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An Integral Formula of Mellin's Type and Some Applications to Microlocal Analysis

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Abstract. We introduce an integral formula of Mellin's type for holomorphic functions without any growth order restriction. We apply this formula to solve some partial differential equations in complex domains. These equations appear in the study of the second microlocal analysis along Lagrangian submanifolds. As an important application, we have an extension of Funakoshi's solvability theorem in [3] to the case without any growth order restriction.

1. Introduction

We introduce an integral formula of Mellin's type for holomorphic functions defined in $D \times \Omega \subset \mathbb{C}^{n-1}_{z'} \times \mathbb{C}_{z_n}$, where Ω is an angled domain with a corner $z_n = 0$ in $\mathbb{C}$. We apply this formula to solve the following partial differential equations in $D \times \Omega$:

$$\sum_{|\alpha| \le m} a_\alpha(z) \partial_{z'}^{\alpha'} (z_n \partial_{z_n})^{\alpha_n} u(z) = f(z),$$

where $a_\alpha(z)$ are holomorphic functions with condition:

$$\Big(a_{(\alpha',0)}(\mathring{x}', 0)\Big)_{|\alpha'|=m} \neq 0, \quad a_{(0,\dots,0,m)}(\mathring{x}', 0) \neq 0.$$

These operators appear in the study of the second microlocal analysis along Lagrangian submanifolds. As an important application, we have an extension of Funakoshi's solvability theorem in [3] to the case without any growth order restriction for $f(z)$; for example, this extension directly implies the solvability of the equation

$$\sum_{|\alpha| \le m} a_\alpha(x) \partial_{x'}^{\alpha'} (x_n \partial_{x_n})^{\alpha_n} u(x) = f(x)$$

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in the space of hyperfunctions for any given hyperfunction $f(x)$ at $(\overset{\circ}{x}{}',0)$ under the transversally elliptic condition:

$$|\sum_{|\alpha|=m} a_\alpha(x)(\eta'/\eta_n)^{\alpha'} x_n{}^{\alpha_n}| \sim (|x_n| + |\eta'/\eta_n|)^m.$$

We give the plan of this paper.

In Section 2, we introduce our main results; an integral formula of Mellin's type for holomorphic functions and the solvability theorems for some partial differential equations. We also review the theory of second microlocal analysis, some existence theorems for the $\bar\partial$ operator and a Cauchy-Kovalevski theorem with a large parameter.

In Sections 3 and 4, we give the proofs of our main theorems.

2. Statements of the Main Theorems

2.1. An integral formula of Mellin's type for holomorphic functions

We denote by $\mathcal{O}$ the sheaf of holomorphic functions on $\mathbb{C}^n = \mathbb{C}^{n-1}_{z'} \times \mathbb{C}_{z_n}$, and by $\mathcal{D}'$ the sheaf of Schwartz distributions on $\mathbb{C}^{n-1}_{z'} \times \mathbb{R}_\rho \times \mathbb{R}_\theta$. Let D, $D' \subset \mathbb{C}^{n-1}$ be pseudoconvex domains with $D' \Subset D$ and let r, $\alpha_\pm$ be constants with $0 < r < 1$, $0 < \alpha_- - \alpha_+ < 2\pi$. We set $I_+ = (0, \pi/2)$, $I_- = (-\pi/2, 0)$.

THEOREM 2.1. *Let $D, D', r, \alpha_\pm$ be as above. Let $f(z)$ be a holomorphic function on $D \times \{z_n \in \mathbb{C}; 0 < |z_n| < r,\ \alpha_+ < \arg z_n < \alpha_-\}$. Then there exist functions $f_0(z) \in \mathcal{O}(D' \times \{z_n \in \mathbb{C}; |z_n| < r\})$ and $g_\pm(z', \lambda) \in \mathcal{D}'(\{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm\})$ with $\lambda = \rho e^{i\theta}$ satisfying the following integral formula for $z' \in D'$ and $z_n \in \mathbb{C}$ with $0 < |z_n| < r$, $\alpha_+ < \arg z_n < \alpha_-$:*

$$f(z) = f_0(z) + \sum_{\sigma=\pm1} \int_{\Gamma_\sigma} (z_n e^{-i\alpha_\sigma})^{i\sigma\lambda} g_\sigma(z', \lambda)\, d\lambda. \tag{2.1}$$

Here, distributions $g_\pm$ satisfy the following conditions:

(1) $\operatorname{supp} g_\pm \subset \{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm; \rho \geq 0\}$.

(2) $(\rho\partial/\partial\rho + i\partial/\partial\theta) g_\pm = 0$, $\partial g_\pm/\partial \bar z_j = 0$ *for* $j = 1, \ldots, n-1$; *in particular, $g_\pm$ are holomorphic functions of (z', λ) in $\{\lambda \neq 0\}$.*

(3) *There exists a positive constant* C_ε*, which is a locally bounded function of* ε *on* $(0, \pi/2)$ *such that we have* $|g_\pm(z', \rho e^{i\theta})| \leq C_{|\theta|}$ *for* $z' \in D'$, $\rho \geq 1$, $\theta \in I_\pm$.

(4) $g_\pm(z', \rho e^{i\theta})$ *converge to limits* $g_\pm(z', \rho e^{\pm i\pi/2})$ *as* $\varepsilon \to \pm\pi/2$ *respectively in the space of holomorphic functions on* D' *with values in* $\mathcal{S}'(\mathbb{R}_\rho)$.

Further we choose the infinite paths $\Gamma_\pm$ *as follows:*

$$\Gamma_\pm \colon \lambda = \lambda_\pm(\rho) = \rho e^{i\theta_\pm(\rho)} \quad (\rho \in \mathbb{R}), \tag{2.2}$$

where each $\theta_\pm(\rho) \in C^\infty(\mathbb{R})$ *satisfies the following conditions respectively:*

$$\begin{cases} 0 < \pm\theta_\pm(\rho) < \pi/2, \\ \pm\theta_\pm(\rho) \downarrow 0, \ \mp\theta'_\pm(\rho) \downarrow 0 \quad \textit{as} \quad \rho \to +\infty, \\ \rho^{-1} \log C_{|\theta_\pm(\rho)|} \to 0 \qquad \textit{as} \quad \rho \to +\infty. \end{cases} \tag{2.3}$$

As for the uniqueness of this decomposition, we have the following: Under the above conditions (1) $\sim$ (4), *another system of distributions* $g^{(1)}_\pm(z', \lambda) \in \mathcal{D}'(D' \times \mathbb{R} \times I_\pm)$ *satisfies* (2.1) *for* $f(z)$ *and* $f_0(z)$ *if and only if*

$$\sum_{\sigma=\pm 1} \sigma e^{i\alpha_\sigma \rho}\Big(g^{(1)}_\sigma(z', \rho e^{i\pi\sigma/2}) - g_\sigma(z', \rho e^{i\pi\sigma/2})\Big) = 0 \tag{2.4}$$

holds on $D' \times \mathbb{R}$.

Remark 2.2. i) Though $g_\pm(z', \rho e^{i\theta})$ are given as distributions of $(\operatorname{Re} z', \operatorname{Im} z', \rho, \theta)$, we use these notations in a symbolic sense. ii) The precise definitions of integrals in (2.1) are given as follows:

$$\int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} g_\pm(z', \lambda)\, d\lambda := \int_{-\infty}^{\infty} e^{\pm i(w - i\alpha_\pm)\lambda_\pm(\rho)} \cdot g_\pm(z', \lambda_\pm(\rho))\lambda'_\pm(\rho) d\rho,$$

where $w = \log z_n$. Indeed, in $\rho > 0$, the integrands are smooth functions of $(\operatorname{Re} z', \operatorname{Im} z', \rho)$ estimated by the following when $|\theta_\pm(\rho)| \leq \pi/4$ and $\pm(\arg z_n - \alpha_\pm) > 0$:

$$C_{|\theta_\pm(\rho)|} \cdot (1 + |\rho\theta'_\pm(\rho)|) \exp\Big(\mp \operatorname{Im}\Big((w - i\alpha_\pm)e^{i\theta_\pm(\rho)}\Big)\rho\Big) \leq$$
$$(1 + |\rho\theta'_\pm(0)|) \exp\Big[-\rho\Big(\frac{|\arg z_n - \alpha_\pm|}{\sqrt{2}} - |\theta_\pm(\rho) \log |z_n|| - \frac{\log C_{|\theta_\pm(\rho)|}}{\rho}\Big)\Big].$$

Hence, these integrals converge locally uniformly on $D' \times \{z_n \in \mathbb{C}; z_n \neq 0, \pm(\arg z_n - \alpha_\pm) > 0\}$, where they are holomorphic in (z', z_n), respectively. Further, as for the convergence of such integrals we can weaken the condition (3) on the uniform-boundedness for $|g_\pm(z', \lambda)|$: For example, we have the following $(3')$ instead of (3);

$(3')$ *For any* $\varepsilon > 0$ *there exists a positive constant* C_ε *such that we have* $|g_\pm(z', \rho e^{i\theta})| \leq C_\varepsilon e^{\varepsilon\rho}$ *for* $z' \in D'$, $\rho \geq 1$, $(\pi/2) - \varepsilon \geq |\theta| \geq \varepsilon$.

Example 2.3. We did not succeed in giving an explicit decomposition (2.1) for $f(z) = z_n^{\lambda_0}$ with a constant $\lambda_0 \notin \{0, 1, 2, ...\}$. Instead, it is easy to give some examples of $g_\pm(z', \lambda) \in \mathcal{D}'(\{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm\})$ satisfying the conditions (1) $\sim$ (4) (or $(3')$ above) of Theorem 2.1. We omit the parameters z' and write $z_n \equiv z$. Let $G_\pm(\lambda)$ be holomorphic functions defined in a neighborhood of

$$\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda \geq 0, \pm \operatorname{Im} \lambda > 0\} \cup \{0\}$$

satisfying the following estimates respectively: There exist some positive constants ℓ, C_ε such that

$$|G_\pm(\rho e^{i\theta})| \leq C_\varepsilon \rho^\ell \quad (\rho \geq 1,\ |\theta| \geq \varepsilon).$$

Then the distributions $g_\pm(\lambda) := G_\pm(\rho e^{i\theta})\mathrm{Y}(\rho)$ satisfy the conditions (1), (2), $(3')$, (4), where $\mathrm{Y}(\rho)$ is the Heaviside function. For example, if we set $g_\pm(\lambda) = c_\pm \mathrm{Y}(\rho)$ with constants $c_\pm$, we have the following decomposition formula satisfying all the conditions in Theorem 2.1:

$$\frac{ic_+}{\log z - i\alpha_+} + \frac{ic_-}{i\alpha_- - \log z} = \sum_{\sigma=\pm 1} \int_{\Gamma_\sigma} (ze^{-i\alpha_\sigma})^{i\sigma\lambda} (c_\sigma \mathrm{Y}(\rho)) d\lambda.$$

2.2. Solvability theorems of some partial differential equations

We apply Theorem 2.1 to the explicit construction of microlocal solutions for some differential operators treated in Funakoshi [3, 5]. Let V and Σ be the following regular involutive and Lagrangian submanifolds of T_M^*X with $M = \mathbb{R}^n, X = \mathbb{C}^n$ respectively:

$$V = \{(x; i\eta) \in T_M^*X; \eta_1 = \cdots = \eta_{n-1} = 0\},$$
$$\Sigma = \{(x; i\eta) \in T_M^*X; \eta_1 = \cdots = \eta_{n-1} = x_n = 0\}.$$

We use the following notations for variables: $x = (x', x_n), z = (z', z_n)$ with $x' = (x_1, ..., x_{n-1}), z' = (z_1, ..., z_{n-1})$, the complex dual variables $\zeta = \xi + i\eta = (\zeta', \zeta_n)$ with $\zeta' = (\zeta_1, ..., \zeta_{n-1})$ and multi-indices $\alpha = (\alpha', \alpha_n)$ with $\alpha' = (\alpha_1, ..., \alpha_{n-1})$.

