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The Corona Type Decomposition of Hardy-Orlicz Spaces

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Abstract. The H^p-corona type problem in several complex variables has been solved affirmatively by Amar [1], Andersson [2], Andersson-Carlesson [3, 4], Krantz-Li [11] and others. In particular, Andersson-Carlsson [4] proved the H^p-norm estimates of the corona solutions which are constructed by a concrete integral representation formula. In this paper, we give some Orlicz space versions for interpolation theorems of Marcinkiewicz type and prove the H_ϕ-norm estimates of the corona solutions for $\phi \in \Delta_2 \cap \nabla_2$. Moreover we also show that the Δ_2-condition is necessary in some interesting cases.

1. Introduction

Let $f_1, \cdots, f_m \in H^\infty(\Omega)$ be corona data and X be a space of holomorphic functions on Ω, where Ω is the bounded strictly pseudoconvex domain with C^3-smooth boundary. As usual, corona data means that $\sum_{i=1}^m |f_i(z)| \geq \delta > 0$ for all $z \in \Omega$.

Then we consider the mapping defined by

$$X \times \cdots \times X \ni (g_1, \cdots, g_m) \mapsto \sum_{i=1}^m f_i g_i \in X.$$

We say that X has X-corona solutions (for any corona data $f_1, \cdots, f_m$) if this mapping is surjective. Then, let $T_k : X \to X$, $(k = 1, \cdots, m)$ be an operator such that

$$h(z) = \sum_{k=1}^m f_k(z) \cdot T_k h(z), \quad (h \in X,\ z \in \Omega)$$

if X has the X-corona solution for the corona data $f_1, \cdots, f_m$. In particular we refer to $T_k h$, $(k = 1, \cdots, m)$ as the X-corona solution if T_k is bounded on X in such sense as $\|T_k h\|_X \leq C\|h\|_X$.

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As is well-known, H^p-corona type theorem says that Hardy space $H^p(\Omega)$ has the $H^p(\Omega)$ -corona solutions for any corona data. (For details, see Amar [1], Andersson [2], Andersson-Carlsson [3, 4], Krantz-Li [11] and others.) In particular, Andersson-Carlsson [4] shows that an explicit integral formula due to Berndtsson [5] provides the corona solutions and admits H^p -estimates.

THEOREM 1 (Andersson-Carlsson [4]). *Let* $1 \leq p < \infty$. *If* $f_1, \cdots, f_m \in H^\infty(\Omega)$ *satisfy that* $\sum_{i=1}^m |f_i(z)| \geq \delta > 0$ *for all* $z \in \Omega$, *then there exist integral operators* $T_i : H^p(\Omega) \to H^p(\Omega)$, $(i = 1, \cdots, m)$ *such that* $\sum_{i=1}^m f_i(z) T_i h(z) = h(z)$, $(z \in \Omega)$ *and* $\|T_i h\|_p \leq C \|h\|_p$ *for a positive constant* C.

In this paper, we show that this integral formula due to Berndtsson [5] admits H_ϕ-estimates and that $H_\phi(\Omega)$ has the $H_\phi(\Omega)$ -corona solutions if $\phi \in \Delta_2 \cap \nabla_2$. On the other hand, we prove that the Δ_2-condition for ϕ is necessary in order that the Szegö projection is of weak type (ϕ, ϕ).

At first, we recall the definition of the Hardy-Orlicz space $H_\phi(\Omega)$ corresponding to an N-function ϕ, where N-function is a continuous Young function $\phi : \boldsymbol{R} \to \boldsymbol{R}_+ \cup \{\infty\}$ such that (1) $\phi(x) = 0$ iff $x = 0$ and (2) $\lim_{x\to 0} \frac{\phi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\phi(x)}{x} = \infty$. $H_\phi(\Omega)$ consists of all holomorphic functions $f \in \mathcal{O}(\Omega)$ such that

$$\limsup_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \phi(|f|) d\sigma_\varepsilon < \infty.$$

We say that an N-function ϕ satisfies the Δ_2 -condition ($\phi \in \Delta_2$) if there exists a positive constant C such that

$$\phi(2x) \leq C\phi(x), \quad (x \geq 0).$$

And we say that an N-function ϕ satisfies the ∇_2 -condition ($\phi \in \nabla_2$) if there exists a constant $a > 1$ such that

$$\phi(x) \leq \frac{1}{2a}\phi(ax), \quad (x \geq 0).$$

Now, our main theorems are as follows.

