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# *Remarks on Analytic Hypoellipticity and Local Solvability in the Space of Hyperfunctions*

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**Abstract.** Let $p(x, D)$ be a pseudodifferential operator on $\mathbb{R}^n$ with a ( formal) analytic symbol $p(x, \xi)$, and let $x^0 \in \mathbb{R}^n$. In this paper we prove that the transposed operator ${}^tp(x, D)$ of $p(x, D)$ is locally solvable at $x^0$ modulo analytic functions in the space of hyperfunctions if $p(x, D)$ is analytic hypoelliptic at $x^0$. We also microlocalize this result.

## 1. Introduction

Let $P$ be a linear partial differential operator on $\mathbb{R}^n$ with $C^\infty$ coefficients, and let $x^0 \in \mathbb{R}^n$. In Treves [10] and Yoshikawa [13] it was proved that if $P$ is hypoelliptic at $x^0$, then there is a neighborhood $U$ of $x^0$ satisfying the following; for every $f \in C^\infty(U)$ there is $u \in \mathcal{D}'(U)$ such that ${}^tPu = f$ in $U$. Here ${}^tP$ denotes the transposed operator of $P$. Recently Albanese, Corli and Rodino proved in [1] that the above result is still valid in the framework of the Gevrey classes and the spaces of ultradistributions. Moreover, Cordaro and Trépreau proved in [2] that $P$ is locally solvable at $x^0$ in the space of hyperfunctions if the coefficients of $P$ are analytic and $P$ is analytic hypoelliptic at $x^0$. Precise definitions of local solvability and analytic hypoellipticity will be given in Definition 1.4 below. They obtained more general results in the first section of [2] which may be a continuation of Schapira [8] and [9]. The aim of this paper is to prove that for a pseudodifferential operator $p(x, D)$ the transposed operator ${}^tp(x, D)$ is locally solvable at $x^0$ modulo analytic functions in the space of hyperfunctions if $p(x, D)$ is analytic hypoelliptic at $x^0$ ( see Theorem 1.6 below). We shall also microlocalize this result, *i.e.*, we shall give the corresponding result in the space of microfunctions ( see Theorem 1.5 below).

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We shall explain briefly about hyperfunctions, microfunctions and pseudodifferential operators acting on them. For the details we refer to [12]. Let $\varepsilon \in \mathbb{R}$, and denote $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ and $|\xi| = (\sum_{j=1}^n |\xi_j|^2)^{1/2}$. We define

$$\widehat{\mathcal{S}}_\varepsilon := \{v(\xi) \in C^\infty(\mathbb{R}^n);\ e^{\varepsilon\langle\xi\rangle}v(\xi) \in \mathcal{S}\},$$

where $\mathcal{S}$ ( $\equiv \mathcal{S}(\mathbb{R}^n)$) denotes the Schwartz space. We introduce the topology to $\widehat{\mathcal{S}}_\varepsilon$ in a natural way. Then the dual space $\widehat{\mathcal{S}}'_\varepsilon$ of $\widehat{\mathcal{S}}_\varepsilon$ can be identified with $\{v(\xi) \in \mathcal{D}';\ e^{-\varepsilon\langle\xi\rangle}v(\xi) \in \mathcal{S}'\}$, since $\mathcal{D}$ ( $= C_0^\infty(\mathbb{R}^n)$) is dense in $\widehat{\mathcal{S}}_\varepsilon$. If $\varepsilon \geq 0$, then $\widehat{\mathcal{S}}_\varepsilon$ is a dense subset of $\mathcal{S}$ and we can define $\mathcal{S}_\varepsilon := \mathcal{F}^{-1}[\widehat{\mathcal{S}}_\varepsilon]$ ( $= \mathcal{F}[\widehat{\mathcal{S}}_\varepsilon]$) ( $\subset \mathcal{S}$), where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transformation and the inverse Fourier transformation on $\mathcal{S}$ ( or $\mathcal{S}'$), respectively. For example, $\mathcal{F}[u](\xi) = \int e^{-ix\cdot\xi}u(x)\,dx$ for $u \in \mathcal{S}$, where $x\cdot\xi = \sum_{j=1}^n x_j\xi_j$ for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. Let $\varepsilon \geq 0$. We introduce the topology in $\mathcal{S}_\varepsilon$ so that $\mathcal{F} : \widehat{\mathcal{S}}_\varepsilon \to \mathcal{S}_\varepsilon$ is homeomorphic. Denote by $\mathcal{S}'_\varepsilon$ the dual space of $\mathcal{S}_\varepsilon$. Since $\mathcal{S}_\varepsilon$ is dense in $\mathcal{S}$, we can regard $\mathcal{S}'$ as a subspace of $\mathcal{S}'_\varepsilon$. We can define the transposed operators ${}^t\mathcal{F}$ and ${}^t\mathcal{F}^{-1}$ of $\mathcal{F}$ and $\mathcal{F}^{-1}$, which map $\mathcal{S}'_\varepsilon$ and $\widehat{\mathcal{S}}'_\varepsilon$ onto $\widehat{\mathcal{S}}'_\varepsilon$ and $\mathcal{S}'_\varepsilon$, respectively. Since $\widehat{\mathcal{S}}_{-\varepsilon} \subset \widehat{\mathcal{S}}'_\varepsilon$ ( $\subset \mathcal{D}'$), we can define $\mathcal{S}_{-\varepsilon} = {}^t\mathcal{F}^{-1}[\widehat{\mathcal{S}}_{-\varepsilon}]$, and introduce the topology in $\mathcal{S}_{-\varepsilon}$ so that ${}^t\mathcal{F}^{-1} : \widehat{\mathcal{S}}_{-\varepsilon} \to \mathcal{S}_{-\varepsilon}$ is homeomorphic. $\mathcal{S}'_{-\varepsilon}$ denotes the dual space of $\mathcal{S}_{-\varepsilon}$. We note that $\mathcal{S}'_{-\varepsilon} = \mathcal{F}[\widehat{\mathcal{S}}'_{-\varepsilon}] \subset \mathcal{S}' \subset \mathcal{S}'_\varepsilon$ and $\mathcal{F} = {}^t\mathcal{F}$ on $\mathcal{S}'$. So we also represent ${}^t\mathcal{F}$ by $\mathcal{F}$. Let $\mathcal{A}(\mathbb{C}^n)$ be the space of entire analytic functions on $\mathbb{C}^n$, and let $K$ be a compact subset of $\mathbb{C}^n$. We denote by $\mathcal{A}'(K)$ the space of analytic functionals carried by $K$, *i.e.*, $u \in \mathcal{A}'(K)$ if and only if (i) $u : \mathcal{A}(\mathbb{C}^n) \ni \varphi \mapsto u(\varphi) \in \mathbb{C}$ is a linear functional, and (ii) for any neighborhood $\omega$ of $K$ in $\mathbb{C}^n$ there is $C_\omega \geq 0$ such that $|u(\varphi)| \leq C_\omega \sup_{z\in\omega}|\varphi(z)|$ for $\varphi \in \mathcal{A}(\mathbb{C}^n)$. Define $\mathcal{A}'(\mathbb{R}^n) := \bigcup_{K \Subset \mathbb{R}^n} \mathcal{A}'(K)$, $\mathcal{S}_\infty := \bigcap_{\varepsilon\in\mathbb{R}} \mathcal{S}_\varepsilon$, $\mathcal{E}_0 := \bigcap_{\varepsilon>0} \mathcal{S}_{-\varepsilon}$ and $\mathcal{F}_0 := \bigcap_{\varepsilon>0} \mathcal{S}'_\varepsilon$. Here $A \Subset B$ means that the closure $\overline{A}$ of $A$ is compact and included in the interior $\overset{\circ}{B}$ of $B$. We note that $\mathcal{F}^{-1}[C_0^\infty(\mathbb{R}^n)] \subset \mathcal{S}_\infty$ and that $\mathcal{S}_\infty$ is dense in $\mathcal{S}_\varepsilon$ and $\mathcal{S}'_\varepsilon$ for $\varepsilon \in \mathbb{R}$. For $u \in \mathcal{A}'(\mathbb{R}^n)$ we can define the Fourier transform $\hat{u}(\xi)$ of $u$ by

$$\hat{u}(\xi)\,( = \mathcal{F}[u](\xi)) = u_z(e^{-iz\cdot\xi}),$$

where $z\cdot\xi = \sum_{j=1}^n z_j\xi_j$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. By definition we have $\hat{u}(\xi) \in \bigcap_{\varepsilon>0} \widehat{\mathcal{S}}_{-\varepsilon}$ ( $= \mathcal{F}[\mathcal{E}_0]$). Therefore, we can regard

$\mathcal{A}'(\mathbb{R}^n)$ as a subspace of $\mathcal{E}_0$, *i.e.*, $\mathcal{A}'(\mathbb{R}^n) \subset \mathcal{E}_0 \subset \mathcal{F}_0$, ( see Lemma 1.1.2 of [12]). The space $\mathcal{F}_0$ plays an important role in our treatment as the space $\mathcal{S}'$ does in the framework of $C^\infty$ and distributions. For a bounded open subset $X$ of $\mathbb{R}^n$ we define the space $\mathcal{B}(X)$ of hyperfunctions in $X$ by

$$\mathcal{B}(X) := \mathcal{A}'(\overline{X})/\mathcal{A}'(\partial X),$$

where $\partial X$ denotes the boundary of $X$.

Let $u \in \mathcal{F}_0$. We define

$$\begin{aligned}&\mathcal{H}(u)(x, x_{n+1}) := (\mathrm{sgn}\ x_{n+1}) \exp[-|x_{n+1}|\langle D\rangle]u(x)/2\\ &( = (\mathrm{sgn}\ x_{n+1})\mathcal{F}_\xi^{-1}[\exp[-|x_{n+1}|\langle\xi\rangle]\hat{u}(\xi)](x)/2 \in \mathcal{S}'(\mathbb{R}^n))\end{aligned}$$

for $x_{n+1} \in \mathbb{R} \setminus \{0\}$, and

$$\begin{aligned}\mathrm{supp}\ u := \bigcap\{F;\ &F \text{ is a closed subset of } \mathbb{R}^n \text{ and there is a real}\\ &\text{analytic function } U(x, x_{n+1}) \text{ in } \mathbb{R}^{n+1} \setminus F \times \{0\}\\ &\text{such that } U(x, x_{n+1}) = \mathcal{H}(u)(x, x_{n+1}) \text{ for } x_{n+1} \neq 0\}.\end{aligned}$$

We note that supp $u$ coincides with the support of $u$ as a distribution if $u \in \mathcal{S}'$ ( see Lemma 1.2.2 of [12]). Moreover, for a compact subset $K$ of $\mathbb{R}^n$, $u \in \mathcal{A}'(K)$ if and only if $u$ is an analytic functional and supp $u \subset K$ ( see Proposition 1.2.6 of [12]). Let $K$ be a compact subset of $\mathbb{R}^n$. It follows from Theorem 1.3.3 of [12] that for any $u$ and $K$ as above there is $v \in \mathcal{A}'(K)$ satisfying supp $(u - v) \cap K \subset \partial K$, and if $v = v_1$, $v_2$ are such functions in $\mathcal{A}'(K)$ we have supp $(v_1 - v_2) \subset \partial K$. Therefore, we can define the restriction map from $\mathcal{F}_0$ to $\mathcal{A}'(K)/\mathcal{A}'(\partial K)$ ( $= \mathcal{B}(\overset{\circ}{K})$) which is surjective. For $x^0 \in \mathbb{R}^n$ we say that $u$ is analytic at $x^0$ if $\mathcal{H}(u)(x, x_{n+1})$ can be continued analytically from $\mathbb{R}^n \times (0, \infty)$ to a neighborhood of $(x^0, 0)$ in $\mathbb{R}^{n+1}$. We define

$$\text{sing supp}\ u := \{x \in \mathbb{R}^n;\ u \text{ is not analytic at } x\}.$$

Next let $u \in \mathcal{B}(X)$, where $X$ is a bounded open subset of $\mathbb{R}^n$. Then there is $v \in \mathcal{A}'(\overline{X})$ such that the residue class of $v$ is $u$ in $\mathcal{B}(X)$. We define

$$\mathrm{supp}\ u := \mathrm{supp}\ v \cap X, \quad \text{sing supp}\ u := \text{sing supp}\ v \cap X.$$

These definitions do not depend on the choice of $v$. So we say that $u$ is analytic at $x^0$ if $x^0 \notin$ sing supp $u$. Let $X$ be an open subset of $\mathbb{R}^n$. We also define $\mathcal{B}(X)$ ( see Definition 1.4.5 of [12]). For open subsets $U$ and $V$ of $X$ with $V \subset U$ the restriction map $\rho_V^U : \mathcal{B}(U) \ni u \mapsto u|_V \in \mathcal{B}(V)$ can be defined so that $\rho_U^U$ is the identity mapping and $\rho_W^V \circ \rho_V^U = \rho_W^U$ for open subsets $U$, $V$ and $W$ of $X$ with $W \subset V \subset U$. By definition we can also define the restriction map from $\mathcal{F}_0$ to $\mathcal{B}(X)$, and we denote by $v|_X$ the restriction of $v \in \mathcal{F}_0$ to $\mathcal{B}(X)$ ( or on $X$). We define the presheaf $\mathcal{B}_X$ by associating $\mathcal{B}(U)$ to every open subset $U$ of $X$. By definition $\mathcal{B}_X$ is a sheaf on $X$.

Next we shall define analytic wave front sets and microfunctions.

DEFINITION 1.1. (i) Let $u \in \mathcal{F}_0$. The analytic wave front set $WF_A(u) \subset T^*\mathbb{R}^n \setminus 0$ ( $\simeq \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$) is defined as follows: $(x^0, \xi^0) \in T^*\mathbb{R}^n \setminus 0$ does not belong to $WF_A(u)$ if there are a conic neighborhood $\Gamma$ of $\xi^0$, $R_0 > 0$ and $\{g^R(\xi)\}_{R \ge R_0} \subset C^\infty(\mathbb{R}^n)$ such that $g^R(\xi) = 1$ in $\Gamma \cap \{\langle \xi \rangle \ge R\}$,

$$|\partial_\xi^{\alpha+\tilde{\alpha}} g^R(\xi)| \le C_{|\tilde{\alpha}|} (C/R)^{|\alpha|} \langle \xi \rangle^{-|\tilde{\alpha}|} \tag{1.1}$$

if $\langle \xi \rangle \ge R|\alpha|$, and $g^R(D)u$ ( $= \mathcal{F}^{-1}[g^R(\xi)\hat{u}(\xi)]$) is analytic at $x^0$ for $R \ge R_0$, where $C$ is a positive constant independent of $R$.

