J. Math. Sci. Univ. Tokyo
9 (2002), 689–746.

Class Field Theory for $\mathbb{F}_q[[X_1, X_2, X_3]]$

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1. Introduction

In this paper we generalize local class field theory from the viewpoint of class field theory for (fractional fields of) complete local rings. First, we review briefly the history of higher-dimensional class field theory. For one-dimensional complete regular local rings having finite residue field, class field theory for their fractional fields is classical. On the other hand for two-dimensional complete regular local rings such as $\mathbb{F}_q[[X_1, X_2]]$ or $\mathbb{Z}_p[[X_1]]$, their class field theory was successfully obtained by A. N. Parshin in [Pa1] and S. Saito in [Sa1] independently by using Milnor K_2^M-idele class group. Such generalizations of local class field theory by using Milnor K-groups

1991 *Mathematics Subject Classification*. Primary 19E15; Secondary 11G25, 14C25.
Key words: class field theory, idele class group, reciprocity homomorphism, étale cohomology, Milnor K-group, Kato-filtration, Grothendieck duality, Hasse principle.
**Supported by* JSPS Research Fellowships for Young Scientists.

originate in the celebrated papers [Ka1], [Pa1], [Pa2] of K. Kato and A. N. Parshin who established so-called higher dimensional local class field theory. Soon thereafter, S. Bloch succeeded in proving class field theory for arithmetic surfaces in [Bl] by the elegant approach through algebraic K-theory. By using this result of Bloch, K. Kato and S. Saito established two-dimensional global class field theory in [Ka-Sa1], where the above result of S. Saito in [Sa1] played an important role. Here, we also remark that in positive characteristic case, two-dimensional global class field theory was also announced by A. N. Parshin in [Pa3].

In [Ka-Sa1], fractional fields of such complete local ring as $\mathbb{F}_q[[X_1, X_2]]$ or $\mathbb{Z}_p[[X_1]]$ are called semi-global fields (there exist other different types of semi-global fields. For more details, we refer [Ka-Sa1]). By generalizing techniques developed in [Bl] or [Ka-Sa1], Kato and Saito finally accomplished so-called higher dimensional global class field theory in [Ka-Sa2] (there, class field theory for arbitrary finitely generated fields over prime fields are proved and established). But surprisingly, they no longer used class field theory of semi-global fields obtained as fractional fields of complete local rings $\mathbb{F}_q[[X_1, \dots, X_n]]$ for $n \geq 3$. So, the motivation for our study can be explained to prove class field theory of fractional fields of complete regular local rings $\mathbb{F}_q[[X_1, \dots, X_n]]$ with arbitrary $n \geq 3$. In this paper we treat the case $n = 3$, and the general case $n \geq 4$ is to be established in the forthcoming paper [Ma2]. We explain more closely our formalism of class field theory. Let us take a three-dimensional complete regular local ring $A := \mathbb{F}_q[[X_1, X_2, X_3]]$ and denote by K its fractional field. We consider the maximal abelian extension K^{ab} of K in the fixed algebraic closure $\overline{K}$ of K. Then, we construct the Milnor K_3^M-idele class group C_K and the reciprocity map $\rho_K\colon C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ through which $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ should be approximated by C_K.

We define the idele class group C_K in Section 1, where at the same time we put a certain nice topology on it. An interesting fact is that our Milnor K_3^M-idele class group C_K quite a lot resembles those of two-dimensional global fields constructed by Kato-Saito (cf. loc.cit.). The reciprocity map ρ_K is defined in rather traditional way in Section 5. For this, we must prove reciprocity laws enjoyed by various local fields $K_{\mathfrak{p}}$ or $K_{\mathfrak{m}}$ in Proposition 5.3 (for the definition of $K_{\mathfrak{p}}, K_{\mathfrak{m}}$, see the notation in Section 1). In Sections 5, 6, ρ_K is proved to describe $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ in a desirable manner.

That is, our Milnor K_3^M-idele class group C_K contains all informations of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ as all previous successful cases. We will state our main results.

THEOREM 1.1 (THEOREM 5.1). *Let $A := \mathbb{F}_q[[X_1, X_2, X_3]]$ and K be its fractional field. We assume $q = p^m$, hence K has characteristic $p > 0$. Then, by using the topological idele class group C_K mentioned above, it holds the following dual reciprocity isomorphism*:

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(C_K, \mathbb{Q}_p/\mathbb{Z}_p),$$

where $\mathrm{Hom}_{\mathrm{cont}}(C_K, \mathbb{Q}_p/\mathbb{Z}_p)$ *denotes the set of all continuous homomorphisms of finite order from C_K to $\mathbb{Q}_p/\mathbb{Z}_p$.*

This theorem is the most essential part in this paper and is established in Section 5 by using the simple commutative diagram (5.71) obtained by using all results in previous sections. In (5.71), the top exact sequence which is proved in Theorem 5.5 plays a crucial role. Next we state prime to p parts.

THEOREM 1.2 (THEOREM 6.1). *Let A, K be as above and $l \neq p$ be an arbitrary prime. Then under the Bloch-Milnor-Kato conjecture (see below), it holds the dual reciprocity isomorphism*

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(C_K, \mathbb{Q}_l/\mathbb{Z}_l).$$

We explain the Bloch-Milnor-Kato conjecture. For an arbitrary field F and a natural number m, it asserts the bijectivity of the Galois symbol $K_m^M(F)/n \xrightarrow{\sim} H^m_{\mathrm{Gal}}(F, \mu_n^{\otimes m})$ for an arbitrary natural number n prime to $\mathrm{ch}(F)$. For the proof of Theorem 1.2, we use this conjecture with $F = K$ and $m = 3, n = l^k$ $(k \geq 1)$. The proof of Theorem 1.2 is much easier than that of Theorem 1.1 except for using some non-trivial results in algebraic K-theory or using Saito's Hasse principle for two-dimensional (not necessarily regular) normal local rings. Combining Theorem 1.1 and Theorem 1.2, we get the following class field theory of K:

Theorem 1.3 (Theorem 6.2). *Let A, K be as in Theorem 1.1. Then under the Bloch-Milnor-Kato conjecture for K, we have the following dual reciprocity isomorphism*:

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(C_K, \mathbb{Q}/\mathbb{Z}) \ \cdots \ (\Diamond)$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the set of all continuous homomorphisms of finite order.*

This theorem proves class field theory for K in a satisfactory manner. In fact, the isomorphism $(\Diamond)$ involves the existence theorem which is treasured in class field theory.

