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Nonlinear Transformation Containing Rotation and Gaussian Measure

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Abstract. The author study the nonlinear transformation of a Gaussian measure and its absolute continuity and singularity relative to the original Gaussian measure. The nonlinear transformation considered contains a rotation part and is not a perturbation of a linear transformation.

1. Introduction

Let (μ, H, B) be an abstract Wiener space, that is, B is a separable real Banach space, H is a separable real Hilbert space which is densely embedded in B, and μ is a Gaussian measure on B such that

$$\int_B \exp(\sqrt{-1}{}_B\langle z, u\rangle_{B^*})\mu(dz) = \exp(-\frac{1}{2} \| u \|_H^2), \qquad u \in B^* \subset H.$$

Here B^* denotes the dual space of the Banach space B. Then B^* can be regarded as a subset of H^*. In this paper, for simplicity of notation, we identify the dual space H^* of the Hilbert space H with H itself.

Let $\Phi : B \to B$ be a measurable map. Our concern is the relation of the measure μ and the image measure $\mu \circ \Phi^{-1}$. The case where there is a measurable map $F : B \to H$ such that $\Phi = I_B + F$ has been studied by many authors ([3], [9], [4], [10], [5], [13], [15]). Here I_B denotes the identity in B. In this paper, we consider the case where Φ is not perturbation of identity.

Let $\mathcal{L}^\infty(H; H)$ denote the Banch space consisting of bounded linear operators with the operator norm $\| \cdot \|_{op}$. Let $O(H)$ denote the set of linear isomorphisms in H, i.e.,

$$O(H) = \{U \in \mathcal{L}^\infty(H; H);\ U^*U = I_H, UU^* = I_H\}.$$

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We regard $O(H)$ as a metric subspace of $\mathcal{L}^\infty(H;H)$.

Let $U : B \to O(H)$ and $F : B \to H$ be measurable maps. We think of the case where $\Phi(z) = U(z)(z + F(z))$, $z \in B$. Such a transformation was already studied in [14]. But we think of this problem from quite different viewpoint. Since it is not clear whether Φ is well-defined, we start with some basic results. We study some regularity problems in Sections 2, 3, 4 and 5. Then we study the relationship between rotation and these regularities in Section 6. In Section 7 we introduce a new notion related to infinite dimensional Lie groups. The main theorems are given in Section 9 (Theorems 49, Corollary 51). We give an example in the last Section.

2. Preliminary from Malliavin Calculus

In this section, we remind some known results and make some preparations. Since we use the notions in Malliavin calculus, we will give definitions of the Ornstein-Uhlenbeck semigroup, the Ornstein-Uhlenbeck generators, and so on.

Let E be a separble real Hilbert space. Let $P_t, t \in [0,\infty)$, denote the Ornstein-Uhlenbeck semigroup, i.e.,

$$P_t f(z) = \int_B f(e^{-t}z + (1-e^{-2t})^{1/2}w)\mu(dw), \qquad z \in B,$$

for $t \geq 0$ and $f \in L^1(B;E,d\mu)$. Let $\mathcal{L}$ denote the infinitesimal generator of the Ornstein-Uhlenbeck semigroup. Let $\mathbf{D}^s_p(E)$, $s \geq 0$, $p \in (1,\infty)$, be a Banach space defined by $(I-\mathcal{L})^{-s/2}L^p(B;E,d\mu)$ with a norm $\| u \|_{s,p;E} = \| (1-\mathcal{L})^{s/2}u \|_{L^p(B;E,d\mu)}$. Let $\mathbf{D}^s_p(E)$, $s < 0$, $p \in (1,\infty)$, be the dual Banach space of $\mathbf{D}^{-s}_q(E^*)$, $1/p + 1/q = 1$. Then identifying $\mathbf{D}^0_p(E)$ with the dual space of $\mathbf{D}^0_q(E^*)$, we may regard $\mathbf{D}^s_p(E)$ as a subset of $\mathbf{D}^t_p(E)$, $-\infty < t < s < \infty$, $p \in (1,\infty)$. We denote $\bigcup_{p\in(1,\infty)} \mathbf{D}^s_p$ by $\mathbf{D}^s_{1+}$, $s \in \mathbf{R}$.

Also, one can define the gradient opertator $D : \mathbf{D}^{s+1}_p(E) \to \mathbf{D}^s_p(H \otimes E)$, $s \in \mathbf{R}$, $p \in (1,\infty)$, and the dual of the gradient operator $D^* : \mathbf{D}^{s+1}_p(H \otimes E) \to \mathbf{D}^s_p(E)$, $s \in \mathbf{R}$, $p \in (1,\infty)$. Then we have $\mathcal{L} = -D^*D$. We also denote by B_r the set $\{h \in H; \| h \|_H \leq r\}$, $r > 0$.

LEMMA 1. *Let E be a separable real Hilbert space.*
(1) *There is an absolute constant $C > 0$ such that*

$$e^t(1-e^{-2t})^{1/2} \| DP_t f \|_{L^\infty(B;H\otimes E)} \leq C \| f \|_{L^\infty(B;E)},$$

for any bounded measurable map $f : B \to E$ and $t \in (0, \infty)$.
(2) *There are absolute constants $\gamma > 0$ and $C > 0$ satisfying the following. If $g : B \to H \otimes E$ is a measurable function satisfying*

$$\| g(z) \|_{H\otimes E} \leq 1, \qquad z \in B,\ h \in H,$$

then

$$\int_B \exp((1 - e^{-2t})\gamma \| (D^* P_t g)(z) \|_E^2)\mu(dz) \leq C, \qquad t > 0.$$

Proof. The assertion (2) is shown in [5] Theorem(4.8). So we only prove the assertion (1). Let $\{h_n\}$ and $\{e_n\}$ be a complete orthonormal basis of H and E respectively. Let $f : B \to E$ and $g : B \to H \otimes E$ be arbitrary bounded measurable maps and let $f_j(z) = (f(z), e_j)_E$, $g_{ij} = (g(z), h_i \otimes e_j)_{H\otimes E}$, $z \in B$. Then by [5] Theorem(4.4), we have

$$\begin{aligned}
&e^t(1 - e^{-2t})^{1/2}(DP_t f(z), g(z))_{H\otimes E} \\
&\quad = \sum_{i,j} \int_B (w, h_i)_H f_j(e^{-t}z + (1 - e^{-2t})^{1/2}w) g_{ij}(z)\mu(dw) \\
&\quad \leq (\sum_j \int_B ((w, \sum_i g_{ij}(z)h_i)_H)^2 \mu(dw))^{1/2} \\
&\quad \times (\sum_j \int_B f_j(e^{-t}z + (1 - e^{-2t})^{1/2}w)^2 \mu(dw))^{1/2} \\
&\quad \leq \| g(z) \|_{H\otimes E} \| f \|_{L^\infty(B;E)} \qquad \mu - a.s.z.
\end{aligned}$$

This implies our assertion. □

Lemma 2. *Let E be a separable real Hilbert space and $f : B \to E$ is a measurable map.*
(1) *For any $p \in (1, \infty)$ and $r > 0$*

$$\begin{aligned}
&\sup\{\| P_t f(z + h) \|_E;\ h \in H,\ \| h \|_H \leq r\} \\
&\quad \leq \exp((p - 1)^{-1} t^{-1} r^2/4) P_t(\| f \|_E^p)(z)^{1/p}, \qquad t > 0,\ z \in B.
\end{aligned}$$

(2) *For any* $r > 0$

$$\sup\{\| P_t f(z+h) \|;\ h \in H,\ \| h \|_H \le r\}$$
$$\le 2P_t(\exp(\| f(\cdot) \|_E^2 /2))(z) + 8t^{-1/2} r \qquad t \in (0,1],\ z \in B.$$

PROOF. First note that

$$\| P_t f(z+h) \|_E \le \int_B \| f(e^{-t}(z+h) + (1-e^{-2t})^{1/2} w) \|_E \mu(dw)$$
$$= \int_B \| f(e^{-t}z + (1-e^{-2t})^{1/2} w) \|_E \exp(a_t (w,h)_H\tilde{} - a_t^2 \| h \|_H^2 /2) \mu(dw).$$

Here $a_t = (e^{-2t}(1-e^{-2t})^{-1})^{1/2} \le (2t)^{-1/2}$. Since

$$\int_B \exp(q(a_t (w,h)_H\tilde{} - a_t^2 \| h \|_H^2 /2)) \mu(dw) = \exp(q(q-1) a_t^2 \| h \|_H^2 /2),$$

we have the assertion (1) by Hölder's inequality.

Note that $xy \le e^x + y \log_+ y$, $x, y \ge 0$. So we have

$$xy \le 2\exp(x^2/2) + 4y(\log_+ y)^{1/2}, \qquad x, y \ge 0.$$

Note that

$$\int_B ((a_t(w,h)_H\tilde{} - a_t^2 \| h \|_H^2 /2) \vee 0)^{1/2} \exp(a_t(w,h)_H\tilde{} - a_t^2 \| h \|_H^2 /2) \mu(dw)$$
$$= \int_B ((a_t(w,h)_H\tilde{} + a_t^2 \| h \|_H^2 /2) \vee 0)^{1/2} \mu(dw)$$
$$\le a_t \| h \|_H + a_t^{1/2} \int_B |(w,h)_H\tilde{}| \mu(dw).$$

This implies the assetion (2). □

COROLLARY 3. *For any* $r > 0$, *there is a* $C_r > 0$ *such that*

$$\int_B \sup\{\| (P_t D^* P_t f)(z+h) \|_E;\ h \in H, \| h \|_H \le r\} \mu(dz)$$
$$\le t^{-1} C_r \| f \|_{L^\infty(B;E)}$$

for any $t \in (0,1]$ and any bounded measurable map $f : B \to H \otimes E$.

PROOF. We may assume that $\| f(z) \|_{H\otimes E} \leq 1$, $z \in B$. Then by Lemma 2 we have

$$\begin{aligned}
&\int_B \sup\{\| (P_t D^* P_t f)(z+h) \|_E;\ h \in H, \| h \|_H \leq r\}\mu(dz) \\
&\quad \leq (1-e^{-2t})^{-1/2}\gamma^{-1} \\
&\quad \times (2\int_B \exp(\frac{\gamma}{2}(1-e^{-2t}) \| D^* P_t f(z) \|_E^2)\mu(dz) + 8t^{-1/2} r).
\end{aligned}$$

By Lemma 1 (2), we have our assertion. □

DEFINITION 4. Let $r > 0$. We say that $(\varphi^{(0)}, \varphi^{(1)})$ is an r-pair, if the following are satisfied.
(1) $\varphi^{(i)} : B \to \mathbf{R}$ is measurable, $0 \leq \varphi^{(i)} \leq 1$, and

$$|\varphi^{(i)}(z+h) - \varphi^{(i)}(z)| \leq \| h \|_H, \qquad z \in B,\ h \in H,$$

for each $i = 0, 1$.
(2) There are σ-compact sets A_0, A_1 in B such that $A_0 + B_r \subset A_1$, $\varphi^{(i)}(z) = 0$, $\mu - a.e. z \in A_1$, and $\varphi^{(1-i)}(z) = 0$, $\mu - a.e. z \in B \setminus A_0$, for $i = 0$ or 1.

LEMMA 5. *Let E be a separable real Hilbert space. Then for $r, \varepsilon > 0$, $\sigma = -1, 0, 1$, and $k = 0, 1, \ldots$ there is a constant $C > 0$ such that*

$$\| \varphi^{(0)} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^\sigma(E)} \leq C \exp(-\frac{r^2}{2t}) \| u \|_{\mathbf{D}_2^\sigma(E)}$$

for any $(r+\varepsilon)$-pair $(\varphi^{(0)}, \varphi^{(1)})$, $u \in \mathbf{D}_2^\sigma(E)$ and $t \in (0,1]$.

PROOF. Let $(\varphi^{(0)}, \varphi^{(1)})$ be an $(r+\varepsilon)$-pair. Note that

$$2 \| u \|^2_{\mathbf{D}_2^1(E)} = 2 \| u \|^2_{\mathbf{D}_2^0(E)} + \| Du \|^2_{\mathbf{D}_2^0(H\otimes E)}, \qquad u \in \mathbf{D}_2^1(E).$$

So we have

$$\begin{aligned}
\| \varphi^{(i)} u \|^2_{\mathbf{D}_2^1(E)} &= 2 \| \varphi^{(i)} u \|^2_{\mathbf{D}_2^0(E)} + \| D\varphi^{(i)} \otimes u + \varphi^{(i)} Du \|^2_{\mathbf{D}_2^0(H\otimes E)} \\
&\leq 4 \| u \|^2_{\mathbf{D}_2^1(E)}, \qquad u \in \mathbf{D}_2^1(E),\ i = 0, 1.
\end{aligned}$$

Also we see that

$$\begin{aligned}\| \cos(s\sqrt{-2\mathcal{L}})u \|_{\mathbf{D}_2^1(E)} &= \| \cos(s\sqrt{-2\mathcal{L}})(1-\mathcal{L})^{1/2}u \|_{\mathbf{D}_2^0(E)} \\ &\leq \| u \|_{\mathbf{D}_2^1(E)}, \qquad u \in \mathbf{D}_2^1(E).\end{aligned}$$

Since the wave has finite propagation speed, we see that

$$\varphi^{(0)} \cos(s\sqrt{-2\mathcal{L}})\varphi^{(1)}u = 0, \qquad |s| \leq r + \varepsilon/2,\ u \in \mathbf{D}_2^1(E).$$

So we have

$$\begin{aligned}&\| \varphi^{(0)}\mathcal{L}^k P_t(\varphi^{(1)}u) \|_{\mathbf{D}_2^1(E)} \\ &= \| 2\int_{r+\varepsilon/2}^{\infty} (\varphi^{(0)} \cos(s\sqrt{-2\mathcal{L}})(\varphi^{(1)}u)(2\pi t)^{-1/2}P_k(s,t)\exp(-\frac{s^2}{2t})ds \|_{\mathbf{D}_2^1(E)} \\ &\leq 8(\int_{r+\varepsilon/2}^{\infty} (2\pi t)^{-1/2}|P_k(s,t)|\exp(-\frac{s^2}{2t})ds) \| u \|_{\mathbf{D}_2^1(E)} .\end{aligned}$$

Here $P_k(s,t)$ is a polynomial in s and $1/t$ of degree $2k$. This proves the case that $\sigma = 1$.