(ii) Let $X$ be an open subset of $\mathbb{R}^n$, and let $u \in \mathcal{B}(X)$ and $(x^0, \xi^0) \in T^*X \setminus 0$ ( $\simeq X \times (\mathbb{R}^n \setminus \{0\})$). Then we say that $(x^0, \xi^0) \notin WF_A(u)$ ( $\subset T^*X \setminus 0$) if there are a bounded open neighborhood $U$ of $x^0$ and $v \in \mathcal{A}'(\overline{U})$ such that $v|_U = u|_U$ in $\mathcal{B}(U)$ and $(x^0, \xi^0) \notin WF_A(v)$

REMARK. (i) $WF_A(u)$ for $u \in \mathcal{B}(X)$ is well-defined. Indeed, it follows from Theorem 2.6.5 in [12] that for any $v \in \mathcal{A}'(\mathbb{R}^n)$ with $x^0 \notin \text{supp } v$ there is $R_1 > 0$ such that $g^R(D)v$ is analytic at $x^0$ if $R \ge R_1$, where $\{g^R(\xi)\}_{R \ge R_0}$ is a family of symbols satisfying (1.1).

(ii) Several remarks on this definition are given in Proposition 3.1.2 of [12].

(iii) From Theorem 3.1.6 in [12] and the results in [3] it follows that our definition of $WF_A(u)$ coincides with the usual definition.

Let $\mathcal{U}$ be an open subset of the cosphere bundle $S^*\mathbb{R}^n$ over $\mathbb{R}^n$, which is identified with $\mathbb{R}^n \times S^{n-1}$. We define

$$\mathcal{C}(\mathcal{U}) := \mathcal{B}(\mathbb{R}^n)/\{u \in \mathcal{B}(\mathbb{R}^n);\ WF_A(u) \cap \mathcal{U} = \emptyset\}.$$

Since $\mathcal{B}$ is a flabby sheaf, we have

$$\mathcal{C}(\mathcal{U}) = \mathcal{B}(U)/\{u \in \mathcal{B}(U);\ WF_A(u) \cap \mathcal{U} = \emptyset\}$$

if $U$ is an open subset of $\mathbb{R}^n$ and $\mathcal{U} \subset U \times S^{n-1}$. Elements of $\mathcal{C}(\mathcal{U})$ are called microfunctions on $\mathcal{U}$. We can define the restriction map $\mathcal{C}(\mathcal{U}) \ni u \mapsto u|_{\mathcal{V}} \in \mathcal{C}(\mathcal{V})$ for open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathbb{R}^n \times S^{n-1}$ with $\mathcal{V} \subset \mathcal{U}$. Let $\Omega$ be an open subset of $\mathbb{R}^n \times S^{n-1}$. We define the presheaf $\mathcal{C}_\Omega$ on $\Omega$ associating $\mathcal{C}(\mathcal{U})$ to every open subset $\mathcal{U}$ of $\Omega$. Then $\mathcal{C}_\Omega$ is a flabby sheaf ( see, *e.g.*, Theorem 3.6.1 of [12]). For each open subset $U$ of $\mathbb{R}^n$ we define the mapping sp: $\mathcal{B}(U) \to \mathcal{C}(U \times S^{n-1})$ such that the residue class in $\mathcal{C}(U \times S^{n-1})$ of $u \in \mathcal{B}(U)$ is equal to sp$(u)$. We also write $u|_{\mathcal{U}} = \mathrm{sp}(u)|_{\mathcal{U}}$ for $u \in \mathcal{B}(U)$ and $v|_{\mathcal{U}} = \mathrm{sp}(v|_U)|_{\mathcal{U}}$ for $v \in \mathcal{F}_0$, where $\mathcal{U}$ is an open subset of $U \times S^{n-1}$.

Assume that $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and there are positive constants $C_k$ ( $k \geq 0$) such that

$$\begin{aligned}(1.2)\qquad & |\partial_\xi^\alpha D_y^{\beta+\tilde{\beta}} \partial_\eta^\gamma a(\xi, y, \eta)| \\ & \leq C_{|\alpha|+|\tilde{\beta}|+|\gamma|} (A/R)^{|\beta|} \langle\xi\rangle^{m_1+|\beta|} \langle\eta\rangle^{m_2} \exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle]\end{aligned}$$

if $\alpha, \beta, \tilde{\beta}, \gamma \in (\mathbb{Z}_+)^n$, $\xi, y, \eta \in \mathbb{R}^n$, $\langle\xi\rangle \geq R|\beta|$, where $D_y = -i\partial_y$, $R \geq 1$, $A \geq 0$, $m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. It should be remarked that some functions satisfying the estimates (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$ are given in Proposition 2.2.3 of [12]. We define pseudodifferential operators $a(D_x, y, D_y)$ and ${}^r a(D_x, y, D_y)$ by

$$a(D_x, y, D_y)u(x) = (2\pi)^{-n} \mathcal{F}_\xi^{-1}\Big[\int\Big(\int e^{-iy\cdot(\xi-\eta)} a(\xi, y, \eta)\hat{u}(\eta)\, d\eta\Big) dy\Big](x)$$

and ${}^r a(D_x, y, D_y)u = b(D_x, y, D_y)$ for $u \in \mathcal{S}_\infty$, respectively, where $b(\xi, y, \eta) = a(\eta, y, \xi)$. Applying the same argument as in the proof of Theorem 2.3.3 of [12] we have the following

Proposition 1.2. *$a(D_x, y, D_y)$ can be extended to a continuous linear operator from $\mathcal{S}_{\varepsilon_2}$ to $\mathcal{S}_{\varepsilon_1}$ and from $\mathcal{S}'_{-\varepsilon_2}$ to $\mathcal{S}'_{-\varepsilon_1}$, respectively, if*

$$(1.3)\qquad \begin{cases} \nu > 1, \quad \varepsilon_2 - \delta_2 = \nu(\varepsilon_1 + \delta_1)_+, \\ \varepsilon_1 + \delta_1 \leq 1/R, \quad R \geq e\sqrt{n}\nu A/(\nu - 1), \end{cases}$$

*where* $c_+ = \max\{c, 0\}$. *Similarly,* ${}^r a(D_x, y, D_y)$ *can be extended to a continuous linear operator from* $\mathcal{S}_{-\varepsilon_1}$ *to* $\mathcal{S}_{-\varepsilon_2}$ *and from* $\mathcal{S}'_{\varepsilon_1}$ *to* $\mathcal{S}'_{\varepsilon_2}$*, respectively, if* (1.3) *is valid.*

Remark. (i) We had a slight improvement in the remark of Theorem 2.3.3 of [12], *i.e.*, we can take $R_1(S, T, \nu) = e\sqrt{n}\nu/(\nu - 1)$ there instead of $R_1(S, T, \nu) = en\nu/(\nu - 1)$ if $n = n' = n''$, $S(y, \xi) = -y \cdot \xi$ and $T(y, \eta) = y \cdot \eta$. This is reflected in the condition (1.3).

(ii) Since for any open sets $X_j$ ( $j = 1, 2$) with $X_1 \Subset X_2$ one can construct a symbol $a(\xi, y, \eta)$ satisfying (1.2) with $m_1 = m_2 = 0$ and $\delta_1 = \delta_2 = 0$, $\operatorname{supp}\, a \subset \mathbb{R}^n \times X_2 \times \mathbb{R}^n$ and $a(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^n$, one can use the operator $a(D_x, y, D_y)$ instead of cut-off functions.

Definition 1.3. Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let $X$ be an open subset of $\mathbb{R}^n$. Moreover, let $R_0 \ge 0$.

(i) Let $R_0 \ge 1$, $m, \delta \in \mathbb{R}$ and $A, B \ge 0$, and let $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(x, \xi) \in S^{m,\delta}(R_0, A, B)$ if $a(x, \xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{(\beta+\tilde{\beta})}(x, \xi)| \le C_{|\tilde{\alpha}|+|\tilde{\beta}|}(A/R_0)^{|\alpha|}(B/R_0)^{|\beta|}\langle\xi\rangle^{m+|\beta|-|\tilde{\alpha}|}e^{\delta\langle\xi\rangle}$$

for any $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in (\mathbb{Z}_+)^n$, $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ with $\langle\xi\rangle \ge R_0(|\alpha| + |\beta|)$, where $a^{(\alpha)}_{(\beta)}(x, \xi) = \partial_\xi^\alpha D_x^\beta a(x, \xi)$ and the $C_k$ are independent of $\alpha$ and $\beta$. We also write $S^m(R_0, A, B) = S^{m,0}(R_0, A, B)$ and $S^m(R_0, A) = S^m(R_0, A, A)$. We define $S^+(R_0, A, B) = \bigcap_{\delta>0} S^{0,\delta}(R_0, A, B)$.

(ii) Let $R_0 \ge 1$, $m_j, \delta_j \in \mathbb{R}$ ( $j = 1, 2$), $A_j \ge 0$ ( $j = 1, 2$) and $B \ge 0$, and let $a(\xi, y, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. We say that $a(\xi, y, \eta) \in S^{m_1,m_2,\delta_1,\delta_2}(R_0, A_1, B, A_2)$ if $a(\xi, y, \eta)$ satisfies

$$\begin{aligned}|\partial_\xi^{\alpha+\tilde{\alpha}} D_y^{\beta^1+\beta^2+\tilde{\beta}} \partial_\eta^{\gamma+\tilde{\gamma}} a(\xi, y, \eta)| &\le C_{|\tilde{\alpha}|+|\tilde{\beta}|+|\tilde{\gamma}|}(A_1/R_0)^{|\alpha|}(B/R_0)^{|\beta^1|+|\beta^2|} \\ &\quad \times (A_2/R_0)^{|\gamma|}\langle\xi\rangle^{m_1+|\beta^1|-|\tilde{\alpha}|}\langle\eta\rangle^{m_2+|\beta^2|-|\tilde{\gamma}|}\exp[\delta_1\langle\xi\rangle + \delta_2\langle\eta\rangle]\end{aligned}$$

for any $\alpha, \tilde{\alpha}, \beta^1, \beta^2, \tilde{\beta}, \gamma, \tilde{\gamma} \in (\mathbb{Z}_+)^n$, $(\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $\langle\xi\rangle \ge R_0(|\alpha| + |\beta^1|)$ and $\langle\eta\rangle \ge R_0(|\gamma| + |\beta^2|)$. We also write $S^{m_1,m_2,\delta_1,\delta_2}(R_0, A) = S^{m_1,m_2,\delta_1,\delta_2}(R_0, A, A, A)$. Similarly, we define $S^+(R_0, A_1, B, A_2) = \bigcap_{\delta>0} S^{0,0,\delta,\delta}(R_0, A_1, B, A_2)$.

(iii) Let $A, B \geq 0$, and let $a(x,\xi) \in C^\infty(\Gamma)$. We say that $a(x,\xi) \in PS^+(\Gamma; R_0, A, B)$ if $a(x,\xi)$ satisfies

$$|a^{(\alpha+\tilde{\alpha})}_{(\beta)}(x,\xi)| \leq C_{|\tilde{\alpha}|,\delta} A^{|\alpha|} B^{|\beta|} |\alpha|!|\beta|!\langle\xi\rangle^{-|\alpha|-|\tilde{\alpha}|} e^{\delta\langle\xi\rangle}$$

for any $\alpha, \tilde{\alpha}, \beta \in (\mathbb{Z}_+)^n$, $(x,\xi) \in \Gamma$ with $|\xi| \geq 1$ and $\langle\xi\rangle \geq R_0|\alpha|$ and $\delta > 0$. We also write $PS^+(\Gamma; R_0, A) = PS^+(\Gamma; R_0, A, A)$. Moreover, we say that $a(x,\xi) \in PS^+(X; R_0, A, B)$ if $a(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$ and $a(x,\xi) \in PS^+(X \times (\mathbb{R}^n \setminus \{0\}); R_0, A, B)$.

(iv) Let $A, C_0 \geq 0$, and let $\{a_j(x,\xi)\}_{j\in\mathbb{Z}_+} \in \prod_{j\in\mathbb{Z}_+} C^\infty(\Gamma)$. We say that $a(x,\xi) \equiv \{a_j(x,\xi)\}_{j\in\mathbb{Z}_+} \in FS^+(\Gamma; C_0, A)$ if $a(x,\xi)$ satisfies

$$|a^{(\alpha)}_{j(\beta)}(x,\xi)| \leq C_\delta C_0^j A^{|\alpha|+|\beta|} j!|\alpha|!|\beta|!\langle\xi\rangle^{-j-|\alpha|} e^{\delta\langle\xi\rangle}$$

for any $j \in \mathbb{Z}_+$, $\alpha, \beta \in (\mathbb{Z}_+)^n$, $(x,\xi) \in \Gamma$ with $|\xi| \geq 1$ and $\delta > 0$, where $C_\delta$ is independent of $\alpha$, $\beta$ and $j$. We also write $a(x,\xi) = \sum_{j=0}^\infty a_j(x,\xi)$ formally. Moreover, we write $FS^+(X; C_0, A) = FS^+(X \times (\mathbb{R}^n \setminus \{0\}); C_0, A)$.

(v) For $a(x,\xi) = \sum_{j=0}^\infty a_j(x,\xi) \in FS^+(\Gamma; C_0, A)$ we define the symbol $({}^t a)(x,\xi)$ by

$$({}^t a)(x,\xi) = \sum_{j=0}^\infty b_j(x,\xi), \quad b_j(x,\xi) = \sum_{k+|\alpha|=j} (-1)^{|\alpha|} a^{(\alpha)}_{k(\alpha)}(x,-\xi)/\alpha!.$$

Remark. It is easy to see that $({}^t a)(x,\xi) \in FS^+(\check{\Gamma}; \max\{C_0, 4nA^2\}, 2A)$, where $\check{\Gamma} = \{(x,\xi);\ (x,-\xi) \in \Gamma\}$. Moreover, we have $({}^t({}^t a))(x,\xi) = a(x,\xi)$.

Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and assume that $a(x,\xi) \in PS^+(\Gamma; R_0, A)$, where $A \geq 0$ and $R_0 \geq 1$. Let $\Gamma_j$ ( $0 \leq j \leq 2$) be open conic subsets of $\Gamma$ such that $\Gamma_0 \Subset \Gamma_1 \Subset \Gamma_2 \Subset \Gamma$, and write $\Gamma^0 = \Gamma \cap (\mathbb{R}^n \times S^{n-1})$, where $\Gamma_2 \Subset \Gamma$ implies that $\Gamma_2^0 \Subset \Gamma$. It follows from Proposition 2.2.3 of [12] that there are symbols $\Phi^R(\xi,y,\eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1,\Gamma_2), C(\Gamma_1,\Gamma_2))$ ( $R \geq 4$) satisfying $0 \leq \Phi^R(\xi,y,\eta) \leq 1$, $\mathrm{supp}\ \Phi^R \subset \mathbb{R}^n \times \Gamma_2$ and $\Phi^R(\xi,y,\eta) = 1$ for $(\xi,y,\eta) \in \mathbb{R}^n \times \Gamma_1$ with $\langle\eta\rangle \geq R$. Put $a^R(\xi,y,\eta) = \Phi^R(\xi,y,\eta)a(y,\eta)$. Then we have $a^R(\xi,y,\eta) \in S^+(R, C_*, 2A + C(\Gamma_1,\Gamma_2), A + C(\Gamma_1,\Gamma_2))$ for $R \geq \max\{4, R_0\}$. Let $u \in \mathcal{C}(\Gamma_0^0)$, and choose $v \in \mathcal{F}_0$ so that $v|_{\Gamma_0^0} = u$. Applying Proposition 1.2

with $a(\xi, y, \eta) = a^R(\eta, y, \xi)$ and noting that $a^R(D_x, y, D_y) = {}^r a(D_x, y, D_y)$, we can see that $a^R(D_x, y, D_y)v$ is well-defined and belongs to $\mathcal{F}_0$ if $R \geq \max\{4, R_0, 2e\sqrt{n}(2A + C(\Gamma_1, \Gamma_2))\}$. Moreover, $a^R(D_x, y, D_y)v$ determines an element $(a^R(D_x, y, D_y)v)|_U \in \mathcal{B}(U)$, where $U$ is a bounded open subset of $\mathbb{R}^n$ satisfying $\Gamma_0^0 \subset U \times S^{n-1}$, and, therefore, an element $\mathrm{sp}((a^R(D_x, y, D_y)v)|_U)|_{\Gamma_0^0}\,(\equiv (a^R(D_x, y, D_y)v)|_{\Gamma_0^0}) \in \mathcal{C}(\Gamma_0^0)$. It follows from Lemma 2.1 below that $(a^R(D_x, y, D_y)v)|_{\Gamma_0^0}$ does not depend on the choice of $\Phi^R(\xi, y, \eta)$ if $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, B)$ and $R \geq R(A, B, \Gamma_0, \Gamma_1)$, where $R(A, B, \Gamma_0, \Gamma_1) > 0$. From Lemma 2.2 it follows that for each conic subset $\Omega$ of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ with $\Omega \Subset \Gamma_0$ there is $R(A, \Omega, \Gamma_0, \Gamma_1, \Gamma_2) > 0$ such that $WF_A(a^R(D_x, y, D_y)w) \cap \Omega = \emptyset$ if $R \geq R(A, \Omega, \Gamma_0, \Gamma_1, \Gamma_2)$, $w \in \mathcal{F}_0$ and $WF_A(w) \cap \Gamma_0 = \emptyset$. Therefore, we can define the operator $a(x, D)$: $\mathcal{C}(\Gamma_0^0) \to \mathcal{C}(\Gamma_0^0)$ by $a(x, D)u = (a^R(D_x, y, D_y)v)|_{\Gamma_0^0}$ for $R \gg 1$, and the operator $a(x, D)$: $\mathcal{C}(\Gamma^0) \to \mathcal{C}(\Gamma^0)$. Moreover, it follows from Lemma 2.2 that

$$a(x, D)(w|_{\mathcal{U}}) = (a(x, D)w)|_{\mathcal{U}} \quad \text{for } w \in \mathcal{C}(\mathcal{V}),$$

where $\mathcal{U}$ and $\mathcal{V}$ are open subsets of $\mathbb{R}^n \times S^{n-1}$ satisfying $\mathcal{U} \subset \mathcal{V} \subset \Gamma^0$. So we can define $a(x, D)$: $\mathcal{C}_{\Gamma^0} \to \mathcal{C}_{\Gamma^0}$, which is a sheaf homomorphism. Let $X$ be an open subset of $\mathbb{R}^n$, and assume that $a(x, \xi) \in PS^+(X; R_0, A)$. Similarly, taking $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$, we can define the operator $a(x, D)$: $\mathcal{B}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$ and the operator $a(x, D)$: $\mathcal{B}(U)/\mathcal{A}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$, where $U$ is a bounded open subset of $X$ and $\mathcal{A}(U)$ denotes the space of all real analytic functions defined in $U$ ( see, also, §2.7 of [12]). In doing so, we may choose $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))$ so that $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times X_1 \times \mathbb{R}^n$, where $\Gamma_j = X_j \times (\mathbb{R}^n \setminus \{0\})$. Moreover, we can define the operator $a(x, D)$: $\mathcal{B}_X \to \mathcal{B}_X/\mathcal{A}_X$ and the operator $a(x, D)$: $\mathcal{B}_X/\mathcal{A}_X \to \mathcal{B}_X/\mathcal{A}_X$, which are sheaf homomorphisms. Here $\mathcal{A}_X$ denotes the sheaf ( of germs) of real analytic functions on $X$.

Assume that $a(x, D)$ is a differential operator in $X$. Let $K$ be a compact subset of $X$. Then, by duality we can define $a(x, D)w \in \mathcal{A}'(K)$ for $w \in \mathcal{A}'(K)$. From Proposition 1.2.6 of [12] and the definition of analytic functionals we have $\mathrm{supp}\ a(x, D)w \subset \mathrm{supp}\ w$ for $w \in \mathcal{A}'(K)$. Therefore, we can define $a(x, D)$: $\mathcal{B}_X \to \mathcal{B}_X$, which is a sheaf homomorphism. From Theorem 2.7.1 of [12] and Lemma 2.5 it follows that two definitions of $a(x, D)$: $\mathcal{B}_X \to \mathcal{B}_X/\mathcal{A}_X$ are consistent.

Next we assume that $a(x, \xi) \equiv \sum_{j=0}^{\infty} a_j(x, \xi) \in FS^+(\Gamma; C_0, A)$. Choose

$\{\phi_j^R(\xi)\}_{j\in\mathbb{Z}_+} \subset C^\infty(\mathbb{R}^n)$ so that $0 \le \phi_j^R(\xi) \le 1$,

$$\phi_j^R(\xi) = \begin{cases} 0 & \text{if } \langle\xi\rangle \le 2Rj, \\ 1 & \text{if } \langle\xi\rangle \ge 3Rj, \end{cases}$$
$$|\partial_\xi^{\alpha+\beta}\phi_j^R(\xi)| \le \widehat{C}_{|\beta|}(\widehat{C}/R)^{|\alpha|}\langle\xi\rangle^{-|\beta|} \quad \text{if } |\alpha| \le 2j,$$

where the $\widehat{C}_{|\beta|}$ and $\widehat{C}$ do not depend on $j$ and $R$ ( see §2.2 of [12]). Then it follows from Lemma 2.2.4 of [12] that

$$\tilde{a}(x,\xi) := \sum_{j=0}^{\infty} \phi_j^{R/2}(\xi)a_j(x,\xi) \in PS^+(\Gamma; R, 2A + 3\widehat{C}, A)$$

if $R > C_0$. So we can define $a(x,D)u \in \mathcal{C}(\Gamma^0)$ by $a(x,D)u = \tilde{a}(x,D)u$. Indeed, applying the same argument as in §3.7 of [12] we can see that $a(x,D)u \in \mathcal{C}(\Gamma^0)$ does not depend on the choice of $\{\phi_j^R(\xi)\}$. Similarly, $a(x,D)$ defines a sheaf homomorphism $a(x,D)\colon \mathcal{C}_{\Gamma^0} \to \mathcal{C}_{\Gamma^0}$.

To state our main results we need the following

Definition 1.4. Let $\Gamma$ be an open subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and let $p(x,\xi) \in PS^+(\Gamma; R_0, A)$ ( or $p(x,\xi) \in FS^+(\Gamma; C_0, A)$), where $R_0 \ge 1$ and $A, C_0 \ge 0$.

(i) For $z^0 = (x^0,\xi^0) \in \Gamma$ we say that $p(x,D)$ is analytic microhypoelliptic at $z^0$ if there is an open neighborhood $\mathcal{U}$ of $(x^0, \xi^0/|\xi^0|)$ in $\Gamma \cap (\mathbb{R}^n \times S^{n-1})$ satisfying supp $u =$ supp $p(x,D)u$ for any $u \in \mathcal{C}(\mathcal{U})$, *i.e.*, the sheaf homomorphism $p(x,D) : \mathcal{C}_\mathcal{U} \to \mathcal{C}_\mathcal{U}$ is injective.

(ii) For $z^0 = (x^0,\xi^0) \in \Gamma$ we say that $p(x,D)$ is microlocally solvable at $z^0$ if there is a open neighborhood $\mathcal{U}$ of $(x^0, \xi^0/|\xi^0|)$ in $\Gamma \cap (\mathbb{R}^n \times S^{n-1})$ satisfying the following; for any $f \in \mathcal{C}(\mathcal{U})$ there is $u \in \mathcal{C}(\mathcal{U})$ such that $p(x,D)u = f$ in $\mathcal{C}(\mathcal{U})$, *i.e.*, $p(x,D)\colon \mathcal{C}(\mathcal{U}) \to \mathcal{C}(\mathcal{U})$ is surjective.

(iii) Assume that $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$, *i.e.*, $p(x,\xi) \in PS^+(X; R_0, A)$ ( or $p(x,\xi) \in FS^+(X; C_0, A)$), where $X$ is an open subset of $\mathbb{R}^n$. Let $x^0 \in X$. We say that $p(x,D)$ is analytic hypoelliptic at $x^0$ if there is an open neighborhood $U$ of $x^0$ in $X$ satisfying supp $u =$ supp $p(x,D)u$ for any $u \in \mathcal{B}(U)/\mathcal{A}(U)$, *i.e.*, the sheaf homomorphism $p(x,D)\colon \mathcal{B}_U/\mathcal{A}_U \to \mathcal{B}_U/\mathcal{A}_U$ is injective. Similarly, we say that $p(x,D)$ is locally solvable at $x^0$ modulo analytic functions if there is an open neighborhood $U$ of $x^0$ in $X$ satisfying the following; for any $f \in \mathcal{B}(U)/\mathcal{A}(U)$ there is $u \in \mathcal{B}(U)/\mathcal{A}(U)$ such that $p(x, D)u = f$ in $\mathcal{B}(U)/\mathcal{A}(U)$, *i.e.*, $p(x, D)$: $\mathcal{B}(U)/\mathcal{A}(U) \to \mathcal{B}(U)/\mathcal{A}(U)$ is surjective. Assume that $p(x, \xi)$ is a polynomial of $\xi$ whose coefficients are real analytic functions of $x$ defined in $X$. Then we say that $p(x, D)$ is locally solvable at $x^0$ if there is an open neighborhood $U$ of $x^0$ in $X$ such that $p(x, D)$: $\mathcal{B}(U) \to \mathcal{B}(U)$ is surjective.

Theorem 1.5. *Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $z^0 = (x^0, \xi^0) \in \Gamma$. Let $p(x, \xi) \in FS^+(\Gamma; C_0, A)$, where $A, C_0 \geq 0$. Then $({}^tp)(x, D)$ is microlocally solvable at $(x^0, -\xi^0)$ if $p(x, D)$ is analytic micro-hypoelliptic at $z^0$.*

Theorem 1.6. *Let $X$ be an open subset of $\mathbb{R}^n$ and $x^0 \in X$. Let $p(x, \xi) \in FS^+(X; C_0, A)$, where $A, C_0 \geq 0$. Then $({}^tp)(x, D)$ is locally solvable at $x^0$ modulo analytic functions if $p(x, D)$ is analytic hypoelliptic at $x^0$.*

In §2 we shall give preliminary lemmas. Theorems 1.5 and 1.6 will be proved in §3.

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## 2. Preliminaries

In this section we shall prepare a series of lemmas for the proofs of Theorems 1.5 and 1.6.

Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. We write $\Gamma_\varepsilon = \{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\});\ |(x, \xi/|\xi|) - (y, \eta/|\eta|)| < \varepsilon$ for some $(y, \eta) \in \Gamma\}$ for $\varepsilon > 0$. For a subset $U$ of $\mathbb{R}^n$ and $\varepsilon > 0$ we write $U_\varepsilon = \{x \in \mathbb{R}^n;\ |x - y| < \varepsilon$ for some $y \in U\}$. We also write $\gamma_\varepsilon = \{\xi \in \mathbb{R}^n \setminus \{0\};\ |\xi/|\xi| - \eta/|\eta|| < \varepsilon$ for some $\eta \in \gamma\}$ for a conic subset $\gamma$ of $\mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$.