Corollary 1.4 (Corollary 6.3). *Let A, K be as in Theorem 1.1. Then under the Bloch-Milnor-Kato conjecture for K, the reciprocity homomorphism*

$$\rho_K \colon C_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

has the dense image in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ *with the Krull topology.*

Finally we show that for certain finite abelian extensions of K, we have the following explicit reciprocity isomorphisms which are familiar in local or global class field theory:

Corollary 1.5 (Corollary 6.4). *Let A, K be as in Theorem 1.3 and we assume the Bloch-Milnor-Kato conjecture for K. Then, for an arbitrary finite abelian extension L/K such that the integral closure of A in L is regular, we have the reciprocity isomorphism*

$$\rho_K \colon C_K/N_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K),$$

where $N_{L/K}$ denotes the norm for Milnor K-groups.

This result can be seen as the first successful explicit description for the Galois group of finite abelian extension of semi-global fields in the sense of Kato-Saito (cf. loc.cit.).

Convention. Through the paper, for an arbitrary commutative ring, we always denote by P^i_R the set of height i prime ideals of R.

2. Construction of the Idele Class Group C_K

In this section we introduce the Milnor K-theoretic topological idele class group C_K which plays the central role in this paper. First, we review basic results. Recall that for an arbitrary field k, the n-th Milnor K-group $K_n^M(k)$ is defined by

Definition 1. We define Minor's K-group $K_q^M(k)$ for a field k as

$$K_q^M(k) := ((k^\times)^{\otimes q})/J,$$

where J is the subgroup of the q-fold tensor product $(k^*)^{\otimes k}$ of $k^\times$ (as a $\mathbb{Z}$-module) generated by elements of the form $a_1 \otimes \ldots \otimes a_q$ satisfying $a_i + a_j = 1$ for some $i \neq j$. For a discrete valuation field F, we will define Kato-filtrations $U^i K_n^M(F)$ for $i \geq 0$ as

$$\begin{aligned} U^i K_n^M(F) := \{\text{Image}\colon x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto K_n^M(F) \mid \\ \text{s.t. } x_1 \in U^{(i)}(F), x_2, \ldots, x_n \in K^*\}, \end{aligned} \tag{2.1}$$

where $U^{(i)}(F)$ denotes the multiplicative group $(1 + u_F^i \mathcal{O}_F)^* \subset \mathcal{O}_F^*$. Here, u_F and $\mathcal{O}_F$ denote the uniformizing parameter and the valuation ring of F, respectively.

Next, we recall class field theory for n-dimensional local fields established by K. Kato and A. N. Parshin in [Ka1], [Pa1]. For this, we first give the definition of higher dimensional local fields.

Definition 2. A complete discrete valuation field k_n is said to be n-dimensional local if there exists the following sequence of fields k_i ($1 \leq i \leq n$):

each k_i is a complete discrete valuation field having k_{i-1} as the residue field of the valuation ring $\mathcal{O}_{k_i}$ of k_i, and k_0 is defined to be a finite field.

For n-dimensional local field k_n, K. Kato and A. N. Parshin established its class field theory as follows:

Theorem 2.1 (Kato, Parshin). *For an arbitrary n-dimensional local field k_n, there exists the canonical reciprocity map*

$$\rho_{k_n}\colon K_n^M(k_n) \to \mathrm{Gal}(k_n^{ab}/k_n)$$

which satisfies the following two conditions:

i) for an arbitrary finite abelian extension k'/k_n, ρ_{k_n} *induces an isomorphism*

$$\rho_{k_n}: K_n^M(k_n)/N_{k'/k_n}(K_n^M(k')) \xrightarrow{\sim} \mathrm{Gal}(k'/k_n),$$

ii) the correspondence $k' \mapsto N_{k'/k_n}(K_n^M(k'))$ *is a bijection between the set of all finite abelian extensions of* k_n *and the set of all open subgroups of* $K_n^M(k_n)$ *of finite index.*

For the convenience, we fix the following notation:

Notations

P_A^2: = the set of all height 2 primes of A,

P_A^1: = the set of all height 1 primes of A,

$P_{\mathfrak{m}}^1$: = the set of all height 1 primes of $A_{\mathfrak{m}}$,

$\mathfrak{p}_{\mathfrak{m}}$: = element of $P_{\mathfrak{m}}^1$,

$A_{\mathfrak{m}} := \varprojlim_n A_{(\mathfrak{m})}/\mathfrak{m}^n$ ($A_{(\mathfrak{m})}$ is a localization, where all elements outside $\mathfrak{m}$ are invertible),

$A_{\mathfrak{p}} := \varprojlim_n A_{(\mathfrak{p})}/\mathfrak{p}^n$,

$A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} := \varprojlim_n A_{\mathfrak{m}(\mathfrak{p}_{\mathfrak{m}})}/\mathfrak{p}_{\mathfrak{m}}^n$ (this is a complete discrete valuation ring)

$K_{\mathfrak{m}} := \mathrm{Frac}\, A_{\mathfrak{m}}$, $K_{\mathfrak{p}} := \mathrm{Frac}\, A_{\mathfrak{p}}$, $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} := \mathrm{Frac}\, A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$.

REMARK 1. In the above notation, $A_{\mathfrak{m}}$ becomes a two-dimensional complete regular local ring whose residue field $\kappa(\mathfrak{m})$ is one-dimensional local. We also remark that $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ is a three-dimensional local field explained in Definition 2.

We will introduce a notion of modulus M.

DEFINITION 3. A modulus M is a formal sum

$$M := \Sigma_{\mathfrak{p}\in P_A^1}\ n_{\mathfrak{p}}\ \overline{(\mathfrak{p})}$$

of prime divisors $\overline{(\mathfrak{p})}$ defined by ($\mathfrak{p} = 0$) in Spec A, and the integer coefficient $n_{\mathfrak{p}}$ is zero except for finitely many $\mathfrak{p}$s.