The proof of the case that $\sigma = 0$ is similar. Taking dual, we have the case that $\sigma = -1$.

This completes the proof. □

For any $\theta \in [0,1]$, we see from [7]that $(\mathbf{D}_2^{s_0}, \mathbf{D}_p^{s_1})_{[\theta]} = \mathbf{D}_r^s$. Here $(\cdot,*)_{[\theta]}$ denotes the complex interpolation space (see [1]), $s = (1-\theta)s_0 + \theta s_1$ and $1/r = (1-\theta)/2 + \theta/p$. By virtue of Stein [11] we see that for each $p \in (1,\infty)$ there is a constant $C > 0$ such that

$$(1) \qquad \| \mathcal{L}P_t u \|_{\mathbf{D}_p^0(E)} \leq Ct^{-1} \| u \|_{\mathbf{D}_p^0(E)}, \qquad t \in (0,1],\ u \in \mathbf{D}_p^0(E).$$

Therefore we see that for any $p \in (1,\infty)$ $-\infty < s < \infty$, $\ell \geq 0$, we have

$$(2) \qquad \| P_t u \|_{\mathbf{D}_p^{s+2\ell}(E)} \leq Ct^{-\ell} \| u \|_{\mathbf{D}_p^s(E)}, \qquad t \in (0,1],\ u \in \mathbf{D}_p^s(E).$$

In particular, there is a constant $C' > 0$ depending only on $p \in (1,\infty)$ and $k = 0, 1, \ldots$ such that

$$\| \varphi^{(0)}\mathcal{L}^k P_t(\varphi^{(1)}u) \|_{\mathbf{D}_p^1(E)} \leq C' t^{-(k+1)} \| u \|_{\mathbf{D}_p^{-1}(E)},$$

for any $t \in (0,1]$, $u \in \mathbf{D}_p^{-1}(E)$, and any $r > 0$ and r-pair $(\varphi^{(0)}, \varphi^{(1)})$.

Therefore as an easy consequence of Lemma 5, we have the following.

PROPOSITION 6. *Let E be a separable real Hilbert space. For any $p \in (1,\infty)$, $k = 0,1,2,\dots$ $r, \varepsilon > 0$ and $\gamma \in (0, 2((1-1/p) \wedge (1/p)))$ there is a constant $C > 0$ such that*

$$\| \varphi^{(0)} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_p^0(E)} \leq C \exp(-\frac{\gamma r^2}{2t}) \| u \|_{\mathbf{D}_p^{-1}(E)},$$

for any $(r+\varepsilon)$-pair $(\varphi^{(0)}, \varphi^{(1)})$, $u \in \mathbf{D}_p^{-1}(E)$, and $t \in (0,1]$.

PROPOSITION 7. *Let E be a separable real Hilbert space. Then for any $r, \varepsilon > 0$ and $k, \ell = 0, 1, \dots$, there is a constant $C > 0$ such that*

$$\| \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^0(H^{\otimes \ell} \otimes E)} \leq C \exp(-\frac{r^2}{2^{\ell+2} t}) \| u \|_{\mathbf{D}_2^{-1}(E)},$$

for any $(r+\varepsilon)$-pair $(\varphi^{(0)}, \varphi^{(1)})$, $u \in \mathbf{D}_2^{-1}(E)$ and $t \in (0,1]$.

PROOF. We prove the assertion by induction in ℓ. By Proposition 6 we see that the assetion holds in the case that $\ell = 0$. Suppose that the assertion holds for ℓ. We have

$$\begin{aligned}
&\| \varphi^{(0)} D^{\ell+1} \mathcal{L}^k P_t(\varphi^{(1)} u) \|^2_{\mathbf{D}_2^0(H^{\otimes(\ell+1)} \otimes E)} \\
&\quad = -(\varphi^{(0)} \mathcal{L} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u), \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u))_{\mathbf{D}_2^0(H^{\otimes \ell} \otimes E)} \\
&\qquad - (D^{\ell+1} \mathcal{L}^k P_t(\varphi^{(1)} u), D\varphi^{(0)} \otimes \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u))_{\mathbf{D}_2^0(H^{\otimes(\ell+1)} \otimes E)} \\
&\quad \leq (\| \varphi^{(0)} D^\ell \mathcal{L}^{k+1} P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^0(H^{\otimes \ell} \otimes E)} \\
&\qquad + \ell \| \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^0(H^{\otimes \ell} \otimes E)}) \\
&\qquad + \| D^{\ell+1} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^0(H^{\otimes(\ell+1)} \otimes E)}) \| \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}_2^0(H^{\otimes \ell} \otimes E)}
\end{aligned}$$

Here we use the fact that $D\mathcal{L} = \mathcal{L}D - D$. So the induction is complete by Inequality (2) and the assumption of induction. This completes the proof. □

For any subset A in B, let us define a function $\rho(\cdot; A) : B \to [0, \infty]$ by

$$\rho(z; A) = \inf\{\| h \|_H ; z + h \in A\}, \qquad z \in B. \tag{3}$$

We remark that if K is a compact set in B, then $\rho(\cdot;K) : B \to [0,\infty]$ is lower-semicontinuous, and that if A is a σ-compact set in B, then $\rho(\cdot;A) : B \to [0,\infty]$ is measurable.

PROPOSITION 8. *Let $r > 0$ and $(\varphi^{(0)},\varphi^{(1)})$ be an $(r+2)$-pair. Then there is a $\psi : B \to \mathbf{R}$ such that $(\psi,\varphi^{(1)})$ be an $(r+1/2)$-pair and that $(1-\psi(z))\varphi^{(0)}(z) = 0$ and $(1-\psi)(z))D\varphi^{(0)}(z) = 0$ $\mu - a.e.z$.*

PROOF. Form the assumption, there are σ-compact sets A_0, A_1 such that $A_0 + B_{r+2} \subset A_1$, $\varphi^{(i)}(z) = 0$ $\mu - a.e.z \in A_1$ and $\varphi^{(1-i)}(z) = 0$ $\mu - a.e.z \in B \setminus A_0$ for $i = 0$ or 1. Let $\psi^{(0)}(z) = (1 - \rho(z;A_0 + B_{1/4})) \vee 0$, and $\psi^{(1)}(z) = (1 - \rho(z;A_0 + B_{r+7/4})) \vee 0$, $z \in B$. Then we see that $\psi^{(i)}$ satisfies our condition for ψ. □

PROPOSITION 9. *Let E be a separable real Hilbert space. For $k,\ell = 0,1,\ldots$ and $r > 0$, there is a constant $C > 0$ such that*

$$\| \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)}u) \|_{\mathbf{D}_2^1(H^{\otimes \ell}\otimes E)} \leq C \exp(-\frac{r^2}{2^{\ell+2}t}) \| u \|_{\mathbf{D}_2^{-1}(E)},$$

for any $(r+2)$-pair $(\varphi^{(0)},\varphi^{(1)})$, $u \in \mathbf{D}_2^{-1}(E)$ and $t \in (0,1]$.

PROOF. We have

$$\begin{aligned} &D(\varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)}u)) \\ &\quad = D\varphi^{(0)} \otimes \psi D^\ell \mathcal{L}^k P_t(\varphi^{(1)}u) + \varphi_n^0 \psi D^{\ell+1} \mathcal{L}^k P_t(\varphi^{(1)}u). \end{aligned}$$

Here ψ is as in Proposition 8. So we have

$$\begin{aligned} &\| \varphi^{(0)} D^\ell \mathcal{L}^k P_t(\varphi^{(1)}u) \|_{\mathbf{D}_2^1(H^{\otimes \ell}\otimes E)} \\ &\quad \leq \| \psi D^\ell \mathcal{L}^k P_t(\varphi^{(1)}u) \|_{\mathbf{D}_2^0(H^{\otimes \ell}\otimes E)} \\ &\qquad + 2 \| \psi D^{\ell+1} \mathcal{L}^k P_t(\varphi^{(1)}u). \|_{\mathbf{D}_2^0(H^{\otimes (\ell+1)}\otimes E)} . \end{aligned}$$

So we have our assertion from Proposition 7. □

LEMMA 10. *Let E be a separable real Hilbert space. Then we have the following.*

(1)*For* $k, \ell = 0, 1, \dots,$ $p \in (1, \infty)$, $r > 0$ *and* $\gamma \in (0, ((1 - 1/p) \wedge (1/p))2^{-(\ell+2)})$ *there is a constant* $C > 0$ *such that*

$$\| \varphi^{(0)} D^{\ell} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}^1_p(H^{\otimes \ell} \otimes E)} \leq C \exp(-\frac{\gamma r^2}{t}) \| u \|_{\mathbf{D}_p^{-1}(E)}$$

for any $(r+2)$*-pair* $(\varphi^{(0)}, \varphi^{(1)})$, $u \in \mathbf{D}_2^{-1}(E)$ *and* $t \in (0, 1]$.
(2)*For* $k, \ell = 0, 1, \dots,$ $p \in (1, \infty)$, $r > 0$ *and* $\gamma \in (0, (1 - 1/p)2^{-(\ell+2)})$ *there is a constant* $C > 0$ *such that*

$$\| \varphi^{(0)} (D^*)^{\ell} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}^1_p(E)} \leq C \exp(-\frac{\gamma r^2}{t}) \| u \|_{\mathbf{D}_p^{-1}(H^{\otimes \ell} E)}$$

for any $(r+2)$*-pair* $(\varphi^{(0)}, \varphi^{(1)})$, $u \in \mathbf{D}_2^{-1}(H^{\otimes \ell} \otimes E)$ *and* $t \in (0, 1]$.

Proof. Let us prove the assertion (1) for $p \in (2, \infty)$. Let $p' \in (p, \infty)$, and let $\theta = (2(p' - p))/(p(p' - 2))$. Then we see that $[\mathbf{D}_2^s, \mathbf{D}_{p'}^s]_{1-\theta} = \mathbf{D}_p^s$, $s \in \mathbf{R}$. By Proposition 9 and Equation 2, we see that there is a constant $C > 0$ such that for any $(r+2)$-pair $(\varphi^{(0)}, \varphi^{(1)})$ and $t \in (0, 1]$.

$$\begin{aligned} &\| \varphi^{(0)} D^{\ell} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}^1_2(H^{\otimes \ell} \otimes E)} \\ &\quad \leq C \exp(-\frac{r^2}{2^{\ell+2} t}) \| u \|_{\mathbf{D}_2^{-1}(E)}, \qquad u \in \mathbf{D}_2^{-1}(E), \end{aligned}$$

and

$$\begin{aligned} &\| \varphi^{(0)} D^{\ell} \mathcal{L}^k P_t(\varphi^{(1)} u) \|_{\mathbf{D}^1_{p'}(H^{\otimes \ell} \otimes E)} \\ &\quad \leq C t^{-(\ell+2k)/2} \| u \|_{\mathbf{D}_{p'}^{-1}(E)}, \qquad u \in \mathbf{D}_{p'}^{-1}(E). \end{aligned}$$

Then we have the assertion (1) by the interpolation theory. The case that $p \in (1, 2)$ is similar.

We have the assertion (2) from the assertion (1) by using the duality.

This completes the proof. □

3. $\mathcal{CH}^\infty$ Maps

Remind that $B_r = \{h \in H;\ \| h \|_H \leq r\}$, $r > 0$. Then, endowing the weak topology of H on B_r, we may regard B_r, $r > 0$, as a compact metric space.

Let K be a compact set in B. Then $\rho(\cdot;K):B\to[0,\infty]$ defined in Equation (3) is lower-semicontinuous. Let $\varphi_n^K:B\to\mathbf{R}$, $n\geq 1$, be given by

$$\varphi_n^K(z)=1-(\rho(z;K+B_n)\wedge 1),\qquad z\in B. \tag{4}$$

Then one can easily see that $(\varphi_n^K,1-\varphi_{n+m+3}^K)$ is an $m+1/2$-pair for any compact set K in B and $n,m\in\mathbf{N}$.