Lemma 2.1. *Let $p(\xi, y, \eta) \in S^+(R_0, A)$. Assume that $p(\xi, y, \eta) = 0$ if $(y, \eta) \in \Gamma_\varepsilon$, $|\xi/|\xi| - \eta/|\eta|| \leq \varepsilon/4$ and $\langle \xi \rangle \geq R_0$, where $\varepsilon > 0$. Then there is $R_0(\varepsilon) > 0$ such that*

$$WF_A(p(D_x, y, D_y)u) \cap \Gamma = \emptyset \quad \textit{for } u \in \mathcal{F}_0$$

*if $R_0 \geq R_0(\varepsilon)A$.*

Proof. It follows from Proposition 1.2 that $p(D_x, y, D_y)u \in \mathcal{F}_0$ if $u \in \mathcal{F}_0$ and $R_0 \geq 2e\sqrt{n}A$. Let $(x^0, \xi^0) \in \Gamma$, and let $U \times \gamma$ be an open conic neighborhood of $(x^0, \xi^0)$ satisfying $U \times \gamma \subset \Gamma$. We choose $\{g^R(\xi)\}_{R \geq R}$ so that supp $g^R \subset \gamma_{\varepsilon/4}$, $g^R(\xi) = 1$ in $\gamma \cap \{\langle \xi \rangle \geq R\}$ and

$$|\partial_\xi^{\alpha+\tilde{\alpha}} g^R(\xi)| \leq C_{|\tilde{\alpha}|}(\varepsilon)(C(\varepsilon)/R)^{|\alpha|}\langle \xi \rangle^{-|\tilde{\alpha}|}$$

if $\langle \xi \rangle \geq R|\alpha|$, where the $C_j(\varepsilon)$ and $C(\varepsilon)$ are positive constants depending on $\varepsilon$. Put

$$\tilde{p}^R(\xi, y, \eta) = g^R(\xi)p(\xi, y, \eta) \,(\in S^+(R, RA/R_0 + C(\varepsilon), RA/R_0, RA/R_0))$$

for $R \geq R_0$. Then we have $\tilde{p}^R(\xi, y, \eta) = 0$ if $y \in U_{\varepsilon/2}$, $|\xi/|\xi| - \eta/|\eta|| \leq \varepsilon/4$ and $\langle \xi \rangle \geq R_0$. Applying Corollary 2.6.3 of [12] we see that there are positive constants $R_j(\varepsilon)$ ( $j = 1, 2$) such that $\tilde{p}^R(D_x, y, D_y)u$ ( $= g^R(D)(p(D_x, y, D_y)u)$) is analytic in $U$ for $u \in \mathcal{F}_0$ if $R \geq R_1(\varepsilon)(RA/R_0 + C(\varepsilon)) + R_2(\varepsilon)$ and $R \geq R_0$. From the definition of $WF_A(\cdot)$ the lemma easily follows. □

Lemma 2.2. *Let $p(\xi, y, \eta) \in S^+(R_0, A)$, and let $\Gamma_1$ be an open conic subset of $\Gamma$ such that $\Gamma_1 \Subset \Gamma$. Then there is $R_0(\Gamma_1, \Gamma) > 0$ such that $WF_A(p(D_x, y, D_y)u) \cap \Gamma_1 = \emptyset$ if $u \in \mathcal{F}_0$, $WF_A(u) \cap \Gamma = \emptyset$ and $R_0 \geq R_0(\Gamma_1, \Gamma)A$.*

Proof. By Proposition 1.2 we have $p(D_x, y, D_y)u \in \mathcal{F}_0$ if $u \in \mathcal{F}_0$ and $R_0 \geq 2e\sqrt{n}A$. Let $u \in \mathcal{F}_0$, and assume that $WF_A(u) \cap \Gamma = \emptyset$. Let $(x^0, \xi^0) \in \Gamma_1$, and let $U \times \gamma$ be an open conic neighborhood of $(x^0, \xi^0)$ satisfying $U \times \gamma \subset \Gamma_1$. Then there is $\varepsilon > 0$ such that $U_{2\varepsilon} \times \gamma_{3\varepsilon} \Subset \Gamma$. We choose $\{g_j^R(\xi)\}_{R \geq R_0}$ ( $j = 1, 2$) so that supp $g_1^R \subset \gamma_\varepsilon$, supp $g_2^R \subset \gamma_{3\varepsilon}$, $g_1^R(\xi) = 1$ in $\gamma \cap \{\langle \xi \rangle \geq R\}$, $g_2^R(\xi) = 1$ in $\gamma_{2\varepsilon} \cap \{\langle \xi \rangle \geq R\}$ and

$$|\partial_\xi^{\alpha+\tilde{\alpha}} g_j^R(\xi)| \leq C_{j,|\tilde{\alpha}|}(\varepsilon)(C(\varepsilon)/R)^{|\alpha|}\langle \xi \rangle^{-|\tilde{\alpha}|}$$

if $\langle \xi \rangle \geq R|\alpha|$ and $j = 1, 2$, where the $C_{j,k}(\varepsilon)$ and $C(\varepsilon)$ are positive constants. Then it follows from Proposition 3.1.2 (i) and (ii) of [12] that there is $R(\varepsilon) > 0$ such that $g_2^R(D)u$ is analytic in $U_\varepsilon$ if $R \geq R(\varepsilon)$. Put

$$p_1^R(\xi, y, \eta) = g_1^R(\xi)p(\xi, y, \eta)g_2^R(\eta) \,(\in S^+(R, RA/R_0 + C(\varepsilon)))$$
$$p_2^R(\xi, y, \eta) = g_1^R(\xi)p(\xi, y, \eta)(1 - g_2^R(\eta)) \,(\in S^+(R, RA/R_0 + C(\varepsilon)))$$

for $R \geq R_0$. Note that $g_1^R(D)(p(D_x, y, D_y)u) = p_1^R(D_x, y, D_y)u + p_2^R(D_x, y, D_y)u$. By Corollary 2.6.6 of [12] there are positive constants $R_1(\varepsilon)$ and $R_2(\varepsilon)$ such that $p_1^R(D_x, y, D_y)u$ is analytic in $U$ if $R \geq R_1(\varepsilon)(RA/R_0 + C(\varepsilon)) + R_2(\varepsilon)$ and $R \geq R_0 \geq 2e\sqrt{n}A$. On the other hand, we have

$$p_2^R(\xi, y, \eta) = 0 \quad \text{if } |\xi/|\xi| - \eta/|\eta|| < \varepsilon \text{ and } \langle\eta\rangle \geq R.$$

Therefore, it follows from Lemma 2.1 ( or Corollary 2.6.3 of [12]) that $p_2^R(D_x, y, D_y)u$ is analytic in $\mathbb{R}^n$ if $R \geq R_0'(\varepsilon)(RA/R_0 + C(\varepsilon))$, where $R_0'(\varepsilon) > 0$. Indeed, one can apply Lemma 2.1 to $p_2^R(\xi, y, \eta)\phi_1^R(\eta)$. Proposition 1.2 implies that $p_2^R(D_x, y, D_y)(1 - \phi_1^R(D))u$ is analytic. This proves the lemma. □

Lemma 2.3. *Let $q(\xi, y, \eta)$ be a symbol in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$|\partial_\xi^{\alpha+\tilde{\alpha}} D_y^\beta \partial_\eta^\gamma q(\xi, y, \eta)| \leq C_{|\tilde{\alpha}|+|\gamma|,\delta}(A/R_0)^{|\alpha|+|\beta|}\langle\eta\rangle^{|\beta|}e^{\delta\langle\xi\rangle+\delta\langle\eta\rangle}$$

*if $\langle\xi\rangle \geq R_0|\alpha|$, $\langle\eta\rangle \geq R_0|\beta|$ and $\delta > 0$, where $A \geq 0$ and $R_0 \geq 1$. Let $U$ be an open subset of $\mathbb{R}^n$, and assume that $q(\xi, y, \eta) = 0$ for $(\xi, y, \eta) \in \mathbb{R}^n \times U_\varepsilon \times \mathbb{R}^n$, where $\varepsilon > 0$. Then there is $R(\varepsilon) > 0$ such that $q(D_x, y, D_y)u$ is analytic in $U$ if $u \in \mathcal{F}_0$ and $R_0 \geq R(\varepsilon)A$.*

Proof. It follows from Proposition 1.2 that $q(D_x, y, D_y)$ is a continuous linear operator on $\mathcal{F}_0$ if $R_0 \geq 2e\sqrt{n}A$. In order to prove the lemma we shall apply the same argument as in the proof of Proposition 3.2.1 of [12]. We may assume that $U$ is bounded. We can write

$$\langle D\rangle^\nu e^{-\rho\langle D\rangle} q(D_x, y, D_y)u = \sum_{k=1}^\infty \sum_{j=1}^\infty \langle e^{-\delta\langle\eta\rangle}\hat{u}(\eta), f_{\nu,\delta,j,k}^R(x, \eta; \rho)\rangle_\eta$$

for $u \in \mathcal{F}_0$, $\nu = 0, 1$, $0 < \rho \leq 1$ and $0 < \delta \leq 1$, where $M \in \mathbb{Z}_+$ satisfies $M > n/2$, $R \geq R_0$, $\psi_j^R(\xi) := \phi_{j-1}^R(\xi) - \phi_j^R(\xi)$ ( $j \in \mathbb{N}$) and

$$f_{\nu,\delta,j,k}^R(x, \eta; \rho) = (2\pi)^{-2n} \int e^{i(x-y)\cdot\xi + iy\cdot\eta + \delta\langle\eta\rangle} \times \psi_k^R(\eta)\langle x - y\rangle^{-2M}\langle D_\xi\rangle^{2M}(\langle\xi\rangle^\nu e^{-\rho\langle\xi\rangle}\psi_j^R(\xi)q(\xi, y, \eta))\, d\xi dy.$$

Here the $\phi_j^R(\xi)$ are symbols as in §1. Since Re $(1 + (x - y)\cdot(x - y)) = 1 + |\,\mathrm{Re}\ x - y|^2 - |\,\mathrm{Im}\ x|^2$ for $x \in \mathbb{C}^n$ and $y \in \mathbb{R}^n$, $f_{\nu,\delta,j,k}^R(x, \eta; \rho)$ is analytic in $x$ if $|\mathrm{Im}\ x| < 1$. Let us first consider the case where $j, k \in \mathbb{N}$ and $2R(k-1)-1 \geq 6Rj$. Then we have $|\eta| \geq 2|\xi|$ if $\psi_j^R(\xi)\psi_k^R(\eta) \neq 0$. Let $K$ be a differential operator defined by

$$ {}^tK = |\xi-\eta|^{-2}\sum_{\ell=1}^n (\eta_\ell - \xi_\ell)D_{y_\ell}. $$

A simple calculation gives

$$ \begin{aligned} &|\partial_\xi^\alpha \partial_\eta^\gamma K^k\{\psi_k^R(\eta)\langle\xi\rangle^\nu e^{-\rho\langle\xi\rangle}\psi_j^R(\xi)q(\xi,y,\eta)\} \\ &\leq C_{|\alpha|+|\gamma|,\delta'}(16nA/R_0)^k\langle\xi\rangle^{\nu-|\alpha|}\langle\eta\rangle^{-|\gamma|}e^{\delta'\langle\xi\rangle+\delta'\langle\eta\rangle} \end{aligned} $$

if $\delta' > 0$. Here we have used the facts given in §2.1 of [12]. Taking $M > (|\gamma|+n)/2$, we can write

$$ \begin{aligned} \langle\eta\rangle^\ell D_\eta^\gamma f_{\nu,\delta,j,k}^R(x,\eta;\rho) &= (2\pi)^{-2n}\int e^{i(x-y)\cdot\xi+iy\cdot\eta+\delta\langle\eta\rangle} \\ &\times \langle x-y\rangle^{-2M}\langle\eta\rangle^\ell \sum_{\gamma'\leq\gamma}\binom{\gamma}{\gamma'} t_{\delta,\gamma-\gamma'}(y,\eta)D_\eta^{\gamma'}\langle D_\xi\rangle^{2M}K^k \\ &\times \{\psi_k^R(\eta)\langle\xi\rangle^\nu e^{-\rho\langle\xi\rangle}\psi_j^R(\xi)q(\xi,y,\eta)\}\,d\xi dy, \end{aligned} $$

where $t_{\delta,\gamma}(y,\eta) = e^{-iy\cdot\eta-\delta\langle\eta\rangle}D_\eta^\gamma e^{iy\cdot\eta+\delta\langle\eta\rangle}$. Therefore, we have

$$ \begin{aligned} |\langle\eta\rangle^\ell D_\eta^\gamma f_{\nu,\delta,j,k}^R(x,\eta;\rho)| &\leq C_{\delta,|\gamma|,\ell,\delta',R}\, j^{-2}k^{-2}\langle \mathrm{Re}\ x\rangle^{|\gamma|} \\ &\qquad \times \exp[(\delta+\delta'+(\rho_1+\delta')/2 - 1/(3R))\langle\eta\rangle] \end{aligned} $$

if $\ell \in \mathbb{Z}_+$, $\gamma \in (\mathbb{Z}_+)^n$, $\delta' > 0$, $x \in \mathbb{C}^n$, $|\mathrm{Im}\ x| \leq \rho_1 \leq 1/2$ and $R_0 \geq 32enA$. Moreover, $\langle e^{-\delta\langle\eta\rangle}\hat{u}(\eta), f_{\nu,\delta,j,k}^R(x,\eta;\rho)\rangle_\eta$ is analytic in $x$ and

$$ (2.1) \qquad |\langle e^{-\delta\langle\eta\rangle}\hat{u}(\eta), f_{\nu,\delta,j,k}^R(x,\eta;\rho)\rangle_\eta| \leq C_{\delta,R,r}(u)j^{-2}k^{-2} $$

if $u \in \mathcal{F}_0$, $x \in \mathbb{C}^n$, $|\mathrm{Re}\ x| \leq r$, $|\mathrm{Im}\ x| \leq \rho_1 \leq 1/2$, $R \geq R_0 \geq 32enA$ and $\delta + \rho_1/2 < 1/(3R)$. Next consider the case where $j, k \in \mathbb{N}$ and $2R(k-1)-1 < 6Rj$. Then we have $2\langle\eta\rangle \leq 9\langle\xi\rangle(1+27R/\langle\xi\rangle)$ if $\psi_j^R(\xi)\psi_k^R(\eta) \neq 0$. Let $L$ be a differential operator defined by

$$ {}^tL = |x-y|^{-2}\sum_{\ell=1}^n (\bar{x}_\ell - y_\ell)D_{\xi_\ell} $$

for $x \in \mathbb{C}^n$ with Re $x \in U$ and $y \notin \mathbb{R}^n \setminus U_\varepsilon$. Then we have

$$\begin{aligned}&|\partial_\eta^\gamma L^{j+M}\{\psi_k^R(\eta)\langle\xi\rangle^\nu e^{-\rho\langle\xi\rangle}\psi_j^R(\xi)q(\xi,y,\eta)\}|\\&\le C_{|\gamma|,M,\delta',R}(\sqrt{n}(A/R_0+(\widehat{C}+6(1+\sqrt{2}))/R)/\varepsilon)^j\\&\quad\times|x-y|^{-M}\langle\xi\rangle^{\nu-M}\langle\eta\rangle^{-|\gamma|}e^{\delta'\langle\xi\rangle+\delta'\langle\eta\rangle}\end{aligned}$$

if $\delta' > 0$, $x \in \mathbb{C}^n$ and Re $x \in U$. Taking $M > |\gamma| + n$, we have

$$|\langle\eta\rangle^\ell D_\eta^\gamma f_{\nu,\delta,j,k}^R(x,\eta;\rho)| \le C_{\delta,|\gamma|,\ell,\varepsilon,R}(U)j^{-2}k^{-2}$$

if $\ell \in \mathbb{Z}_+$, $\gamma \in (\mathbb{Z}_+)^n$, $x \in \mathbb{C}^n$, Re $x \in U$, $|\text{Im } x| \le \rho_1$ and

$$\begin{cases}R_0 \ge 4e\sqrt{n}A/\varepsilon, \quad R \ge 4e\sqrt{n}(\widehat{C}+6(1+\sqrt{2}))/\varepsilon,\\ 9\delta+\rho_1 < 1/(3R).\end{cases} \tag{2.2}$$