Theorem 2. *Let* $\phi, \phi_2 \in \Delta_2 \cap \nabla_2$ *satisfy that* $\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} < 1$, *where* φ *and* φ_2 *are the left derivatives of* ϕ *and* ϕ_2 *respectively. We suppose that a sublinear operator* B *defined on* $H^1(\Omega)$ *and* $H_{\phi_2}(\Omega)$ *is of weak type* $(1,1)$ *and of weak type* (ϕ_2, ϕ_2). *Then* B *is defined on* $H_\phi(\Omega)$ *and the following holds:*

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma \le C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma \,:\, g \in L_\phi(\partial\Omega)\; \textit{such that}\, f = Sg\right\}, \quad (f \in H_\phi(\Omega)),$$

where S *is the Szegö projection.*

By combining the Theorems 1 and 2, we obtain the following corona type decomposition of Hardy-Orlicz spaces:

Corollary 1. *Let* $\phi \in \Delta_2 \cap \nabla_2$. *If* $f_1, \cdots, f_m \in H^\infty(\Omega)$ *is a corona data, that is, they satisfy that* $\sum_{i=1}^m |f_i(z)| \ge \delta > 0$ *for all* $z \in \Omega$, *then there exist integral operators* $T_i : H_\phi(\Omega) \to H_\phi(\Omega)$, $(i = 1, \cdots, m)$ *such that* $\sum_{i=1}^m f_i(z)T_i h(z) = h(z)$, $(z \in \Omega)$. *Furthermore it follows that there exists a positive constant* C *such that*

$$\int_{\partial\Omega} \phi(|T_i h|)d\sigma \le C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma : g \in L_\phi(\partial\Omega)\; \textit{such that}\, h = Sg\right\},$$

where S *is the Szegö projection.*

On the other hand, we show that the Δ_2-condition is necessary in the following sense.

Theorem 3. *Let* ϕ *be an* N*-function. We suppose that* S *is the Szegö projection on* Ω. *If* S *is of weak type* (ϕ, ϕ), *that is*

$$\phi(\lambda)\sigma(\{|Sf| > \lambda\}) \le C_1 \int_{\partial\Omega} \phi(C_2|f|)d\sigma, \quad (\lambda > 0,\ f \in L_\phi(\partial\Omega)),$$

then ϕ *satisfies the* Δ_2*-condition.*

Theorem 4. *Let* $f_1, \cdots, f_m \in H^\infty(\Omega)$ *be a corona data satisfying that* $\sum_{i=1}^m \|f_i\|_\infty < 1$. *We suppose that* $T_i : H^\infty(\Omega) \to H^1(\Omega)$, $(i = 1, \cdots, m)$ *is* *an operator which satisfies that* $h(z) = \sum_{i=1}^{m} f_i(z)T_ih(z)$, $(z \in \Omega)$. *If every operator* T_i *satisfies that*

$$\phi(\lambda)\sigma(\{|T_ih| > \lambda\}) \leq C \int_{\partial\Omega} \phi(|h|)d\sigma, \quad (\lambda > 0,\, h \in H_\phi(\Omega)),$$

then ϕ *satisfies the* Δ_2*-condition.*

2. Preliminaries

As shown in Imai [8], we can identify the Hardy-Orlicz space $H_\phi(\Omega)$ with a subspace of the associated Orlicz space $L_\phi(\partial\Omega)$ on the boundary $\partial\Omega$. It may be noted that the boundary $\partial\Omega$ endowed with a non-isotropic distance is a space of homogeneous type. (For details, see Stein [15].) Thus, we will show a variant of the interpolation thorem of Marcikiewicz type on a space X of homogeneous type below. In what follows, we denote the quasi-distance over X by d and Borel regular measure on X with doubling condition by μ:

$$\begin{aligned} d(x,y) &\leq K\{d(x,z) + d(z,y)\}, \quad (x,y,z \in X), \\ \mu(S_{2r}(x)) &\leq A\mu(S_r(x)), \quad (x \in X, r > 0), \end{aligned}$$

where $S_r(x) = \{y \in X : d(x,y) < r\}$ is a sphere at center x with radius r.