Proposition 11. *For any $\ell,k=0,1,\dots,$ $p\in(1,2]$, $m^2(p-1)^2>2^{\ell+2}pn^2$, $n,m\in\mathbf{N}$, and any compact set K in B, there is a constant $C>0$ such that*

$$\begin{aligned}&\|\sup\{\|(\varphi_{2n}P_{2t}(D^\ell\mathcal{L}^k((1-\varphi_{3n+2m+9})u)))(\cdot+h)\|_{H^{\otimes\ell}\otimes E};\\&\qquad h\in H,\ \|h\|_H\leq n\}\|_{L^p(B)}\\&\leq\|u\|_{\mathbf{D}_p^{-1}(E)},\qquad t\in(0,1],\ u\in\mathbf{D}_p^{-1}(E),\end{aligned}$$

and that

$$\begin{aligned}&\|\sup\{\|(\varphi_{2n}(P_{2t}(D^*)^\ell\mathcal{L}^k((1-\varphi_{3n+2m+9})u)))(\cdot+h)\|_E;\\&\qquad h\in H,\ \|h\|_H\leq n\}\|_{L^p(B)}\\&\leq\|u\|_{\mathbf{D}_p^{-1}(H^{\otimes\ell}\otimes E)},\qquad t\in(0,1],\ u\in\mathbf{D}_p^{-1}(H^{\otimes\ell}\otimes E).\end{aligned}$$

Proof. Since the proofs are similar, we prove only the first assertion. Let

$$\begin{aligned}g(z)=\sup\{&\|(\varphi_{2n}P_tD^\ell\mathcal{L}^kP_t((1-\varphi_{3n+2m+9})u))(z+h);\|_E;\\&h\in H,\ \|h\|_H\leq n\}\end{aligned}$$

Then by Lemma 2 (1) we have

$$g(z)\leq\exp(\frac{n^2}{2t(p-1)})\varphi_{3n+3}(z)P_t(\|D^\ell\mathcal{L}^kP_t((1-\varphi_{3n+2m+9})u))(\cdot)\|_E^p)(z)^{1/p}.$$

Note that

$$\begin{aligned}&\varphi_{3n+3}(z)P_t(\|D^\ell\mathcal{L}^kP_t((1-\varphi_{3n+2m+9})u))(\cdot)\|_E^p)(z)^{1/p}\\&\quad\leq 2\varphi_{3n+3}(z)P_t((1-\varphi_{3n+m+6})(\|D^\ell\mathcal{L}^kP_t((1-\varphi_{3n+2m+9})u))\|_E^p)(z)^{1/p}\\&\qquad+2P_t(\|\varphi_{3n+m+6}D^\ell\mathcal{L}^kP_t((1-\varphi_{3n+2m+9})u))\|_E^p)(z)^{1/p}\end{aligned}$$

Therefore by Lemma 10 we have

$$\begin{aligned}&\| g \|_{L^p(B)}\\ &\leq 2C\exp(\frac{n^2}{2t(p-1)})\\ &\times \exp(-\frac{\gamma m^2}{2t})(\| D^\ell \mathcal{L}^k P_t((1-\varphi_{3n+2m+9})u)) \|_{\mathbf{D}_p^0} + \| u \|_{\mathbf{D}_p^{-1}}).\end{aligned}$$

This completes the proof. □

Definition 12. Let M be a Polish space.
(1) We say that a measurable map $f : B \to M$ is a compact HC map, if $f(z+\cdot) : B_r \to M$ is continuous for any $z \in B$ and $r > 0$.
(2) We say that a measurable map $f : B \to M$ is a $\mathcal{CH}$ map, if there is a compact HC map $\tilde{f} : B \to M$ such that $f(z) = \tilde{f}(z)$ $\mu - a.e.z.$

Definition 13. Let M be a Polishi space.
(1) We say that a map $f : B \to M$ is H-regular, if f is measurable and if there is a compact set K in B with $\mu(K) > 0$ such that $f|_{K+B_r}$ is a continuous map from $K + B_r$ into M for any $r > 0$.
(2) Let N be a separable metric space. We say that a map $f : N \times B \to M$ is H-regular, if f is measurable and if there is a compact set K in B with $\mu(K) > 0$ such that $f|_{N\times(K+B_r)}$ is a continuous map from $N \times (K + B_r)$ into M for any $r > 0$.

Proposition 14. *Let M_n, $n \in \mathbf{N}$, be Polish spaces, and $f_n : B \to H$, $n \in \mathbf{N}$, be H-regular maps. Then there is a compact set K in B with $\mu(K) > 0$ such that $f_n|_{K+B_r}$ is a continuous map from $K+B_r$ into M_n for all $r > 0$ and $n \in \mathbf{N}$.*

Proof. For each $n \in \mathbf{N}$, there is a compact set K_n in B with $\mu(K_n) > 0$ such that $f_n|_{K_n+B_r}$ is a continuous map from $K_n + B_r$ into M_n for any $r > 0$. Because of H-ergordicity of μ we have $\mu(\bigcup_{m=1}^\infty (K_n + B_m)) = 1$. So there is an $m_n \geq 1$ such that $\mu(K_n + B_{m_n}) \geq 1 - 2^{-n-1}$. Letting $K = \bigcap_{n=1}^\infty (K_n + B_{m_n})$, we have our assertion. □

Proposition 15. *If $f : B \to M$ is a $\mathcal{CH}$-map, then there is an H-regular map $\tilde{f} : B \to M$ such that $f(z) = \tilde{f}(z)$, $\mu - a.e.z.$*

Proof. We may assume that f is a compact HC map. For any $n \geq 1$ let $f_n : B \to C(B_n; M)$ be given by $f_n(z)(h) = f(z+h)$, $z \in B, h \in B_n$. Here $C(B_n; M)$ denotes the Polish space consisting of continuous maps from B_n into M. Since f_n is measurable, there is a compact set K_n in B such that $\mu(K_n) \geq 1 - 3^{-n}$ and $f_n|_{K_n} : K_n \to C(B_n; M)$ is continuous. Then we see that f is a continuous map from $K_n + B_n$ into M. Letting $K = \bigcap_{n=1}^{\infty} K_n$ and $\tilde{f} = f$, we have our assertion. □

Definition 16. Let E be a separable real Hilbert space. $\mathcal{D}(\mathcal{L}; E)$ (resp. $\mathcal{D}(D; E)$, $\mathcal{D}(D^*; H \otimes E)$) is defined to be a set of measurable maps $u : B \to E$ (resp. $u : B \to E$, $u : B \to H \otimes E$) such that there are a compact set K in B with $\mu(K) > 0$ and a measurable map $v : B \to E$ (resp. $v : B \to H \otimes E$, $v : B \to E$) satisfying the following.
(1) For each $n \geq 1$ $\varphi_n^K u \in \mathbf{D}_{1+}^1(E)$, (resp. $\varphi_n^K u \in \mathbf{D}_{1+}^1(E)$, $\varphi_n^K u \in \mathbf{D}_{1+}^1(H \otimes E)$).
(2) For each $n \geq 1$ $\varphi_n^K v \in \mathbf{D}_{1+}^0(E)$, (resp. $\varphi_n^K v \in \mathbf{D}_{1+}^0(H \otimes E)$, $\varphi_n^K v \in \mathbf{D}_{1+}^0(E)$).
(3) $\varphi_n^K \mathcal{L}(\varphi_{n+2}^K u) = \varphi_n^K v$, (resp. $\varphi_n^K D(\varphi_{n+2}^K u) = \varphi_n^K v$, $\varphi_n^K D^*(\varphi_{n+2}^K u) = \varphi_n^K v$) in $\mathbf{D}_{1+}^{-1}(E)$, $n \geq 1$.

Proposition 17. *v in Definition 16 is uniquely determined μ-a.s.*

Proof. Let $u \in \mathcal{D}(\mathcal{L}; E)$, K^i, $i = 1,2$ are compact sets in B, and $v^i : B \to E$ are measurable maps such that $\varphi_n^{K^i} u \in \mathbf{D}_{p_{i,n}}^1(E)$, $\varphi_n^{K^i} v^i \in \mathbf{D}_{p_{i,n}}^0(E)$, and $\varphi_n^{K^i} \mathcal{L}(\varphi_{n+2}^{K^i} u) = \varphi_n^{K^i} v_i$, $i = 1,2$, $n \in \mathbf{N}$. We may assume that $\mu(K^i) \geq 2/3$.

Let $K = K^1 \cap K^2$. Then we have $\varphi_n^K u = \varphi_n^K(\varphi_{n+2}^{K^i} u)$. So we see that $\varphi_{n-2}^K v^i = \varphi_{n-2}^K \mathcal{L}(\varphi_n^K u) = \varphi_n^{K^i} v^i$, $n \geq 1$. So we have $v^1 = v^2$. The proof for D and D^* are similar. □

We denote the v in Definition 16 by $\mathcal{L}u$, Du, and D^*u, respectively.

Proposition 18. *$\mathcal{D}(\mathcal{L}; E) \subset \mathcal{D}(D; E)$ and $\mathcal{D}(\mathcal{L}; E) \subset \mathcal{D}(D^*; H \otimes E)$.*

Proof. Let $u \in \mathcal{D}(\mathcal{L}; E)$ and K is a compact set in B such that $\varphi_n^K u \in \mathbf{D}_{1+}^1(E)$, $n \in \mathbf{N}$. Then we have $\varphi_m^K D(\varphi_{n+2}^K u) = \varphi_m^K D(\varphi_{m+2}^K u)$ for $n \geq m \geq 1$. Thus there is a measurable map $v : B \to H \times E$ such that

$\varphi_n^K D(\varphi_{n+2}^K u) = \varphi_n^K v$. So $u \in \mathcal{D}(D; E)$. This proves the first assertion. The proof for the second assertion is similar. $\square$

PROPOSITION 19. *Let E be a separable Hilbert space.*
(1) *Let $u \in \mathcal{D}(\mathcal{L}; E)$ and suppose that $\mathcal{L}u \in \mathcal{D}(\mathcal{L}; E)$. Then $Du \in \mathcal{D}(\mathcal{L}; H \otimes E)$ and $D\mathcal{L}u = \mathcal{L}Du - Du$.*
(2) *Let $u \in \mathcal{D}(\mathcal{L}; H \otimes E)$ and suppose that $\mathcal{L}u \in \mathcal{D}(\mathcal{L}; H \otimes E)$. Then $D^*u \in \mathcal{D}(\mathcal{L}; E)$ and $D^*\mathcal{L}u = \mathcal{L}D^*u + D^*u$.*

PROOF. Since the proofs of the assertions (1) and (2) are similar, we prove only (1). Similarly to the proof of Proposition 14, we see that there is a compact set K in B with $\mu(K) > 0$ satisfying the following.

$$\varphi_n^K u, \varphi_n^K \mathcal{L}u \in \mathbf{D}_{1+}^1(E), \quad \varphi_n^K Du \in \mathbf{D}_{1+}^0(H \otimes E),$$

$$\begin{aligned}\varphi_n^K \mathcal{L}(\varphi_{n+2}^K u) &= \varphi_n^K \mathcal{L}u, \varphi_n^K \mathcal{L}(\varphi_{n+2}^K \mathcal{L}u) \\ &= \varphi_n^K \mathcal{L}u, \text{ and } \varphi_n^K D(\varphi_{n+2}^K u) = \varphi_n^K Du\end{aligned}$$

for all $n \geq 1$. Note that

$$\varphi_{n+8}^K u = e^{-1}P_1(\varphi_{n+8}^K u) + \int_0^1 e^{-t}P_t((I - \mathcal{L})(\varphi_{n+8}^K u))dt$$

and so we have

$$\begin{aligned}\varphi_n^K Du = e^{-1}\varphi_n^K DP_1(\varphi_{n+8}^K u) + \varphi_n^K D(\int_0^1 e^{-t}P_t(\varphi_{n+6}^K(u - \mathcal{L}u))dt) \\ + \int_0^1 e^{-t}\varphi_n^K DP_t((1 - \varphi_{n+6}^K)(I - \mathcal{L})(\varphi_{n+8}^K u))dt.\end{aligned}$$

Then by Lemma 10 and the fact that

$$\int_0^1 e^{-t}P_t dt = (I - \mathcal{L})^{-1}(I - e^{-1}P_1),$$

we see that $\varphi_n^K Du \in \mathbf{D}_{1+}^1(H \otimes E)$.

Also, we see that

$$\begin{aligned}\varphi_n^K \mathcal{L}P_t(\varphi_{n+6}^K Du) &= \varphi_n^K \mathcal{L}P_t D(\varphi_{n+8}^K u) - \varphi_n^K \mathcal{L}P_t(1 - \varphi_{n+6}^K)D(\varphi_{n+8}^K u) \\ &= e^t \varphi_n^K DP_t(\varphi_{n+6}^K(\mathcal{L}u - u)) \\ &\quad + e^t \varphi_n^K DP_t((1 - \varphi_{n+6}^K)(\mathcal{L} - I)(\varphi_{n+8}^K u)) \\ &\quad - \varphi_n^K \mathcal{L}P_t(1 - \varphi_{n+6}^K)D(\varphi_{n+8}^K u).\end{aligned}$$

Letting $t \to 0$, we see by Lemma 10 that

$$\varphi_n^K \mathcal{L}(\varphi_{n+6}^K Du) = \varphi_n^K D(\varphi_{n+6}^K(\mathcal{L}u + u) = \varphi_n^K D(\mathcal{L}u + u)$$

in $\mathbf{D}_{1+}^{-1}(H \otimes E)$. Since $\varphi_n^K \mathcal{L}(\varphi_{n+2}^K - \varphi_{n+6}^K) = 0$ as an operator from $\mathbf{D}_p^1$ to $\mathbf{D}_p^{-1}$, we have

$$\varphi_n^K \mathcal{L}(\varphi_{n+2}^K Du) = \varphi_n^K D(\mathcal{L}u + u) \text{ and } \varphi_n^K D(\mathcal{L}u + u) \in \mathbf{D}_{1+}^0.$$

So we have our assertion. □

Definition 20. Let E be a separable real Hilbert space. We say that $f : B \to E$ is a $\mathcal{CH}^1$ map, if $f \in \mathcal{D}(\mathcal{L}; E)$ and $f : B \to E$ and $\mathcal{L}f : B \to E$ are $\mathcal{CH}$ maps.

We say that $f : B \to E$ is a $\mathcal{CH}^n$ map, $n \geq 2$, if f is a $\mathcal{CH}^{n-1}$ map and $\mathcal{L}f : B \to E$ is a $\mathcal{CH}^{n-1}$ map. Also, we say that $f : B \to E$ is a $\mathcal{CH}^\infty$ map, if f is a $\mathcal{CH}^n$ map for all $n \geq 1$.

Proposition 21. *For any $u \in \mathbf{D}_p^s(E)$, $p \in (1, \infty)$, $s \in \mathbf{R}$, and $t > 0$, $P_t u : B \to E$ is a $\mathcal{CH}^\infty$ map.*

Proof. Since we have $\mathcal{L}^n P_t u = P_{t/2}(\mathcal{L}^n P_{t/2} u))$, it is sufficient to prove that $P_t f : B \to E$ is $\mathcal{CH}$ map for any $t > 0$ and $f \in \mathbf{D}_p^0(E)$. It is easy to see that $P_t f : B \to E$ is continuous if $f : B \to E$ is continuous and bounded. Since the set of bounded continuous functions is dense in $\mathbf{D}_p^0(E)$, we have our assertion by Lemma 2. □

Our main result in this section is the following.