Let $\overset{\circ}{p} = (\overset{\circ}{x}; i\overset{\circ}{\eta})$ be a point of Σ with $\overset{\circ}{x} = (\overset{\circ}{x}{}', 0), \overset{\circ}{\eta} = (0, ..., 0, 1)$. After Funakoshi [3] we consider the following differential operator of order m with analytic coefficients defined in a neighborhood of $\overset{\circ}{x}$:

$$P(x, D_{x'}, x_n D_{x_n}) = \sum_{|\alpha| \le m} a_\alpha(x) D_{x'}^{\alpha'} (x_n D_{x_n})^{\alpha_n}, \tag{2.5}$$

where $D_{x_j} = \partial / \partial x_j \ \ (j = 1, ..., n)$. The operators of this type with our main condition

$$\Big(a_{(\alpha',0)}(\overset{\circ}{x}) \Big)_{|\alpha'|=m} \neq 0, \quad a_{(0,...,0,m)}(\overset{\circ}{x}) \neq 0 \tag{2.6}$$

cover the transversally elliptic operators along Σ treated by Grigis-Schapira-Sjöstrand [6] at least as for the symbols. Indeed this class of operators has an essential importance in our theorems. Firstly we consider the following Cauchy problem for holomorphic functions:

$$\begin{cases} P(z, D_{z'}, z_n D_{z_n}) u(z) = f(z), \\ \partial_{z_1}^j u(\overset{\circ}{x}_1, z_2, ..., z_n) = h_j(z_2, ..., z_n) \ \ j = 0, ..., m-1, \end{cases} \tag{2.7}$$

where $P(z, D_{z'}, z_n D_{z_n})$ is the complexification of P at (2.5). We set complex submanifolds Y, X', Y' of X as follows:

$$Y = \{z \in X; z_n = 0\}, \tag{2.8}$$

$$X' = \{z \in X; z_1 = \overset{\circ}{x}_1\} \supset Y' = Y \cap X' = \{z \in X'; z_n = 0\}. \tag{2.9}$$

We denote by r_0, α_-, α_+ some real constants such that $r_0 > 0, \alpha_+ < \alpha_-$ with $\alpha_- - \alpha_+ < 2\pi$.

Theorem 2.4. *Let $P(z, D_{z'}, z_n D_{z_n})$, $\overset{\circ}{x}$ be as above. We assume that the coefficients $a_\alpha(z)$ of P are holomorphic in $\{z \in X; |z - \overset{\circ}{x}| < r_0\}$ with condition*

$$a_{(m,0,...,0)}(\overset{\circ}{x}) \neq 0. \tag{2.10}$$

Then there exist positive constants r_1 $(< r_0)$ *and* K *satisfying the following: For any holomorphic functions* $f(z) \in \mathcal{O}(\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_0,\ \alpha_+ < \arg z_n < \alpha_-\})$ *and* $h_j(z_2, ..., z_n) \in \mathcal{O}(\{z \in X' \setminus Y'; |z - \overset{\circ}{x}| < r_0,\ \alpha_+ < \arg z_n < \alpha_-\})$ $(j = 0, ..., m-1)$*, we have a unique solution* $u(z)$ *of Cauchy problem* (2.7)*, which is holomorphic in*

$$\text{(2.11)} \qquad \{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_1, \\ \alpha_+ + K|z - \overset{\circ}{x}| < \arg z_n < \alpha_- - K|z - \overset{\circ}{x}|\}.$$

REMARK 2.5. We can obtain this result only by the Cauchy-Kovalevski theorem and analytic continuation. The advantage of our proof is to give an integral expression of $u(z)$; indeed we use only once the Cauchy-Kovalevski theorem with a large parameter due to Funakoshi [3] and we obtain the domain (2.11) directly from this integral expression of $u(z)$.

The domain (2.11) is essentially smaller than the original domains of holomorphy of $f, h_0, ..., h_{m-1}$ because of the terms $\pm K|z - \overset{\circ}{x}|$. However under a stronger condition (2.6) we can remove these terms about the solvability property. The following theorem is our second main result, which is a generalization of Funakoshi [3].

THEOREM 2.6. *Let* P *be the following differential operator of order* m *defined in a neighborhood of* $\overset{\circ}{x} = (\overset{\circ}{x}{}', 0) \in M$:

$$\text{(2.12)} \qquad P(z, D_{z'}, z_n D_{z_n}) = \sum_{|\alpha| \le m} a_\alpha(z) D_{z'}^{\alpha'} (z_n D_{z_n})^{\alpha_n},$$

where $a_\alpha(z)$ *are holomorphic in* $\{z \in X; |z - \overset{\circ}{x}| < r_0\}$ *with condition:*

$$\Big(a_{(\alpha',0)}(\overset{\circ}{x})\Big)_{|\alpha'|=m} \neq 0, \quad a_{(0,\dots,0,m)}(\overset{\circ}{x}) \neq 0.$$

Then there exists a small constant $r_1 > 0$ *satisfying the following: For any holomorphic function* $f(z) \in \mathcal{O}(\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_0,\ \alpha_+ < \arg z_n < \alpha_-\})$ *we have a solution* $u(z)$ *of* $Pu = f$*, which is holomorphic in*

$$\text{(2.13)} \qquad \{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_1,\ \alpha_+ < \arg z_n < \alpha_-\}.$$

REMARK 2.7. In Theorem 3.2 of the former paper of Funakoshi [3], he essentially treated the case that $\alpha_+ = 0, \alpha_- = \pi$ under an additional hypothesis concerning the growth order of $f(z)$:

$$|f(z)| \leq C|\operatorname{Im} z_n|^{-q}$$

for some $q \in (0,1)$ as $\operatorname{Im} z_n \to +0$. We can remove this condition by the new idea in the decomposition of holomorphic functions as stated in Theorem 2.1.

Before giving the statements of microlocal versions of Theorems 2.4, 2.6, we recall 2 sheaves related to second microlocal analysis: The sheaf $\mathcal{C}^{\mathbb{R}}_{Y|X}$ on $T^*_Y X$ of holomorphic microfunctions defined by

$$\mathcal{C}^{\mathbb{R}}_{Y|X} = \mu_Y(\mathcal{O}_X)[1], \tag{2.14}$$

and the sheaf $\mathcal{A}^2_V$ on V of second analytic functions defined by

$$\mathcal{A}^2_V = \mu_N(\mathcal{O}_X)[1]|_V, \tag{2.15}$$

where $Y = \{z \in X; z_n = 0\}, N = \{z \in X; \operatorname{Im} z_n = 0\}$. Here we denote by $\mu_*(\cdot)$ the microlocalization functor by Kashiwara-Schapira [9, 10]. Both sheaves are defined on Σ and we have natural inclusion morphisms:

$$\mathcal{C}^{\mathbb{R}}_{Y|X}|_\Sigma \hookrightarrow \mathcal{A}^2_V|_\Sigma \hookrightarrow \mathcal{C}_M|_\Sigma,$$

where $\mathcal{C}_M$ is the sheaf on $T^*_M X$ of microfunctions. Indeed, the stalks at $\mathring{p} = (\mathring{x}; i dx_n)$ with $\mathring{x} = (\mathring{x}', 0)$ are given as follows:

(2.16)
$$\mathcal{C}^{\mathbb{R}}_{Y|X}|_{\mathring{p}} = \Big(\varinjlim_{r \to +0} \mathcal{O}(\{z \in \mathbb{C}^n; \operatorname{Im} z_n > -r|\operatorname{Re} z_n|, |z - \mathring{x}| < r\})\Big)/\mathcal{O}|_{\mathring{x}},$$

$$\mathcal{A}^2_V|_{\mathring{p}} = \Big(\varinjlim_{r \to +0} \mathcal{O}(\{z \in \mathbb{C}^n; \operatorname{Im} z_n > 0, |z - \mathring{x}| < r\})\Big)/\mathcal{O}|_{\mathring{x}}. \tag{2.17}$$

Further we set

$$\begin{aligned} \Sigma' =& \{(z_2, ..., z_n; \zeta_2, ..., \zeta_n) \in T^*X'; \operatorname{Im} z_2 = \cdots = \operatorname{Im} z_{n-1} = 0, \\ & z_n = 0, \zeta_2 = \cdots = \zeta_{n-1} = \operatorname{Re} \zeta_n = 0\} \simeq \Sigma \cap \pi^{-1}(X') \end{aligned} \tag{2.18}$$

with a natural projection $\pi : T^*X \to X$. The following theorems are direct corollaries of Theorems 2.4 and 2.6.