At first, in order to improve our interpolation theorem which is a variant of one in Gallardo [7], let us describe a definition of weak type inequality in $L_\phi(\partial\Omega)$. Let us recall that an operator T is said to be quasi-additive if $|T(f+g)| \leq C(|Tf| + |Tg|)$ for a constant $C > 0$. If $C = 1$ here, then T is called sublinear.

Definition 1. A sublinear operator T defined on an Orlicz space $L_\phi(X)$ is of weak type (ϕ, ϕ) if there exists positive constants C_1 and C_2 such that

$$\phi(\lambda)\mu(\{x \in X : |Tf| > \lambda\}) \leq C_1 \int_X \phi(C_2|f|)d\mu, \quad (f \in L_\phi(X),\, \lambda > 0).$$

Lemma 1. *Let* ϕ, ϕ_1 *and* ϕ_2 *be three N-functions satisfying the following growth conditions:*

$$\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} \quad < \quad 1,$$

$$\inf_{\lambda>0} \frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} \quad > \quad 1,$$

where φ,φ_1 *and* φ_2 *are the left derivatives of* ϕ, ϕ_1 *and* ϕ_2 *respectively. Then, there exist positive constants* C_1 *and* C_2 *such that*

$$\begin{aligned} \int_u^\infty \frac{\varphi(t)}{\phi_1(t)}dt &\leq C_1\frac{\phi(u)}{\phi_1(u)}, \quad (u>0), \\ \int_0^u \frac{\varphi(t)}{\phi_2(t)}dt &\leq C_2\frac{\phi(u)}{\phi_2(u)}, \quad (u>0). \end{aligned}$$

PROOF. We may take a positive number r such that

$$\sup_{\lambda>0} \frac{\varphi(\lambda)\phi_1(\lambda)}{\phi(\lambda)\varphi_1(\lambda)} < r < 1.$$

Then it follows that

$$\frac{\varphi(\lambda)}{\phi_1(\lambda)} < r\phi(\lambda)\frac{\varphi_1(\lambda)}{\phi_1(\lambda)^2} = -r\phi(\lambda)\frac{d}{d\lambda}\left(\frac{1}{\phi_1(\lambda)}\right), \quad (\lambda>0).$$

On the other hand, for any $\lambda_0 > 0$, the following holds:

$$\log\frac{\phi(\lambda)}{\phi(\lambda_0)} = \int_{\lambda_0}^\lambda \frac{\varphi(t)}{\phi(t)}dt \leq r\int_{\lambda_0}^\lambda \frac{\varphi_1(t)}{\phi_1(t)}dt = \log\left(\frac{\phi_1(\lambda)}{\phi_1(\lambda_0)}\right)^r, \quad (\lambda \geq \lambda_0).$$

Hence we obtain that

$$\begin{aligned} \int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda &\leq -r\left[\frac{\phi(\lambda)}{\phi_1(\lambda)}\right]_u^\infty + r\int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda \\ &= r\frac{\phi(u)}{\phi_1(u)} + r\int_u^\infty \frac{\varphi(\lambda)}{\phi_1(\lambda)}d\lambda, \quad (u>0), \end{aligned}$$

since $\frac{\phi(\lambda)}{\phi_1(\lambda)} \leq \frac{\phi(\lambda_0)}{\phi_1(\lambda_0)^r}\phi_1(\lambda)^{r-1} = C\phi_1(\lambda)^{r-1} \to 0$, $(\lambda \to \infty)$. Thus we conclude that

$$\int_u^\infty \frac{\varphi(\lambda)}{\phi(\lambda)}d\lambda \leq \frac{r}{1-r}\frac{\phi(u)}{\phi_1(u)}, \quad (u>0).$$

We can show the another inequality in the same way as above. □

Using the lemma above, we can prove a variant of the interpolation theorem in Gallardo [7] to prove the next theorem.

THEOREM 5. *Let ϕ, ϕ_1 and ϕ_2 be as in the lemma above and $\phi_1, \phi_2 \in \Delta_2$. We suppose that a sublinear operator $T : L_{\phi_1}(X) + L_{\phi_2}(X) \to M(X)$ is of weak type (ϕ_1, ϕ_1) and of weak type (ϕ_2, ϕ_2), where $M(X)$ is the set of all measurable functions on X. Then T is bounded on the Orlicz space $L_\phi(X)$:*

$$\int_X \phi(|Tf|)d\mu \le C_1 \int_X \phi(C_2|f|)d\mu, \quad (f \in L_\phi(X)).$$

Moreover we can obtain the same conclusion if T is of type (∞, ∞) and of weak type (ϕ_2, ϕ_2).