Lemma 22. *Let E be a separable real Hilbert space.*
(1)*If $f : B \to E$ be a $\mathcal{CH}^1$ map, then $Df : B \to H \otimes E$ is a $\mathcal{CH}$ map.*
(2)*If $f : B \to H \otimes E$ be a $\mathcal{CH}^2$ map, then $D^* f : B \to E$ is a $\mathcal{CH}$ map.*

We have the following as an easy consequence of Lemma 22 and Proposition 19.

Theorem 23. *Let E be a separable real Hilbert space.*
(1)*If $f : B \to E$ be a $\mathcal{CH}^\infty$ map, then $Df : B \to H \otimes E$ is a $\mathcal{CH}^\infty$ map.*
(2)*If $f : B \to H \otimes E$ be a $\mathcal{CH}^\infty$ map, then $D^* f : B \to E$ is a $\mathcal{CH}^\infty$ map.*

Proof of Lemma 22. Since the proofs of the assertion (1) and (2) are similar and the proof of the assertion (2) is more delicate, we prove the assertion (2) only. Let $f : B \to H \otimes E$ be an $\mathcal{CH}^2$-map. By Proposition 15, we may assume that there is a compact subset K in B with $\mu(K) > 0$ satisfying the following.
(i) $f, \mathcal{L}f, \mathcal{L}^2 f$ are continuous on $K + B_n$,
(ii) $\varphi_n^K f, \varphi_n^K \mathcal{L}f \in \mathbf{D}_{1+}^1$,
and
(iii) $\varphi_n^K \mathcal{L}f = \varphi_n^K \mathcal{L}(\varphi_{n+2}^K f)$ and $\varphi_n^K \mathcal{L}^2 f = \varphi_n^K \mathcal{L}(\varphi_{n+2}^K \mathcal{L}f)$ for any $n \geq 1$.
Then we see that $\varphi_n^K f$, $\varphi_n^K \mathcal{L}f$ and $\varphi_n^K \mathcal{L}^2 f$ are bounded. Note that

$$\begin{aligned}\varphi_{2n}^K D^* f &= \varphi_{2n}^K D^*(\varphi_{3n+2m+8}^K f) \\ &= 4\int_0^\infty te^{-2t}\varphi_{2n}^K D^*(I-\mathcal{L})^2 P_{2t}(\varphi_{3n+2m+8}^K f)dt.\end{aligned}$$

Let

$$\begin{aligned}u_{n,m,k} &= 4\int_{2^{-k}}^\infty te^{-2t} D^*(I-\mathcal{L})^2 P_{2t}(\varphi_{3n+2m+8}^K f)dt \\ &= 4P_{2^{-k+1}}(\int_{2^{-k}}^\infty te^{-2t+2^{-k+2}} D^*(I-\mathcal{L})^2 P_{2t-2^{-k+1}}(\varphi_{3n+2m+8}^K f)dt).\end{aligned}$$

Then we see that $u_{n,m,k} : B \to E$, $k \in \mathbf{N}$, are $\mathcal{HC}^0$ maps and so we may assume that $u_{n,m,k}$ is compact HC map. Note that

$$\begin{aligned}&(I-\mathcal{L})^2 P_t(\varphi_{3n+2m+8}^K f) \\ &\quad = (I-\mathcal{L})P_t(\varphi_{3n+2m+6}^K (f-\mathcal{L}f)) \\ &\qquad + (I-\mathcal{L})P_t((1-\varphi_{3n+2m+6})(I-\mathcal{L})(\varphi_{3n+2m+8}^K f)) \\ &\quad = P_t(\varphi_{3n+2m+4}^K (f-2\mathcal{L}f+\mathcal{L}^2 f)) \\ &\qquad + P_t((1-\varphi_{3n+2m+4}^K)(I-\mathcal{L})(\varphi_{3n+2m+6}^K (f-\mathcal{L}f))) \\ &\qquad + (I-\mathcal{L})P_t((1-\varphi_{3n+2m+6}^K)(I-\mathcal{L})(\varphi_{3n+2m+8}^K f)).\end{aligned}$$

Let

$$\begin{aligned}v_{n,m}(z;t) = 4tP_{2t}D^*&((1-\varphi_{3n+2m+4}^K)(I-\mathcal{L})(\varphi_{3n+2m+6}^K (f-\mathcal{L}f))) \\ &+ 4tP_{2t}D^*((1-\varphi_{3n+2m+4}^K)(I-\mathcal{L})((1-\varphi_{3n+2m+6}^K) \\ &\times (I-\mathcal{L})(\varphi_{3n+4m+8}^K f)))(z).\end{aligned}$$

and

$$w_{n,m}(z;t) = 4te^{-t}P_tD^*P_t(\varphi_{3n+2m+4}\ K(f - 2\mathcal{L}f + \mathcal{L}^2 f))$$

Note that $(I-\mathcal{L})(\varphi^K_{3n+2m+6}(f-\mathcal{L}f))$ and $(I-\mathcal{L})(\varphi^K_{3n+4m+8}f)$ belongs to $\mathbf{D}_2^{-1}(H\otimes E)$. Then we have

$$\varphi^K_{2n}D^*f = \varphi^K_{2n}u_{n,m,k} + \int_0^{2^{-k}} \varphi^K_{2n}(v_{n,m}(\cdot,t) + w_{n,m}(\cdot,t))dt$$

Let

$$v^*_{n,m}(z;t) = \sup\{\| \varphi^K_{2n}v_{n,m}(z+h;t) \|_E; h \in B_n\},$$

and

$$w^*_{n,m}(z;t) = \sup\{\| w_{n,m}(z+h;t) \|_E; h \in B_n\}, \qquad z \in B,\ t > 0.$$

Since $\varphi_{3n+2m+4}(f - 2\mathcal{L}f + \mathcal{L}^2 f)$ is bounded, we have by Corollary 3

$$\int_0^{2^{-k}} w^*_n(z;t)dt \to 0, \qquad \mu - a.e.z,\ k \to \infty.$$

By Proposition 11, if $(m-3)^2 > 16n^2$, there is a $C_1 > 0$ such that

$$\| \varphi_{2n}(\cdot)v^*_n(\cdot;t) \|_{L^2} \le C_1, \qquad t \in (0,1].$$

Thus we see that

$$\varphi_{2n}(z)\int_0^{2^{-k}} v^*_n(z;t)dt \to 0, \qquad \mu - a.e.z,\ k \to \infty.$$

Let $m_n = 4n+4$. These imply that

$$\begin{aligned}&\varphi_{2n}(z)\sup\{|u_{n,m_n,k}(z+h) - u_{n,m_n,k'}(z+h)|;\\ &\qquad h \in H, \| h \|_H \le n\} \to 0, \quad k,k' \to \infty, \quad \mu - a.e.z,\end{aligned}$$

for any $n \in \mathbf{N}$. Let

$$\begin{aligned}A_n = \{z \in K + B_n;\ &\sup\{|u_{n,m_n,k}(z+h) - u_{n,m_n,k'}(z+h)|;\\ &h \in H, \| h \|_H \le n\} \to 0,\ k,k' \to \infty\},\end{aligned}$$

$n \in \mathbf{N}$. Then we see that $\mu(A_n) = \mu(K+B_n) \to 1$. Let K_n be a compact suset of A_n satisfying $\mu(A_n \setminus K_n) \le 2^{-n}$. Let $u_n : B \to E$ be given by

$u_n(z) = \lim_{k\to\infty} u_{n,m_n,k}(z)$, $z \in K_n + B_n$ and $u_n(z) = 0$, $z \in B \setminus (K_n + B_n)$ Then we see that $D^*f(z+h) = u_n(z+h)$, $\mu - a.e.z \in K_n$ for each $h \in B_n$, and that $u_n(z+\cdot) : B_n \to E$ is continuous for each $z \in K_n$.

Let us take a subsequence $\{n_k\}$ such that $\mu(K_{n_k}) \geq 1 - 2^{-k}$. Let $A = \bigcup_{j=1}^{\infty}(\bigcap_{k=j}^{\infty} K_{n_k})$, and V be a dense subset of H. Let

$$\begin{aligned}\tilde{A}_\ell = \{z \in A;\ \sup\{|u_{n_k}(z+h) - u_{n_{k'}}(z+h)|;\\ h \in V, \| h \|_H \leq \ell\} \to 0,\ k, k' \to \infty\}.\end{aligned}$$

Then we see that $\mu(\tilde{A}_\ell) = 1$. Let $\tilde{A} = \bigcap_{\ell=1}^{\infty} \tilde{A}_\ell$, and let $u : B \to E$ be given by $u(z) = \lim_{k\to\infty} u_{n_k}(z)$, $z \in \tilde{A}$ and $u(z) = 0$, $z \in B \setminus \tilde{A}$. Then we see that u is a compact HC-map and $D^*f(z) = u(z)$, $\mu - a.e.z$. So we see that D^*f is an $\mathcal{HC}$ map. This completes the proof. □

For separable real Hilbert spaces E_i, $i = 0, 1$, let $C^\infty(E_0; E_1)$ be the space of smooth maps from E_0 to E_1, i.e., $C^\infty(E_0; E_1)$ is the space of continuous maps $F : E_0 \to E_1$ such that for all $n \geq 1$, $e_0, e_1, \ldots, e_n \in E_0$, the map $(x_1, \ldots, x_n) \to F(e_0 + \sum_{k=1}^n x_k e_k)$ is a smooth map from $\mathbf{R}^n$ to E_1 and that there is a continuous map $F^{(n)}$ from E_0 to the space of continuous n-multilinear maps $M_n(E_0^n; E_1)$ for which

$$\frac{\partial}{\partial x_1 \cdots \partial x_n} F(e_0 + \sum_{k=1}^n x_i e_i)|_{x_1=\cdots=x_n=0} = F^{(n)}(e_0)(e_1, \ldots, e_n).$$

Let $F \in C^\infty(E_0, E_1)$, and $(e_0, \tilde{e}_1, \tilde{e}_2) \in E_1 \times \mathcal{L}^2(H; E_1)^2 = E_1 \times (H \otimes E_1)^2$. Then for any complete orthnormal basis $\{h_j\}_{j=1}^\infty$, we see that $\sum_{j=1}^\infty F^{(2)}(e_0)(\tilde{e}_1(h_j), \tilde{e}_2(h_j))$ converges in E_1 and does not depend on the choice of basis $\{h_j\}_{j=1}^\infty$. Thus, we can define $\langle F^{(2)} \rangle : E_0 \to M_2((H \otimes E_0)^2; E_1)$ to be the sum of this series. Then the map $(e_0, \tilde{e}_1, \tilde{e}_2) \to \langle F^{(2)} \rangle(e_0)(\tilde{e}_1, \tilde{e}_2)$ can be an element of $C^\infty(E_0 \oplus (H \otimes E_0) \oplus (H \otimes E_0); E_1)$.

THEOREM 24. *Let E_i, $i = 0, 1$ be separable real Hilbert spaces and $F \in C^\infty(E_0; E_1)$. If $u : B \to E_0$ is a $\mathcal{CH}^\infty$ map, then $F \circ u : B \to E_1$ is also a $\mathcal{CH}$ map and we have*

$$D(F \circ u)(z) = F^{(1)}(u(z))(Du(z)),$$

and

$$\mathcal{L}(F\circ u)(z)=F^{(1)}(u(z))(\mathcal{L}u(z))+\frac{1}{2}\langle F^{(2)}\rangle(u(z))(Du(z),Du(z)),\ \mu\text{-}a.e.z.$$

Proof. Let $u:B\to E_0$ is a $\mathcal{CH}^\infty$ map. Then it is obvious that $F\circ u:B\to E_1$ is a $\mathcal{CH}$ map. Let K be a compact set in B such that $\varphi_n^K u\in\mathbf{D}^1_{1+}(E_0)$, $\varphi_n^K u\in\mathbf{D}^1_{1+}(H\otimes E_0)$ and $\varphi_n^K D(\varphi_{n+2}^K u)=\varphi_n^K Du$, $n\in\mathbf{N}$. Then one can easily see that

$$\varphi_n^K D(\varphi_{n+2}^K(F\circ u))=\varphi_n^K F^{(1)}(\varphi_{n+2}^K u)(D(\varphi_{n+2}^K u)).$$

So we see that $F\circ u\in\mathcal{D}(D;E_1)$ and $D(F\circ u)=F^{(1)}\circ uDu$. Similarly we have $D(F\circ u)\in\mathcal{D}(D^*;H\otimes E_1)$ and

$$D^*D(F\circ u)=F^{(1)}\circ u(D^*Du)-\langle F^{(2)}\rangle\circ u(Du,Du).$$

(See the proof of [8] Theorem(1.9).)

By a little discussion we see that $F\circ u\in\mathcal{D}(\mathcal{L},E_1)$ and

$$\mathcal{L}(F\circ u)=(F^{(1)}\circ u)(\mathcal{L}u)+\frac{1}{2}(\langle F^{(2)}\rangle\circ u)(Du,Du).\tag{5}$$

So we see that $F\circ u:B\to E_1$ is a $\mathcal{CH}^1$ map. By Equation (5) and Theorem 23, we see that $F\circ u:B\to E_1$ is a $\mathcal{CH}^n$ map, $n\geq 1$, inductively.