Moreover, $\langle e^{-\delta\langle\eta\rangle}\hat{u}(\eta), f_{\nu,\delta,j,k}^R(x,\eta;\rho)\rangle_\eta$ is analytic in $x$ and

$$|\langle e^{-\delta\langle\eta\rangle}\hat{u}(\eta), f_{\nu,\delta,j,k}^R(x,\eta;\rho)\rangle_\eta| \le C_{\delta,\varepsilon,R}(U,u)j^{-2}k^{-2} \tag{2.3}$$

if $u \in \mathcal{F}_0$, $x \in \mathbb{C}^n$, Re $x \in U$, $|\text{Im } x| \le \rho_1 \le 1/2$ and (2.2) is valid. We put

$$V(x,x_{n+1}) = \mathcal{H}(q(D_x,y,D_y)u)(x,x_{n+1})$$

and assume that

$$\begin{aligned}&R_0 \ge \max\{32enA, 4e\sqrt{n}A/\varepsilon\},\\&0<\rho_1<\min\{1/2, 1/(3R_0), \varepsilon/(12e\sqrt{n}(\widehat{C}+6(1+\sqrt{2}))\}.\end{aligned}$$

Then it follows from (2.1) and (2.3) that $\langle D_x\rangle^\nu V(x,\rho)$ ( $\nu = 0,1$) can be continued analytically to $\{x \in \mathbb{C}^n;$ Re $x \in U$ and $|\text{Im } x| < \rho_1\}$. Applying Lemma 1.2.4 of [12] to the Cauchy problem

$$\begin{cases}(1-\Delta_{x,x_{n+1}})v(x,x_{n+1}) = 0,\\ v(x,\rho) = V(x,\rho), \quad (\partial v/\partial x_{n+1})(x,\rho) = -\langle D_x\rangle V(x,\rho),\end{cases}$$

we can show that $V(x,x_{n+1})$ can be continued analytically from $\mathbb{R}^n \times (0,\infty)$ to $U \times (\rho-\rho_1,\infty)$. This implies that $q(D_x,y,D_y)u$ is analytic in $U$. □

Lemma 2.4. *Let $a(x,\xi)$ be a symbol satisfying*

$$|a^{(\alpha)}_{(\beta+\tilde{\beta})}(x,\xi)| \le C_{|\alpha|+|\tilde{\beta}|,\delta}(A/R_0)^{|\beta|}\langle\xi\rangle^{|\beta|}e^{\delta\langle\xi\rangle}$$

*if $\langle\xi\rangle \ge R_0|\beta|$ and $\delta > 0$, where $R_0 > 0$ and $A \ge 0$. Let $U$ be an open subset of $\mathbb{R}^n$, and assume that*

$$|a^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{|\alpha|}B^{|\beta|}|\beta|!\, e^{-c\langle\xi\rangle}$$

*for $x \in U_\varepsilon$, where $B$, $c$ and $\varepsilon$ are positive constants. Then there is $C > 0$, which is independent of $A$, $R_0$, $B$, $c$ and $\varepsilon$, such that $a(x,D)u$ is analytic in $U$ if $u \in \mathcal{F}_0$ and $R_0 \ge CA$.*

Proof. Choose symbols $\varphi^R(x,\xi) \in S^0(R, C_*, C(\varepsilon))$ ( $R \ge 4$) so that $0 \le \varphi^R(x,\xi) \le 1$, supp $\varphi^R \subset U_\varepsilon \times \mathbb{R}^n$ and $\varphi^R(x,\xi) = 1$ for $x \in U_{2\varepsilon/3}$. We put

$$a_1^R(x,\xi) = \varphi^R(x,\xi)a(x,\xi), \quad a_2^R(x,\xi) = (1-\varphi^R(x,\xi))a(x,\xi).$$

Then we have

$$\begin{aligned} &|a^{R(\alpha)}_{1(\beta)}(x,\xi)| \le C_{|\alpha|+|\beta|,\varepsilon}e^{-c\langle\xi\rangle}, \\ &|a^{R(\alpha)}_{1(\beta)}(x,\xi)| \le C_{|\alpha|}B^{|\beta|}|\beta|!e^{-c\langle\xi\rangle} \quad \text{for } x \in U_{2\varepsilon/3}. \end{aligned}$$

Since $e^{-c\langle\xi\rangle/2}\hat{u}(\xi) \in \mathcal{S}'$ and

$$a_1^R(x,D)u(x) = (2\pi)^{-n}\langle e^{-c\langle\xi\rangle/2}\hat{u}(\xi), e^{ix\cdot\xi+c\langle\xi\rangle/2}a_1^R(x,\xi)\rangle_\xi$$

for $u \in \mathcal{F}_0$, $a_1^R(x,D)u(x)$ is analytic in $U_{2\varepsilon/3}$. Moreover, we have supp $a_2^R \cap \overline{U}_{\varepsilon/3} \times \mathbb{R}^n = \emptyset$ and

$$|a^{R(\alpha)}_{2(\beta+\tilde{\beta})}(x,\xi)| \le C_{|\alpha|+|\tilde{\beta}|,\delta}(A/R_0 + C(\varepsilon)/R)^{|\beta|}\langle\xi\rangle^{|\beta|}e^{\delta\langle\xi\rangle}$$

if $R \ge R_0$, $\langle\xi\rangle \ge R|\beta|$ and $\delta > 0$. It follows from Theorem 2.6.1 of [12] that there are $C > 0$ and $R(\varepsilon) > 0$ such that supp $a_2^R(x,D)u \cap U = \emptyset$ if $R_0 \ge CA$, $R \ge R(\varepsilon)$ and $u \in \mathcal{F}_0$. This proves the lemma. □

Let $\Gamma$ be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and assume that $a(x,\xi) \in PS^+(\Gamma; R_0, A)$, where $A \ge 0$ and $R_0 \ge 4$. Let $\Gamma_j$ ( $j = 1, 2$)

be open conic subsets of $\Gamma$ such that $\Gamma_1 \Subset \Gamma_2 \Subset \Gamma$. Moreover, let $\varepsilon > 0$, and let $X \times \gamma$ be an open conic subset of $\Gamma_1$ such that $X_{2\varepsilon} \times \gamma_{2\varepsilon} \subset \Gamma_1$. We choose symbols $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))$ and $\varphi^R(x, \xi) \in S^{0,0}(R, C_*, C(\varepsilon))$ and $g^R(\xi) \in S^{0,0}(R, C(\varepsilon))$ ( $R \geq 4$) so that $0 \leq \Phi^R(\xi, y, \eta), \varphi^R(x, \xi), g^R(\xi) \leq 1$, supp $\Phi^R \subset \mathbb{R}^n \times \Gamma_2$, supp $\varphi^R \subset X_\varepsilon \times \mathbb{R}^n$, supp $g^R \subset \gamma_\varepsilon \cap \{|\xi| \geq R\}$, $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1$ with $\langle \eta \rangle \geq R$, $\varphi^R(x, \xi) = 1$ for $(x, \xi) \in X_{\varepsilon/2} \times \mathbb{R}^n$ and $g^R(\xi) = 1$ for $\xi \in \gamma_{\varepsilon/2}$ with $|\xi| \geq 2R$ ( see Proposition 2.2.3 in [12]). Put $a^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta) a(y, \eta)$ and $A^R(x, \xi) = \varphi^R(x, \xi) g^R(\xi) a(x, \xi)$. We denote $\gamma^0 = \gamma \cap S^{n-1}$. Then we have the following

Lemma 2.5. *There is* $R_1(A, \Gamma_1, \Gamma_2, \varepsilon) \geq 4$ *such that*

$$(A^R(x, D)u)|_{X \times \gamma^0} = (a^R(D_x, y, D_y)u)|_{X \times \gamma^0} \quad \textit{in } \mathcal{C}(X \times \gamma^0),$$

*i.e.,*

$$(A^R(x, D)u)|_{X \times \gamma^0} = a(x, D)(u|_{X \times \gamma^0}) \quad \textit{in } \mathcal{C}(X \times \gamma^0),$$

*if* $R \geq \max\{R_0, R_1(A, \Gamma_1, \Gamma_2, \varepsilon)\}$ *and* $u \in \mathcal{F}_0$.

Proof. It suffices to show that there is $R_1(A, \Gamma_1, \Gamma_2, \varepsilon) \geq 4$ such that

$$WF_A(a^R(D_x, y, D_y)u - A^R(x, D)u) \cap X \times \gamma = \emptyset$$

if $R \geq \max\{R_0, R_1(A, \Gamma_1, \Gamma_2, \varepsilon)\}$ and $u \in \mathcal{F}_0$. Write

$$a^R(D_x, y, D_y) - A^R(x, D) = a_1^R(D_x, y, D_y) + a_2^R(D_x, y, D_y) \quad \text{on } \mathcal{F}_0,$$

where

$$\begin{aligned} a_1^R(\xi, y, \eta) &= (\Phi^R(\xi, y, \eta) g^R(\eta) - \varphi^R(y, \eta) g^R(\eta)) a(y, \eta), \\ a_2^R(\xi, y, \eta) &= \Phi^R(\xi, y, \eta)(1 - g^R(\eta)) a(y, \eta). \end{aligned}$$

We note that

$$\begin{aligned} &|\partial_\xi^{\alpha+\tilde{\alpha}} D_y^\beta \partial_\eta^\gamma a_1^R(\xi, y, \eta)| \\ &\leq C_{|\tilde{\alpha}|+|\gamma|,\delta} (C_*/R)^{|\alpha|} ((A + C(\Gamma_1, \Gamma_2) + C(\varepsilon))/R)^{|\beta|} \langle \eta \rangle^{|\beta|} e^{\delta \langle \eta \rangle} \end{aligned}$$

if $\langle\xi\rangle \geq R|\alpha|$, $\langle\eta\rangle \geq R|\beta|$ and $\delta > 0$, and that $a_1^R(\xi, y, \eta) = 0$ if $y \in X_{\varepsilon/2}$. By Lemma 2.3 there is $R_1(A, \Gamma_1, \Gamma_2, \varepsilon) \geq 4$ such that $a_1^R(D_x, y, D_y)u$ is analytic in $X$ if $u \in \mathcal{F}_0$ and $R \geq R_1(A, \Gamma_1, \Gamma_2, \varepsilon)$. It is easy to see that $a_2^R(\xi, y, \eta) \in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), A + C(\Gamma_1, \Gamma_2) + C(\varepsilon))$ if $R \geq R_0$, and that $a_2^R(\xi, y, \eta) = 0$ if $\eta \in \gamma_{\varepsilon/2}$ and $|\eta| \geq 2R$. Therefore, from Lemma 2.1 there is $R_2(A, \Gamma_1, \Gamma_2, \varepsilon) \geq 4$ such that

$$WF_A(a_2^R(D_x, y, D_y)u) \cap \mathbb{R}^n \times \gamma = \emptyset \quad \text{for } u \in \mathcal{F}_0$$

if $R \geq \max\{R_0, R_2(A, \Gamma_1, \Gamma_2, \varepsilon)\}$, which proves the lemma. □

Next assume that $a(x, \xi) \equiv \sum_{j=0}^{\infty} a_j(x, \xi) \in FS^+(\Gamma; C_0, A)$. We put $\tilde{a}(x, \xi) = \sum_{j=0}^{\infty} \phi_j^{R/2}(\xi) a_j(x, \xi)$ ( $\in PS^+(\Gamma; R, 2A+3\widehat{C}, A)$) and $\tilde{a}^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta)\tilde{a}(y, \eta)$ ( $\in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), 2A + 3\widehat{C} + C(\Gamma_1, \Gamma_2)))$ for $R > C_0$.

Lemma 2.6. *There is $R(A, \Gamma_1, \Gamma_2, \varepsilon) \geq 4$ such that*

$$({}^t\tilde{a}^R(D_x, y, D_y)u)|_{X \times (-\gamma)^0} = ({}^t a)(x, D)(u|_{X \times (-\gamma)^0}) \quad \textit{in } \mathcal{C}(X \times (-\gamma)^0)$$

*if $R \geq R(A, \Gamma_1, \Gamma_2, \varepsilon)$ and $u \in \mathcal{F}_0$, where $-\gamma = \{\xi;\ -\xi \in \gamma\}$.*

Proof. Note that ${}^t\tilde{a}^R(D_x, y, D_y)u = B^R(D_x, y, D_y)u$ for $u \in \mathcal{F}_0$, where $B^R(\xi, y, \eta) = \tilde{a}^R(-\eta, y, -\xi)$. It follows from Corollary 2.4.7 in [12] that there are symbols $q_j(x, \xi)$ ( $j = 1, 2$) and $R(C_0, A_1) > \max\{4, C_0\}$ such that ${}^t\tilde{a}^R(D_x, y, D_y) = q_1(x, D) + q_2(x, D)$ on $\mathcal{S}_\infty$, $q_1(x, \xi) \in S^+(4R, \widehat{C}_* + 10A_1)$ and

$$|q_{2(\beta)}^{(\alpha)}(x, \xi)| \leq C_{|\alpha|, R}(4R + 1)^{|\beta|}|\beta|! e^{-\langle\xi\rangle/R}$$

if $R \geq R(C_0, A_1)$, where $A_1 = \max\{C_*, 2A + 3\widehat{C} + C(\Gamma_1, \Gamma_2)\}$ and $\widehat{C}_*$ is a positive constant. There is $R(C_0, A_1, \varepsilon) \geq R(C_0, A_1)$ such that

$$|\partial_\xi^\alpha D_x^\beta \{q_1(x, \xi) - q(x, \xi)\}| \leq C_{|\alpha|, R}(R + 1)^{|\beta|}|\beta|! e^{-\langle\xi\rangle/R}$$

if $(x, -\xi) \in X_\varepsilon \times \gamma_\varepsilon$ and $R \geq R(C_0, A_1, \varepsilon)$, where

$$b_j(x, \xi) = \sum_{k+|\alpha|=j} (-1)^{|\alpha|} a_{k(\alpha)}^{(\alpha)}(x, -\xi)/\alpha! \quad ( j \in \mathbb{Z}_+),$$

$$q(x, \xi) = \sum_{j=0}^{\infty} \phi_j^{4R}(\xi) b_j(x, \xi) \quad \text{for } (x, -\xi) \in \Gamma.$$