PROOF. From the weak type inequality and the sublinearity in the hypothesis, we can assume that

$$\begin{aligned} |T(f+g)| &\le |Tf| + |Tg|, \\ \phi_i(\lambda)\nu(|Tf| > \lambda) &\le C_i \int \phi_i(|f|)d\mu, \quad (i = 1, 2). \end{aligned}$$

For any $f \in L_\phi(X)$ and any $\lambda > 0$, we take f_λ and f^λ as follows:

$$\begin{aligned} f_\lambda &= f\chi_{\{|f| > \frac{\lambda}{2}\}}. \\ f^\lambda &= f - f_\lambda. \end{aligned}$$

Then, since $\nu(|Tf| > \lambda) \le \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) + \nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right)$, the following holds.

$$\begin{aligned} \int \phi(|Tf|)d\nu &= \int_0^\infty \varphi(\lambda)\nu(|Tf| > \lambda)d\lambda \\ &\le \int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right)d\lambda \\ &\quad + \int_0^\infty \varphi(\lambda)\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right)d\lambda. \end{aligned}$$

It may be noted that $f_\lambda \in L_{\phi_2}$ and $f^\lambda \in L_{\phi_1}$. In fact, $\phi_2(x) \le C_R\phi(x)$, $(\frac{\lambda}{2} = R \le x)$ and $\phi_1(x) \le C'_R\phi(x)$, $(x \le R = \frac{\lambda}{2})$, it follows that $\phi_2(|f_\lambda|) \le C_R\phi(|f|)$ and $\phi_1(|f^\lambda|) \le C'_R\phi(|f|)$. From the weak type inequality, the first term in the right hand side above is less than

$$\int_0^\infty \varphi(\lambda)d\lambda \int C_2\frac{\phi_2(|f_\lambda|)}{\phi_2(\frac{\lambda}{2})}d\mu \;\;\le\;\; C_2\int \phi_2(|f|)d\mu \int_0^{2|f|}\frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda.$$

We note that there exists $K > 0$ such that $K\phi_2(\frac{\lambda}{2}) \ge \phi_2(\lambda)$ since $\phi_2 \in \Delta_2$. Then, by using Lemma 1 we obtain that

$$\begin{aligned}\int_0^{2|f|}\frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda &\le K\int_0^{2|f|}\frac{\varphi(\lambda)}{\phi_2(\lambda)}d\lambda \\ &\le K'\frac{\phi(2|f|)}{\phi_2(2|f|)} \\ &\le K'\frac{\phi(2|f|)}{\phi_2(|f|)}.\end{aligned}$$

Hence the following holds.

$$\begin{aligned}\int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right)d\lambda &\le C_2K'\int \phi_2(|f|)\frac{\phi(2|f|)}{\phi_2(|f|)}d\mu \\ &\le C_2K'\int \phi(2|f|)d\mu.\end{aligned}$$

In a similar way as above, we can obtain that

$$\int_0^\infty \varphi(\lambda)\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right)d\lambda \le C_1K'\int \phi(2|f|)d\mu.$$

In the case that T is of type (∞,∞), we may assume that

$$\begin{aligned}\|Tf\|_\infty &\le C_1\|f\|_\infty. \\ \phi_2(\lambda)\nu(|Tf| > \lambda) &\le C_2\int \phi_2(|f|)d\mu.\end{aligned}$$

For any $f \in L_\phi(X)$ and any $\lambda > 0$, we take f_λ and f^λ as follows:

$$\begin{aligned}f_\lambda &= f\chi_{\{|f|>\frac{\lambda}{2C_1}\}}. \\ f^\lambda &= f - f_\lambda.\end{aligned}$$

We note that $\nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right) = 0$ since $\|Tf^\lambda\|_\infty \le C_1\|f^\lambda\|_\infty \le C_1\frac{\lambda}{2C_1} = \frac{\lambda}{2}$. Thus we obtain that

$$\nu(|Tf| > \lambda) \le \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right) + \nu\left(|Tf^\lambda| > \frac{\lambda}{2}\right) = \nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right).$$