Next, for an arbitrary modulus M and each $\mathfrak{m} \in P_A^2$, we define the group $C_{\mathfrak{m}}(M)$ by

(2.2)

$$C_{\mathfrak{m}}(M) := \operatorname{Coker}\left(K_3^M(K_{\mathfrak{m}}) \xrightarrow{diagonal} \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P_{\mathfrak{m}}^1} \left(K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) / U^{M(\mathfrak{p}_{\mathfrak{m}})} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \right) \right),$$

where $\mathfrak{p}_{\mathfrak{m}}$ denotes a height one prime of two-dimensional complete local ring $A_{\mathfrak{m}}$ and the positive integer $M(\mathfrak{p}_{\mathfrak{m}})$ is defined to be $n_{\mathfrak{p}}$ if $\mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p}$ under the canonical map $\operatorname{Spec} A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} \to \operatorname{Spec} A$. Gathering $C_{\mathfrak{m}}(M)$ over all elements $\mathfrak{m} \in P_A^2$, we define the group $C_K(M)/n$ for $n \geq 2$ as

$$C_K(M)/n := \operatorname{Coker}\left(\bigoplus_{\mathfrak{p} \in P_A^1} K_3^M(K_{\mathfrak{p}}) \to \bigoplus_{\mathfrak{m} \in P_A^2} C_{\mathfrak{m}}(M)/n \right), \tag{2.3}$$

and put the *discrete* topology on $C_K(M)/n$. The reason why we take modulo n in (2.3) is that we should prove its well-definedness. That is, the image of each group $K_3^M(K_{\mathfrak{p}})$ in $\prod_{\mathfrak{m} \in P_A^2} C_{\mathfrak{m}}(M)/n$ actually lives in the *direct sum.* We prove this as the following lemma:

LEMMA 2.2. *For each $\mathfrak{p} \in P_A^1$, the image of $K_3^M(K_{\mathfrak{p}})$ in $\prod_{\mathfrak{m} \in P_A^2, \mathfrak{m} \supset \mathfrak{p}} C_{\mathfrak{m}}(M)/n$ lies in $\bigoplus_{\mathfrak{m} \in P_A^2, \mathfrak{m} \supset \mathfrak{p}} C_{\mathfrak{m}}(M)/n$.*

PROOF. We will only treat the case $n = p$ to fix ideas. Hereafter, we will denote by $k_3^M(K_\lambda)$ the group $K_3^M(K_\lambda)/p$ and by $U^i k_3^M(K_\lambda)$ the image $U^i K_3^M(K_\lambda)$ in $k_3^M(K_\lambda)$.

Take an arbitrary height one prime $\lambda \in P_A^1$. Then, each element of $K_3^M(K_\lambda)/p$ lands in only such component $C_{\mathfrak{m}}(M)/p$ of $\prod_{\mathfrak{m} \in P_A^2} C_{\mathfrak{m}}(M)/p$ as $\mathfrak{m} \supset \lambda$. So, we have only to prove that an arbitrary element $\alpha \in k_3^M(K_\lambda)$ vanishes in $C_{\mathfrak{m}}(M)/p$ for almost all $\mathfrak{m}$ which contains λ. If we denote by $n_{\mathfrak{p}}$ the coefficient of $(\mathfrak{p})$ in M, the definition of $C_{\mathfrak{m}}(M)/p$ immediately shows that $U^{n_\lambda} k_3^M(K_\lambda)$ vanishes in $C_{\mathfrak{m}}(M)/p$. On the other hand, Theorem 2.7 at the end of this section shows the surjection $k_3^M(K) \twoheadrightarrow k_3^M(K_\lambda)/U^{n_\lambda} k_3^M(K_\lambda)$, which allow us to consider $\alpha \in k_3^M(K)$. So, we can write

$$\alpha = (a_1, b_1, c_1) \cdots (a_n, b_n, c_n) \tag{2.4}$$

with $a_i, b_i, c_i \in K$. Further as A is uniquely factorized domain, we may assume that all a_i, b_i, c_i lie in $A[\frac{1}{u_{\mathfrak{p}_1}}, \dots, \frac{1}{u_{\mathfrak{p}_m}}]$ with finitely many height one primes $\mathfrak{p}_j$ ($j = 1, \dots, m$). The problematic height two primes can be classified into the following three types:

i) height two prime $\mathfrak{m}$ which lie over both λ and some $\mathfrak{p}_j$ ($\neq \lambda$).

ii) height two prime $\mathfrak{m}$ such that λ splits into several primes in $A_{\mathfrak{m}}$. Indeed, this is the case if the curve λ has the nodal singularity at $\mathfrak{m}$ (we can consider λ as a curve in the surface $\mathrm{Spec}A \setminus \mathfrak{m}_A$).

iii) height two primes which lie over both λ and some height one prime $\mathfrak{p}$ having its modulus number $n_{\mathfrak{p}} > 0$.

We will exclude these three cases in the below. The point is that only finitely many height two primes are candidates for each of these three types. So even in total, only finitely many height two primes are excluded. In the next Sub-lemma, we prove that except for finitely many $\mathfrak{m}$s excluded just now, an arbitrary element $\alpha \in k_3^M(K)$ vanishes in $C_{\mathfrak{m}}(M)/p$.

The situation is as follows: $\alpha \in k_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}])$, $M(\mathfrak{p}_{\mathfrak{m}}) = 0$ for every height one prime $\mathfrak{p}_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$ except for λ (λ remains prime because we excluded the type (ii)).