THEOREM 2.8. *Let $P(z, D_{z'}, z_n D_{z_n}), X', Y', \Sigma', \overset{\circ}{p}$ be as above. We suppose the condition* (2.10) *for P. Then for any germs*

$$f(z) \in \mathcal{C}^{\mathbb{R}}_{Y|X}|_{\overset{\circ}{p}}\ ,\ h_j(z_2, ..., z_n) \in \mathcal{C}^{\mathbb{R}}_{Y'|X'}|_{\overset{\circ}{p}'}\ (j = 0, ..., m-1)$$

with $\overset{\circ}{p}' = (\overset{\circ}{x}_2, ..., \overset{\circ}{x}_{n-1}, 0; idx_n) \in T^*_{Y'}X'$ the Cauchy problem (2.7) *has a unique solution $u(z) \in \mathcal{C}^{\mathbb{R}}_{Y|X}|_{\overset{\circ}{p}}$. In other words, we have the following exact sequence and isomorphism in a neighborhood of $\overset{\circ}{p}$:*

$$0 \longrightarrow {\mathcal{C}^{\mathbb{R}}_{Y|X}}^{P}|_{\Sigma} \longrightarrow \mathcal{C}^{\mathbb{R}}_{Y|X}|_{\Sigma} \xrightarrow{P} \mathcal{C}^{\mathbb{R}}_{Y|X}|_{\Sigma} \longrightarrow 0, \tag{2.19}$$

$${\mathcal{C}^{\mathbb{R}}_{Y|X}}^{P}|_{\Sigma \cap \pi^{-1}(X')} \xrightarrow{\sim} \left(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\right)^m|_{\Sigma'}, \tag{2.20}$$

where ${\mathcal{C}^{\mathbb{R}}_{Y|X}}^{P} :=$ kernel$(\mathcal{C}^{\mathbb{R}}_{Y|X} \xrightarrow{P} \mathcal{C}^{\mathbb{R}}_{Y|X})$ and a natural trace morphism:

$$\mathcal{C}^{\mathbb{R}}_{Y|X}|_{\Sigma \cap \pi^{-1}(X')} \ni u(z) \longmapsto \left(\partial^j_{z_1} u(\overset{\circ}{x}_1, z_2, ..., z_n)\right)_{j=0}^{m-1} \in \left(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\right)^m|_{\Sigma'}.$$

REMARK 2.9. According to Professor M. Uchida, this result is obtained also by the usual Cauchy-Kovalevski theorem and the method of the micro-support theory. However, our aim is to give an explicit formula for the solution (also see Remark 2.5).

PROOF. Choose some holomorphic functions $F(z), H_j(z_2, ..., z_n)$ $(j = 0, ..., m-1)$ such that $f = [F], h_j = [H_j]$ $(j = 0, ..., m-1)$: Here, $F \in \mathcal{O}(\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_0, -\delta < \arg z_n < \pi + \delta\})$ and $H_j \in \mathcal{O}(\{z \in X' \setminus Y'; |z - \overset{\circ}{x}| < r_0, -\delta < \arg z_n < \pi + \delta\})$ $(j = 0, ..., m-1)$ for some small positive constants r_0, δ. By Theorem 2.4 we get a solution $U(z)$ of the Cauchy problem (2.7) where u, f and h_j $(j = 0, ..., m-1)$ should be replaced by U, F and H_j $(j = 0, ..., m-1)$ respectively. Then there exists a sufficiently small $r_1 > 0$ such that the domain (2.11) of holomorphy of U includes

$$\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_1, -\frac{\delta}{2} < \arg z_n < \pi + \frac{\delta}{2}\}.$$

Therefore $u = [U]$ is a solution of (2.7) in $\mathcal{C}^{\mathbb{R}}_{Y|X}|_{\overset{\circ}{p}}$. The uniqueness of u follows directly from the Cauchy-Kovalevski theorem for holomorphic functions at $\overset{\circ}{x}$. □

THEOREM 2.10. *Let $P(x, D_{x'}, x_n D_{x_n}), \overset{\circ}{p}$ be as above. We suppose our main condition (2.6) for P. Then we have the following exact sequence and isomorphism in a neighborhood of $\overset{\circ}{p}$:*

$$0 \longrightarrow \mathcal{A}^{2}_{V}{}^{P}|_{\Sigma} \longrightarrow \mathcal{A}^{2}_{V}|_{\Sigma} \xrightarrow{P} \mathcal{A}^{2}_{V}|_{\Sigma} \longrightarrow 0, \tag{2.21}$$

$$\mathcal{C}^{\mathbb{R}}_{Y|X}{}^{P}|_{\Sigma} \xrightarrow{\sim} \mathcal{A}^{2}_{V}{}^{P}|_{\Sigma}, \tag{2.22}$$

where $\mathcal{A}^{2}_{V}{}^{P} := kernel(\mathcal{A}^{2}_{V} \xrightarrow{P} \mathcal{A}^{2}_{V})$.

REMARK 2.11. The last isomorphism is already obtained in Theorem 3.1 of Funakoshi [3]. We quoted it here for the reader's convenience. Further, as in Remark 2.7, Funakoshi [3] obtained also the surjectivity of (2.21) under a growth order restriction.

PROOF. We have only to prove the surjectivity of $P : \mathcal{A}^{2}_{V}|_{\overset{\circ}{p}} \longrightarrow \mathcal{A}^{2}_{V}|_{\overset{\circ}{p}}$. A germ $f(x)$ of $\mathcal{A}^{2}_{V}|_{\overset{\circ}{p}}$ is expressed as an equivalence class $f(x) = [F(z)]$ by using some holomorphic $F(z)$ defined in

$$\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_0,\ 0 < \arg z_n < \pi\}$$

with a small $r_0 > 0$. By Theorem 2.6 we get a holomorphic solution $U(z)$ of $PU = F$ defined in a domain similar to the above. Hence $[U(z)] \in A^{2}_{V}|_{\overset{\circ}{p}}$ is a solution of $Pu = f$. □

Together with Funakoshi's former results concerning the solvability in small second microfunctions in [5], we obtain the following theorem as a direct corollary of Theorems 2.8 and 2.10.

THEOREM 2.12. *Let $P(x, D_{x'}, x_n D_{x_n}), \overset{\circ}{p}, X', Y', \Sigma'$ be as above. We suppose the transversal ellipticity for the principal symbol $\sigma(P)$:*

$$|\sigma(P)(x, i\eta'/\eta_n, ix_n)| \sim (|x_n| + |\eta'/\eta_n|)^m \tag{2.23}$$

in a neighborhood of $\overset{\circ}{p}$ *in* $T^*_M X$*. Then we have the following exact sequence and isomorphisms in a neighborhood of* $\overset{\circ}{p}$*:*

$$0 \longrightarrow {\mathcal{C}_M}^P|_\Sigma \longrightarrow \mathcal{C}_M|_\Sigma \xrightarrow{P} \mathcal{C}_M|_\Sigma \longrightarrow 0, \tag{2.24}$$

$$ {\mathcal{C}_M}^P|_{\Sigma\cap\pi^{-1}(X')} \xleftarrow{\sim} {\mathcal{C}^{\mathbb{R}}_{Y|X}}^P|_{\Sigma\cap\pi^{-1}(X')} \xrightarrow{\sim} \Big(\mathcal{C}^{\mathbb{R}}_{Y'|X'}\Big)^m|_{\Sigma'}, \tag{2.25}$$

where ${\mathcal{C}_M}^P := \mathit{kernel}(\mathcal{C}_M \xrightarrow{P} \mathcal{C}_M)$*.*

PROOF. By the solvability result in small second microfunctions of Funakoshi [5] for a transversal elliptic equation $Pu = f$, we have the isomorphisms

$$ {\mathcal{A}^2_V}^P|_\Sigma \xrightarrow{\sim} {\mathcal{C}_M}^P|_\Sigma,\ \Big(\mathcal{A}^2_V/P\mathcal{A}^2_V\Big)|_\Sigma \xrightarrow{\sim} \Big(\mathcal{C}_M/P\mathcal{C}_M\Big)|_\Sigma $$

in a neighborhood of $\overset{\circ}{p}$. We remark here that condition (2.23) implies our main condition (2.6) and also the condition (2.10) for P. Therefore the exactness of (2.24) follows from Theorem 2.10. Further the isomorphisms (2.25) follow from Theorems 2.10 and 2.8. □

REMARK 2.13. The operators of the above class are called transversally elliptic operators along the Lagrangian submanifold Σ. In [6, 19], Grigis-Schapira-Sjöstrand proved a propagation theorem on analytic wavefront sets of solutions for those operators by using their FBI-method [18]. On the other hand, as for the solvability property of those operators, we had a Funakoshi's result in small second microfunctions for more general operators ([5]), and also a Wakabayashi's result for some class of transversally elliptic operators ([21]). However, to get a solvability theorem in microfunctions or hyperfunctions Funakoshi needed also any solvability result in second analytic functions $\mathcal{A}^2_V$. Our last theorem covers most of transversally elliptic operator along Σ, but we restricted the lower order parts of such operators in some special forms. We refer to [1, 2, 4, 8, 10, 11, 12, 13, 14, 15, 16, 17, 20] for second microlocal analysis in hyperfunction theory.

2.3. Preliminaries on existence theorems for the $\bar{\partial}$ operator and a Cauchy-Kovalevski theorem with a large parameter

Let Ω be an open set in $\mathbb{C}^n$ and φ a real-valued continuous function in Ω. Recall the $L^2(\Omega, \varphi)$ space of Hörmander, that is, $f \in L^2(\Omega, \varphi)$ if and only if

$$\|f\|_\varphi^2 := \int_\Omega |f|^2 e^{-\varphi}\, dV(z) < \infty.$$

Here the symbol $dV(z)$ is the standard Euclidean volume element on $\mathbb{C}^n$. This is a subspace of the space $L^2(\Omega, \mathrm{loc})$ of functions in Ω which are locally square integrable with respect to the Lebesgue measure, and it is clear that every function in $L^2(\Omega, \mathrm{loc})$ belongs to $L^2(\Omega, \varphi)$ for some φ. By $L^2_{(p,q)}(\Omega, \varphi)$ we denote the space of forms of type (p, q) with coefficients in $L^2(\Omega, \varphi)$,

$$f = {\sum_{|I|=p}}' {\sum_{|J|=q}}' f_{I,J}\, dz^I \wedge d\bar{z}^J,$$

where $\sum'$ means that the summation is performed only over strictly increasing multi-indices. We set

$$|f|^2 = {\sum_{I,J}}' |f_{I,J}|^2, \quad \|f\|_\varphi^2 = \int |f|^2 e^{-\varphi}\, dV = {\sum_{I,J}}' \|f_{I,J}\|_\varphi^2.$$

Note that $L^2(\Omega, \varphi)$ is a Hilbert space with this norm. Similarly we define the space $L^2_{(p,q)}(\Omega, \mathrm{loc})$.

The following theorem is of fundamental importance.