Therefore it follows that

$$\begin{aligned}\int \phi(|f|)d\nu &= \int_0^\infty \varphi(\lambda)\nu(|Tf| > \lambda)d\lambda \\ &\le \int_0^\infty \varphi(\lambda)\nu\left(|Tf_\lambda| > \frac{\lambda}{2}\right)d\lambda \\ &\le C_2\int_0^\infty \varphi(\lambda)d\lambda\frac{\int \phi_2(|f_\lambda|)d\mu}{\phi_2(\frac{\lambda}{2})} \\ &\le C_2\int \phi_2(|f|)d\mu\int_0^{2C_1|f|}\frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda.\end{aligned}$$

Since $\phi_2 \in \Delta_2$, there exists $K > 0$ such that $K\phi_2(\frac{\lambda}{2}) \ge \phi_2(\lambda)$. Then, using Lemma 1, the following holds.

$$\begin{aligned}C_2\int \phi_2(|f|)d\mu\int_0^{2C_1|f|}\frac{\varphi(\lambda)}{\phi_2(\frac{\lambda}{2})}d\lambda &\le C_2K\int \phi_2(|f|)d\mu\int_0^{2C_1|f|}\frac{\varphi(\lambda)}{\phi_2(\lambda)}d\lambda \\ &\le C_2K\int \phi_2(|f|)\frac{\phi(2C_1|f|)}{\phi_2(2C_1|f|)}d\mu.\end{aligned}$$

Now we should note that $\phi_2(|f|) \le \phi_2(2C_1|f|)$ if $2C_1 \ge 1$ and that $\phi_2(|f|) \le L\phi_2(2C_1|f|)$ for an $L > 0$ if $2C_1 < 1$ since $\phi_2 \in \Delta_2$. Hence we obtain that

$$C_2K\int \phi_2(|f|)\frac{\phi(2C_1|f|)}{\phi_2(2C_1|f|)}d\mu \le C_2KL\int \phi(2C_1|f|)d\mu.$$

This completes the proof. □

Furthermore, a small modification of the proof in Coifman-Weiss [6] leads us to the following.

Theorem 6. *Let* $\phi \in \Delta_2 \cap \nabla_2$ *and* ϕ_2 *be an N-function . We suppose that* $\sup_{\lambda>0}\frac{\varphi(\lambda)\phi_2(\lambda)}{\phi(\lambda)\varphi_2(\lambda)} < 1$ *and that a sublinear operator* $B : H^1_{Re}(X) +$

$L_{\phi_2}(X) \to M(X)$ *is of weak type* $(H^1_{Re}, 1)$ *and of weak type* (ϕ_2, ϕ_2)*, where* $M(X)$ *is the set of all measurable functions on* X*. If* X *is bounded, then the following holds:*

$$\int_X \phi(|Bf|)d\mu \leq C \int_X \phi(|f|)d\mu, \quad (f \in L_\phi(X)).$$

Before giving the proof of this theorem, we show the boundedness of a modified maximal operator $M_q f = M(|f|^q)^{\frac{1}{q}}$, $(q > 1)$, where M is the Hardy-Littlewood maximal operator.

LEMMA 2. *Let* $\phi \in \Delta_2 \cap \nabla_2$*. We suppose that* $1 < q < \inf_{s>0} \frac{s\varphi(s)}{\phi(s)}$*. (For details, see Rao-Ren [14].) Then, we obtain the following:*

$$\int_X \phi(M_q f)d\mu \leq C \int_X \phi(|f|)d\mu \quad (f \in L_\phi(X)).$$

PROOF. Since the Hardy-Littlewood maximal operator M is of weak type $(1,1)$, a modified maximal operator M_q is of weak type (q,q). And it is trivial that M_q is of type (∞, ∞). Therefore we can apply the interpolation theorem 5 to M_q in order to obtain the following.

$$\int_X \phi(M_q f)d\mu \leq C \int_X \phi(|f|)d\mu \quad (f \in L_\phi(X)). \ \Box$$

PROOF OF THEOREM 6. It may be noted that the outline of this proof is similar to one in Coifman-Weiss [6].