SUB-LEMMA 2.3. *For an element $\mathfrak{m} \in P_A^2$, the group $k_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}])$ vanishes in $C_{\mathfrak{m}}(M)/p$ if each modulus $M(\mathfrak{p})$ is zero for all $\mathfrak{p}_{\mathfrak{m}} \in P_{\mathfrak{m}}^1$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda$.*

PROOF. Consider the Gersten-Quillen complex

$$k_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}]) \to k_3^M(K_{\mathfrak{m}}) \overset{\oplus\partial_{\mathfrak{p}_{\mathfrak{m}}}}{\to} \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \neq \lambda} k_2^M(\kappa(\mathfrak{p}_{\mathfrak{m}})) \to 0, \tag{2.5}$$

where $\partial_{\mathfrak{p}_{\mathfrak{m}}}$ is the boundary map $\partial_{\mathfrak{p}_{\mathfrak{m}}}$: $k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to k_2^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))$ in algebraic K-theory. By Theorem 2.7 below, the kernel of each boundary map $\partial_{\mathfrak{p}_{\mathfrak{m}}}$ coincides with $U^0 k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$. So the above sequence (2.5) shows that $k_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}])$ lies in $U^0 k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ for all $\mathfrak{p}_{\mathfrak{m}}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda$. So immediately follows that if we move an arbitrary element $\alpha \in k_3^M(A_{\mathfrak{m}}[\frac{1}{\lambda}]) \subset k_3^M(K_{\mathfrak{m},\lambda})$ into $\bigoplus_{\mathfrak{p}_{\mathfrak{m}} \neq \lambda} \big(k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/U^0 k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})\big)$ using the diagonal image of $\alpha \in k_3^M(K_{\mathfrak{m}})$ in $C_{\mathfrak{m}}(M)/p$, α becomes 0 in $k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/U^0 k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ for all $\mathfrak{p}_{\mathfrak{m}}$ such that $\mathfrak{p}_{\mathfrak{m}} \neq \lambda$. Hence, we get the desired assertion in the sub-lemma. $\square$

We are in the stage to define the idele class group C_K for K.

DEFINITION 4. We define the topological idele class group C_K by

$$C_K := \varprojlim_{M,\, n\geq 2} C_K(M)/n, \tag{2.6}$$

where $\varprojlim_M$ is taken by the surjection $C_K(M')/n \twoheadrightarrow C_K(M)/n$ if $M' - M$ is effective ($\Leftrightarrow n'_{\mathfrak{p}} - n_{\mathfrak{p}} \geq 0$), and we endow the inverse limit topology on C_K induced from the *discrete* topology on each $C_K(M)/n$ (by definition, the fundamental open subsets of C_K are the inverse images of all subgroups of $C_K(M)/n$ in C_K with M running over all moduli and $n \geq 2$).

We have a useful lemma.

LEMMA 2.4 (Explicit Representation of C_K). *The above idele class group C_K can be also represented explicitly as follows*:

$$C_K/n \xrightarrow{\sim} \varprojlim_M D_{K,n}/F^M D_{K,n}, \tag{2.7}$$

where $F^M D_{K,n} :=$ Image $\left({\prod_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}' U^{M(\mathfrak{p}_{\mathfrak{m}})} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to D_{K,n} \right)$ ($M(\mathfrak{p}_{\mathfrak{m}})$ *is defined in* (2.2)) *and* $D_{K,n}$ *is defined by*

$$D_{K,n} := \left({\prod_{\mathfrak{m}\in P_A^2,\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1}}' K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/n \right) \Big/ \prod\nolimits_{\mathfrak{m}\in P_A^2} K_3^M(K_{\mathfrak{m}}) \prod\nolimits_{\mathfrak{p}\in P_A^1} K_3^M(K_{\mathfrak{p}}).$$

$\left({\prod_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}' K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/n \right)$ *is the subgroup of the direct product* $\left(\prod_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/n \right)$ *such that an arbitrary element* $a \in \left({\prod_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}' K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \right)$ *satisfies the following two conditions*:

1) there can be associated the set S of finitely many height-one prime ideals of A such that $(\mathfrak{m}, \mathfrak{p}_{\mathfrak{m}})$*-component* $a_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ *of a lies in* $U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/n$ *if* $\mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p}$ *such that* $\mathfrak{p} \notin S$, *2) for any element* $\mathfrak{p} \in P^1_A$, $(\mathfrak{m}, \mathfrak{p}_\mathfrak{m})$*-component* $a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$ *of* a *lies in* $K^M_3(A_\mathfrak{m}[\frac{1}{u_{\mathfrak{p}_\mathfrak{m}}}])/n$ *for almost all* $\mathfrak{p}_\mathfrak{m}$ *such that* $\mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p}$.

Finally, it also holds that

$$D_{K,n}/F^M D_{K,n} \xrightarrow{\sim} C_K(M)/n. \tag{2.8}$$

Proof. This is easily observed in the similar way as in Sub-lemma 2.3. □

Next, we define the subgroup F^0C_K of C_K which plays an important role in Section 5.

$$F^0C_\mathfrak{m}(M) := \mathrm{Image}\left(\bigoplus_{\mathfrak{p}_\mathfrak{m} \in P^1_\mathfrak{m}} U^0K^M_3(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \to C_\mathfrak{m}(M)\right)$$

$$F^0C_K(M)/n := \mathrm{Image}\left(\bigoplus_{\mathfrak{m} \in P^2_A} F^0C_\mathfrak{m}(M) \to C_K(M)/n\right). \tag{2.9}$$

By an easy check, we see an isomorphism $F^0D_{K,n}/F^MD_{K,n} \cong F^0C_K(M)/n$. Using this, we have

Definition 5. We define F^0C_K by the inverse limit

$$F^0C_K := \varprojlim_{M,\, n \geq 2} F^0C_K(M)/n. \tag{2.10}$$

In Section 5 and Section 6, it is proved that this group F^0C_K corresponds to the maximal unramified extension of K by the reciprocity map ρ_K. So, if the residue field of A is $\mathbb{F}_q$, it follows that $\mathrm{Coker}(F^0C_K \to C_K) \cong \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$.

Here, we also review the Bloch-Milnor-Kato conjecture which plays an important role in the proof for prime to p part in Section 6.