THEOREM 2.14 (Hörmander [7]). *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and φ any continuous plurisubharmonic function in Ω. For every $g \in L^2_{(p,q+1)}(\Omega, \varphi)$ with $\bar\partial g = 0$ there is a solution $u \in L^2_{(p,q)}(\Omega, \mathrm{loc})$ of the equation $\bar\partial u = g$ such that*

$$\int_\Omega |u|^2 e^{-\varphi}(1 + |z|^2)^{-2}\, dV \leq \int_\Omega |g|^2 e^{-\varphi}\, dV.$$

Next, we consider the Cauchy-Kovalevski theorem with a complex parameter λ for equations

$$P(z, D_{z'}, z_n D_{z_n} \pm i\lambda) U_\pm(z, \lambda) = G_\pm(z, \lambda),$$

where P is the differential operator at (2.5). First of all we set

$$\Omega_r = \{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r\}$$

for $r > 0$. We suppose that the coefficients $\{a_\alpha(z)\}_\alpha$ of P are holomorphic in Ω_{r_1} with some $r_1 > 0$, and that

$$a_{(m,0,\ldots,0)}(\overset{\circ}{x}) \neq 0.$$

Note that the principal symbol of $P(z, D_{z'}, z_n D_{z_n} \pm i\lambda)$ does not depend on the parameter λ. Let $\Gamma_\pm$ be the infinite paths defined at (2.2), and $G_\pm(z,\lambda)$ be functions in $L^\infty_{\mathrm{loc}}(\Omega_{r_1} \times \Gamma_\pm)$ satisfying the following conditions on $\Omega_{r_1} \times \Gamma_\pm$ for some positive valued function $M(\lambda) \in L^\infty_{\mathrm{loc}}(\Gamma_\pm)$:

$$\begin{cases} \bar{\partial}_z G_\pm(z,\lambda) = 0, \\ |G_\pm(z,\lambda)| \leq M(\lambda) \ \ (\operatorname{Re}\lambda > 0). \end{cases} \tag{2.26}$$

Then we consider the Cauchy problem:

$$\begin{cases} P(z, D_{z'}, z_n D_{z_n} \pm i\lambda) U_\pm(z,\lambda) = G_\pm(z,\lambda), \\ D^j_{z_1} U_\pm(\overset{\circ}{x}_1, z_2, \ldots, z_n) = 0 \quad j = 0, \ldots, m-1. \end{cases} \tag{2.27}$$

PROPOSITION 2.15 (Funakoshi [3]). *There exist some positive constants r_2, K which are depending only on r_1 and P such that the Cauchy problem* (2.27) *has a unique solution $U_\pm \in L^\infty_{loc}(\Omega_{r_2} \times \Gamma_\pm)$ satisfying the following conditions on $\Omega_{r_2} \times \Gamma_\pm$:*

$$\begin{cases} \bar{\partial}_z U_\pm(z,\lambda) = 0, \\ |U_\pm(z,\lambda)| \leq K M(\lambda) e^{K|z-\overset{\circ}{x}|\cdot\operatorname{Re}\lambda} \ \ (\operatorname{Re}\lambda > 0). \end{cases} \tag{2.28}$$

REMARK 2.16. In the paper of Funakoshi [3], this result is stated in a more specialized version. That is, for any $M(\lambda) = Ce^{-\delta \operatorname{Re}\lambda}$ with some positive constants C, δ, we have a sufficiently small $\delta' > 0$ and a positive constant C' such that $|U_\pm(z,\lambda)| \leq C' e^{-\delta' \operatorname{Re}\lambda}$ $(\operatorname{Re}\lambda > 0)$. However after replacing $G_\pm(z,\lambda)$ by $M(\lambda)^{-1} G_\pm(z,\lambda)$ we can directly obtain the result above from the original proof of Funakoshi based on the method of majorant series.

3. Proof of Theorem 2.1

We can suppose from the beginning that $0 \le \alpha_+ < \alpha_- < 2\pi$. Further we choose a pseudoconvex open set D'' as $D' \Subset D'' \Subset D$. We set:

$$U_0 = \{z_n \in \mathbb{C}; |z_n| < r\},$$
$$U_1 = \mathbb{P}^1 \setminus \{z_n \in \mathbb{C}; |z_n| \le r,\ \alpha_- \le \arg z_n \le \alpha_+ + 2\pi\}.$$

PROPOSITION 3.1. *We can find functions $f_j(z) \in \mathcal{O}(D \times U_j)$ for $j = 1, 2$ such that $f = f_0 + f_1$ in $D \times \{z_n \in \mathbb{C}; 0 < |z_n| < r,\ \alpha_+ < \arg z_n < \alpha_-\}$ and $f_1(z', \infty) \equiv 0$.*

PROOF. Note that the open set $(D \times U_0) \cup (D \times U_1) \subset \mathbb{C}^{n-1} \times \mathbb{P}^1$ is a Stein manifold. Therefore we can find the above functions by the solvability of the first Cousin problem. □

Next, choose the system of local coordinates $(z', w) = (z_1, \dots, z_{n-1}, w)$ with

$$w = \log z_n, \quad \alpha_+ < \arg z_n < \alpha_-,$$

and set $w = u + iv$. Then we will decompose the second function $f_1(z', e^w)$ into a sum $f_+(z', w) + f_-(z', w)$ of holomorphic functions $f_\pm \in \mathcal{O}(D'' \times \Omega_\pm)$ satisfying some growth order conditions. Here we set:

$$\begin{aligned}
\Omega\ &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \alpha_+ < \operatorname{Im} w < \alpha_-\},\\
\Omega^+ &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \operatorname{Im} w > \alpha_+\},\\
\Omega^- &= \{w \in \mathbb{C}; \operatorname{Re} w > \log r \ \text{ or } \ \operatorname{Im} w < \alpha_-\},
\end{aligned}$$

To this end, we will solve a $\bar\partial$-equation under some growth order condition as follows: We choose a C^∞-function $\psi\colon \mathbb{R} \to \mathbb{R}$ such that $0 \le \psi(v) \le 1$ for $\forall v \in \mathbb{R}$, $\psi(v) = 0$ for $\forall v \le \alpha_+ + \delta_1$ and $\psi(v) = 1$ for $\forall v \ge \alpha_- - \delta_1$, where $\delta_1 > 0$ is a small constant. Using this function, we define:

$$g(z', w) = \frac{\partial}{\partial \bar w}(f_1(z', e^w)\psi(v)) = \frac{i}{2} f_1(z', e^w)\psi'(v)$$

for $\alpha_+ < v < \alpha_-$. We can consider $g(z', w)$ as a C^∞-function on $D \times \mathbb{C}$ by setting $g(z', w) \equiv 0$ for $\operatorname{Im} w \in \mathbb{R} \setminus (\alpha_+, \alpha_-)$.

LEMMA 3.2. *There exists a C^∞-function $\chi\colon \mathbb{R} \to \mathbb{R}$ such that $g(z', w) \in L^2(D'' \times \mathbb{C}, \chi)$, $\chi'(u) < 0$, $\chi''(u) \geq 0$ for any $u \in \mathbb{R}$ and that $\chi(u) = 1/2 - u$ for $u > 0$.*

PROOF. Because of the fact that $f_1 \in \mathcal{O}(D \times U_1)$ and that $f_1(z', \infty) \equiv 0$, there exists a constant C_1 such that we have the inequality

$$|f_1(z)| \leq C_1 |z_n|^{-1} \quad \text{for} \quad z' \in D'', \quad |z_n| \geq 1.$$

Then we have the inequality:

$$|f_1(z', e^w)| \leq C_1 e^{-u} \quad \text{for} \quad z' \in D'', \quad u \geq 0. \tag{3.1}$$

We set $\chi(u) = 1/2 - u$ for $u \geq 0$. By (3.1), we get for $u \geq 0$

$$|g(z', w)|^2 e^{-\chi(u)} = \frac{1}{4} |\psi'(v) f_1(z', e^w)|^2 e^{-1/2+u} \leq \frac{1}{4} |\psi'(v)|^2 C_1^2 e^{-1/2-u}.$$

On the other hand, we can extend $\chi(u)$ to $u < 0$ so that $\chi'(u) < 0$, $\chi''(u) \geq 0$ and we have the inequalities

$$\sup\{|f_1(z', e^w)|; z' \in D'',\ \alpha_+ + \delta_1 \leq v \leq \alpha_- - \delta_1\} \leq C_2 e^{(\chi(u)+u)/2}$$

with some constant $C_2 > 0$. Then we have an inequality for $u < 0$ similar to the above:

$$|g(z', w)|^2 e^{-\chi(u)} \leq \frac{1}{4} |\psi'(v)|^2 C_2^2 e^{-|u|}. \ \square$$

LEMMA 3.3. *There exists a subharmonic function $\varphi(w) \in C^2(\mathbb{C})$ such that $\varphi(w) \geq \chi(u)$ for $\alpha_+ + \delta_1 \leq v \leq \alpha_- - \delta_1$ and $\varphi(w) = 0$ for $w \notin \{w \in \mathbb{C}; u < 1,\ \alpha_+ < v < \alpha_-\}$.*

PROOF. We construct $\varphi(w)$ as the product $\varphi(w) = \Phi(u)\Psi(v)$ of real and non-negative-valued C^2-functions $\Phi(u)$ and $\Psi(v)$. Choose $\Psi(v) \in C^\infty(\mathbb{R})$ so that $0 \leq \Psi(v) \leq 1$ for $v \in \mathbb{R}$, $\Psi(v) = 0$ for $v \leq \alpha_+$ or $v \geq \alpha_-$ and $\Psi(v) = 1$ for $\alpha_+ + \delta_1 \leq v \leq \alpha_- - \delta_1$. Since we can choose Ψ so that

$\Psi''(v) \geq 0$ in a neighborhood of $\{v \in \mathbb{R}; \Psi(v) = 0\}$, we may suppose that $\Psi''(v)/\Psi(v)$ is bounded from below. Hence, there exists a positive constant C_3 such that

$$\Psi''(v) + C_3\Psi(v) \geq 0 \quad \text{for} \quad v \in \mathbb{R}. \tag{3.2}$$

Assume a C^2-function $\Phi(u)$ satisfies the conditions:

(1) $\Phi''(u) - C_3\Phi(u) \geq 0$ for $u \in \mathbb{R}$.

(2) $\Phi(u) = 0$ for $u \geq 1$.

(3) $\Phi(u) \geq \max\{\chi(u), 0\}$ for $u \in \mathbb{R}$.

In this case, we have by (3.2)

$$\Delta\varphi(w) = \Phi''(u)\Psi(v) + \Phi(u)\Psi''(v) \geq 0$$

and then we can obtain the required function $\varphi(w) = \Phi(u)\Psi(v)$. So it is enough to construct a C^2-function $\Phi(u)$ with these properties. Let C_4 and C_5 be positive constants. We define

$$\Phi(u) = C_4 e^{-C_5/(1-u)} \ (0 \leq u < 1), \quad \Phi(u) = 0 \ (u \geq 1).$$

Setting $C_5 = 1 + \sqrt{1 + C_3}$, we have for $0 \leq u < 1$

$$\begin{aligned}\Phi''(u) - C_3\Phi(u) &= C_4 e^{-C_5/(1-u)} \left\{ \frac{C_5^2 - 2C_5(1-u)}{(1-u)^4} - C_3 \right\} \\ &\geq C_4 e^{-C_5/(1-u)} \frac{C_5^2 - 2C_5 - C_3}{(1-u)^4} = 0.\end{aligned}$$

Hence $\Phi(u)$ satisfies the condition (1) for $u \geq 0$. Moreover, if we choose a sufficiently large constant C_4, we have:

$$\Phi\left(\frac{1}{2}\right) = C_4 e^{-2C_5} \geq \frac{1}{2} = \chi(0).$$

Therefore $\Phi(u)$ satisfies the condition (3) for $u \geq 0$.