We take $q > 1$ such that $1 < q < \inf_{s>0} \frac{s\varphi(s)}{\phi(s)}$ (For details, see Gallardo [7].) It may be assumed that there exist some sphere S_0 such that $X = S_0$ if X is bounded. Then we let $f \in L_\phi(X)$ and $U^\alpha = \{M_q f > \alpha\}$, $(\alpha > 0)$. Since M_q is bounded on $L_\phi(X)$ from Lemma 2, the following holds:

$$\mu(U^\alpha) \leq C \frac{\int_X \phi(|f|)d\mu}{\phi(\alpha)}.$$

Hence, if $\alpha > \alpha_0 = \phi^{-1}\left(2C\frac{\int_X \phi(|f|)d\mu}{\mu(S_0)}\right)$, we obtain that $\mu(U^\alpha) \leq \frac{\mu(S_0)}{2} < \mu(X)$. Thus U^α is a bounded open subset of X and $U^\alpha \neq X$ if $\alpha > \alpha_0$.

Now we can apply the Whitney covering lemma to U^α for $\alpha > \alpha_0$ in order to choose some sequence of spheres S_j such that the following holds:

(1) $U^\alpha = \bigcup_j S_j$.

(2) No points belong to more than m of spheres S_j.

We let $\chi_j = \chi_{S_j}$ be the characteristic function of a sphere S_j and $\eta_j(x) = \frac{\chi_j(x)}{\sum_k \chi_k(x)}\chi_{U^\alpha}(x)$. Using these functions, we construct the Calderón-Zygmund decomposition $f = g_\alpha + h_\alpha$ for $\alpha > \alpha_0$ as follows:

$$
\begin{aligned}
g_\alpha(x) &= \begin{cases} f(x), & (x \notin U^\alpha) \\ \sum_j m_{S_j}(\eta_j f)\chi_j(x), & (x \in U^\alpha), \end{cases} \\
h_\alpha(x) &= \sum_j h_j^\alpha(x), \\
h_j^\alpha(x) &= f(x)\eta_j(x) - m_{S_j}(\eta_j f)\chi_j(x),
\end{aligned}
$$

where $m_{S_j}(\eta_j f) = \mu(S_j)^{-1}\int_{S_j} \eta_j f d\mu$. Then it is shown that there exist positive constant C such that

$$\left(\frac{1}{\mu(S_j)}\int_{S_j} |h_j^\alpha|^q d\mu\right)^{\frac{1}{q}} \leq C\alpha.$$

(For details, see Coifman-Weiss [6].) Hence $a_j = \frac{1}{C\alpha\mu(S_j)}h_j^\alpha$ is a $(1,q)$-atom and $h_\alpha = C\alpha\sum_j \mu(S_j)a_j \in H^1_{Re}(X) = H^{1,q}_{Re}(X)$ for $\alpha > \alpha_0$. Then, from the m-disjointness of S_j and the definition of the norm in $H^1_{Re}(X) = H^{1,q}_{Re}(X)$, it follows that

$$\|h_\alpha\|_{H^1_{Re}} \leq C\alpha\sum_j \mu(S_j) \leq mC\alpha\mu(U^\alpha).$$

Using this decomposition $f = g_\alpha + h_\alpha$ for $\alpha > \alpha_0$ and the definition of α_0, we obtain that

$$
\begin{aligned}
\int_X \phi(|Bf|)d\mu &= \int_0^\infty \varphi(\alpha)\mu(|Bf| > \alpha)d\alpha \\
&= \int_0^{\alpha_0} \varphi(\alpha)\mu(|Bf| > \alpha)d\alpha + \int_{\alpha_0}^\infty \varphi(\alpha)\mu(|Bf| > \alpha)d\alpha \\
&\leq \phi(\alpha_0)\mu(X) + \int_{\alpha_0}^\infty \varphi(\alpha)\mu(|Bf| > \alpha)d\alpha
\end{aligned}
$$

$$\begin{aligned}
&\leq 2C\int_X \phi(|f|)d\mu + \int_{\alpha_0}^{\infty}\varphi(\alpha)\mu\left(|Bg_\alpha| > \frac{\alpha}{2}\right)d\alpha \\
&\quad + \int_{\alpha_0}^{\infty}\varphi(\alpha)\mu\left(|Bh_\alpha| > \frac{\alpha}{2}\right)d\alpha \\
&= 2C\int_X \phi(|f|)d\mu + I_1 + I_2
\end{aligned}$$