Conjecture (Bloch-Milnor-Kato). *For an arbitrary field* K *and a natural number* n *prime to the characteristic of* K*, there holds the following Galois symbol isomorphism*:

$$K^M_n(K)/m \xrightarrow{\sim} H^n_{\mathrm{Gal}}(K, \mu_m^{\otimes n}).$$

For this conjecture, recent development of motivic cohomology by Voevodsky or Rost and others seems to bring the final solution in the near future. For the p primary version of this conjecture, we have

THEOREM 2.5 (Bloch-Gabber-Kato). *For an arbitrary field K of positive characteristic p, the following differential symbol becomes an isomorphism*:

$$K_n^M(K)/p^m \xrightarrow{\sim} H^0_{\mathrm{Gal}}(K,\, W_m\Omega^n_{K,\log}),$$

where $W_m\Omega^n_{K,\log}$ denotes the logarithmic Hodge-Witt sheaves of length m.

This theorem works often useful at several places in Section 5. We also review the purity theorem for Logarithmic Hodge-Witt sheaves established by Shiho.

THEOREM 2.6 (Shiho [Sh2], Theorem 3.2). *Let X be an arbitrary regular scheme X of dimension n such that any closed point $x \in X^{(n)}$ satisfies $[\kappa(x)\colon \kappa(x)^p] = p^i$. Then, for any regular closed subscheme $Z \hookrightarrow X$ of codimension c, it holds the following purity isomorphism*:

$$H^1_{\acute{e}t}(Z,\, \Omega^{n+i-c}_{Z,\log}) \cong H^{1+c}_Z(X,\, \Omega^{n+i}_{X,\log}).$$

This theorem is also indispensable in Section 5 in proving isomorphisms between local cohomologies of henselian local rings and those of complete local rings.

Finally, we recall Kato's calculation of Milnor K-groups for discrete valuation rings which are basic throughout this paper.

THEOREM 2.7 (Kato [Ka1] II, Lemma 6, p.616). *Let L be a discrete valuation field with residue field F such that $(F : F^p) = p^d$, and $K_n^M(L), U^iK_n^M(L)$ be as in Definition 1. We denote by $k_n^M(L)$ the group $K_n^M(L)/K_n^M(L)^p$, and denote by $U^ik_n^M(L)$ the image of $U^iK_n^M(L)$ in $k_n^M(L)$. Then, each sub-quotient $Gr^ik_n^M(L)\colon = U^ik_n^M(L)/U^{i+1}k_n^M(L)$ is calculated as follows*:

(1) *There exists an exact sequence*

$$0 \to K_n^M(F) \to K_n^M(L)/U^1K_n^M(L) \xrightarrow{\partial} K_{n-1}^M(F) \to 0$$

and $\mathrm{Ker}(K_n^M(L) \xrightarrow{\partial} K_{n-1}^M(F))$ *coincides with* $U^0K_n^M(L)$. *Consequently, it holds*

$$U^0K_n^M(L)/U^1K_n^M(L) \cong K_n^M(F).$$

Hereafter, we assume $i > 0$, *and* π_L *denotes a uniformizing parameter of the valuation ring* $\mathcal{O}_L$ *of* L.

(2) *if* $p \nmid i$, *there exists an isomorphism*

$$\xi_i \colon \Omega_F^{n-1} \xrightarrow{\sim} Gr^i k_n^M(L),$$

where ξ_i *sends* $\omega = s\, dt_1/t_1 \wedge \ldots \wedge dt_{n-1}/t_{n-1} \mapsto \overline{(1 + s\,\pi_L^i, t_1, \ldots, t_{n-1})} \in Gr^i k_n^M(L)$.

(3) *if* $p \mid i$, *there exists an isomorphism*

$$\xi_i \colon \Omega_F^{n-1}/\Omega_{F,d=0}^{n-1} \oplus \Omega_F^{n-2}/\Omega_{F,d=0}^{n-2} \xrightarrow{\sim} Gr^i k_n^M(L),$$

where $\Omega^i_{F,d=0}$ *denotes the set of* d*-closed* i*-forms of* F. ξ_i *sends* $\omega \in \Omega_F^{n-1}$ *in the same way as in* (2) *and sends* $\omega = s\, dt_1/t_1 \wedge \ldots \wedge dt_{n-2}/t_{n-2} \mapsto \overline{(1 + s\,\pi_L^i, t_1, \ldots, t_{n-2}, \pi_L)} \in Gr^i k_n^M(L)$.

(4) *if* $\mathrm{ch}(k) = 0$ *and* $i = e\,p/(p-1)$, *there exists an isomorphism*

$$\xi_{e\,p/(p-1)} \colon \Omega_F^{n-1}/(1 + aC)\Omega_{F,d=0}^{n-1} \oplus \Omega_F^{n-2}/(1 + aC)\Omega_{F,d=0}^{n-2} \xrightarrow{\sim} Gr^i k_n^M(L),$$

where $a := \overline{\pi_L^{-e}p}$ *is the residue class of* $\pi_L^{-e}p$ *in* F, *and maps are defined in the same way in the above.*

Remark 2. In this paper, we use Theorem 2.7 for $d = 2, n = 3$. So in this case, the term $\Omega_F^{n-1}/\Omega_{F,d=0}^{n-1}$ in (3) vanishes, which follows from the fact all two-forms in Ω_F^2 are d-closed (note that $(F\colon F^p) = p^2$).

3. Duality for Two Dimensional Complete Gorenstein Local Rings

In this section, we establish explicit duality results for two-dimensional complete normal Gorenstein local rings having finite residue field. We begin with the Grothendieck duality theorem.

