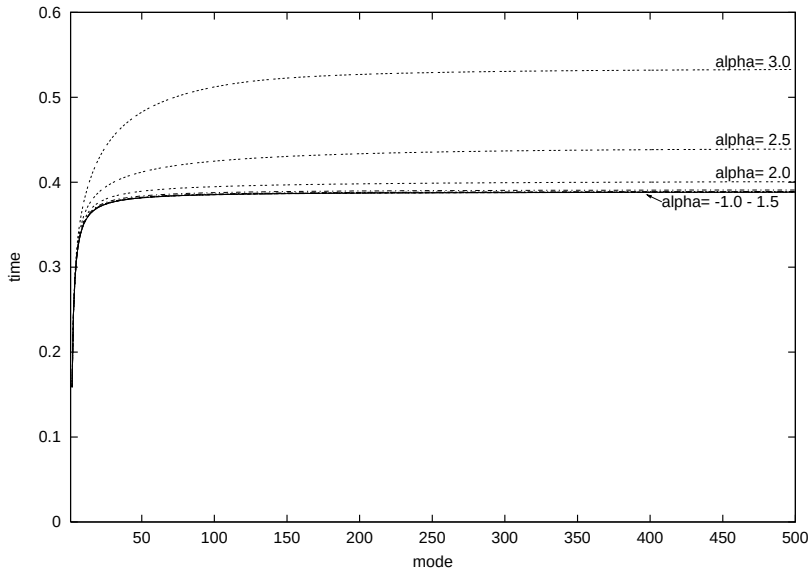

Fig. 11. Estimated times $t^{(n)}$ when $\mathrm{i}\omega_n\left(t^{(n)}\right) > 1$ for $\alpha = -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5$ and 3. Initial data are $A_n = 2e^{\frac{(-1)^{n+1}}{n^4}}$ and $\nu = 0.01$.

a small range of the viscosity coefficient if all the initial Fourier coefficients are non-negative. For other range of initial data, we show some numerical examples that indicate blow-up of the solution.

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