This completes the proof. □

4. Continuity of Stochastic Processes

Let M be a totally bounded metric space with a metric function d_M. For any $t>0$, $N(t;M,d_M)$ denote the minimum of cardinals of t-d_M nets of M. Let us define $\varepsilon(M,d_M)$ to be

$$\varepsilon(M,d_M)=\limsup_{t\downarrow 0}\frac{\log\log N(t;M,d_M)}{\log(1/t)}.$$

We call $\varepsilon(M,d_M)$ an ε-entropy of the metric space M.

The following is somehow well-known, but we give a proof.

LEMMA 25. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* E *be a separable Banch space and* $X : M \times \Omega \to E$ *be a measurable map. Suppose that* $\alpha > \varepsilon(M, d_M)$, *and that there are* $\gamma > 0$ *and* $C < \infty$ *such that*

$$\sup_{x,y \in M, x \neq y} E^P[\exp(\gamma \frac{\| X(x) - X(y) \|_E^2}{d_M(x,y)^\alpha})] \leq C.$$

Then there is a sequence $\{c_n\}_{n=1}^\infty$ *of positive numbers depending only on the metric space* (M, d_M) *and* α, γ, C *such that* $c_n \to 0, n \to \infty$, *and that*

$$E[\sup_{x,y \in A} \{\| X(x) - X(y) \|_E;\ d_M(x,y) \leq 2^{-n}\}] \leq c_n, \qquad n \geq 1$$

for any countable subset A *of* M.

PROOF. Let $N_n = N(2^{-n}; M, d_M)$, $n = 1, 2, \ldots$ We see that

$$P(\| X(x) - X(y) \|_E > t) \leq C \exp(-\frac{\gamma t^2}{d_M(x,y)^\alpha}), \qquad t > 0,\ x, y \in M.$$

Let $B(x, r) = \{y \in M;\ d_M(y, x) < r\}$, $x \in M$, $r > 0$. Then there are $x_{n,k}$, $n \geq 1$, $k = 1, 2, \ldots, N_n$, such that

$$\bigcup_{k=1}^{N_n} B(x_{n,k}, 2^{-n}) = M.$$

Let Z_n, $n \geq 1$, be given by

$$Z_n = \max_{k=1,\ldots,N_n} \max_{\ell=1,\ldots,N_{n+1}} \{ \| X(x_{n,k}) - X(x_{n+1,\ell}) \|_E; \\ d_M(x_{n,k}, x_{n+1,\ell}) \leq 2^{-(n-1)}\}$$

Then we have

$$P(Z_n > t) \leq N_n N_{n+1} C \exp(-\gamma t^2 2^{\alpha(n-1)}).$$

Then we have

$$E[Z_n] \leq \frac{1}{n^2} + \int_{1/n^2}^\infty P(Z_n > t) dt \\ \leq \frac{1}{n^2} + N_n N_{n+1} C C_0 \exp(-\gamma \frac{2^{\alpha(n-1)}}{n^4}),$$

where

$$C_0 = \int_0^\infty \exp(-\gamma t^2)dt.$$

Let Z'_n, $n \geq 1$, be given by

$$Z'_n = \max_{k,\ell=1,\dots,N_n} \{\| X(x_{n,k}) - X(x_{n,\ell}) \|_E;\ d_M(x_{n,k}, x_{n,\ell}) \leq 2^{-(n-2)}\}$$

Then we have

$$E[Z'_n] \leq \frac{1}{n^2} + N_n^2 C C_1 \exp(-\gamma \frac{2^{\alpha(n-2)}}{n^4}),$$

where

$$C_1 = \int_0^\infty \exp(-\gamma 2^{-\alpha} t^2/2)dt.$$

One can easily see that for any countable subset A of B

$$\begin{aligned}
&\sup_{x,y \in A} \{\| X(x) - X(y) \|_E;\ d_M(x,y) \leq 2^{-n}\} \\
&\quad \leq \sup\{\| X(x) - X(y) \|_E; \\
&\qquad\qquad x, y \in \bigcup_{m=n}^\infty \{x_{m,k};\ k = 1, \dots, N_m\}, d_M(x,y) \leq 2^{-n}\} \\
&\quad \leq Z'_n + 2\sum_{k=n}^\infty Z_k, \quad a.s. \qquad n \geq 1.
\end{aligned}$$

So we have our assertion. □

The following is an easy consequence of the previous Lemma.

THEOREM 26. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, E be a separable Banach space and $X : M \times \Omega \to E$ be a measurable map. Suppose that $\alpha > \varepsilon(M, d_M)$, and there are $\gamma, C > 0$ such that*

$$\sup_{x,y \in M, x \neq y} E^P[\exp(\gamma \frac{\| X(x) - X(y) \|_E^2}{d_M(x,y)^\alpha})] < \infty,$$

Then there is a measurable map $\tilde{X} : M \times \Omega \to E$ satisfying the following.
(1) *$\tilde{X}(\cdot, \omega) : M \to E$ is continuous for all $\omega \in \Omega$.*
(2) *$P(\tilde{X}(x) = X(x)) = 1$, for all $x \in M$.*

Also, we have the following.

THEOREM 27. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* $X : M \times \Omega \to [0, \infty)$ *be a measurable map such that* $X(\cdot, \omega) : M \to [0, \infty)$ *is lower semi-continuous. Let* $\alpha > \varepsilon(M, d_M)$*. If there is a* $\gamma > 0$ *such that*

$$\sup_{x,y \in M, x \neq y} E^P[\exp(\gamma \frac{|X(x) - X(y)|^2}{d_M(x,y)^\alpha})] < \infty,$$

then

$$P(\sup_{x \in M} X(x) < \infty) = 1.$$

PROOF. Let A be a countable dense subset of M. Then we see that $\sup_{x \in M} X(x) = \sup_{x \in A} X(x)$. Then we have our assertion from Lemma 25. □

LEMMA 28. *Let* H_0, H_1 *be separable real Hilbert spaces such that* H_0 *is densely continuously embedded in* H_1*. Let* $U = \{h \in H_0; \| h \|_{H_0} \leq 1\}$*. Assume that* $\varepsilon(U; \| \cdot \|_{H_1}) < 2$*. Then the inclusion map* $i : H_0 \to H_1$ *is a Hilbert-Schmidt operator.*

PROOF. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X(h)$, $h \in H_0$ be mean zero Gaussian system of random variables such that $E[X(h)X(h')] = (i(h), i(h'))_{H_1}$, $h, h' \in H_0$. Then we see that

$$E[\exp(\frac{|X(h) - X(h')|^2}{4 \| h - h' \|_{H_1}^2})] \leq 2, \qquad h, h' \in H_0$$

Since $\varepsilon(nU; \| \cdot \|_{H_1}) < 2$ for any $n \in \mathbf{N}$, by Theorem 26 we see that there is a $\tilde{X} : H_0 \times \Omega \to \mathbf{R}$ such that $P(\tilde{X}(h) = X(h)) = 1$, $h \in H_0$, and that $\tilde{X}(\cdot, \omega) : H_0 \to \mathbf{R}$ is continuous for all $\omega \in \Omega$. One can easily check that

$$P(\tilde{X}(ah + bh') = a\tilde{X}(h) + b\tilde{X}(h') \text{ for all } h, h' \in H_0,\ a, b \in \mathbf{R}) = 1.$$

So we may regard $\tilde{X}$ as a H_0^*-valued random variable. Let ν be a probability law in H_0^* of $\tilde{X}$. Then (ν, H_1^*, H_0^*) is an abstract Wiener space. Therefore $i^* : H_1^* \to H_0^*$ is a Hilbert-Schmidt operator. This implies our assertion. □

Lemma 29. *Let H_0, E be separable real Hilbert spaces and B_0 be a Banach space such that H_0 is densely continuously embedded in B_0. Let $T : B_0 \to E$ be a bounded linear operator. Let $U = \{h \in H_0;\ \| h \|_{H_0} \leq 1\}$ and assume that $\varepsilon(U; \| \cdot \|_{B_0}) < 2$. Then $T|_{H_0} : H_0 \to E$ is a Hilbert-Schmidt operator.*

Proof. Let H_0' be the orthogonal subspace of $H \cap \ker T$ in H_0. Define an inner product $(\cdot,\cdot)_1$ on H_0' by $(h_1, h_2)_1 = (Th_1, Th_2)_E$, $h_1, h_2 \in H_0'$. Then this inner product is closable. Let H_1 be the completion of H_0' with respect to the inner product $(\cdot,\cdot)_1$. Then we see that $\| h \|_{H_1} \leq \| Th \|_E \leq C \| h \|_{B_0}$, $h \in H_0'$ for some constant $C > 0$. Therefore we have $\varepsilon(U \cap H_0', \| \cdot \|_{H_1}) < 2$. So we see that the inclusion map $i : H_0' \to H_1$ is Hilbert-Schmidt type. Let $\{e_n'\}$ and $\{e_n\}$ be complete orthonormal bases of H_0' and $H_0 \cap \ker T$. Then we see that

$$\sum_n \| Te_n' \|_E^2 + \sum_n \| Te_n \|_E^2 = \sum_n \| e_n' \|_{H_1}^2 < \infty.$$

So we have our assertion. □

5. Continuity of Stochastic Extension

Remind that $\mathcal{L}^\infty(H; H)$ is the Banach space consisting of bounded linear operators in H with an operator norm $\| \cdot \|_{op}$. Let $\{P_n\}_{n=1}^\infty$ be a sequence of orthogonal projections such that the image of P_n, $n \geq 1$, is a finite dimensional vector subspace in B^* and $P_n \uparrow I_H$, strongly as $n \to \infty$. Then we can extend the operator P_n to a bounded linear operator $\tilde{P}_n$ from B into H. We may assume that

$$\int_B \| z - \tilde{P}_n z \|_B^2 \,\mu(dz) \leq 2^{-n}, \qquad n = 1, 2, \dots .$$

Let $A \in \mathcal{L}^\infty(H; H)$. Then by [7] Theorem(1.14), we see that there is a measurable map $\tilde{A} : B \to B$ such that

$$\| \tilde{A}(z) - A\tilde{P}_n z \|_B \to 0, \qquad \mu - a.s.z$$

By the argument [7] Lemma(1.11), Theorems (1.13) and (1.14), we have the following.

THEOREM 30. *There is a $\gamma > 0$ such that*

$$\sup\{\int_B \exp(\gamma \parallel \tilde{A}(z) \parallel_B^2)\mu(dz);\ A \in \mathcal{L}^\infty(H;H),\ \parallel A \parallel_{op} \leq 1\} < \infty.$$

Moreover, we have the following.

THEOREM 31. *Let M be a compact subset in $\mathcal{L}^\infty(H;H)$, and assume that $\varepsilon(M, \|\cdot\|_{op}) < 2$. Then there is a measurable mapping $\Psi : M \times B \to B$ and a compact set K in B satisfying the following.*
(1) $\Psi(A, z) = \tilde{A}(z)$, $\mu - a.e.z$, for any $A \in M$.
(2) $\Psi(\cdot, z + \cdot) : M \times H \to B$ is continuous for all $z \in B$.
(3) $\mu(K) > 0$ and $\Psi(A, z + h) = \Psi(A, z) + Ah$ for all $A \in M$, $z \in K$ and $h \in H$.
(4) $\Psi(\cdot, \cdot) : M \times (K + B_r) \to B$ is continuous for all $r > 0$.

PROOF. Let M_0 be a countable dense subset in M. Let Ω_0 be the set of $z \in B$ such that $\{A\tilde{P}_n z\}_{n=1}^\infty$ is a Cauchy sequence in B for all $A \in M_0$. Then we see that $\mu(\Omega_0) = 1$ and $\Omega_0 + H = \Omega_0$.

Let $F : M_0 \times \Omega_0 \to B$ be given by

$$F(A, z) = \lim_{n\to\infty} A\tilde{P}_n z, \qquad A \in M_0,\ z \in \Omega_0.$$

By Theorem 30 we see that there is a $\gamma > 0$ such that

$$\sup\{\int_B \exp(\gamma \frac{\parallel \tilde{A}(z) - \tilde{A}'(z) \parallel_B^2}{\parallel A - A' \parallel_{op}^2})\mu(dz);\ A, A' \in M\} < \infty.$$

So there is a measurable map $X : M \times B \to B$ such that $X(A, z) = \tilde{A}(z)$, $\mu - a.e.z$, for any $A \in M$, and $X(\cdot, z) : M \to B$ is continuous. Let Ω_1 be the set of $z \in \Omega_0$ such that $F(A, z) = X(A, z)$ for all $A \in M_0$. Then $\mu(\Omega_1) = 1$ and $F(\cdot, z) : M_0 \to B$ is uniformly continuous for all $z \in \Omega_1$. Let Ω_2 be a σ-compact subset of Ω_1 such that $\mu(\Omega_2) = 1$. Since $F(A, z+h) = F(A, z)+Ah$, $z \in \Omega_2$, $h \in H$, we see that $F(\cdot, z) : M_0 \to B$ is uniformly continuous for all $z \in \Omega_2 + H$. So we can extend $F(\cdot, z)$ to be a continuous map from M to B for all $z \in \Omega_2 + H$. Let $\Psi : M \times B \to B$ by $\Psi(A, z) = F(A, z)$, $A \in M$, $z \in \Omega_2 + H$, and $\Psi(A, z) = 0$, $A \in M$, $z \in B \setminus (\Omega_2 + H)$. Then Ψ can be regarded as a measurable map from B into $C(M; B)$. Let K be a compact set in B such that $\mu(K) > 0$, $K \subset \Omega_2$, and Ψ is a continuous map from K into $C(M; B)$. Then we see that the assertions (1), (2) and (3) hold. Also, by the compactness of B_r in B and the assertion (3), we see that $\Psi : M \times (K + B_r) \to B$ is continuous.

This completes the proof. □

6. Rotation

Remind that $O(H)$ denotes the set of linear isomorphism in H. Let $\mathcal{A}(H)$ denote the set of anti-symmetric bounded linear operators in H, that is, $A \in \mathcal{A}(H)$ if A is a bounded linear operator in H satisfying $A^* = -A$.