THEOREM (Grothendieck [Ha], Theorem 6. 2, p. 278). *For an arbitrary normal local ring R over a field k and an arbitrary finite R-module M, the following isomorphisms hold*:

$$\mathbb{E}\mathrm{xt}_R^{-i}(M, D_R^\bullet) \otimes_R \widehat{R} \cong \mathrm{Hom}_R(H^i_{\mathfrak{m}_R}(R, M), I_R) \tag{3.1}$$

$$H^i_{\mathfrak{m}_R}(R, M) \cong \mathrm{Hom}_R(\mathbb{E}\mathrm{xt}_R^{-i}(M, D_R^\bullet), I_R), \tag{3.2}$$

where $D_R^\bullet$, I_R denote the normalized dualizing complex and the injective hull of R, respectively. For a gorenstein local ring R, we have $D_R^\bullet \cong \Omega_R^n[n]$ and for a regular local ring R, I_R is given by $I_R := \varinjlim_{i>0} R/\mathfrak{m}_R^i$. Hence for a gorenstein local ring R, (3.1) *reads*

$$\mathrm{Ext}_R^{n-i}(M, \Omega_R^n) \otimes_R \widehat{R} \cong \mathrm{Hom}_R(H^i_{\mathfrak{m}_R}(R, M), I_R). \tag{3.3}$$

The following corollaries are very useful which follow from the definition of I_R:

COROLLARY 3.2. *Under the same assumption in Theorem 3.1, if R is Gorenstein, it holds*

$$\mathrm{Ext}_R^{n-i}(M, \Omega_R^n) \otimes_R \widehat{R} \cong \mathrm{Hom}(H^i_{\mathfrak{m}_R}(R, M), \kappa(\mathfrak{m}_R)),$$

where Hom *denotes the set of all homomorphisms between discrete groups and $\kappa(\mathfrak{m}_R) := R/\mathfrak{m}_R$.*

COROLLARY 3.3. *Under the same assumption in Corollary 3.2, it holds*

$$H^i_{\mathfrak{m}_R}(R, M) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Ext}_R^{n-i}(M, \Omega_R^n), \kappa(\mathfrak{m}_R)),$$

where cont *means the set of continuous homomorphism with respect to $\mathfrak{m}_R$-*adic *topology on $\mathrm{Ext}_R^{n-i}(M, \Omega_R^n)$.*

By using these results, we will prove

Theorem 3.4. *Let R be an arbitrary two-dimensional normal Gorenstein local ring with residue field $\mathbb{F}_q$, and F be its fractional field. We attach the discrete group*

$$D_1 \colon = \operatorname{Coker}\left(\Omega_F^2 \xrightarrow{\text{diagonal}} \bigoplus_{\mathfrak{q}\in P_R^1} \left(\Omega_{F_\mathfrak{q}}^2/\Omega_{R_\mathfrak{q}}^2\right)\right),$$

where $F_\mathfrak{q}$, $R_\mathfrak{q}$ denote completions of F, R at height one prime $\mathfrak{q}$, respectively. Then with this module, we have the following canonical isomorphism:

$$\widehat{R} \xrightarrow{\sim} \operatorname{Hom}(D_1, \mathbb{Z}/p), \tag{3.4}$$

where Hom *denotes the set of homomorphisms between discrete groups.*

Proof. We will use Corollary 3.2. We put $M = \Omega_R^2$, $n = 2$, $i = 2$. Then, it reads

$$\widehat{R} \cong \operatorname{Hom}(H_{\mathfrak{m}_R}^2(R, \Omega_R^2), \mathbb{F}_q) \cong \operatorname{Hom}(H_{\mathfrak{m}_R}^2(R, \Omega_R^2), Z/p),$$

where the second isomorphism comes from trace homomorphism $\operatorname{Tr}\colon \mathbb{F}_q \to \mathbb{Z}/p$. Thus, the proof of the isomorphism (3.4) is reduced to

Claim 3.5. There holds an isomorphism $H_{\mathfrak{m}_R}^2(R, \Omega_R^2) \cong D_1$.

Proof. Let us consider the following localization sequence which is exact:

$$\begin{aligned} \cdots \to & H_{\mathfrak{m}_R}^1(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2) \to H^1(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2) \to H^1(X, \Omega_X^2) \to \\ & H_{\mathfrak{m}_R}^2(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2) \to H^2(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2) \to H^2(X, \Omega_X^2) \cdots, \end{aligned} \tag{3.5}$$

where $X = \operatorname{Spec} R \setminus \mathfrak{m}_R$ with the maximal ideal $\mathfrak{m}_R$ of R. We see that both groups $H^1(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2)$ and $H^2(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2)$ in (3.5) vanish, so we obtain the isomorphism

$$H^1(X, \Omega_X^2) \cong H_{\mathfrak{m}_R}^2(\operatorname{Spec} R, \Omega_{\operatorname{Spec} R}^2). \tag{3.6}$$

We analyze the group $H^1(X, \Omega_X^2)$. First we notice that the Krull dimension of X is 1. By considering the localization sequence in étale cohomology of X

with the attention to the fact that each height one prime of X corresponds bijectively to the unique height one prime of R, we obtain the following exact sequence:

$$\cdots \to H^0(X, \Omega_X^2) \to H^0(F, \Omega_F^2) \to \bigoplus_{\mathfrak{q}\in X^{(1)}} H^1_{\mathfrak{q}}(X, \Omega_X^2) \to H^1(X, \Omega_X^2) \to 0,$$

where the final 0 is obtained by replacing the group $H^1(F, \Omega_F^2)$ by 0. From this sequence, we can find that the group $H^1(X, \Omega_X^2)$ is explicitly expressed as

$$H^1(X, \Omega_X^2) \cong \left(\bigoplus_{\mathfrak{q}\in X^{(1)}} H^1_{\mathfrak{q}}(X, \Omega_X^2) \right) / \left(H^0(F, \Omega_F^2)/H^0(X, \Omega_X^2) \right).$$

By replacing $H^0(F, \Omega_F^2)$ and $H^0(X, \Omega_X^2)$ with Ω_F^2 and Ω_R^2 respectively, we obtain

$$H^1(X, \Omega_X^2) \cong \left(\bigoplus_{\mathfrak{q}\in X^{(1)}} H^1_{\mathfrak{q}}(X, \Omega_X^2) \right) /(\Omega_F^2 /\Omega_R^2). \tag{3.7}$$