We substitute the weak type inequality hypothesis

$$\mu\left(|Bh_\alpha| > \frac{\alpha}{2}\right) \leq \frac{M_1\|h_\alpha\|_{H^1_{Re}}}{\frac{\alpha}{2}} = \frac{2M_1\|h_\alpha\|_{H^1_{Re}}}{\alpha}$$

and the H^1_{Re}-norm estimate into I_2 to obtain that

$$I_2 \leq 2mM_1C\int_{\alpha_0}^{\infty}\varphi(\alpha)\mu(U^\alpha)d\alpha \leq 2mM_1C\int_X \phi(M_q f)d\mu.$$

Applying Lemma 2, it follows that $I_2 \leq C\int_X \phi(|f|)d\mu$. In order to estimate I_1, we substitute the weak type inequality hypothesis

$$\mu\left(|Bg_\alpha| > \frac{\alpha}{2}\right) \leq M_2\frac{\int_X \phi_2(|g_\alpha|)d\mu}{\phi_2(\frac{\alpha}{2})} = CM_2\frac{\int_X \phi_2(|g_\alpha|)d\mu}{\phi_2(\alpha)}$$

into I_1 and obtain that

$$\begin{aligned}
I_1 &\leq CM_2\int_{\alpha_0}^{\infty}\frac{\varphi(\alpha)}{\phi_2(\alpha)}d\alpha\int_X \phi_2(|g_\alpha|)d\mu \\
&= CM_2\int_{\alpha_0}^{\infty}\frac{\varphi(\alpha)}{\phi_2(\alpha)}d\alpha\left(\int_{U^\alpha}\phi_2(|g_\alpha|)d\mu + \int_{X\setminus U^\alpha}\phi_2(|g_\alpha|)d\mu\right) \\
&= CM_2(I_{1,1} + I_{1,2}).
\end{aligned}$$

Since $|g_\alpha| \leq C\alpha$, we apply Lemma 2 to show that

$$I_{1,1} = C\int_{\alpha_0}^{\infty}\varphi(\alpha)\mu(U^\alpha)d\alpha \leq C\int_X \phi(M_q f)d\mu \leq C\int_X \phi(|f|)d\mu.$$

In order to estimate $I_{1,2}$, we apply Lemma 1 and the fact that $|f| \leq M_q f$, *a.e.* as follows:

$$\begin{aligned}
I_{1,2} &= \int_{\alpha_0}^{\infty}\frac{\varphi(\alpha)}{\phi_2(\alpha)}d\alpha\int_{X\setminus U^\alpha}\phi_2(|g_\alpha|)d\mu = \int_X \phi_2(|f|)d\mu\int_{M_q f}^{\infty}\frac{\varphi(\alpha)}{\phi_2(\alpha)}d\alpha \\
&\leq \int_X \phi_2(|f|)d\mu\int_{|f|}^{\infty}\frac{\varphi(\alpha)}{\phi_2(\alpha)}d\alpha \leq C\int_X \phi(|f|)d\mu.
\end{aligned}$$

Therefore, we conclude that $\int_X \phi(|Bf|)d\mu \leq C\int_X \phi(|f|)d\mu$, $(f \in L_\phi(X))$. $\square$

3. Proofs

As shown in Imai [8], we recall that the Szegö projection S is bounded on the Orlicz space $L_\phi(\partial\Omega)$, that is,

$$\int_{\partial\Omega} \phi(|Sf|)d\sigma \le C \int_{\partial\Omega} \phi(|f|)d\sigma, \quad (f \in L_\phi(\partial\Omega)),$$

if $\phi \in \Delta_2 \cap \nabla_2$. And it is shown that S is bounded from $H^1_{Re}(\partial\Omega)$ to $H^1(\Omega)$ in Krantz [10]. Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let $A = B \circ S$. Then, since A is bounded on real Hardy space $H^1_{Re}(\partial\Omega)$ and on an Orlicz space $L_{\phi_2}(\partial\Omega)$, we can apply our interpolation theorem, Theorem 6, in the previous section to the operator A in order to show that

$$\int_{\partial\Omega} \phi(|Ag|)d\sigma \le C \int_{\partial\Omega} \phi(|g|)d\sigma, \quad (g \in L_\phi(\partial\Omega)).$$

For any $f \in H_\phi(\Omega)$ there exists $g \in L_\phi(\partial\Omega)$ such that $f = Sg$ since $H_\phi(\Omega) = SL_\phi(\partial\Omega)$ as shown in Imai [8]. Hence we have that $Bf = (B \circ S)g = Ag$ in order to obtain that