Proposition 32. *Let $U \in O(H)$. Then the probability law of $\tilde{U}(z)$ under $\mu(dz)$ is μ.*

Proof. Note that for any $\xi \in B^*$

$$\int_B \exp(\sqrt{-1}{}_B\langle \tilde{U}(z), \xi\rangle_{B^*})\mu(dz)$$
$$= \lim_{n\to\infty} \int_B \exp(\sqrt{-1}(\tilde{P}_n z, U^*\xi)_H)\mu(dz) = \exp(-\frac{1}{2} \parallel \xi \parallel_H^2).$$

Thus we have our proposition. □

Let $U \in O(H)$ and E be a separable real Hilbert space. Then by Proposition 32, we can define an isometric linear operator T_U in $\mathbf{D}_p^s(E)$, $p \in (1, \infty)$, $s \in \mathbf{R}$, by

$$(T_U u)(z) = u(\tilde{U} z), \qquad z \in B.$$

Proposition 33. *Let E be a separable real Hilbert space.*
(1) For any $U \in O(H)$,

$$P_t T_U = T_U P_t \text{ and } (1 - \mathcal{L})^{-s} T_U = T_U (1 - \mathcal{L})^{-s}$$

in $L^p(B; E, d\mu)$, $p \in (1, \infty)$, $t, s > 0$.
(2) Let $U_n \in O(H)$, $n \geq 1$, and $U \in O(H)$, and assume that $U_n \to U$ strongly in H as $n \to \infty$. Then $T_{U_n} \to T_U$ strongly in $\mathbf{D}_p^s(E)$ as $n \to \infty$, $p \in (1, \infty)$, $s \in \mathbf{R}$.

PROOF. The assertion (1) follows from the following computation.

$$\begin{aligned} P_t(T_U f)(z) &= \int_B f(\tilde{U}(e^{-t}z + (1-e^{-2t})^{1/2}w))\mu(dw) \\ &= \int_B f(e^{-t}\tilde{U}(z) + (1-e^{-2t})^{1/2}w)\mu(dw) = T_U(P_t f)(z). \end{aligned}$$

To prove the assertion (2), it suffices us to prove the case that $s = 0$ because of the assertion (1). Let $\mathcal{V}$ be the set of E-valued functions u such that there is an $m \geq 1$ and a bounded continuous function $f : B \to E$ such that $u(z) = f(\tilde{P}_m z)$, $z \in B$. Then $\mathcal{V}$ is dense in $L^p(B; E, d\mu)$, $p \in (1, \infty)$. Since we have

$$\int_B |(\xi, \tilde{U}(z)) - (\xi, \tilde{U}_n(z))|^2 \mu(dz) = \| U^*\xi - U_n^*\xi \|_H^2 \to 0, \quad n \to \infty, \quad \xi \in B^*,$$

we see that $T_{U_n}u$ converges to $T_U u$ in probability and so in L^p for any $u \in \mathcal{V}$. Since T_U is isometric, we have our assertion (2). □

PROPOSITION 34. *Let E be a separable real Hilbert space and let $u \in \mathbf{D}_p^2(E)$ for some $p \in (1, \infty)$. Also let $A \in \mathcal{A}(H)$. Then we have*

$$(T_{e^A}u)(z) = u(z) - \int_0^1 (T_{e^{tA}}(D^*(ADu)))(z)dt, \qquad \mu - a.e.z.$$

PROOF. Let $A \in \mathcal{A}$. Let $m \geq 1$ and V_m be the image of P_m. Let $f : V_m \to E$ be a bounded smooth function with bounded derivatives of any order, and $u : B \to E$ be given by $u(z) = f(\tilde{P}_m z)$, $z \in B$.

Let $A_n = P_n A P_n$, $n \geq m$. Then we have

$$(T_{e^{A_n}}u)(z) = f(P_m e^{A_n}\tilde{P}_n z), \qquad \mu - a.e.z.$$

Also, we see that

$$\begin{aligned} \frac{d}{dt} f(P_m e^{tA_n}\tilde{P}_n z) &= (P_m A_n e^{tA_n}\tilde{P}_n z, \nabla f(P_m e^{tA_n}\tilde{P}_n z))_H \\ &= -{}_B\langle (e^{tA_n})\ z, A_n \nabla f(P_m e^{tA_n}\tilde{P}_n z)\rangle_{B^*} \\ &= -(T_{e^{tA_n}}(D^*(A_n Du)))(z), \end{aligned}$$

since $Du(z) = \nabla f(P_m z)$ and $\mathrm{trace}(D(A_n Du)(z)) = \mathrm{trace}(A_n D^2 u(z)) = 0$. So we have

$$(6) \qquad (T_{e^{A_n}} u)(z) = u(z) - \int_0^1 (T_{e^{tA_n}}(D^*(A_n Du)))(z)dt, \qquad \mu - a.e.z.$$

Then we see that Equation (6) hold for all $u \in \mathbf{D}_p^2$. Since $A_n \to A$ strongly in H as $n \to \infty$, we have our assertion from Proposition 33(2) by letting $n \to \infty$ in Equation (6). □

PROPOSITION 35. *There are $\gamma > 0$ and $C > 0$ satisfying the following. If $\varphi : B \to \mathbf{R}$ is a measurable function satisfying*

$$|\varphi(z+h) - \varphi(z)| \leq \| h \|_H, \qquad z \in B, h \in H,$$

then

$$\int_B \exp(\gamma t^{-1} |\varphi(z) - (P_t\varphi)(z)|^2)\mu(dz) \leq C, \qquad t \in (0,1].$$

PROOF. Note that

$$\frac{d}{dt} P_t \varphi = -D^* D P_t \varphi = -e^{-t} D^* P_t(D\varphi).$$

Note that

$$1 - e^{-t} = \int_0^t e^{-s} ds \geq \frac{t}{4}, \qquad t \in [0,1].$$

So we have

$$\begin{aligned} &t^{-1/2} |\varphi - P_t \varphi| \\ &\quad \leq 4(2t^{1/2})^{-1} \Big(\int_0^t s^{-1/2} (1 - e^{-s})^{1/2} |D^* P_s(D\varphi)| ds \Big), \qquad t \in (0,1]. \end{aligned}$$

Observing $\| D\varphi \|_H \leq 1$, we have

$$\begin{aligned} &\int_B \exp(ct^{-1} |\varphi - P_t \varphi|^2) d\mu \\ &\quad \leq (2t^{1/2})^{-1} \Big(\int_0^t s^{-1/2} ds \int_B \exp(16c(1 - e^{-s}) |D^* P_s(D\varphi)|^2) d\mu \Big), \\ &\quad\ t \in (0,1]. \end{aligned}$$

So by Lemma 1 we have our assertion. □

LEMMA 36. *There are $\gamma > 0$ and $C > 0$ satisfying the following. If $\varphi : B \to \mathbf{R}$ is a measurable function satisfying*

$$|\varphi(z+h) - \varphi(z)| \leq \| h \|_H, \qquad z \in B, h \in H,$$

then

$$\int_B \exp(\gamma \parallel A \parallel_{op}^{-1} |\varphi(\tilde{e^A}(z)) - \varphi(z)|^2)|)\mu(dz) \leq C$$

for any $A \in \mathcal{A}(H)$ with $A \neq 0$ and $\| A \|_{op} \leq 1$.

PROOF. Let $A \in \mathcal{A}(H)$ such that $A \neq 0$ and $\| A \|_{op} \leq 1$, and let $t = \| A \|_{op}$. Then we have

$$\begin{aligned} &|\varphi(\tilde{e^A}(z)) - \varphi(z)| \\ &\quad \leq |\varphi(\tilde{e^A}(z)) - (P_t\varphi)(\tilde{e^A}(z))| + |\varphi(z) - P_t\varphi(z)| \\ &\qquad + |(P_t\varphi)(\tilde{e^A}(z)) - P_t\varphi(z)| \end{aligned}$$

and

$$|(P_t\varphi)(\tilde{e^A}(z)) - P_t\varphi(z)| \leq \int_0^1 t|D^*P_t(t^{-1}AD\varphi)(\tilde{e^{sA}}(z))|ds$$

So we have

$$\begin{aligned} &\int_B \exp(ct^{-1}|\varphi(\tilde{e^A}(z)) - \varphi(z)|^2)|)\mu(dz) \\ &\leq 2\int_B \exp(3ct^{-1}|\varphi - P_t\varphi|^2))\mu(dz) + \int_B \exp(3ct|D^*P_t(t^{-1}AD\varphi)|^2)\mu(dz) \end{aligned}$$

Since $\| t^{-1}AD\varphi \|_H \leq 1$, we have our assetion from Lemma 1(2) and Proposition 35. □

By [7]Theorem(4.9) and Proposition 32, we have the following.

PROPOSITION 37. *There are $\gamma > 0$ and $C > 0$ satisfying the following. If $\varphi : B \to \mathbf{R}$ is a measurable function satisfying*

$$|\varphi(z+h) - \varphi(z)| \leq \| h \|_H, \qquad z \in B, h \in H,$$

then

$$\int_B \exp(\gamma|\varphi(\tilde{U}(z)) - \varphi(z)|^2)\mu(dz) \le C$$

for any $U \in O(H)$.

By Lemma 36 and Proposition 37, we have the following.

THEOREM 38. *There is a* $\gamma > 0$ *satisfying the following. If* $\varphi : B \to \mathbf{R}$ *is a measurable function satisfying*

$$|\varphi(z+h) - \varphi(z)| \le \| h \|_H, \qquad z \in B, h \in H,$$

then

$$\sup\{\int_B \exp(\gamma \| U - I_H \|_{op}^{-1} |\varphi(\tilde{U}(z)) - \varphi(z)|^2)\mu(dz); \\ U \in O(H), U \neq I_H\} < \infty.$$

COROLLARY 39. *Let* K *be a compact set in* B *with* $\mu(K) > 0$. *Let* M *be a totally bounded subset of* $O(H)$ *such that* $\varepsilon(M; \| \cdot \|_{op}) < 1$. *Then*

$$\sup\{\rho(\Psi(U,z);K);\ U \in M\} < \infty, \quad \mu - a.s.z.$$

Here Ψ *is as in Theorem 31.*

PROOF. Since $\rho(\cdot;K) : B \to [0,\infty]$ is lower semi-continuous, we see that $\rho(\Psi(\cdot,z);K) : M \to [0,\infty]$ is lower semi-continuous. We also see that

$$|\rho(z+h;K) - \rho(z;K)| \le \| h \|_H .$$

So by Theorem 38 we see that there is a $\gamma > 0$ such that

$$\sup\{\int_B \exp(\gamma \| U_1 - U_0 \|_{op}^{-1} |\rho(\Psi(U_1;z);K) - \rho(\Psi(U_0;z);K)|^2)\mu(dz); \\ U_0, U_1 \in M, U_1 \neq U_0\} < \infty.$$

Then by Theorem 27, we have our assertion. □

THEOREM 40. *Let* M *be a totally bounded subset of* $O(H)$ *such that* $\varepsilon(M; \| \cdot \|_{op}) < 1$. *Let* N *be a Polish space and* $f : B \to N$ *be a* $\mathcal{CH}$ *map.*

Then there is a H-regular map $\tilde{f} : M \times B \to N$ such that $\tilde{f}(U, z) = f(\tilde{U}(z)))$ $\mu - a.e.z$ for all $U \in M$.

Proof. Since $f : B \to N$ is $\mathcal{CH}$ map, we may assume that there is a compact set K' in B such that $\mu(K') > 0$ and $f : K' + B_r \to N$ is continuous for all $r > 0$. Also by Theorem 27 we may assume that $\Psi : M \times (K' + B_r) \to B$ is continuous for all $r > 0$. Then by Corollary 39 we have $\sup\{\rho(\Psi(U, z); K'); U \in M\} < \infty$ $\mu - a.e.z$. Let $R > 0$ and K be a compact subset of K' such that $\mu(K) > 0$ and $\sup\{\rho(\Psi(U, z); K'); U \in M, z \in K\} \leq R$. Then we see that $\Psi(U, z) \in K' + B_{R+r}$ for any $U \in M$, $z \in K + B_r$ and $r > 0$. Let $\tilde{f}(U, z) = f(\Psi(U, z))$, $U \in M$, $z \in B$. Then we see that $\tilde{f} : M \times (K + B_r) \to N$ is continuous for all $r > 0$.

This completes the proof. □

7. Polish Subgroups

Definition 41. We say that G is a Polish subgroup of $O(H)$, if the following are satisfied.
(1) G is a subgroup of $O(H)$.
(2) G has a metric function d_G such that (G, d_G) is a Polish space, the inclusion map from G into $O(H)$ is continuous, and

$$d_G(g_1, g_2) = d_G(I_H, g_1^{-1} g_2), \qquad g_1, g_2 \in G.$$

For a Polish subgroup G of $O(H)$, we define $\varepsilon(G)$ by

$$\varepsilon(G) = \lim_{\delta \downarrow 0} \varepsilon(\{g \in G;\ d_G(I_H, g) \leq \delta\}, \| \cdot \|_{op}).$$

Definition 42. We say that G is a Hilbert-Lie subgroup of $O(H)$, if G is a Polish subgroup of $O(H)$ and if there is a Hilbert space $\mathcal{G}$ satisfying the following.
(1) $\mathcal{G}$ is continuously embedded in $\mathcal{A}(H)$ as a vector space.
(2) There are neighborhood U_0 of 0 in $\mathcal{G}$ and a neighborhood U_1 of I_H in G such that
(i) $\psi : U_0 \to U_1$ given by $\psi(A) = \exp(A)$, $A \in U_0 \subset \mathcal{A}$, is a homeomorphism and that

(ii) for any $g \in G$ the map $\psi^{-1}(g\psi(\cdot)) : \psi^{-1}(U_1 \cap gU_1) \to \mathcal{G}$ is smooth.
We call $\mathcal{G}$ the Lie algebra of G.