Write

$$ {}^t\tilde{a}^R(D_x, y, D_y) = \tilde{q}_1(x, D) + \tilde{q}_2(x, D) + \widetilde{B}^R(D_x, y, D_y) \quad \text{on } \mathcal{S}_\infty, $$

where $\tilde{q}_j(x,\xi) = q_j(x,\xi)g^R(-\xi)$ ( $j = 1, 2$) and $\widetilde{B}^R(\xi, y, \eta) = \tilde{a}^R(-\eta, y, -\xi)(1-g^R(-\xi))$. Proposition 1.2 implies that $\tilde{q}_2(x, D)u$ is analytic if $u \in \mathcal{F}_0$. It follows from Lemma 2.1 that there is $R_1(C_0, A_1, \varepsilon) \geq 4$ such that

$$ WF_A(\widetilde{B}^R(D_x, y, D_y)u) \cap \mathbb{R}^n \times (-\gamma) = \emptyset \quad \text{for } u \in \mathcal{F}_0 $$

if $R \geq R_1(C_0, A_1, \varepsilon)$. We note that $b_j(x,\xi) \in FS^+(\check{\Gamma}; C_0', 2A)$, where $C_0' = \max\{C_0, 4nA^2\}$. Put

$$ \begin{aligned} \tilde{b}(x,\xi) &= \sum_{j=0}^{\infty} \phi_j^{R/2}(\xi) b_j(x,\xi) \, (\in PS^+(\check{\Gamma}; R, 4A + 3\widehat{C}, 2A)), \\ b^R(x,\xi) &= \varphi^R(x,\xi) g^R(-\xi) \tilde{b}(x,\xi) \\ &\qquad (\in S^+(R, C_* + 4A + 3\widehat{C} + C(\varepsilon), 2A + C(\varepsilon))), \end{aligned} $$

where $R > C_0'$. Then we can see that $\tilde{q}_1(x,\xi) - b^R(x,\xi) \in S^+(4R, A_2)$ and

$$ |\partial_\xi^\alpha D_x^\beta \{\tilde{q}_1(x,\xi) - b^R(x,\xi)\}| \leq C_{|\alpha|,R} A_R^{|\beta|} |\beta|! e^{-\langle\xi\rangle/(24R)} \tag{2.4} $$

if $x \in X_{\varepsilon/2}$ and $R \geq \max\{R_1(C_0, A_1, \varepsilon), eC_0'/2\}$, where $A_2 = \max\{\widehat{C}_* + 10A_1 + 4C(\varepsilon), 4C_* + 16A + 12\widehat{C} + 4C(\varepsilon)\}$ and $A_R = \max\{R+1, 2A\}$. Indeed, we have

$$ \begin{aligned} & b^R(x,\xi) - q(x,\xi)g^R(-\xi) = g^R(-\xi) \sum_{j=0}^{\infty} (\phi_j^{R/2}(\xi) - \phi_j^{4R}(\xi)) b_j(x,\xi) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } x \in X_{\varepsilon/2}, \\ & \operatorname{supp}\, (\phi_j^{R/2} - \phi_j^{4R}) \subset \{\xi;\ Rj \leq \langle\xi\rangle \leq 12Rj\}, \\ & |\partial_\xi^\alpha D_x^\beta \{b^R(x,\xi) - q(x,\xi)g^R(-\xi)\}| \\ & \leq C_{|\alpha|,R,\varepsilon} \sum_{j=0}^{\infty} (j!/(1+j^j))(C_0'/R)^j \chi_j(\xi) (2A)^{|\beta|} |\beta|! e^{\langle\xi\rangle/(24R)} \\ & \leq C'_{|\alpha|,R,\varepsilon} (2A)^{|\beta|} |\beta|! e^{-\langle\xi\rangle/(24R)} \quad \text{if } x \in X_{\varepsilon/2} \text{ and } R \geq eC_0', \end{aligned} $$

where $\chi_j(\xi) = \begin{cases} 1 & \text{if } Rj \leq \langle\xi\rangle \leq 12Rj, \\ 0 & \text{otherwise.} \end{cases}$ The estimates (2.4) and Lemma 2.4 implies that there is $C > 0$ such that $\tilde{q}_1(x, D)u - b^R(x, D)u$ is analytic in $X$ if $u \in \mathcal{F}_0$ and $R \geq CA_2$. This gives

$$WF_A({}^t\tilde{a}^R(D_x, y, D_y)u - b^R(x, D)u) \cap X \times (-\gamma) = \emptyset \quad \text{for } u \in \mathcal{F}_0$$

if $R \geq \max\{R_1(C_0, A_1, \varepsilon), CA_2\}$. So the lemma easily follows from Lemma 2.5. □

For $\varepsilon, \nu \in \mathbb{R}$ we can define

$$L^2_{\varepsilon,\nu} := \{f \in \mathcal{S}'_{-\varepsilon};\ \langle x\rangle^\nu e^{\varepsilon\langle D\rangle} f(x) \in L^2(\mathbb{R}^n)\}.$$

Indeed, $e^{\varepsilon\langle D\rangle} f(x) \in \mathcal{S}'$ and $\langle x\rangle^\nu e^{\varepsilon\langle D\rangle} f(x)$ is well-defined in $\mathcal{S}'$ if $f \in \mathcal{S}'_{-\varepsilon}$. $L^2_{\varepsilon,\nu}$ is a Hilbert space in which the scalar product is given by

$$(f, g)_{L^2_{\varepsilon,\nu}} := (\langle x\rangle^\nu e^{\varepsilon\langle D\rangle} f, \langle x\rangle^\nu e^{\varepsilon\langle D\rangle} g)_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ denotes the scalar product of $L^2(\mathbb{R}^n)$.

Lemma 2.7. *Let $a(\xi, y, \eta)$ be a symbol satisfying*

$$\begin{aligned} &|\partial_\xi^\alpha D_y^{\beta+\tilde{\beta}} \partial_\eta^\gamma a(\xi, y, \eta)| \\ &\leq C_{|\alpha|+|\tilde{\beta}|+|\gamma|} (A/R_0)^{|\beta|} \langle\xi\rangle^{-|\alpha|+|\beta|} \langle\eta\rangle^{-|\gamma|} \exp[\delta_1\langle\xi\rangle - \delta_2\langle\eta\rangle] \end{aligned}$$

*for any $\alpha, \beta, \tilde{\beta}, \gamma \in (\mathbb{Z}_+)^n$ and $(\xi, y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $\langle\xi\rangle \geq R_0|\beta|$, where $A \geq 0$, $R_0 \geq 1$ and $\delta_1, \delta_2 \in \mathbb{R}$.*

(i) *$a(D_x, y, D_y)$ is well-defined on $L^2_{\varepsilon_2,\nu}$ and maps continuously $L^2_{\varepsilon_2,\nu}$ to $L^2_{\varepsilon_1,\nu}$ if $R_0 \geq 25e\sqrt{n}A$, $2(\varepsilon_1 + \delta_1)_+ < \varepsilon_2 + \delta_2$ and $3(\varepsilon_1 + \delta_1) + 2(\varepsilon_2 + \delta_2)_- < 1/R_0$.*

(ii) *If $\varepsilon_1 < \varepsilon_2$ and $\nu_1 < \nu_2$, then $L^2_{\varepsilon_2,\nu_2} \subset L^2_{\varepsilon_1,\nu_1}$ and the inclusion map $L^2_{\varepsilon_2,\nu_2} \ni u \mapsto u \in L^2_{\varepsilon_1,\nu_1}$ is compact.*

Remark. The assertion (i) is given in Lemma 5.1.6 of [12] when $\nu = 0$.

Proof. (i) Choose a symbol $g(\xi,\eta)$ so that $|\partial_\xi^\alpha \partial_\eta^\gamma g(\xi,\eta)| \le C_{|\alpha|+|\gamma|}\langle\xi\rangle^{-|\alpha|}\langle\eta\rangle^{-|\gamma|}$, $g(\xi,\eta)=1$ if $|\xi|\le 3|\eta|/2$ or $|\xi|\le 1$, and $g(\xi,\eta)=0$ if $|\xi|\ge 2|\eta|$ and $|\xi|\ge 2$. We put

$$a_1(\xi,y,\eta)=g(\xi,\eta)a(\xi,y,\eta), \quad a_2(\xi,y,\eta)=(1-g(\xi,\eta))a(\xi,y,\eta).$$

Let $\varepsilon_1,\varepsilon_2\in\mathbb{R}$ satisfy $2(\varepsilon_1+\delta_1)_+<\varepsilon_2+\delta_2$. Then we have

$$|\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma\{\exp[\varepsilon_1\langle\xi\rangle-\varepsilon_2\langle\eta\rangle]a_1(\xi,y,\eta)\}| \le C_{|\alpha|+|\beta|+|\gamma|}\langle\xi\rangle^{-|\alpha|}\langle\eta\rangle^{-|\gamma|}.$$

Therefore, there is $b_1(x,\xi)\in S^0_{1,0}$ such that

$$\exp[\varepsilon_1\langle D\rangle]a_1(D_x,y,D_y)\exp[-\varepsilon_2\langle D\rangle]=b_1(x,D) \quad \text{on } \mathcal{S}_\infty.$$

Moreover, we have

$$\begin{aligned}&|\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma\{\exp[-\delta\langle\xi\rangle+\delta_2\langle\eta\rangle]a_2(\xi,y,\eta)\}|\\ &\le C_{|\alpha|+|\beta|+|\gamma|}\langle\xi\rangle^{-|\alpha|}\langle\eta\rangle^{-|\gamma|}\exp[-(\delta-\delta_1)\langle\xi\rangle/2]\end{aligned}$$

if $\delta>\delta_1$. This gives $a_2(D_x,y,D_y)v\in\mathcal{S}_{-\delta}$ and $\sum_{j=1}^\infty \psi_j^{R_0}(D)a_2(D_x,y,D_y)v=a_2(D_x,y,D_y)v$ in $\mathcal{S}_{-\delta}$ if $v\in\mathcal{S}_\infty$ and $\delta>\delta_1$, where $\psi_j^R(\xi)=\phi_{j-1}^R(\xi)-\phi_j^R(\xi)$. Put

$$\tilde a_2(\xi,y,\eta)=\sum_{j=1}^\infty \psi_j^{R_0}(\xi)K^j a_2(\xi,y,\eta),$$

where $K=|\xi-\eta|^{-2}\sum_{k=1}^n(\xi_k-\eta_k)D_{y_k}$. Then we have

$$\begin{aligned}&a_2(D_x,y,D_y)=\tilde a_2(D_x,y,D_y) \quad \text{on } \mathcal{S}_\infty,\\ &|\partial_\xi^\alpha D_y^\beta \partial_\eta^\gamma\{\exp[\varepsilon_1\langle\xi\rangle-\varepsilon_2\langle\eta\rangle]\tilde a_2(\xi,y,\eta)\}|\\ &\le C_{|\alpha|+|\beta|+|\gamma|}\exp[(\delta_1-1/(3R_0)+\varepsilon_1+2(\varepsilon_2+\delta_2)_-/3)\langle\xi\rangle]\end{aligned}$$

if $R_0\ge 25e\sqrt{n}A$, where $c_-=\max\{-c,0\}$ ( see the proof of Lemma 5.1.6 of [12]). Now assume that $R_0\ge 25e\sqrt{n}A$ and $3(\varepsilon_1+\delta_1)+2(\varepsilon_2+\delta_2)_-<1/R_0$. Then there is $b_2(x,\xi)\in S^{-\infty}$ ( $\subset S^0_{1,0}$) such that

$$\exp[\varepsilon_1\langle D\rangle]a_2(D_x,y,D_y)\exp[-\varepsilon_2\langle D\rangle]=b_2(x,D) \quad \text{on } \mathcal{S}_\infty.$$

Putting $b(x,\xi) = b_1(x,\xi) + b_2(x,\xi)$ ( $\in S^0_{1,0}$), we have

$$\exp[\varepsilon_1\langle D\rangle]a(D_x, y, D_y)\exp[-\varepsilon_2\langle D\rangle] = b(x,D) \quad \text{on } \mathcal{S}_\infty.$$

Let $\nu \in \mathbb{R}$, and put

$$\tilde{b}_\nu(x,\xi) = (2\pi)^{-n} \text{ Os-}\int e^{-y\cdot\eta}\langle x\rangle^\nu b(x,\xi+\eta)\langle x+y\rangle^{-\nu}\, dyd\eta,$$

where Os-$\int$ denotes an oscillatory integral. Then we have $\tilde{b}_\nu(x,\xi) \in S^0_{1,0}$ and

$$\langle x\rangle^\nu b(x,D)(\langle x\rangle^{-\nu}v) = \tilde{b}_\nu(x,D)v \quad \text{on } \mathcal{S}.$$

Let $\chi(\xi)$ be a function in $C_0^\infty(\mathbb{R}^n)$ such that $\chi(\xi) = 1$ if $|\xi| \le 1$. Then we have $\langle x\rangle^\nu\chi(D/j)(\langle x\rangle^{-\nu}f(x)) \to f(x)$ in $\mathcal{S}$ as $j \to \infty$ for $f \in \mathcal{S}$. This implies that $\{\langle x\rangle^\nu f(x);\ f \in \mathcal{S}_\infty\}$ is dense in $L^2(\mathbb{R}^n)$. Therefore, $\langle x\rangle^\nu\exp[\varepsilon_1\langle D\rangle]a(D_x,y,D_y)\exp[-\varepsilon_2\langle D\rangle]\langle x\rangle^{-\nu}$ can be extended to a bounded operator on $L^2(\mathbb{R}^n)$, *i.e.*, $a(D_x,y,D_y)$ maps continuously $L^2_{\varepsilon_2,\nu}$ to $L^2_{\varepsilon_1,\nu}$.