We have $H^1_{\mathfrak{q}}(X, \Omega_X^2) \cong H^1_{\mathfrak{q}}(\operatorname{Spec} R^{\mathrm{h}}_{\mathfrak{q}}, \Omega^2_{\operatorname{Spec} R^{\mathrm{h}}_{\mathfrak{q}}})$, where $R^{\mathrm{h}}_{\mathfrak{q}}$ denotes the henselization of R at $\mathfrak{q}$. So, it holds that

$$H^1_{\mathfrak{q}}(X, \Omega_X^2) \cong H^1_{\mathfrak{q}}(\operatorname{Spec} R^{\mathrm{h}}_{\mathfrak{q}}, \Omega^2_{\operatorname{Spec} R^{\mathrm{h}}_{\mathfrak{q}}}) \cong \Omega^2_{F^{\mathrm{h}}_{\mathfrak{q}}}/\Omega^2_{R^{\mathrm{h}}_{\mathfrak{q}}} \cong \Omega^2_{F_{\mathfrak{q}}}/\Omega^2_{R_{\mathfrak{q}}}, \tag{3.8}$$

where $R_{\mathfrak{q}}$, $F_{\mathfrak{q}}$ denote the completion of $R^{\mathrm{h}}_{\mathfrak{q}}$, $F^{\mathrm{h}}_{\mathfrak{q}}$ at $\mathfrak{q}$, respectively. From (3.6), (3.7) and (3.8), the desired isomorphism $H^2_{\mathfrak{m}_R}(R, \Omega_R^2) \cong D_1$ follows. □

Next, we state the duality theorem for R/R^p.

Theorem 3.6. *Let R be same as in Theorem 3.4. We will define the discrete module D_2 by*

$$D_2 \colon = \operatorname{Coker}\left(\Omega_F^1 \xrightarrow{\text{diagonal}} \bigoplus_{\mathfrak{q}\in P_R^1} (\Omega^1_{F_{\mathfrak{q}}}/(\Omega^1_{F_{\mathfrak{q}},d=0}, \Omega^1_{R_{\mathfrak{q}}})) \right).$$

Then, there exists the following isomorphism:

$$(R/R^p) \cong \mathrm{Hom}(D_2,\, \mathbb{Z}/p).$$

Proof. For the proof, we need the Cartier operator C. Let us consider the exact sequences

$$0 \to R \overset{x \mapsto x^p}{\to} R \to R/R^p \to 0 \tag{3.9}$$

$$0 \to (R/R^p)^* \to \left(\bigoplus_{\mathfrak{q} \in P_R^1} \left(\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}} \right) \right) /\Omega^2_F \overset{C}{\to} \left(\bigoplus_{\mathfrak{q} \in P_R^1} \left(\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}} \right) \right) /\Omega^2_F \to 0, \tag{3.10}$$

where (3.10) is obtained by taking the Pontryagin dual of (3.9) together with Theorem 3.4. But the property of the Cartier operator C shows that

$$\mathrm{Ker}(C\colon \Omega^2_{F_\mathfrak{q}} \to \Omega^2_{F_\mathfrak{q}}) = d\,\Omega^1_{F_\mathfrak{q}} \cong (\Omega^1_{F_\mathfrak{q}}/\Omega^1_{F_\mathfrak{q},\, d=0}) \tag{3.11}$$

$$\mathrm{Ker}(C\colon \Omega^2_F \to \Omega^2_F) = d\,\Omega^1_F \cong (\Omega^1_F/\Omega^1_{F, d=0}), \tag{3.12}$$

where d denotes the differential operator. So by (3.11), it follows that

$$\mathrm{Ker}(C\colon \Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}} \to \Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}) = \Omega^1_{F_\mathfrak{q}}/(\Omega^1_{F_\mathfrak{q},\, d=0}, \Omega^1_{R_\mathfrak{q}}) \tag{3.13}$$

Now from (3.12) and (3.13), easy arguments show

$$\begin{aligned} &\mathrm{Ker}\Big(C\colon (\bigoplus_{\mathfrak{q} \in P_R^1} (\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}))/\Omega^2_F \to (\bigoplus_{\mathfrak{q} \in P_R^1} (\Omega^2_{F_\mathfrak{q}}/\Omega^2_{R_\mathfrak{q}}))/\Omega^2_F\Big) \\ &= \left(\bigoplus_{\mathfrak{q} \in P_R^1} (\Omega^1_{F_\mathfrak{q}}/(\Omega^1_{F_\mathfrak{q},\, d=0}, \Omega^1_{R_\mathfrak{q}})) \right) /\Omega^1_F, \end{aligned} \tag{3.14}$$

where the right hand side of (3.14) is nothing but D_2. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R/R^p & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(D_1,\, \mathbb{Z}/p) & \longrightarrow & \mathrm{Hom}(D_1,\, \mathbb{Z}/p) & \longrightarrow & \mathrm{Hom}(D_2,\, \mathbb{Z}/p) & \longrightarrow & 0 \end{array}$$

from which we get the desired bijectivity $R/R^p \cong \mathrm{Hom}(D_2, \mathbb{Z}/p)$. $\square$

Here, we state the duality theorem for F, F/F^p where F is the fractional field of R. For this, we explain the differential idele class groups. It is defined by

$$E_1 := (\overline{\prod_{\mathfrak{q}\in P_R^1}} \Omega^2_{F_\mathfrak{q}})/\Omega^2_F \tag{3.15}$$

$$E_2 := (\overline{\prod_{\mathfrak{q}\in P_R^1}} (\Omega^1_{F_\mathfrak{q}}/\Omega^1_{F_\mathfrak{q},d=0}))/(\Omega^1_F/\Omega^1_{F,d=0}), \tag{3.16}$$

where Ω^2_F, $(\Omega^1_F/\Omega^1_{F,d=0})$ are embedded diagonally into the numerators of (3.15) and (3.16), respectively. The restricted product in (3.15) is defined by the condition that any element in it lies in the group $\left(\prod_{\mathfrak{q}\in U} \Omega^2_{R_\mathfrak{q}}\right) \oplus \left(\bigoplus_{\mathfrak{q}\notin U} \Omega^2_{F_\mathfrak{q}}\right)$ for some open $U \subset \operatorname{Spec} R$. The restricted product in (3.16) is in the same way defined as (3.15).