Remark. A Hilbert-Lie subgroup G of $O(H)$ is a C^∞ Hilbert manifold (c.f.[12]).

Theorem 43. *Let G be a Polish subgroup of $O(H)$ such that $\varepsilon(G) < 2$. Then there is a measurable mapping $\Psi : G \times B \to B$ and a compact set K in B satisfying the following.*
(1) $\Psi(g, z) = \tilde{g}(z)$, $\mu - a.e.z$, for any $g \in G$.
(2) $\Psi(\cdot, z + \cdot) : G \times H \to B$ is continuous for all $z \in B$.
(3) $\mu(K) > 0$ and $\Psi(g, z + h) = \Psi(g, z) + gh$ for all $g \in G$ and $z \in K$.
(4) $\Psi(\cdot, \cdot) : G \times (K + B_r) \to B$ is continuous for all $r > 0$.

Proof. By the definition of $\varepsilon(G)$, we see that there is a $\delta > 0$ such that $\varepsilon(U, \|\cdot\|_{op}) < 2$, where $U = \{g \in G;\ d_G(I_H, g) \le \delta\}$. Let $\{g_n\}_{n=1}^\infty$ be a dense subset of G. It is easy to see that $\varepsilon(g_n U, \|\cdot\|_{op}) = \varepsilon(U, \|\cdot\|_{op}) < 2$. So by Theorem 31 there are $\Phi_n : (g_n U) \times B \to B$ and compact sets K_n in B, $n = 1, 2 \dots$, satisfying the following.
(1) $\Psi_n(g, z) = \tilde{g}(z)$, $\mu - a.e.z$, for any $g \in g_n U$,
(2) $\mu(K_n) > 0$ and $\Psi_n(g, z + h) = \Psi(g, z) + gh$ for all $g \in g_n U$ and $z \in K$.
(3) $\Psi(\cdot, \cdot) : (g_n U) \times (K_n + B_r) \to B$ is continuous for all $r > 0$.
We may assume that $\mu(K_n) \ge 1 - 2^{-(n+1)}$, $n = 1, 2, \dots$. Let $A_{n,m}$, $n, m = 1, 2, \dots$, be the set of $z \in B$ such that $\Psi_n(g_k, z) = \Psi_m(g_k, z)$ for all $g_k \in (g_n U) \cap (g_m U)$. Then we see that $\mu(A_{n,m}) = 1$.

Let $A = (\bigcap_{n,m=1}^\infty A_{n,m}) \cap (\bigcap_{n=1}^\infty K_n)$. Then we have $\mu(A) > 0$. Also we see the following.
$\Psi_n(g, z) = \Psi_m(g, z)$ for $g \in (g_n U) \cap (g_m U)$, $z \in A$, and $n, m = 1, 2, \dots$,
$\Psi_n(g, z + h) = \Psi_n(g, z) + gh$ for $g \in (g_n U)$, $z \in A$, and $n = 1, 2, \dots$, and
$\Psi_n(\cdot, z + *) : (g_n U) \times H \to B$ is continuous for $z \in A$.
Let K be a compact set in B such that $\mu(K) > 0$ and $K \subset A$. Let $\Psi : G \times B \to B$ be given by $\Psi(g, z) = 0$, if $g \in G$ and $z \in B \setminus (K + H)$, and $\Psi(g, z) = \Psi_n(g, z)$, if $g \in (g_n U)$, and $z \in K + H$. Then we have our assetion. □

We have the following by a similar proof of Theorems 40 and 43.

THEOREM 44. *Let G_1 be a Polish subgroup of $O(H)$ such that $\varepsilon(G_1) < 2$ and a measurable mapping $\Psi : G_1 \times B \to B$ be as in Theorem 43. Let G_0 be a Hilbert-Lie subgroup of $O(H)$ such that G_0 is continuously imbedded in G_1 and that $\varepsilon(G_0) < 1$. Let M be a Polish space and $f : B \to M$ be an H-regular map. Then for any $g_1 \in G_1$ the map $(g, z) \to f(\Psi(gg_1, z))$ is an H-regular map from $G_0 \times B$ to M.*

LEMMA 45. *Let G be a Hilbert-Lie subgroup of $O(H)$ with its Lie algebra $\mathcal{G}$ such that $\varepsilon(G) < 1$. Let E be a separable real Hilbert space. Then we have the following.*
(1) There is a bounded linear operator $T : H \otimes E \to H \otimes \mathcal{G}^* \otimes E \cong \mathcal{L}^2(\mathcal{G}; H \otimes E)$ such that $T(h \otimes u)(A) = (Ah) \otimes u$, $h \in H$, $u \in E$, and $A \in \mathcal{G} \subset \mathcal{A}(H)$.
(2) For any $\mathcal{CH}^\infty$ map $f : B \to E$ there is an $\mathcal{CH}$ map $F : B \to \mathcal{G}^* \otimes E$ such that

$$f((e^A)\ (z+h)) = f(z) + \int_0^1 (F((e^{tA})\ (z+th))(A) + Df((e^{tA})\ (z+th))(h))dt \quad \mu - a.e.z$$

$$F(z)(A) = D^*(ADf)(z), \qquad \mu - a.e.z$$

for any $A \in \mathcal{G}$ and $h \in H$.

PROOF. (1) For each $h \in H$ let $S(h)$ be a map from $\mathcal{L}^\infty(H; H)$ into H given by $S(h)(A) = Ah$, $A \in \mathcal{L}^\infty(H; H)$. Then $S(h)$ is obviously a bounded linear operator.

Let $U = \{A \in \mathcal{G};\ \| A \|_{\mathcal{G}} \leq 1\}$. Then by the definition of Hilbert-Lie subgroup and the assumption, we see that $\varepsilon(U; \| \cdot \|_{op}) < 1$. So by Lemma 29 we see that the restriction of $S(h)$ on $\mathcal{G}$ is a Hilbert-Schmidt type for each $h \in H$.

Let S be a map from H into $\mathcal{L}^2(\mathcal{G}; H) \cong H \otimes \mathcal{G}^*$. Then by Banach-Steinhaus' Theorem, we see that S is a bounded operator. So $S \otimes I_E$ is a bounded linear operator from $H \otimes E$ into $(H \times \mathcal{G}^*) \otimes E$. Letting $T = S \otimes I_E$, we have the assertion (1).

(2) Similarly to the proof of Proposition 34, we see that

$$f((e^{A})\ (z+h)) = f(z) + \int_0^1 (D^*ADf)((e^{tA})\ (z+th)) \\ + Df((e^{tA})\ (z+th))(h))dt \quad \mu - a.e.z$$

for any $A \in \mathcal{G}$ and $h \in H$. By Theorem 23 we see that $Df : B \to H \otimes E$ is an $\mathcal{CH}^\infty$ map. So we see that $T(Df) : B \to H \otimes \mathcal{G}^* \otimes E$ is an $\mathcal{CH}^\infty$ map. Thus again by Theorem 23 we see that $D^*(T(Df)) : B \to \mathcal{G}^* \otimes E$ is an $\mathcal{CH}^\infty$ map. Letting $F = D^*(T(Df))$, we have our assertion.

This completes the proof. □

Definition 46. We say that a Hilbert-Lie subgroup G_0 of $O(H)$ is admissible, if there are a Polish subgroup G_1 of $O(H)$, an abstract Wiener space (ν, H_0, W), a diffeomorphism $R_{00} : G_0 \to H_0$, and H_0-regular maps $S : W \to G_1$, $R_0 : G_0 \times W \to H_0$, and $R_1 : H_0 \times W \to G_0$, satisfying the following.
(1) G_0 is included in G_1 continuously, $\varepsilon(G_0) < 1$ and $\varepsilon(G_1) < 2$.
(2) $S(w + R_0(g, w)) = gS(w)$, $\nu - a.e.w$, for any $g \in G_0$.
(3) $R_0(\cdot, w + \cdot) : G_0 \times H_0 \to H_0$ and $R_1(\cdot, w + \cdot) : H_0 \times H \to G_0$, are continuously Frechét differentiable, $R_0(R_1(h, w), w) = h$, $w \in W$, $h \in H_0$, and $R_1(R_0(g, w), w) = g$, $w \in W$, $g \in G_0$.

The following is an easy consequence of Theorem 43 and Lemma 45.

Theorem 47. *Let G_0 be an admissible Hilbert-Lie subgroup of $O(H)$. Let (ν, H_0, W) be an abstract Wiener space, G_1 be a Polish subgroup of $O(H)$ and an H_0-regular map $S : W \to G_1$ be as in Definition 46. Let $\Psi : G_1 \times B \to B$ be as in Theorem 43. Let E be a separable real Hilbert space and $f : B \to E$ be an H-regular $\mathcal{CH}^\infty$-map. Then the map $f(\Psi(S(\cdot), \cdot)) : W \oplus B \to E$ is an $(H_0 \oplus H)$-regular $\mathcal{CH}^1$-map.*

8. Main Theorems

For a map $\Phi : B \to B$ and a subset A in B, let $N(\cdot\, ; A, \Phi) : B \to [0, \infty]$ be given by

$$N(z; A, \Phi) = \#\{z' \in A; \Phi(z') = z\}, \qquad z \in B.$$

Here $\#A$ denotes the cardinal of the set A.

Then we have the following.

THEOREM 48. *Let $F : B \to H$ be an H-regular and $\mathcal{CH}^1$-map, and let $\Phi : B \to B$ be given by $\Phi = I_B + F$. Then we have*

$$\int_A f(\Phi(z))|d(z;F)|\mu(dz) = \int_B N(z;A,\Phi)f(z)\mu(dz),$$

for any non-negative measurable function $f : B \to \mathbf{R}$ and a Borel set A in B. Here

$$d(z;F) = \det{}_2(I_H + DF(z))\exp(-D^*F(z) - \frac{1}{2} \parallel F(z) \parallel_H^2), z \in B.$$

PROOF. We give only a sketch of a proof, because this theorem is a version of a change of variables formula and the Sard theorem (c.f. [15] and its references). Since F is an H regular and $\mathcal{CH}^1$-map, $DF : B \to H$ is a $\mathcal{CH}$-map by Lemma 22. So there are a compact set K in B and a measurable map $G : B \to \mathcal{L}^2(H;H)$ such that $F : K + B_r \to H$, and $G : K + B_r \to \mathcal{L}^2(H;H)$ are continuous for all $R > 0$, and that

$$\lim_{t\to 0} \frac{1}{t}(F(z+th) - F(z) - tG(z)h) = 0, \qquad z \in K + H.$$

Let M_1 be the set of $z \in K + H$ for which $I_H + DF(z) : H \to H$ is bijective, $M_0 = (K+H) \setminus M_1$ and let $N_0 = B \setminus (K+H)$. Then by the Sard theorem (c.f.[6]), we have $\mu(\Phi(M_0)) = 0$. Also, by the change of variables formula, we see that

$$\int_{A\cap M_1} f(\Phi(z))|d(z;F)|\mu(dz) = \int_B N(z;A\cap M_1,\Phi)f(z)\mu(dz).$$

Noting that $\Phi(N_0) \subset N_0$ and $\mu(N_0) = 0$, we have our assertion. □

In this section, we extend this theorem.

Let G_0 be an admissible Hilbert Lie subgroup of $O(H)$ Let (ν, H_0, W) be an abstract Wiener space, G_1 be a Polish subgroup of $O(H)$, an H_0-regular map $S : W \to G_1$ and a diffeomorphism $R_{00} : G_0 \to H_0$ be as in Definition 46. Let $\Psi : G_1 \times B \to B$ be as in Theorem 43.

Let $U : B \to G_0$ and $F : B \to H$ be H-regular maps such that $R_{00} \circ U : B \to H_0$ and F are $\mathcal{CH}^\infty$-maps. Let $\Phi : B \to B$ be given by $\Phi(z) = \Psi(U(z), z + F(z))$, $z \in B$.

The following is our main thereom.

Theorem 49. *Suppose that $\tilde{\Omega}_0$ be a Borel set in $W \times B$ satisfying the following.*
(1) $\tilde{\Omega}_0 + (H_0 \otimes H) = \tilde{\Omega}_0$, and $(\nu \otimes \mu)(\tilde{\Omega}_0) = 1$.
(2) $\Psi(g, \Psi(S(w), z)) = \Psi(gS(w), z)$ for any $g \in G_0$ and $(w, z) \in \tilde{\Omega}_0$.
(3) $\Omega_0 = \{(\Psi(S(w), z));\ (w, z) \in \tilde{\Omega}_0\}$ is a Borel set in B.
Then there is a measurable map $\delta : B \to [0, \infty]$ such that

$$\int_A f(\Phi(z))\delta(z)\mu(dz) = \int_B N(z; A \cap \Omega_0, \Phi) f(z)\mu(dz),$$

for any non-negative measurable function $f : B \to \mathbf{R}$ and a Borel set A in B.

Proof.
Step 1. Let $\pi : W \times B \to B$ be defined by $\pi(w, z) = \Psi(S(w), z)$, $(w, z) \in W \times B$. Since $\mu \circ \Psi(g, \cdot)^{-1} = \mu$, $g \in G_1$, we see that $(\nu \otimes \mu) \circ \pi^{-1} = \mu$. Also, let $\tilde{G} : W \times B \to H_0$, and $\tilde{F} : W \times B \to H$ be given by $\tilde{G}(w, z) = R_0(U(\pi(w, z)), w)$, and $\tilde{F}(w, z) = S(w)^{-1} F(\pi(w, z))$, $(w, z) \in W \times B$. Then we see that $\tilde{G}$ and $\tilde{F}$ are $(H_0 \oplus H)$-regular $\mathcal{CH}^1$-maps.