(ii) Assume that $\varepsilon_1 < \varepsilon_2$ and $\nu_1 < \nu_2$. Then there is $c(x,\xi) \in S^{-1}_{1,0}$ such that $\langle x\rangle^{\nu_2}\exp[(\varepsilon_1-\varepsilon_2)\langle D\rangle](\langle x\rangle^{-\nu_2}u) = c(x,D)u$ for $u \in \mathcal{S}$. Therefore, the operator: $L^2(\mathbb{R}^n) \ni u \mapsto \langle x\rangle^{\nu_1}\exp[(\varepsilon_1-\varepsilon_2)\langle D\rangle](\langle x\rangle^{-\nu_2}u) \in L^2(\mathbb{R}^n)$ is compact ( see, *e.g.*, Theorem 5.14 of [5]). This proves the assertion (ii). □

Lemma 2.8. *Let $X$ and $X_1$ be bounded open subsets of $\mathbb{R}^n$ satisfying $X_1 \Subset X$, and let $a(\xi,y,\eta)$ be a symbol such that* supp $a \subset \mathbb{R}^n\times X_1\times\mathbb{R}^n$ *and*

$$\begin{aligned}(2.5)\quad &|\partial_\xi^\alpha D_y^{\beta+\tilde\beta}\partial_\eta^{\gamma+\tilde\gamma}a(\xi,y,\eta)| \\ &\le C_{|\alpha|+|\tilde\beta|+|\tilde\gamma|}(A/R_0)^{|\beta|+|\gamma|}\langle\xi\rangle^{m_1-|\alpha|+|\beta|}\langle\eta\rangle^{m_2-|\tilde\gamma|}\exp[\delta_1\langle\xi\rangle+\delta_2\langle\eta\rangle]\end{aligned}$$

*if $\langle\xi\rangle \ge R_0|\beta|$ and $\langle\eta\rangle \ge R_0|\gamma|$, where $A \ge 0$, $R_0 \ge 1$ and $m_1, m_2, \delta_1, \delta_2 \in \mathbb{R}$. Put $\varepsilon = \text{dis}(X_1, \mathbb{R}^n\setminus X)$, and assume that $u \in \mathcal{F}_0$ and that $u$ is analytic in a neighborhood of $\overline{X}$, where $\text{dis}(Y_1,Y_2) := \inf\{|x-y|;\ x\in Y_1 \text{ and } y \in Y_2\}$ for $Y_1, Y_2 \subset \mathbb{R}^n$. Then there are positive constants $\delta(\varepsilon,u)$ and $\delta_j(\varepsilon,u)$ ( $j=1,2$) such that $a(D_x,y,D_y)u \in \mathcal{S}_\delta$ if $R_0 \ge 4e\sqrt{n}\max\{1,2/\varepsilon\}A$, $2\delta_1+(\delta_2)_+ < 1/R_0$, $\delta_j \le \delta_j(\varepsilon,u)$ ( $j=1,2$) and $\delta < \min\{1/(2R_0), \delta(\varepsilon,u)\}$.*

Proof. We shall prove the lemma in the same way as Theorem 2.6.7 of [12]. Put $u_\rho(x) = e^{-\rho\langle D\rangle}u(x)$ for $\rho > 0$. Then we have $u_\rho(x) \in C^\infty(\mathbb{R}^n)$ for $\rho > 0$ and

$$\begin{aligned} &|D^\beta u_\rho(x)| \le C(u)A(u)^{|\beta|}|\beta|! \quad \text{for } x \in X \text{ and } 0 < \rho \le 1, \\ &|u_\rho(x)| \le C_\rho(1+|x|)^\ell \quad \text{for } x \in \mathbb{R}^n \text{ and } \rho > 0, \end{aligned} \tag{2.6}$$

where $C(u)$, $A(u)$ and $C_\rho$ are positive constants and $\ell \in \mathbb{Z}_+$. Let $X_2$ be an open subset of $X$ satisfying $X_1 \Subset X_2 \Subset X$ and $\mathrm{dis}(X_1, \mathbb{R}^n \setminus X_2) = \varepsilon/2$. We choose a family $\{\chi_j\}_{j\in\mathbb{N}}$ of $C_0^\infty(X)$ so that $\chi_j(x) = 1$ in $X_2$ and $|D^\beta\chi_j(x)| \le C(C_* j/\varepsilon)^{|\beta|}$ for $|\beta| \le j$. Then (2.6) yields

$$|\mathcal{F}[\chi_j u_\rho](\xi)| \le C'(u)(1+\sqrt{n}(C_*/\varepsilon + A(u))j)^j\langle\xi\rangle^{-j}$$

for $0 < \rho \le 1$. Note that

$$\begin{aligned} &\partial_\xi^\alpha \mathcal{F}[a(D_x, y, D_y)\psi_j^R(D)e^{\rho\langle D\rangle}(\chi_j u_\rho)](\xi) \\ &= (2\pi)^{-n}\sum_{\alpha^1+\alpha^2=\alpha}\frac{\alpha!}{\alpha^1!\alpha^2!}\int e^{-iy\cdot(\xi-\eta)}a_{\alpha^1,\alpha^2}(\xi, y, \eta)\psi_j^R(\eta) \\ &\qquad\qquad\qquad\qquad \times e^{\rho\langle\eta\rangle}\mathcal{F}[\chi_j u_\rho](\eta)\, d\eta dy, \end{aligned}$$

where $a_{\alpha^1,\alpha^2}(\xi, y, \eta) = (-iy)^{\alpha^1}\partial_\xi^{\alpha^2}a(\xi, y, \eta)$. Replacing $p(\xi, y, \eta)$ by $a_{\alpha^1,\alpha^2}(\xi, y, \eta)$ in the proof of Theorem 2.6.7 of [12], we have

$$\begin{aligned} &|\partial_\xi^\alpha \mathcal{F}[a(D_x, y, D_y)\psi_j^R(D)e^{\rho\langle D\rangle}(\chi_j u_\rho)](\xi)| \\ &\le C_{R,R_0,\alpha}(u)j^{n+m_2}2^{-j}\langle\xi\rangle^{m_1}e^{-\delta\langle\xi\rangle} \end{aligned} \tag{2.7}$$

if $\rho > 0$, $R \ge 2e(1+\sqrt{n}(C_*/\varepsilon + A(u)))$, $R_0 \ge 2e\sqrt{n}A$, $\rho + \delta_2 + 2(\delta_1+\delta)_+ \le 1/(3R)$, $\delta_1 \le 1/(2R_0)$ and $\delta \le 1/(2R_0)$. Similarly, we have

$$\begin{aligned} &|\partial_\xi^\alpha \mathcal{F}[a(D_x, y, D_y)\psi_j^R(D)e^{\rho\langle D\rangle}((1-\chi_j)u_\rho)](\xi)| \\ &\le C_{\rho,A,R,R_0,\alpha}(u)j^{-2}\langle\xi\rangle^{m_1}e^{-\delta\langle\xi\rangle} \end{aligned}$$

if $\rho > 0$, $R \ge 8e\sqrt{n}(C_* + \widehat{C} + 6(1+\sqrt{2}))/\varepsilon$, $R_0 \ge 4e\sqrt{n}\max\{1, 2/\varepsilon\}A$, $\delta \le 1/(2R_0)$, $2\delta_1 + (\rho+\delta_2)_+ \le 1/R_0$, $\rho + \delta_2 \le 1/(3R)$ and $\delta \le 1/(12R) - \delta_1 - (\rho+\delta_2)/4$. This, together with (2.7), yields

$$|\partial_\xi^\alpha \mathcal{F}[a(D_x, y, D_y)u](\xi)| \le C_{R_0,\alpha}(u, a)\langle\xi\rangle^{m_1}e^{-\delta\langle\xi\rangle}$$

if $R_0 \geq 4e\sqrt{n}\max\{1, 2/\varepsilon\}A$, $\delta_2+2(\delta_1+\delta)_+ < c(\varepsilon, u)/3$, $2\delta_1+(\delta_2)_+ < 1/R_0$, $\delta \leq 1/(2R_0)$ and $\delta+\delta_1+\delta_2/4 < c(\varepsilon, u)/12$, where $c(\varepsilon, u) = \min\{1/(2e(1+\sqrt{n}(C_*/\varepsilon + A(u)))), \varepsilon/(8e\sqrt{n}(C_* + \widehat{C} + 6(1+\sqrt{2})))\}$, which proves the lemma. □

LEMMA 2.9. *Let* $\Gamma$ *be an open conic subset of* $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ *satisfying* $\Gamma \Subset \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, *and let* $a(\xi, y, \eta)$ *be a symbol such that* $\mathrm{supp}\ a \subset \mathbb{R}^n \times \Gamma$ *and* $a(\xi, y, \eta)$ *satisfies the estimates* (2.5) *if* $\langle\xi\rangle \geq R_0|\beta|$ *and* $\langle\eta\rangle \geq R_0|\gamma|$. *Let* $\varepsilon > 0$, *and assume that* $u \in \mathcal{F}_0$ *and that* $WF_A(u) \cap \Gamma_\varepsilon = \emptyset$. *Then there are positive constants* $R_0(\varepsilon)$, $\delta(\varepsilon, u)$ *and* $\delta_j(\varepsilon, u)$ ( $j = 1, 2$) *such that* $a(D_x, y, D_y)u \in \mathcal{S}_\delta$ *if* $R_0 \geq R_0(\varepsilon)A$, $2\delta_1 + (\delta_2)_+ < 1/R_0$, $\delta_j \leq \delta_j(\varepsilon, u)$ ( $j = 1, 2$) *and* $\delta < \min\{1/(2R_0), \delta(\varepsilon, u)\}$.

PROOF. One can prove the lemma in the same way as in the proof of Lemma 4.1.1 of [12], using Lemma 2.8 instead of Theorem 2.6.7 of [12]. □

It follows from Lemma 2.7(ii) that $\{L^2_{1/j,1/j}\}_{j\in\mathbb{N}}$ is a compact injective sequence of Hilbert spaces, *i.e.*, the inclusion maps: $L^2_{1/j,1/j} \ni u \mapsto u \in L^2_{1/(j+1),1/(j+1)}$ ( $j \in \mathbb{N}$) are compact. We denote by $\mathcal{X}$ the inductive limit $\varinjlim L^2_{1/j,1/j}$ of the sequence $\{L^2_{1/j,1/j}\}$ ( as a locally convex space). Then $\mathcal{X}$ is a separable complete bornologic (DF) Montel space and for any bounded subset $B$ of $\mathcal{X}$ there is $j \in \mathbb{N}$ such that $B \subset L^2_{1/j,1/j}$ and $B$ is bounded in $L^2_{1/j,1/j}$ ( see, *e.g.*, Theorems 6 and 6′ in [4]). For terminology we refer to Schaefer [7]. Moreover, $S$ is open ( resp. closed) in $\mathcal{X}$ if and only if $S \cap L^2_{1/j,1/j}$ is open ( resp. closed) in $L^2_{1/j,1/j}$ for each $j \in \mathbb{N}$, *i.e.*, the topology of $\mathcal{X}$ is the inductive limit topology of $\{L^2_{1/j,1/j}\}$ as a topological space ( see Theorem 6 in [4]). By Theorem 9 of [4] we have

$$L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X} = \varinjlim (L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}), \tag{2.8}$$

where the inductive limit on the right-hand side is the inductive limit as a locally convex space.

LEMMA 2.10. *Let* $F$ *be a closed subspace of* $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$, *and put*

$$F_j = F \cap (L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}).$$

*Then we have* $F = \varinjlim F_j$ ( *as a locally convex space*).

PROOF. By Proposition 8.6.8(i) of [6] it suffices to show that $S$ is open in $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$ if $S \cap L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}$ is open in $L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}$ for each $j \in \mathbb{N}$, *i.e.*, the topology of $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$ is the inductive limit topology of a sequence $\{L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}\}$ of topological spaces. We note that (2.8) is also valid if the inductive limits $\varinjlim L^2_{1/j,1/j}$ $( = \mathcal{X})$ and $\varinjlim (L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j})$ are replaced by the inductive limits as topological spaces. Recall that the topology of $\mathcal{X}$ coincides with the inductive limit topology of $\{L^2_{1/j,1/j}\}$ as a topological space. Therefore, the topology of $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$ coincides with the inductive limit topology of $\{L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j}\}$ as a topological space, which proves the lemma. $\square$

## 3. Proof of Theorems 1.5 and 1.6

First we shall prove Theorem 1.5. Assume that $p(x, D)$ is analytic microhypoelliptic at $z^0$. Let $\Gamma_j$ $( 0 \le j \le 2)$ be open conic subsets of $\Gamma$ such that $z^0 \in \Gamma_0 \Subset \Gamma_1 \Subset \Gamma_2 \Subset \Gamma$. By assumption we may assume that

$$\text{supp } p(x, D)u = \text{supp } u \quad \text{for } u \in \mathcal{C}(\Gamma_0^0), \tag{3.1}$$

where $\Gamma_0^0 = \Gamma_0 \cap (\mathbb{R}^n \times S^{n-1})$. Choose $\Phi^R(\xi, y, \eta) \in S^{0,0,0,0}(R, C_*, C(\Gamma_1, \Gamma_2), C(\Gamma_1, \Gamma_2))$ $( R \ge 4)$ so that $0 \le \Phi^R(\xi, y, \eta) \le 1$, supp $\Phi^R \subset \mathbb{R}^n \times \Gamma_2$ and $\Phi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Gamma_1$ with $\langle \eta \rangle \ge R$. We put

$$p^R(\xi, y, \eta) = \Phi^R(\xi, y, \eta) \sum_{j=0}^{\infty} \phi_j^{R/2}(\eta) p_j(y, \eta),$$

where $R > \max\{4, C_0\}$. Then we have

$$p^R(\xi, y, \eta) \in S^+(R, C_*, 2A + C(\Gamma_1, \Gamma_2), 2A + 3\widehat{C} + C(\Gamma_1, \Gamma_2)).$$

By definition there is $R(A, \Gamma_0, \Gamma_1, \Gamma_2) > \max\{4, C_0\}$ such that

$$\begin{aligned} &(p^R(D_x, y, D_y)v)|_{\Gamma_0^0} = p(x, D)(v|_{\Gamma_0^0}) \quad \text{in } \mathcal{C}(\Gamma_0^0), \\ &WF_A(p^R(D_x, y, D_y)v) \cap \Gamma_0 = WF_A(v) \cap \Gamma_0 \end{aligned} \tag{3.2}$$

if $R \ge R(A, \Gamma_0, \Gamma_1, \Gamma_2)$ and $v \in \mathcal{F}_0$. Let $\Omega_j$ $( j = 1, 2)$ be open conic neighborhoods of $z^0$ satisfying $\Omega_2 \Subset \Omega_1 \Subset \Gamma_0$, and let $\Psi^R(\xi, y, \eta) \in$

$S^{0,0,0,0}(R, C_*, C(\Omega_2, \Omega_1), C(\Omega_2, \Omega_1))$ ( $R \geq 4$) satisfy supp $\Psi^R \subset \mathbb{R}^n \times \Omega_1$ and $\Psi^R(\xi, y, \eta) = 1$ for $(\xi, y, \eta) \in \mathbb{R}^n \times \Omega_2$ with $\langle \eta \rangle \geq R$. We assume that $R \geq \max\{R(A, \Gamma_0, \Gamma_1, \Gamma_2), 25e\sqrt{n} \max\{2A + C(\Gamma_1, \Gamma_2), C(\Omega_2, \Omega_1)\}\}$. Let $\mathcal{X}$ be the locally convex space defined in §2, *i.e.*, $\mathcal{X} = \varinjlim L^2_{1/j,1/j}$. We define an operator $T : L^2(\mathbb{R}^n) \to \mathcal{X} \times \mathcal{X}$ as follows;

(i) the domain $D(T)$ of $T$ is given by

$$D(T) = \{f \in L^2(\mathbb{R}^n);\ (1 - \Psi^R(D_x, y, D_y))f \in \mathcal{X} \text{ and } p^R(D_x, y, D_y)f \in \mathcal{X}\},$$

(ii) $Tf = ((1 - \Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f)$ for $f \in D(T)$.