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma = \int_{\partial\Omega} \phi(|Ag|)d\sigma \le C \int_{\partial\Omega} \phi(|g|)d\sigma.$$

Since g is an arbitrary function in $L_\phi(\partial\Omega)$ such that $f = Sg$, we can conclude that

$$\int_{\partial\Omega} \phi(|Bf|)d\sigma \le C \inf\left\{\int_{\partial\Omega} \phi(|g|)d\sigma \,:\, g \in L_\phi(\partial\Omega)\,\text{such that}\, f = Sg\right\}. \ \square$$

Proof of Corollary 1. Since $\phi \in \Delta_2 \cap \nabla_2$, there exists an N-function $\phi_2 \in \Delta_2 \cap \nabla_2$ such that $\sup_{x>0} \frac{\varphi(x)\phi_2(x)}{\phi(x)\varphi_2(x)} < 1$. (For details, see Rao-Ren [14].) Then we may apply Theorem 2 to operators T_i in Theorem 1 in order to complete the proof. $\square$

Before giving the proofs of Theorems 3 and 4, we show the following lemma.

LEMMA 3. *Let ϕ be an N-function. We suppose that a sublinear operator T on $L_\phi(\partial\Omega)$ is of weak type (ϕ, ϕ), that is,*

$$\phi(\lambda)\sigma(|Tf| > \lambda) \leq C_1 \int_{\partial\Omega} \phi(C_2|f|)d\sigma, \quad (f \in L_\phi(\partial\Omega),\, \lambda > 0).$$

If $\sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty > C_2$, then ϕ satisfies the Δ_2-condition.

PROOF. From the hypothesis, there exist $r > 1$ and $\|f\|_\infty \leq 1$ such that

$$K = \sigma(\{|Tf| > rC_2\}) > 0.$$

Then, for any $\lambda > 0$, we define a function $g \in L_\phi(\partial\Omega)$ by

$$g(\zeta) = \frac{\lambda}{rC_2} f(\zeta).$$

By applying the inequality of weak type to g, we obtain that

$$\phi(\lambda)\sigma\left\{|Tg| > \lambda\right\} \leq C_1 \int_{\partial\Omega} \phi(C_2|g|)d\sigma.$$

Since $\{|Tg| > \lambda\} = \{|Tf| > rC_2\}$, we have that $\sigma\left(\{|Tg| > \lambda\}\right) = \sigma\left(\{|Tf| > rC_2\}\right) = K > 0$. Therefore, we have that

$$\begin{aligned} \phi(\lambda) &\leq \sigma\left(\{|Tf| > rC_2\}\right)^{-1} C_1 \int_{\partial\Omega} \phi\left(C_2 \frac{\lambda}{rC_2} \|f\|_\infty\right) d\sigma \\ &\leq C_1 K^{-1} \|\sigma\| \cdot \phi\left(\frac{\lambda}{r}\right). \end{aligned}$$

This inequality shows that ϕ satisfies the Δ_2 -condition. □

Now we are ready to prove Theorems 3 and 4.

PROOF OF THEOREM 3. Since $SL^\infty(\partial\Omega) = BMOA \supset H^\infty$, it follows that

$$\sup\left\{\|Sf\|_\infty : f \in L^\infty \text{such that } \|f\|_\infty \leq 1\right\} = \infty.$$

Therefore we can apply Lemma 3 to the Szegö projection S. □

PROOF OF THEOREM 4. We suppose that $\sup\{\|T_i f\|_\infty : f \in H^\infty$ such that $\|f\|_\infty \leq 1\} \leq 1$ for every $i = 1, \cdots, m$. Now we choose a bounded holomorphic function $h \in H^\infty(\Omega)$ such that $\sum_{i=1}^m \|f_i\|_\infty < \|h\|_\infty \le 1$. Then we have that

$$\begin{aligned} \|h\|_\infty &\le \sum_{i=1}^m \|f_i\|_\infty \|T_i h\|_\infty \\ &\le \sum_{i=1}^m \|f_i\|_\infty \\ &< \|h\|_\infty. \end{aligned}$$

This is a contradiction. Therefore there exist a certain $k \in \{1, \cdots, m\}$ such that

$$\sup\{\|T_k f\|_\infty : f \in H^\infty \text{such that } \|f\|_\infty \le 1\} > 1.$$

Then we can apply Lemma 3 to the operators T_k. □

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