Next, we construct $\varPhi(u)$ for $u \leq 0$. We consider the ordinary differential equation:

$$\begin{cases} \varPhi''(u) = \left(C_3 + \dfrac{\chi''(u)}{\chi(u)} \right) \varPhi(u) \\ \varPhi'(0) = -C_4 C_5 e^{-C_5} \\ \varPhi(0) = C_4 e^{-C_5}. \end{cases} \tag{3.3}$$

There exists a unique solution $\varPhi(u) \in C^2((-\infty, 0))$ of (3.3), since $\chi(u) > 0$ for $u \leq 0$. In this case, we have $\varPhi(u) > 0$, $\varPhi''(u) > 0$ and

$$\varPhi''(u) - C_3 \varPhi(u) = \frac{\chi''(u)}{\chi(u)} \varPhi(u) \geq 0$$

for $u \leq 0$. Furthermore, we have

$$(\chi(u)\varPhi'(u) - \chi'(u)\varPhi(u))' = \chi(u)\varPhi''(u) - \chi''(u)\varPhi(u) = C_3 \chi(u)\varPhi(u) \geq 0$$

and

$$\chi(0)\varPhi'(0) - \chi'(0)\varPhi(0) = C_4 e^{-C_5}(-\frac{1}{2}C_5 + 1) \leq 0.$$

Then we obtain $\chi(u)\varPhi'(u) - \chi'(u)\varPhi(u) \leq 0$ and $(\chi(u)^{-1}\varPhi(u))' \leq 0$ for $u \leq 0$. From the fact that $(\chi(0)^{-1}\varPhi(0)) = 2C_4 e^{-C_5} > 1$ for a sufficiently large constant C_4, we can get $\chi(u)^{-1}\varPhi(u) > 1$ for $u \leq 0$. Therefore $\varPhi(u)$ satisfies the conditions (1) and (3) for $u \leq 0$.

Finally, we can claim that $\varPhi(u) \in C^2(\mathbb{R})$ because of the equality $C_5^2 - 2C_5 - C_3 = 0$. $\square$

From Lemmas 3.2 and 3.3, it follows that $g(z', w) \in L^2(D'' \times \mathbb{C}, \varphi)$. Then we can apply Theorem 2.14 due to Hörmander [7] to $g(z', w)\, d\bar{w} \in L^2_{(0,1)}(D'' \times \mathbb{C}, \varphi)$, that is to say, there is a solution $h(z', w) \in L^2(D'' \times \mathbb{C}, \mathrm{loc})$ of the equation $\bar{\partial} h = g\, d\bar{w}$ such that

$$\int_{D'' \times \mathbb{C}} |h|^2 e^{-\varphi} \left(1 + |(z', w)|^2\right)^{-2} dV \leq \int_{D'' \times \mathbb{C}} |g|^2 e^{-\varphi}\, dV.$$

In fact, $h \in L^2(D'' \times \mathbb{C}, \phi)$, where $\phi(z', w) := \varphi(w) + 2\log(1 + |(z', w)|^2)$.

Set:

$$f_+(z', w) = f_1(z', e^w)(1 - \psi(v)) + h(z', w),$$
$$f_-(z', w) = f_1(z', e^w)\psi(v) - h(z', w).$$

We find immediately that $f_\pm \in \mathcal{O}(D'' \times \Omega^\pm)$ and that

$$f_1(z', e^w) = f_+(z', w) + f_-(z', w) \quad \text{for} \quad (z', w) \in D'' \times \Omega.$$

Now we estimate values of the holomorphic functions $f_\pm$.

Proposition 3.4. *There exist positive-valued locally bounded functions $C_\theta^\pm$ on $\{0 < \pm\theta \leq \pi/2\}$ such that we have*

$$|f_\pm(z', w)| \leq C_\theta^\pm \left(1 + |w|^2\right) \quad \textit{for}\, \forall z' \in D',\, w = i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta}$$

with $\mu \in \mathbb{R}$, $\nu \geq 0$.

Proof. Take an arbitrary $\theta \in \{0 < \pm\theta \leq \pi/2\}$. Set:

$$\delta_2 = \delta_2(\theta) := \frac{1}{2} \min\{\mathrm{dis}(\bar{D}', \partial D''), |\log r| \sin|\theta|, 1\}$$
$$V_\theta^\pm = \{i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta} \in \mathbb{C}; \mu \in \mathbb{R}, \nu \geq -\delta_2(\theta)\} \subset \Omega^\pm$$

We can claim that

$$f_\pm|_{D'' \times V_\theta^\pm} \in L^2(D'' \times V_\theta^\pm, \phi).$$

Indeed, we have $h \in L^2(D'' \times V_\theta^\pm, \phi)$, and by the inequality (3.1):

$$\begin{aligned}
&\int_{D'' \times V_\theta^+} |f_1(1 - \psi)|^2 e^{-\phi}\, dV \\
&\quad \leq \int_{D'' \times \{w \in V_\theta^+; u \leq 0,\, v \leq \alpha_- - \delta_1\}} |f_1|^2 e^{-\phi}\, dV + \int_{D'' \times \{w \in \mathbb{C}; u \geq 0\}} |f_1|^2 e^{-\phi}\, dV \\
&\quad \leq \int_{D'' \times \{w \in V_\theta^+; u \leq 0,\, v \leq \alpha_- - \delta_1\}} |f_1|^2 e^{-\phi}\, dV \\
&\qquad\qquad + \int_{D'' \times \{w \in \mathbb{C}; u \geq 0\}} C_1^2 e^{-2u} \left(1 + v^2\right)^{-2}\, dV < +\infty.
\end{aligned}$$

Here, since $D'' \times \{w \in V_\theta^+; u \leq 0,\, v \leq \alpha_- - \delta_1\} \Subset D \times \Omega$, the first integral is finite. Thus, we get $f_+|_{D''\times V_\theta^+} \in L^2(D'' \times V_\theta^+, \phi)$. It is all the same to the case of $f_-|_{D''\times V_\theta^-}$.

Since $f_\pm$ is holomorphic on $D'' \times \Omega^\pm$, we have

$$f_\pm(z'_\circ, w_\circ) = \frac{1}{\mathrm{vol}(B((z'_\circ, w_\circ), \delta_2))} \int_{B((z'_\circ,w_\circ),\delta_2)} f_\pm(z', w)\, dV(z', w)$$

for any $(z'_\circ, w_\circ) \in D' \times \Omega^\pm$ with $w_\circ = i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta}$, $\mu \in \mathbb{R}$, $\nu \geq 0$. Here we set the open ball:

$$B((z'_\circ, w_\circ), \delta_2) = \{(z', w) \in \mathbb{C}^n; |(z', w) - (z'_\circ, w_\circ)| < \delta_2\}.$$

Note that $\bar{B}((z'_\circ, w_\circ), \delta_2) \subset D'' \times V_\theta^\pm$. Then we have the inequalities:

$$\begin{aligned}
|f_\pm(z'_\circ, w_\circ)| &\leq \frac{1}{\mathrm{vol}(B((z'_\circ, w_\circ), \delta_2))} \int_{B((z'_\circ,w_\circ),\delta_2)} |f_\pm| e^{-\phi/2} e^{\phi/2}\, dV \\
&\leq \frac{1}{\mathrm{vol}(B(0, \delta_2))} \left(\int_{B((z'_\circ,w_\circ),\delta_2)} |f_\pm|^2 e^{-\phi}\, dV \right)^{1/2} \cdot \left(\int_{B((z'_\circ,w_\circ),\delta_2)} e^{\phi}\, dV \right)^{1/2} \\
&\leq \frac{1}{\mathrm{vol}(B(0, \delta_2))^{1/2}} \left\| f_\pm|_{D''\times V_\theta^\pm} \right\|_\phi \sup\{e^{\phi/2}; (z', w) \in B((z'_\circ, w_\circ), \delta_2)\}.
\end{aligned}$$

From Lemma 3.3 and the fact that $e^{\phi/2} = e^{\varphi(w)/2}(1 + |(z', w)|^2)$, there exists a constant $A > 0$ such that

$$\sup_{B((z'_\circ,w_\circ),\delta_2)} e^{\phi/2} \leq \sup_{V_\theta^\pm} e^{\varphi(w)/2} \cdot A\left(1 + |w_\circ|^2\right).$$

We note here that the maximum of $\varphi(w)/2$ over $V_\theta^\pm$ is taken on a compact set $V_\theta^\pm \cap \{u \leq 1, \alpha_+ \leq v \leq \alpha_-\}$. Therefore if we set

$$C_\theta^\pm = \frac{A}{\mathrm{vol}(B(0, \delta_2(\theta)))^{1/2}} \sup_{V_\theta^\pm} e^{\varphi(w)/2} \cdot \left\| f_\pm|_{D''\times V_\theta^\pm} \right\|_\phi,$$

we can get the required inequalities. Further, we easily find the locally-boundedness on $\{0 < \pm\theta \leq \pi/2\}$ as functions of θ by these explicit definitions of $C_\theta^\pm$. □

Now, we define the following holomorphic functions:

$$F_{\pm}(z', w) = \frac{f_{\pm}(z', w)}{(w - i\alpha_{\pm} \pm i + 1)^4}. \tag{3.4}$$

By Proposition 3.4, we can get the following estimates.

COROLLARY 3.5. *There exist positive-valued locally bounded functions* $C_{\theta}^{\pm\prime}$ *on* $\{0 < \pm\theta \leq \pi/2\}$ *such that we have*

$$|F_{\pm}(z', w)| \leq \frac{C_{\theta}^{\pm\prime}}{1 + \mu^2 + \nu^2} \quad \textit{for} \quad z' \in D',\, w = i\alpha_{\pm} \pm (\mu + i\nu)e^{-i\theta}$$

with $\mu \in \mathbb{R}$, $\nu \geq 0$.

DEFINITION 3.6. We define

$$G_{\pm}(z', \lambda) = e^{-i\theta} \int_{-\infty}^{\infty} F_{\pm}(z', i\alpha_{\pm} \pm \mu e^{-i\theta}) e^{-i\mu\rho}\, d\mu, \tag{3.5}$$

for $z' \in D'$, $\lambda = \rho e^{i\theta}$ with $\rho \in \mathbb{R}$, $\theta \in \{0 < \pm\theta \leq \pi/2\}$.

Note that the integrals in (3.5) absolutely converge by Corollary 3.5 and that these functions are continuous in (z', ρ, θ). Note, moreover, that $G_{\pm}$ are written as:

$$G_{\pm}(z', \lambda) = \pm \int_{\gamma_{\pm}(\theta)} F_{\pm}(z', w) e^{\mp i(w - i\alpha_{\pm})\lambda}\, dw, \tag{3.6}$$

where $\gamma_{\pm}(\theta)$ is the path $\gamma_{\pm}(\theta)\colon w = i\alpha_{\pm} \pm \mu e^{-i\theta}$, $\mu \in \mathbb{R}$.