It follows from Lemma 2.9 and the analytic microhypoellipticity of $p$ that $\mathcal{X} = D(T)$ if $R \geq R(\Omega_2, \Omega_1, \Gamma_0)$, where $R(\Omega_2, \Omega_1, \Gamma_0)$ is a positive constant depending on $\Omega_2$, $\Omega_1$ and $\Gamma_0$. Indeed, let $u \in D(T)$. Then $u \in L^2(\mathbb{R}^n)$ and there is $j \in \mathbb{N}$ such that $(1 - \Psi^R(D_x, y, D_y))u \in L^2_{1/j,1/j}$. Since $p^R(D_x, y, D_y)u$ is analytic in $\mathbb{R}^n$, (3.2) gives $WF_A(u) \cap \Gamma_0 = \emptyset$. It follows from Lemma 2.9 that there are $R(\Omega_2, \Omega_1, \Gamma_0) > 0$ and $\delta(u, \Omega_1, \Gamma_0) > 0$ such that $\Psi^R(D_x, y, D_y)u \in L^2_{\delta,\nu}$ if $R \geq R(\Omega_2, \Omega_1, \Gamma_0)$, $\nu \in \mathbb{R}$, $\delta < \min\{1/(2R), \delta(u, \Omega_1, \Gamma_0)\}$. This implies that $u \in \mathcal{X}$.

We next show that $T$ is a closed operator. Assume that $R \geq R(\Omega_2, \Omega_1, \Gamma_0)$. Let $A$ be a directed set, and let $\{w_a\}_{a \in A}$ be a directed family of points in $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$ satisfying $w_a \to w \equiv (f, g, h)$ in $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$, where $w_a = (f_a, (1 - \Psi^R(D_x, y, D_y))f_a, p^R(D_x, y, D_y)f_a) \in \text{graph}(T)$. Define $\mathcal{Z} = \varprojlim L^2_{-1/j,-1/j}$. Then $\mathcal{Z}$ is a reflexive Fréchet space and $\mathcal{Z}' = \mathcal{X}$ with obvious identification ( see, *e.g.*, Theorems 1 and 11 of [4]). Moreover, we have also $\mathcal{X} \subset \mathcal{Z} \subset \mathcal{F}_0$ with obvious identification and the inclusion map $\iota : \mathcal{X} \ni v \mapsto v \in \mathcal{Z}$ is continuous. Indeed, let $B$ be a bounded subset of $\mathcal{X}$. Then there is $j \in \mathbb{N}$ such that $B$ is bounded in $L^2_{1/j,1/j}$ ( see Theorem 6 of [4]). This implies that there is $C_B > 0$ such that $\|\langle x \rangle^{1/j} e^{\langle D \rangle/j} v\| \leq C_B$ for $v \in B$, where $\|f\|$ denotes the $L^2$-norm of $f \in L^2(\mathbb{R}^n)$. Therefore, $B$ is bounded in $\mathcal{Z}$. Since $\mathcal{X}$ is bornologic, the inclusion map $\iota$ is continuous ( see Theorem 6 in [4]). Noting that $\mathcal{Z}$ and $L^2(\mathbb{R}^n)$ are metric spaces and that $(1 - \Psi^R(D_x, y, D_y))f_a \to g$ in $\mathcal{Z}$ and $f_a \to f$ in $L^2(\mathbb{R}^n)$, we have $(1 - \Psi^R(D_x, y, D_y))f = g$ ( in $\mathcal{Z}$). Similarly, we have $p^R(D_x, y, D_y)f = h$. This implies that $f \in D(T)$ and $Tf = ((1 - \Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f)$. Therefore, $T$ is a closed operator.

Let $\{p_i\}_{i\in I}$ be a fundamental system of semi-norms on $\mathcal{X}$, *i.e.*, for any continuous semi-norm $q$ on $\mathcal{X}$ there are $i \in I$ and $C > 0$ satisfying $q(f) \le Cp_i(f)$ for $f \in \mathcal{X}$. graph$(T)$ is a closed subspace of $L^2(\mathbb{R}^n) \times \mathcal{X} \times \mathcal{X}$ and its topology ( the induced topology) is generated by a family of semi-norms $\{q_i\}_{i\in I}$, where

$$q_i(w) = \|f\| + p_i((1 - \Psi^R(D_x, y, D_y))f) + p_i(p^R(D_x, y, D_y)f)$$

for $w = (f, (1-\Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f) \in \mathrm{graph}(T)$. From Lemma 2.10 we have

$$\mathrm{graph}(T) = \varinjlim\, (\,\mathrm{graph}(T) \cap (L^2(\mathbb{R}^n) \times L^2_{1/j,1/j} \times L^2_{1/j,1/j})).$$

It is obvious that the projection: $\mathrm{graph}(T) \ni (f, (1 - \Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f) \mapsto f \in \mathcal{X}$ is closed. Since the injective limit of ( weakly) compact sequence of locally convex spaces is barreled, the strong dual of a reflexive Fréchet space and B-complete, it follows from the closed graph theorem that for any $i \in I$ there are $j \in I$ and $C > 0$ such that

$$\begin{aligned} &p_i(f) \le Cq_j(w) \\ &\text{for } w = (f, (1 - \Psi^R(D_x, y, D_y))f, p^R(D_x, y, D_y)f) \in \mathrm{graph}(T). \end{aligned} \tag{3.3}$$

For terminology and the closed graph theorem we refer to §8 of chapter IV in [7].

LEMMA 3.1. *For any $i \in I$ there are $j \in I$ and $C > 0$ such that*

$$\begin{aligned} p_i(f) \le\, &C(p_j((1 - \Psi^R(D_x, y, D_y))f) + p_j(p^R(D_x, y, D_y)f) \\ &+ \|e^{-\langle D\rangle} f\|) \quad \textit{for } f \in \mathcal{X}. \end{aligned}$$

PROOF. The inclusion map $\iota : \mathcal{X} \ni f \mapsto f \in H^1(\mathbb{R}^n)$ is continuous, where $H^1(\mathbb{R}^n)$ denotes the Sobolev space of order 1. Indeed, let $B$ be a bounded subset of $\mathcal{X}$. Then there are $j \in \mathbb{N}$ and $C_B > 0$ such that $\|\langle x\rangle^{1/j} e^{\langle D\rangle/j} f\| \le C_B$ for $f \in B$. It is obvious that $\|\langle D\rangle f\| \le (j/e)\|\langle x\rangle^{1/j} e^{\langle D\rangle/j} f\|$ for $f \in L^2_{1/j,1/j}$. So $B$ is bounded in $H^1(\mathbb{R}^n)$ and $\iota$ is continuous. Thus there are $i_0 \in I$ and $C_0 > 0$ satisfying

$$\|\langle D\rangle f\| \le C_0 p_{i_0}(f) \quad \text{for } f \in \mathcal{X}. \tag{3.4}$$

On the other hand, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\|f\| \le \varepsilon \|\langle D\rangle f\| + C_\varepsilon \|e^{-\langle D\rangle} f\| \quad \text{for } f \in H^1(\mathbb{R}^n). \tag{3.5}$$

Therefore, from (3.3) with $i = i_0$, (3.4) and (3.5) there are $j_0 \in I$ and $C_1 > 0$ such that

$$\begin{aligned} \|f\| &\le C_0 p_{i_0}(f) \\ &\le C_1(p_{j_0}((1-\Psi^R(D_x,y,D_y))f) + p_{j_0}(p^R(D_x,y,D_y)f) + \|e^{-\langle D\rangle} f\|) \end{aligned}$$

for $f \in \mathcal{X}$. This, together with (3.3), proves the lemma. □

Let $f \in \mathcal{A}'(\mathbb{R}^n)$. We shall show that there are an open neighborhood $\mathcal{U}$ of $(x^0, \xi^0/|\xi^0|)$ in $\mathbb{R}^n \times S^{n-1}$, which is independent of $f$, and $u \in \mathcal{X}'$ such that $({}^t p)(x,D)(u|_\mathcal{U}) = f|_\mathcal{U}$ in $\mathcal{C}(\mathcal{U})$. We note that $f \in \mathcal{A}'(\mathbb{R}^n) \subset \mathcal{X}' \subset \mathcal{F}_0 \subset \mathcal{S}'_\delta$ and $\mathcal{S}_\infty \subset \mathcal{S}_\delta \subset \mathcal{X}$ for $\delta > 0$. Moreover, we have

$$\begin{aligned} \langle g, v\rangle_{\mathcal{X}',\mathcal{X}} &= \langle g, v\rangle_{\mathcal{S}'_\delta,\mathcal{S}_\delta} \quad \text{for } \delta > 0,\ g \in \mathcal{X}' \text{ and } v \in \mathcal{S}_\delta, \\ \langle g, v\rangle_{\mathcal{S}'_\varepsilon,\mathcal{S}_\varepsilon} &= \langle g, v\rangle_{\mathcal{S}'_\delta,\mathcal{S}_\delta} \quad \text{for } \varepsilon \ge \delta,\ g \in \mathcal{S}'_\delta \text{ and } v \in \mathcal{S}_\varepsilon, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{X}',\mathcal{X}}$ ( resp. $\langle\cdot,\cdot\rangle_{\mathcal{S}'_\delta,\mathcal{S}_\delta}$) denotes the duality between $\mathcal{X}'$ and $\mathcal{X}$ ( resp. $\mathcal{S}'_\delta$ and $\mathcal{S}_\delta$). Therefore, we denote simply by $\langle\cdot,\cdot\rangle$ these dualities. Define

$$\begin{aligned} \mathcal{M} &:= L^2_{-1} \times \mathcal{X} \times \mathcal{X}, \\ \mathcal{N} &:= \{(v, (1-\Psi^R(D_x,y,D_y))v, p^R(D_x,y,D_y)v) \in \mathcal{M};\ v \in \mathcal{S}_\infty\}, \end{aligned}$$

where $L^2_\varepsilon = L^2_{\varepsilon,0}$. Let $F$ be a linear functional on $\mathcal{N}$ defined by $F(w) = \langle f, v_1\rangle$ for $w = (v_1, v_2, v_3) \in \mathcal{N}$. Note that there are $i_1 \in I$ and $C_2 > 0$ satisfying $|\langle f, v_1\rangle| \le C_2 p_{i_1}(v_1)$ for $v_1 \in \mathcal{X}$. By Lemma 3.1 there are $j_1 \in I$ and $C_3 > 0$ such that

$$|F(w)| \le C_3(p_{j_1}(v_2) + p_{j_1}(v_3) + \|e^{-\langle D\rangle} v_1\|) \quad \text{for } w \equiv (v_1, v_2, v_3) \in \mathcal{N}.$$

Therefore, it follows from the Hahn-Banach theorem that there is $\widetilde{F} \equiv (-\psi, -\varphi, u) \in \mathcal{M}'$ ( $= L^2_1 \times \mathcal{X}' \times \mathcal{X}'$) such that $\widetilde{F}|_\mathcal{N} = F$, *i.e.*,

$$\begin{aligned} \langle f, v\rangle = &- \langle \psi, v\rangle - \langle \varphi, (1-\Psi^R(D_x,y,D_y))v\rangle \\ &+ \langle u, p^R(D_x,y,D_y)v\rangle \quad \text{for } v \in \mathcal{S}_\infty. \end{aligned}$$

This yields

$$\langle {}^t p^R(D_x, y, D_y)u, v\rangle = \langle f + \psi + (1 - {}^t\Psi^R(D_x, y, D_y))\varphi, v\rangle$$

for $v \in \mathcal{S}_\infty$, *i.e.*,

$${}^t p^R(D_x, y, D_y)u = f + \psi + (1 - {}^t\Psi^R(D_x, y, D_y))\varphi \quad \text{in } \mathcal{F}_0.$$

Note that $\psi \in \mathcal{A}(\mathbb{R}^n)$. Let $\Omega_3$ be an open conic neighborhood of $(x^0, -\xi^0)$ satisfying $\Omega_3 \Subset \check{\Omega}_2$, where $\check{\Omega}_2 = \{(x, \xi);\ (x, -\xi) \in \Omega_2\}$. From Lemma 2.1 there is $R_1(\Omega_3, \Omega_2, \Omega_1) > 0$ such that

$$WF_A((1 - {}^t\Psi^R(D_x, y, D_y))\varphi) \cap \Omega_3 = \emptyset \quad \text{if } R \geq R_1(\Omega_3, \Omega_2, \Omega_1).$$

Therefore, Lemma 2.6 gives

$$({}^t p)(x, D)(u|_{\Omega_3^0}) = f|_{\Omega_3^0} \quad \text{in } \mathcal{C}(\Omega_3^0),$$

where $\Omega_3^0 = \Omega_3 \cap (\mathbb{R}^n \times S^{n-1})$, which proves Theorem 1.5.

Similarly, one can prove Theorem 1.6 if one choose $\Gamma = X \times (\mathbb{R}^n \setminus \{0\})$.

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