Let $\tilde{\Phi} : W \times B \to W \times B$ be given by $\tilde{\Phi}(w, z) = (w + \tilde{G}(w, z), z + \tilde{F}(w, z))$. Then by Theorem 48, we see that there is a measurable map $\tilde{\delta} : W \times B \to [0, \infty)$ such that

$$\begin{aligned} &\int_{\tilde{A}} \tilde{f}(\tilde{\Phi}(w, z))\tilde{\delta}(w, z)\nu(dw) \otimes \mu(dz) \\ &\qquad = \int_{W \times B} N((w, z); \tilde{A}, \tilde{\Phi}) \tilde{f}(w, z)\nu(dw) \otimes \mu(dz) \end{aligned}$$

for any measurable function $\tilde{f} : W \times B \to [0, \infty)$ and any Borel set $\tilde{A}$ in $W \times B$.

Step 2. We will prove the following.

Claim. (i) If $(w', z') \in W \times B$ and $(w, z) = \tilde{\Phi}(w', z') \in \tilde{\Omega}_0$, then $(w', z') \in \tilde{\Omega}_0$ and $\Phi(\pi(w', z')) = \pi(w, z) \in \Omega_0$.

(ii) If $(w,z) \in \tilde{\Omega}_0$, $\xi \in \Omega_0$ and $\Phi(\xi) = \pi(w,z)$, then there is a unique $(k,h) \in H_0 \oplus H$ such that $\pi(w+k, z+h) = \xi$ and $\tilde{\Phi}(w+k, z+h) = (w,z)$.

Let $(w', z') \in W \times B$, and $(w,z) = \tilde{\Phi}(w', z') \in \tilde{\Omega}_0$. Then we see that $(w', z') \in \tilde{\Omega}_0 + (H_0 \oplus H) = \tilde{\Omega}_0$. So we have

$$\begin{aligned}\pi(w,z) &= \pi(\tilde{\Phi}(w',z')) = \Psi(U(\pi(w',z'))S(w'), z + S(w')^{-1}F(\pi(w',z'))) \\ &= \Psi(U(\pi(w',z')), \Psi(S(w'), z + S(w')^{-1}F(\pi(w',z')))) \\ &= \Psi(U(\pi(w',z')), \pi(w',z') + F(\pi(w',z'))) = \Phi(\pi(w',z')).\end{aligned}$$

Thus we have the Claim (i).

Let $(w,z) \in \tilde{\Omega}_0$, $\xi \in \Omega_0$, and assume that $\Phi(\xi) = \pi(w,z)$. Then there is a $(w_0, z_0) \in \tilde{\Omega}_0$ such that $\xi = \pi(w_0, z_0)$. Let $w' = w + R_0(U(\xi)^{-1}, w)$, and $z' = z - S(w')^{-1}F(\xi)$. Then we see that $(w', z') \in \tilde{\Omega}_0$. Moreover we see that

$$\begin{aligned}\pi(w',z') &= \Psi(U(\xi)^{-1}S(w), z) - F(\xi) = \Psi(U(\xi)^{-1}, \pi(w,z)) - F(\xi) \\ &= \Psi(U(\xi)^{-1}, \Psi(U(\xi), \Psi(S(w_0), z_0) + F(\xi))) - F(\xi) \\ &= \pi(w_0, z_0) = \xi.\end{aligned}$$

Note that

$$\begin{aligned}S(w) &= U(\xi)S(w') = S(w' + R_0(U(\xi), w')) \\ &= R_1(w' - w + R_0(U(\xi), w'), w)S(w).\end{aligned}$$

Since $R_0(\cdot, w) : H_0 \to O(H)$ is one to one, we see that $w = w' + R_0(U(\xi), w')$. So we have

$$\tilde{\Phi}(w', z') = (w' + R_0(U(\pi(w',z'), w'), z' + S(w')^{-1}F(\pi(w',z')) = (w,z)$$

So we see that the existence of such $(k,h) \in H_0 \oplus H$ in the Claim (ii).

Let $(k,h) \in H_0 \otimes H$ and suppose that $\pi(w+k, z+h) = \xi$ and $\tilde{\Phi}(w + k, z+h) = (w,z)$. Then we have

$$S(w) = S((w+k) + R_0(U(\pi(w+k, z+h)), w+k)) = U(\xi)S(w+k).$$

So we have

$$S(w+k) = U(\xi)^{-1}S(w) = S(w + R_0(U(\xi)^{-1}, w)).$$

So we see that $k = R_0(w, U(\xi)^{-1})$. Also, we see that

$$z = (z+h) + S(w+k)^{-1}F(\pi(w+k, z+h)) = z + h + S(w+k)^{-1}F(\xi).$$

So we see that $h = -S(w+k)^{-1}F(\xi)$. This shows the uniqueness of (k, h). This completes the proof of our Claim.

Step 3. Now we prove our theorem. Let A be a Borel set in B and $f : B \to [0, \infty)$ be a measurable function. Let $\tilde{A} = \pi^{-1}(A)$, and $\tilde{f} = f \circ \pi$. Then our Claim implies that

$$N((w,z); \tilde{A} \cap \tilde{\Omega}_0, \tilde{\Phi}) = N(\pi(w,z); A \cap \Omega_0, \Phi)$$

for any $(w, z) \in \tilde{\Omega}_0$. Then we have

$$\begin{aligned}
&\int_B N(z; A \cap \Omega_0, \Phi) f(z) \mu(dz) \\
&= \int_{W \times B} N(\pi(w,z); A \cap \Omega_0, \Phi) f(\pi(w,z)) \nu(dw) \otimes \mu(dz) \\
&= \int_{W \times B} N((w,z); \tilde{A} \cap \tilde{\Omega}_0, \tilde{\Phi}) \tilde{f}(w,z) \nu(dw) \otimes \mu(dz) \\
&= \int_{\tilde{A}} \tilde{f}(\tilde{\Phi}(w,z)) \tilde{\delta}(w,z) \nu(dw) \otimes \mu(dz) = \int_A f(\Phi(z)) \delta(z) \mu(dz).
\end{aligned}$$

Here $\delta : B \to [0, \infty]$ is given by $\delta \circ \pi = E^{\nu \otimes \mu}[\tilde{\delta} | \pi(\cdot)]$.

This completes the proof. □

Theorem 50. *There is a σ-compact set $\tilde{\Omega}_0$ in $W \times B$ satisfying the following.*
(1) $\tilde{\Omega}_0 + (H_0 \oplus H) = \tilde{\Omega}_0$, *and* $(\nu \otimes \mu)(\tilde{\Omega}_0) = 1$.
(2) $\Psi(g, \Psi(S(w), z)) = \Psi(gS(w), z)$ *for any* $g \in G_0$ *and* $(w, z) \in \tilde{\Omega}_0$.
(3) $\Omega_0 = \{(\Psi(S(w), z));\ (w, z) \in \tilde{\Omega}_0\}$ *is a σ-compact set in* B.

Proof. We see that there is a compact set K_1 in B such that $\mu(K_1) > 0$, $\Psi : G_1 \times (K_1 + B_r) \to B$ is continuous for any $r > 0$, and $\Psi(g, z + h) = \Psi(g, z) + gh$, $g \in G_1$, $z \in K_1$, $h \in H$. Also, there is a compact set K_0 in W such that $\nu(K_0) > 0$, $S : K_0 + B'_r \to G_1$ is continuous for $r > 0$ and $S(w + k) = R_1(k, w)S(w)$, $w \in K_0$, $k \in H_0$. Here $B'_r = \{k \in H_0;\ \| k \|_{H_0} \leq r\}$.

From the definition of $\varepsilon(G_0)$, we see that there is a countable open covering $\{U_n\}_{n=1}^{\infty}$ of G_0 such that $\varepsilon(U_n; \| \cdot \|_{op}) < 1$. Then similarly to the proof of Corollary 39, we see that

$$\mu(\sup\{\rho(\Psi(g\tilde{g}, z), K_1); g \in U_n\} < \infty, \text{ for all } n \geq 1) = 1,$$

for any $\tilde{g} \in G_1$. So we see that

$$(\nu \otimes \mu)(\sup\{\rho(\Psi(gS(w), z), K_1); g \in U_n\} < \infty, \text{ for all } n \geq 1) = 1.$$

Hence there is a compact set $\tilde{K}$ in $W \times B$ such that $(\nu \otimes \mu)(\tilde{K}) > 0$, $\tilde{K} \subset K_0 \times K_1$ and $\sup\{\rho(\Psi(gS(w), z), K_1); g \in U_n\} < \infty$ for any $n \geq 1$ and $(w, z) \in \tilde{K}$. Then we see that a map $\Psi(\cdot, \Psi(S(\cdot), \cdot)) : G_1 \times (\tilde{K} + (B'_r \times B_r)) \to B$ is continuous for any $r > 0$. It is obvious that

$$\Psi(gS(w + k), z + h) = \Psi(g, \Psi(S(w + k), z + h)), \qquad \nu \otimes \mu - a.e.(w, z)$$

for all $(g, k, h) \in G_1 \times H_0 \times H$.

So there is a compact set $\tilde{K}_0$ in $W \times B$ such that $(\nu \otimes \mu)(\tilde{K}_0) > 0$, $\tilde{K}_0 \subset \tilde{K}$ and

$$\Psi(gS(w), z) = \Psi(g, \Psi(S(w), z)),$$

for all $(g, k, h) \in G_1 \times \tilde{K}_0 + (B'_r \times B_r)$, $r > 0$. Letting $\tilde{\Omega}_0 = \tilde{K}_0 + (H_0 \oplus H)$, we have our assertion. □

From Theorems 49 and 50, we have the following.

COROLLARY 51. *There is a σ compact set Ω_0 and a measurable map $\delta : B \to [0, \infty]$ satisfying the following.*
(1) $\mu(\Omega_0) = 1$ *and* $\Omega_0 + H = \Omega_0$.
(2) *For any non-negative measurable function $f : B \to \mathbf{R}$ and a Borel set A in B,*

$$\int_A f(\Phi(z))\delta(z)\mu(dz) = \int_B N(z; A \cap \Omega_0, \Phi)f(z)\mu(dz).$$

9. Example

Let d be an integer. Let $B = \{z \in C([0, 1]; \mathbf{R}^d);\ z(0) = 0\}$, μ be the standard Wiener measure in B and H be the Cameron-Martin space of μ, i.e.,

$$H = \{h \in B; h(t) \text{ is absolutely continuous in } t, \int_0^1 |\frac{dh}{dt}(t)|^2 dt < \infty\}.$$

Then (μ, H, B) is an abstract Wiener space. We take this as a basic abstract Wiener space.

Let $O(d)$ be the set of $d \times d$ orthogonal matrices, and $o(d)$ be the space of $d \times d$ skew symmetric matrices. We introduce an inner product on $o(d)$ by $(A_0, A_1) = \text{trace}\,(A_0^* A_1)$. Let $W = \{w \in C([0,1]; o(d));\ w(0) = 0\}$ and ν be the standard Wiener measure on W, i.e., ν is a mean 0 Gaussian measure with

$$\int_W (w(t), A_0)(w(s), A_1)\nu(dw) = (A_0, A_1)(t \wedge s),$$
$$t, s \in [0,1],\ A_0, A_1 \in o(d).$$

Let H_0 be the Cameron-Martin space of ν, i.e.,

$$H_0 = \{h \in W; h(t) \text{ is absolutely continuous in } t, \int_0^1 |\frac{dh}{dt}(t)|^2 dt < \infty\}.$$

For each $g \in C^1([0,1]; O(d))$, we denote $g(t)^{-1}\frac{dg}{dt}$ by $\frac{Dg}{dt}$. Then $\frac{Dg}{dt} \in C([0,1]; o(d))$. Let G_1 denotes the set of $g \in C^1([0,1]; O(d))$ such that $g(0) = I_d$ and $\frac{Dg}{dt} \in W$. For each $h \in H$ and $g \in G_1$, we define

$$(gh)(t) = \int_0^t g(s)\frac{dh}{ds}(s)ds, \qquad t \in [0,1].$$

Then we may regard G_1 as a Polish subgroup of $O(H)$. Moreover, we see that the map $h \to gh$ in H can be extended a bounded linear operator in B. Let G_0 be the set of $g \in G_1$ such that $\frac{Dg}{dt} \in H_0$. Then we see that G_0 is a Hilbert-Lie subgroup of $O(H)$. By [2], we see that $\varepsilon(G_0) = 1/2$, and $\varepsilon(G_1) = 1$. Let $S(w) = \{S_t(w);\ t \in [0,1]\}$, $w \in W$, be the solution of the following ODE.

$$\frac{d}{dt}S_t(w) = S_t(w)w(t),\ t \in [0,1], \qquad S_0(w) = 0.$$

Then the map $S : W \to G_1$ is continuous. Note that

$$\frac{d}{dt}(g(t)S_t(w)) = (g(t)S_t(w))(w(t) + S_t(w)^{-1}\frac{Dg}{dt}(t)S_t(w)),$$
$$g \in G_0,\ w \in W.$$

So we see that

$$S(w + S.(w)^{-1}\frac{Dg}{dt}(\cdot)S.(w)) = gS(w), \qquad g \in G_0,\ w \in W.$$

These show that G_0 is admissible. So we have the following from Theorem 49.

Corollary 52. *Let $U : B \to G_0$ and $F : B \to H$ be H-regular maps such that $\frac{DU}{dt} : B \to H_0$ and $F : B \to H$ are $\mathcal{CH}^{\infty}$-maps. Let $\Psi : B \to B$ be defined by $\Psi(z) = U(z)(z + F(z))$, $z \in B$. Then there is a measurable function $\delta : B \to [0, \infty]$ such that*

$$\int_A f(\Phi(z))\delta(z)\mu(dz) = \int_B N(z; A, \Phi)f(z)\mu(dz),$$

for any non-negative measurable function $f : B \to \mathbf{R}$ and a Borel set A in B.

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