LEMMA 3.7.

(1) $\operatorname{supp} G_{\pm} \subset \{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_{\pm}; \rho \geq 0\}$.

(2) $(\rho\partial/\partial\rho + i\partial/\partial\theta)G_{\pm} = 0$, $\partial G_{\pm}/\partial\bar{z}_j = 0$ *for* $j = 1, \ldots, n-1$*; in particular,* $G_{\pm}$ *are holomorphic functions of* (z', λ) *in* $\{\lambda \neq 0\}$.

(3) $|G_{\pm}(z', \rho e^{i\theta})| \leq \pi C_{\theta}^{\pm\prime}$ *for* $\forall z', \forall\rho, \forall\theta \in \{0 < \pm\theta \leq \pi/2\}$.

PROOF. (1). We can deform the path of integration in (3.6) into $w = i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta}$, $\mu \in \mathbb{R}$ for any $\nu > 0$. Therefore we have:

$$G_\pm(z', \lambda) = e^{-i\theta} \int_{-\infty}^{\infty} F_\pm(z', i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta})e^{-i(\mu+i\nu)\rho}\, d\mu. \tag{3.7}$$

Then we have by Corollary 3.5

$$|G_\pm(z', \lambda)| \le \int_{-\infty}^{\infty} |F_\pm(z', i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta})|e^{\nu\rho}\, d\mu \le e^{\nu\rho} \int_{-\infty}^{\infty} \frac{C_\theta^{\pm\prime} d\mu}{1 + \mu^2}.$$

Hence $G_\pm(z', \lambda) = 0$ for $\rho < 0$, since ν is arbitrary as long as $\nu \ge 0$ in the preceding inequalities.

(2). We get the equalities:

$$\begin{aligned}
&\left(\rho\frac{\partial}{\partial\rho} + i\frac{\partial}{\partial\theta}\right)\left(F_\pm(z', i\alpha_\pm \pm \mu e^{-i\theta})e^{-i(\mu\rho+\theta)}\right) \\
&= \Big(\frac{\partial F_\pm}{\partial w}(z', i\alpha_\pm \pm \mu e^{-i\theta})(\pm\mu e^{-i\theta}) \\
&\qquad + F_\pm(z', i\alpha_\pm \pm \mu e^{-i\theta})(1 - i\mu\rho)\Big)e^{-i(\mu\rho+\theta)} \\
&= \frac{\partial}{\partial\mu}\left(F_\pm(z', i\alpha_\pm \pm \mu e^{-i\theta})\mu e^{-i(\mu\rho+\theta)}\right)
\end{aligned}$$

and

$$\frac{\partial}{\partial\bar{z}_j}\left(F_\pm(z', i\alpha_\pm \pm \mu e^{-i\theta})e^{-i(\mu\rho+\theta)}\right) = 0, \quad j = 1, \dots, n-1.$$

Then we have the required equalities immediately.

(3). It is easy to get the required inequality by the estimation in Corollary 3.5. □

DEFINITION 3.8. We set the distributions $g_\pm(z', \lambda) \in \mathcal{D}'(\{(z', \rho, \theta) \in D' \times \mathbb{R} \times I_\pm\})$ with $\lambda = \rho e^{i\theta}$ in the statement of Theorem 2.1 by

$$g_\pm(z', \lambda) = \frac{1}{2\pi}\Big(e^{-i\theta}\frac{\partial}{\partial\rho} + 1 \mp i\Big)^4 G_\pm(z', \lambda). \tag{3.8}$$

Further we give the constant C_ε by

$$C_\varepsilon := \frac{4!}{2\pi}\Big(\frac{1}{\min\{\sin(\frac{\varepsilon}{2}), \sin(\frac{\pi-2\varepsilon}{4})\}} + 2\Big)^4 \cdot \sup\{\pi C_\theta^{\pm\prime}; \frac{\varepsilon}{2} \le |\theta| \le \frac{\pi}{2}\} \tag{3.9}$$

for $0 < \varepsilon < \pi/2$. Note that $\pi C_\theta^{\pm'} \le C_{|\theta|}$.

Then, since $\rho\partial_\rho + i\partial_\theta$ commutes with $e^{-i\theta}\partial_\rho$, we obtain the conditions (1) $\sim$ (3) for $g_\pm$ in Theorem 2.1 directly from Lemma 3.7 and the Cauchy estimates. Further, $e^{-i\theta}F_\pm(z', i\alpha_\pm \pm \mu e^{i\theta})$ converges as $\theta \to \pm\pi/2$ respectively in the space of holomorphic functions on D' with values in $L^2(\mathbb{R}_\mu)$ because of the uniform estimates in Proposition 3.4. Therefore the distributions $g_\pm(z', \rho e^{i\theta})$ induced by $e^{-i\theta}F_\pm(z', i\alpha_\pm \pm \mu e^{i\theta})$ converge as $\theta \to \pm\pi/2$ respectively in the space of holomorphic functions on D' with values in $\mathcal{S}'(\mathbb{R}_\rho)$. Thus we also obtain the condition (4) for $g_\pm$ in Theorem 2.1.

Hereafter, let $\Gamma_\pm$ be any paths satisfying conditions (2.2), (2.3).

Lemma 3.9. *For any $z' \in D'$, $w = i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta}$ with $\mu \in \mathbb{R}$, $\nu > 0$ and with $\theta \in I_\pm$, we have in a classical sense*

$$F_\pm(z', w) = \frac{e^{i\theta}}{2\pi}\int_{-\infty}^{\infty} G_\pm(z', \rho e^{i\theta})e^{i(\mu+i\nu)\rho}\,d\rho. \tag{3.10}$$

Further by the change of the path of the integration we finally obtain that

$$F_\pm(z', w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_\pm(z', \rho e^{i\theta_\pm(\rho)}) \times e^{\pm i(w - i\alpha_\pm)\rho e^{i\theta_\pm(\rho)}}(1 + i\rho\theta'_\pm(\rho))e^{i\theta_\pm(\rho)}d\rho \tag{3.11}$$

for any $z' \in D', w \in \{\pm \operatorname{Im} w > \pm\alpha_\pm\}$.

Proof. By (3.7) we have the equations

$$e^{-\nu\rho + i\theta}G_\pm(z', \lambda) = \int_{-\infty}^{\infty} F_\pm(z', i\alpha_\pm \pm (\mu + i\nu)e^{-i\theta})e^{-i\mu\rho}\,d\mu.$$

Further by Lemma 3.7 we find that $e^{-\nu\rho+i\theta}G_\pm(z', \rho e^{i\theta})$ are integrable with respect to ρ over $\mathbb{R}$ (because $\nu > 0$). Therefore we obtain the equalities (3.10) by the inverse transformation. Since $\operatorname{supp} G_\pm \subset \{\rho \ge 0\}$, we have

$$F_\pm(z', w) = \frac{1}{2\pi}\int_0^{\infty} G_\pm(z', \rho e^{i\theta})e^{\pm i(w - i\alpha_\pm)\rho e^{i\theta}} \cdot e^{i\theta}d\rho,$$

where $z' \in D', w \in \{\operatorname{Re} w > 0\} \cup \{\pm \operatorname{Im} w > \pm\alpha_\pm\}$ and any constant $\theta \in I_\pm$ satisfying $\pm \operatorname{Im}((w - i\alpha_\pm)e^{i\theta}) > 0$. Note that $G_\pm(z', \rho e^{i\varphi})$ are holomorphic functions of $\rho e^{i\varphi}$ in $\{\rho > 0\}$ and that they are continuous at $\{\rho = 0\}$. Thus, for $z' \in D', w \in \{\pm \operatorname{Im}(w - i\alpha_\pm) > 0\}$ we obtain

$$
\begin{aligned}
(3.12) \qquad & F_\pm(z', w) = \\
& \lim_{R \to +\infty} \Big(\int_0^R G_\pm(z', \rho e^{i\theta_\pm(\rho)}) e^{\pm i(w - i\alpha_\pm)\rho e^{i\theta_\pm(\rho)}} \frac{(1 + i\rho\theta'_\pm(\rho))}{2\pi} e^{i\theta_\pm(\rho)} d\rho \\
& \qquad - \frac{1}{2\pi} \int_{Re^{i\theta}}^{Re^{i\theta_\pm(R)}} G_\pm(z', \lambda) e^{\pm i(w - i\alpha_\pm)\lambda} d\lambda \Big).
\end{aligned}
$$

The integrand of the first term is estimated by

$$
\begin{aligned}
& \pi {C^\pm_{\theta_\pm(\rho)}}' \cdot \frac{(1 + \rho|\theta'_\pm(\rho)|)}{2\pi} \exp\Big(\mp \operatorname{Im}\Big((w - i\alpha_\pm)e^{i\theta_\pm(\rho)}\Big)\rho\Big) \\
& \le \frac{(1 + \rho|\theta'_\pm(0)|)}{2\pi} \exp\Big[-\rho\Big(\pm \operatorname{Im}\Big((w - i\alpha_\pm)e^{i\theta_\pm(\rho)}\Big) - \frac{\log C_{|\theta_\pm(\rho)|}}{\rho}\Big)\Big].
\end{aligned}
$$

Hence, the first integral converges locally uniformly on $D' \times \{\pm \operatorname{Im}(w - i\alpha_\pm) > 0\}$ to the left side of (3.11) as $R \to +\infty$. Taking the path of the second integral of (3.12) on a circle $|\lambda| = R$, we have the following estimate for the second term of (3.12):

$$
\begin{aligned}
& \frac{\pi R}{2\pi \cdot 2} \sup \Big\{ \pi C^{\pm\prime}_\varphi \exp\Big(\mp R \operatorname{Im}((w - i\alpha_\pm)e^{i\varphi})\Big); \pm\theta_\pm(R) \le \pm\varphi \le \pm\theta \Big\} \\
& \qquad \le \frac{R}{4} \exp\Big[-R\Big(|\operatorname{Im}(w - i\alpha_\pm)| \cos\theta \\
& \qquad\qquad - |\operatorname{Re}(w - i\alpha_\pm) \sin\theta| - \frac{\log C_{|\theta_\pm(R)|}}{R}\Big)\Big]
\end{aligned}
$$

for any sufficiently large R. Since $|\theta|$ can be chosen small enough for each w, the second term of (3.12) converges to 0 as $R \to +\infty$. Thus we completed the proof of the lemma. □

Here, recalling the relationship (3.4) between $f_\pm$ and $F_\pm$, we have

$$\begin{aligned} f_\pm(z',w) &= (w - i\alpha_\pm \pm i + 1)^4 F_\pm(z',w) \\ &= \int_{-\infty}^{\infty} \frac{\lambda'_\pm(\rho)}{2\pi} G_\pm(z',\lambda_\pm(\rho)) \cdot \Big(1 \mp i - \frac{1}{\lambda'_\pm(\rho)}\frac{\partial}{\partial\rho}\Big)^4 e^{\pm i(w-i\alpha_\pm)\lambda_\pm(\rho)} d\rho \\ &= \int_{-\infty}^{\infty} e^{\pm i(w-i\alpha_\pm)\lambda_\pm(\rho)} \Big(1 \mp i + \frac{\partial}{\partial\rho}\frac{1}{\lambda'_\pm(\rho)}\Big)^4 \Big(\frac{\lambda'_\pm(\rho)}{2\pi} G_\pm(z',\lambda_\pm(\rho))\Big) d\rho \\ &= \int_{-\infty}^{\infty} \frac{\lambda'_\pm(\rho)}{2\pi} e^{\pm i(w-i\alpha_\pm)\lambda_\pm(\rho)} \cdot \Big(1 \mp i + \frac{1}{\lambda'_\pm(\rho)}\frac{\partial}{\partial\rho}\Big)^4 G_\pm(z',\lambda_\pm(\rho)) d\rho. \end{aligned}$$

Since

$$\begin{aligned} \partial_\rho[G_\pm(z',\lambda_\pm(\rho))] &= [\partial_\rho G_\pm(z',\rho e^{i\theta}) + \theta'_\pm(\rho)\partial_\theta G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} \\ &= [\partial_\rho G_\pm(z',\rho e^{i\theta}) + i\rho\theta'_\pm(\rho)\partial_\rho G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} \\ &= e^{-i\theta_\pm(\rho)}\lambda'_\pm(\rho)[\partial_\rho G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)}, \end{aligned}$$

we have

$$(\lambda'_\pm(\rho)^{-1}\partial_\rho)^j[G(z',\lambda_\pm(\rho))] = [(e^{-i\theta}\partial_\rho)^j G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)}.$$

Hence we get

$$\begin{aligned} &(1 \mp i + \lambda'_\pm(\rho)^{-1}\partial_\rho)^4 G_\pm(z',\lambda_\pm(\rho)) \\ &= [(1 \mp i + e^{-i\theta}\partial_\rho)^4 G_\pm(z',\rho e^{i\theta})]|_{\theta=\theta_\pm(\rho)} = 2\pi g_\pm(z',\lambda_\pm(\rho)). \end{aligned}$$

Therefore we have

$$\begin{aligned} f_\pm(z',w) &= \int_{-\infty}^{\infty} e^{\pm i(w-i\alpha_\pm)\lambda_\pm(\rho)} \cdot g_\pm(z',\lambda_\pm(\rho))\lambda'_\pm(\rho) d\rho \\ &= \int_{\Gamma_\pm} e^{\pm i(w-i\alpha_\pm)\lambda} \cdot g_\pm(z',\lambda) d\lambda. \end{aligned}$$

Thus we obtain our main decomposition formula (2.1). Let $g^{(1)}_\pm(z',\lambda) \in \mathcal{D}'(D' \times \mathbb{R} \times I_\pm)$ be another system of distributions satisfying the conditions (1) $\sim$ (4) in Theorem 2.1. Then, the equation (2.1) is equivalent to the following:

$$\sum_{\sigma=\pm 1} \int_{\Gamma_\sigma} (z_n e^{-i\alpha_\sigma})^{i\sigma\lambda} \Big(g^{(1)}_\sigma(z',\lambda) - g_\sigma(z',\lambda)\Big) d\lambda = 0 \tag{3.13}$$

for any $z \in D' \times \{z_n \in \mathbb{C}; 0 < |z_n| < r,\ \alpha_+ < \arg z_n < \alpha_-\}$. Since each term of (3.13) is holomorphic in $D' \times \{z_n \in \mathbb{C}; z_n \neq 0,\ \alpha_+ < \arg z_n < \alpha_-\}$, the equation (3.13) reduces to the one for any $z \in D' \times \{z_n = e^{u+i(\alpha_+ + \alpha_-)/2} \in \mathbb{C}; u > 0\}$:

$$\sum_{\sigma=\pm 1} \int_{\Gamma_\sigma} e^{i\sigma\lambda u - \lambda\frac{(\alpha_- - \alpha_+)}{2}} \Big(g_\sigma^{(1)}(z', \lambda) - g_\sigma(z', \lambda) \Big) d\lambda = 0. \tag{3.14}$$

Since $\mathrm{Re}(i\sigma\lambda u - \lambda(\alpha_- - \alpha_+)/2) = -\rho(|\sin\theta| \cdot u + \cos\theta \cdot |\alpha_- - \alpha_+|/2)$ for any $\theta \in I_\sigma$, we can deform the path Γ_σ to $\{\lambda = \rho e^{i\sigma\pi/2}; \rho \in \mathbb{R}\}$ under the conditions (1) $\sim$ (4) in Theorem 2.1. Therefore the left side of (3.14) is equal to

$$\sum_{\sigma=\pm 1} i\sigma \int_{-\infty}^{+\infty} e^{-\rho u - i\sigma\rho\frac{(\alpha_- - \alpha_+)}{2}} \Big(g_\sigma^{(1)}(z', \rho e^{i\frac{\sigma\pi}{2}}) - g_\sigma(z', \rho e^{i\frac{\sigma\pi}{2}}) \Big) d\rho. \tag{3.15}$$

Because each term of (3.15) extends holomorphically with respect to u to $\{w \in \mathbb{C}; \mathrm{Re}\, w > 0\}$, the equation (3.14) is equivalent to

(3.16)
$$\sum_{\sigma=\pm 1} \sigma \int_{-\infty}^{+\infty} e^{-i\rho v - i\sigma\rho\frac{(\alpha_- - \alpha_+)}{2}} \Big(g_\sigma^{(1)}(z', \rho e^{i\frac{\sigma\pi}{2}}) - g_\sigma(z', \rho e^{i\frac{\sigma\pi}{2}}) \Big) d\rho = 0$$

for any $(z', v) \in D' \times \mathbb{R}$. It is clear that the equation (3.16) is equivalent to (2.4). This completes the proof of Theorem 2.1.

4. Proofs of Theorems 2.4, 2.6

Firstly we prove Theorem 2.4. Replacing $f(z)$ by

$$f(z) - P(z, D_{z'}, z_n D_{z_n}) \Big(\sum_{j=0}^{m-1} \frac{(z_1 - \mathring{x}_1)^j}{j!} h_j(z_2, ..., z_n) \Big),$$

we can take the Cauchy data $\partial_{z_1}^j u(\mathring{x}_1, z_2, ..., z_n)$ $(j = 0, ..., m-1)$ as 0. Then by Theorem 2.1 we have a decomposition for $f(z)$:

$$f(z) = f_0(z) + \int_{\Gamma_+} (z_n e^{-i\alpha_+})^{i\lambda} g_+(z', \lambda) d\lambda + \int_{\Gamma_-} (z_n e^{-i\alpha_-})^{-i\lambda} g_-(z', \lambda) d\lambda$$

in $\{z' \in \mathbb{C}^{n-1}; |z' - \overset{\circ}{x}{}'| < r_1\} \times \{z_n \in \mathbb{C} \setminus \{0\}; |z_n| < r_1, \alpha_+ < \arg z_n < \alpha_-\}$ for a smaller positive r_1 ($< r_0$). Here, $f_0(z)$ is holomorphic at $\overset{\circ}{x} = (\overset{\circ}{x}{}', 0)$, $g_\pm(z', \rho e^{i\theta}) \in \mathcal{D}'(\{|z' - \overset{\circ}{x}{}'| < r_1\} \times \mathbb{R} \times \{0 < \pm\theta < \pi/2\})$ are distributions satisfying conditions (1) $\sim$ (4) in Theorem 2.1, and $\Gamma_\pm$ are infinite paths introduced at (2.2). It is sufficient to find holomorphic functions $U_\pm(z)$ which are defined in a domain like (2.11) such that

$$\begin{cases} P(z, D_{z'}, z_n D_{z_n}) U_\pm(z) = \int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} g_\pm(z', \lambda) d\lambda, \\ \partial_{z_1}^j U_\pm(\overset{\circ}{x}_1, z_2, ..., z_n) = 0 \ \ (j = 0, ..., m-1). \end{cases} \tag{4.1}$$

Choose a function $\varphi(\rho) \in C^\infty(\mathbb{R})$ as

$$0 \le \varphi(\rho) \le 1, \ \ \varphi(\rho) = 0 \ (\rho \le 1), \ \ \varphi(\rho) = 1 \ (\rho \ge 2). \tag{4.2}$$

and set

$$h_\pm(z', \lambda) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(1 - \varphi(\rho))}{\lambda - \rho e^{i\theta_\pm(\rho)}} g_\pm(z', \rho e^{i\theta_\pm(\rho)})(1 + i\rho\theta'_\pm(\rho)) e^{i\theta_\pm(\rho)} d\rho. \tag{4.3}$$

Then, it is easy to see that $h_\pm(z', \lambda)$ are holomorphic in

$$\{z' \in \mathbb{C}^{n-1}; |z' - \overset{\circ}{x}{}'| < r_1\} \times \Big(\mathbb{C} \setminus \{\lambda = \rho e^{i\theta_\pm(\rho)}; 0 \le \rho \le 2\}\Big).$$

Therefore we can divide $\int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} g_\pm(z', \lambda) d\lambda$ into sums:

$$\int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda_\pm(\rho)} \varphi(\rho) g_\pm(z', \lambda_\pm(\rho)) d\lambda_\pm(\rho) \\ + \int_{\Gamma_\pm^0} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} h_\pm(z', \lambda) d\lambda,$$

where $\Gamma_\pm^0$ are some closed curves turning around $\{\lambda = \rho e^{i\theta_\pm(\rho)}; 0 \le \rho \le 2\}$ counter-clockwise, respectively. Then after the method in Funakoshi [3] we put the solutions $U_\pm(z)$ of the form

$$U_\pm(z) = \int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} V_\pm(z, \lambda) d\lambda + \int_{\Gamma_\pm^0} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} W_\pm(z, \lambda) d\lambda \tag{4.4}$$

for (4.1). Then, it is sufficient to pose the following conditions:

$$(4.5)\qquad \begin{cases} \bar{\partial}_z V_\pm(z,\lambda_\pm(\rho)) = 0, \\ P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda_\pm(\rho)\Big) V_\pm(z,\lambda_\pm(\rho)) = \varphi(\rho) g_\pm(z',\lambda_\pm(\rho)), \\ \partial_{z_1}^j V_\pm(\mathring{x}_1, z_2, ..., z_n, \lambda_\pm(\rho)) = 0 \ \ (j = 0, ..., m-1), \end{cases}$$

and

$$(4.6)\qquad \begin{cases} \bar{\partial}_z W_\pm(z,\lambda) = 0, \\ P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda\Big) W_\pm(z,\lambda) = h_\pm(z',\lambda), \\ \partial_{z_1}^j W_\pm(\mathring{x}_1, z_2, ..., z_n, \lambda) = 0 \ \ (j = 0, ..., m-1), \end{cases}$$

for any $(\lambda, \rho, z) \in \mathit{\Gamma}_\pm^0 \times \mathbb{R} \times \{z \in \mathbb{C}^n; |z - \mathring{x}| < r_2\}$ with a smaller positive $r_2 \le r_1$. Since $\varphi(\rho) g_\pm(z', \lambda_\pm(\rho))$ is a continuous function on $\{z' \in \mathbb{C}^{n-1}; |z' - \mathring{x}'| < r_1\} \times \mathbb{R}_\rho$ with support in $\{\rho \ge 1\}$ satisfying an estimate by $C_{|\theta_\pm(\rho)|}$ respectively, we can solve (4.5) by Funakoshi's Proposition 2.15. Further, we can directly apply Cauchy-Kovalevski theorem to the problem (4.6) because the parameter λ moves only over compact sets $\mathit{\Gamma}_\pm^0$. Therefore, there exist some positive constants r_2 $(< r_1)$, K and solutions $V_\pm(z, \lambda_\pm(\rho)) \in L^\infty_{\mathrm{loc}}(\{z \in \mathbb{C}^n; |z - \mathring{x}| < r_2\} \times \mathbb{R})$, $W_\pm(z,\lambda) \in \mathcal{O}(\{z \in \mathbb{C}^n; |z - \mathring{x}| < r_2\} \times \mathit{\Gamma}_\pm^0)$ such that

$$(4.7)\qquad \begin{cases} \operatorname{supp} V_\pm(z,\lambda_\pm(\rho)) \subset \{\rho \ge 1\}, \\ \bar{\partial}_z V_\pm(z,\lambda_\pm(\rho)) = 0, \\ |V_\pm(z,\lambda_\pm(\rho))| \le K C_{|\theta_\pm(\rho)|} e^{K|z-\mathring{x}|\cdot \operatorname{Re}\lambda_\pm(\rho)}. \end{cases}$$

Hence, the integrals at (4.4) converge locally uniformly on

$$\{z \in X \setminus Y; |z - \mathring{x}| < r_2, \alpha_+ + K|z - \mathring{x}| < \arg z_n < \alpha_- - K|z - \mathring{x}|\}.$$

This completes the proof of Theorem 2.4.

Secondly we prove Theorem 2.6. By Theorem 2.1 we have a decomposition for $f(z)$:

$$f(z) = f_0(z) + \int_{\mathit{\Gamma}_+} (z_n e^{-i\alpha_+})^{i\lambda} g_+(z',\lambda) d\lambda + \int_{\mathit{\Gamma}_-} (z_n e^{-i\alpha_-})^{-i\lambda} g_-(z',\lambda) d\lambda$$

in $\{z' \in \mathbb{C}^{n-1}; |z' - \mathring{x}'| < r_1\} \times \{z_n \in \mathbb{C} \setminus \{0\}; |z_n| < r_1, \alpha_+ < \arg z_n < \alpha_-\}$ for a smaller positive r_1 $(< r_0)$. Here, $f_0(z)$ is holomorphic at $\mathring{x}$, $g_\pm(z', \rho e^{i\theta}) \in$

$\mathcal{D}'(\{|z' - \overset{\circ}{x}'| < r_1\} \times \mathbb{R} \times \{0 < \pm\theta < \pi/2\})$ are distributions satisfying conditions (1) $\sim$ (4) in Theorem 2.1, and $\Gamma_\pm$ are infinite paths introduced at (2.2). Choosing a cut-off function $\varphi(\rho)$ as in (4.2), we can use the cut-off arguments in the proof of Theorem 2.4. Hence we have only to find some holomorphic solutions $U_\pm(z)$ of

$$P(z, D_{z'}, z_n D_{z_n}) U_\pm(z) = \int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda_\pm(\rho)} \varphi(\rho) g_\pm(z', \lambda_\pm(\rho)) d\lambda_\pm(\rho).$$

Setting $U_\pm(z)$ in the form

$$U_\pm(z) = \int_{\Gamma_\pm} (z_n e^{-i\alpha_\pm})^{\pm i\lambda} V_\pm(z, \lambda) d\lambda, \tag{4.8}$$

we will find some solutions $V_\pm(z, \lambda_\pm(\rho)) \in L^\infty_{\mathrm{loc}}(\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_2\} \times \mathbb{R}_\rho)$ of the equations

$$\begin{cases} \bar{\partial}_z V_\pm(z, \lambda_\pm(\rho)) = 0, \\ P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda_\pm(\rho)\Big) V_\pm(z, \lambda_\pm(\rho)) = \varphi(\rho) g_\pm(z', \lambda_\pm(\rho)) \end{cases} \tag{4.9}$$

for a sufficiently small $r_2 > 0$. By performing some suitable linear coordinate transformation in z' under the condition (2.6) we may assume that

$$a_{(m,0,\dots,0)}(z) \neq 0, \quad a_{(0,\dots,0,m)}(z) \neq 0 \ \text{ in } \{|z - \overset{\circ}{x}| \leq r_1\}. \tag{4.10}$$

Write

$$P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda\Big) = a_{(0,\dots,m)}(z) \Big((\pm i\lambda)^m + \sum_{j=0}^{m-1} (\pm i\lambda)^j P_j(z, D_z) \Big),$$

where $P_j(z, D_z)$ is a differential operator of order $\leq m - j$ for every j. After Funakoshi's method of asymptotic solutions in [3], we set

$$\begin{aligned} V_\pm^{(1)}(z, \rho) = \sum_{k=0}^{\infty} & (\pm i\lambda_\pm(\rho))^{-m} \chi_{\{Ak<\rho\}}(\rho) \\ \times \Big(- \sum_{j=0}^{m-1} & (\pm i\lambda_\pm(\rho))^{j-m} P_j(z, D_z) \Big)^k \Big(\frac{\varphi(\rho)}{a_{(0,\dots,m)}(z)} g_\pm(z', \lambda_\pm(\rho)) \Big) \end{aligned} \tag{4.11}$$

with a large positive constant A. Note that $\varphi(\rho)g_{\pm}(z', \lambda_{\pm}(\rho))/a_{(0,\dots,m)}(z)$ have uniform infra-exponential growth orders as $\rho \to +\infty$ since we have uniform estimates on $\{|z' - \overset{\circ}{x}{}'| < r_1\} \times \{\rho > 0\}$:

$$|\varphi(\rho)g_{\pm}(z', \lambda_{\pm}(\rho))| \leq \exp\Big[\rho\Big(\frac{\log C_{|\theta_{\pm}(\rho)|}}{\rho}\Big)\Big].$$

Therefore by the propositions on some Cauchy-estimates in Section 4.2 of Funakoshi [3] we obtain the following lemma:

LEMMA 4.1. *There exist a small constant $r_2 > 0$, and a large constant $A_0 > 0$ such that for any $A \geq A_0$ and any $\rho \in \mathbb{R}$ the series at* (4.11) *converge absolutely and uniformly on $\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_2\}$ with estimates:*

$$|V_{\pm}^{(1)}(z, \rho)| \leq A_0 \exp\Big[\rho\Big(\frac{\log C_{|\theta_{\pm}(\rho)|}}{\rho}\Big)\Big] \quad (\rho > 0). \tag{4.12}$$

Further, the differences

$$\begin{aligned} R_{\pm}(z, \rho) &:= \varphi(\rho)g_{\pm}(z', \lambda_{\pm}(\rho)) - P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda_{\pm}(\rho)\Big)V_{\pm}^{(1)}(z, \rho) \\ &= \sum_{k=1}^{\infty} a_{(0,\dots,m)}(z)\chi_{\{A(k-1)<\rho\leq Ak\}}(\rho) \\ &\quad \times \Big(-\sum_{j=0}^{m-1}(\pm i\lambda_{\pm}(\rho))^{j-m}P_j(z, D_z)\Big)^k \Big(\frac{\varphi(\rho)}{a_{(0,\dots,m)}(z)}g_{\pm}(z', \lambda_{\pm}(\rho))\Big) \end{aligned}$$

have some uniform exponentially decreasing estimates

$$|R_{\pm}(z, \rho)| \leq A_0 e^{-\delta\rho} \quad (\rho \geq 1) \tag{4.13}$$

on $\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_2\}$ with a constant $\delta > 0$ independent of ρ, respectively.

The proof of this lemma goes in the same way as in Funakoshi [3]. Note that $R_{\pm}(z, \rho) \in L^{\infty}_{\mathrm{loc}}(\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_2\} \times \mathbb{R}_{\rho})$ satisfy $\bar{\partial}_z R_{\pm}(z, \rho) = 0$. Therefore by the condition (4.10), we can apply Proposition 2.15 to get a solution $V_{\pm}^{(2)}(z, \rho) \in L^{\infty}_{\mathrm{loc}}(\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_3\} \times \mathbb{R}_{\rho})$ of

$$\begin{cases} \bar{\partial}_z V_{\pm}^{(2)}(z, \rho) & = 0, \\ P\Big(z, D_{z'}, z_n D_{z_n} \pm i\lambda_{\pm}(\rho)\Big)V_{\pm}^{(2)}(z, \rho) & = R_{\pm}(z, \rho), \\ \partial_{z_1}^j V_{\pm}^{(2)}(\overset{\circ}{x}_1, z_2, \dots, z_n, \rho) & = 0 \quad (j = 0, \dots, m-1) \end{cases}$$

satisfying the uniform exponentially decreasing estimates

$$|V_{\pm}^{(2)}(z,\rho)| \le A_1 e^{-\frac{\delta}{2}\rho} \quad (\rho \ge 1) \tag{4.14}$$

on $\{z \in \mathbb{C}^n; |z - \overset{\circ}{x}| < r_3\}$ for some positive constants $r_3\ (< r_2), A_1\ (> A_0)$. It is clear that $\operatorname{supp}(V_{\pm}^{(2)}(z,\rho)) \subset \{\rho \ge 1\}$. Hence, we get some solutions $V_{\pm}(z,\lambda_{\pm}(\rho)) := V_{\pm}^{(1)}(z,\rho) + V_{\pm}^{(2)}(z,\rho)$ of (4.9), which are holomorphic in z and have supports contained in $\{\rho \ge 1\}$. By (4.12), (4.14) we also have the following estimates:

$$|V_{\pm}(z,\lambda_{\pm}(\rho))| \le (A_0 + A_1)\exp\Big[\rho\Big(\frac{\log C_{|\theta_{\pm}(\rho)|}}{\rho}\Big)\Big] \quad (\rho \ge 1).$$

Thus the functions $U_{\pm}(z)$ defined at (4.8) are holomorphic in $\{z \in X \setminus Y; |z - \overset{\circ}{x}| < r_3,\ \pm(\arg z_n - \alpha_{\pm}) > 0\}$ respectively. This completes the proof of Theorem 2.6.

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