Remark 3. We want to mention that the above defined differential idele class groups E_1, E_2 are also defined in terms of the differential idele class groups D_1, D_2. For this, we will prepare the modulus $M := \sum_{\mathfrak{q}\in P_R^1} n_\mathfrak{q}\overline{(\mathfrak{q})}$, where $n_\mathfrak{q} = 0$ for almost all $\mathfrak{q}$. Then, E_1, E_2 are given by

$$E_1 \cong \varprojlim{}_M \mathrm{Coker}\left(\Omega^2_F \xrightarrow{\text{diagonal}} \bigoplus_{\mathfrak{q}\in P_R^1} \left(\Omega^2_{F_\mathfrak{q}}/\mathfrak{q}^{n_\mathfrak{q}}\Omega^2_{R_\mathfrak{q}}\right)\right)$$

$$E_2 \cong \varprojlim{}_M \mathrm{Coker}\left(\Omega^1_F \xrightarrow{\text{diagonal}} \bigoplus_{\mathfrak{q}\in P_R^1} (\Omega^1_{F_\mathfrak{q}}/(\Omega^1_{F_\mathfrak{q},d=0}, \mathfrak{q}^{n_\mathfrak{q}}\Omega^1_{R_\mathfrak{q}}))\right),$$

where we put the discrete topology on $\mathrm{Coker}\left(\Omega^2_F \xrightarrow{\text{diagonal}} \bigoplus_{\mathfrak{q}\in P_R^1} \times \left(\Omega^2_{F_\mathfrak{q}}/\mathfrak{q}^{n_\mathfrak{q}}\Omega^2_{R_\mathfrak{q}}\right)\right)$ and put the inverse limit topology on E_1. We also treat in the same manner for E_2. The proof of these isomorphisms are understood without difficulty by direct calculation.

The duality results for F, F/F^p are stated as follows:

Theorem 3.7. *Let R be the same ring as in Theorem 3.4 and F be its fractional field. Then, there exists the following isomorphism*:

$$F \cong \mathrm{Hom}_{\mathrm{c}}(E_1, \mathbb{Z}/p),$$

where $\mathrm{Hom}_{\mathrm{c}}$ *denotes the set of homomorphism* $\chi\colon (\underline{\prod}_{\mathfrak{q}\in P^1_R} \Omega^2_{F_{\mathfrak{q}}})/\Omega^2_F \to \mathbb{Z}/p$ *such that* χ *annihilates* $\mathfrak{q}^{n_{\mathfrak{q}}}\Omega^2_{R_{\mathfrak{q}}}$ *for each* $\mathfrak{q}$ *with some* $n_{\mathfrak{q}} \geq 0$ *and almost all* $n_{\mathfrak{q}} = 0$.

Theorem 3.8. *For an arbitrary complete local ring R with fractional field F which satisfies the condition in Theorem 3.4, there exists the following canonical isomorphism*:

$$(F/F^p) \cong \mathrm{Hom}_{\mathrm{c}}(E_2, \mathbb{Z}/p),$$

where $\mathrm{Hom}_{\mathrm{c}}$ *denotes the same meaning as in Theorem 3.7.*

These theorems are immediately obtained from Theorem 3.4, Theorem 3.6 respectively by considering the fact that $F = \varinjlim_{f\in R} R[\frac{1}{f}]$ and the isomorphisms explained in Remark 3.

4. The Complete Discrete Valuation Field $K_{\mathfrak{p}}$

In this section, we will devote ourselves to prove the following class field theory for $K_{\mathfrak{p}}$:

Theorem 4.1. *For each height one prime $\mathfrak{p}$ of A, we consider the complete discrete valuation field $K_{\mathfrak{p}}$ (cf. Notations in page 5). Then, the filtered topological idele class group $F^iC_{K_{\mathfrak{p}}}$ $(i \geq -1)$ can be associated for $K_{\mathfrak{p}}$ and it holds the following dual reciprocity isomorphism*:

$$\rho^*_{K_{\mathfrak{p}}}\colon H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(F^0C_{K_{\mathfrak{p}}}, \mathbb{Z}/p), \tag{4.1}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the set of continuous homomorphisms of finite orders.*

We begin with the definition of the idele class group $C_{K_{\mathfrak{p}}}$ for $K_{\mathfrak{p}}$. Firstly, we define the three-dimensional local field $K_{\mathfrak{p},\mathfrak{q}}$ for each height one prime

$\mathfrak{q}$ of two-dimensional complete normal local ring $\widetilde{A/\mathfrak{p}}$ ($\widetilde{A/\mathfrak{p}}$ denotes the normalization of $A/\mathfrak{p}$). $K_{\mathfrak{p},\mathfrak{q}}$ is the complete discrete valuation field defined uniquely up to isomorphism by the following two conditions:

1) $K_{\mathfrak{p}} \subset K_{\mathfrak{p},\mathfrak{q}}$ and $\mathfrak{m}_{K_{\mathfrak{p}}} \mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}} = \mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$, where $\mathfrak{m}_{K_{\mathfrak{p}}}$ and $\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ denote the maximal ideals of the valuation rings $\mathcal{O}_{K_{\mathfrak{p}}}$ and $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$, respectively.

2) The residue field $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ of the valuation ring $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$ of $K_{\mathfrak{p},\mathfrak{q}}$ coincides with the fractional field $\kappa(\mathfrak{p})_{\mathfrak{q}}$ of the complete discrete valuation ring $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}} := \varprojlim_n (\widetilde{A/\mathfrak{p}})_{(\mathfrak{q})}/\mathfrak{q}^n$. Equivalently explained, $\kappa(\mathfrak{p})_{\mathfrak{q}}$ is the completion of $\kappa(\mathfrak{p})$ at $\mathfrak{q}$.

We consider the Milnor K-group $K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$ of the two-dimensional complete local ring $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]$ and put the filtration $F^i K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$ for an integer $i \geq 0$ by

$$F^i K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) := \mathrm{Ker}\left(K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \to K_3^M(((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}/\mathfrak{q}^i)[[u_{\mathfrak{p}}]])\right).$$

More explicitly, we have

(4.2)
$$F^i K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \cong \{1 + \mathfrak{q}^i(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]], (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]^*, (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]^*\}$$

which, for example, is obtained by faithfully following arguments of Denis-Stein [De-St] who proved $\mathrm{Ker}\left(K_2^M(R) \to K_2^M(R/a)\right) \cong \{1 + a\,R,\ R^*\}$. We return to the definition of $C_{K_{\mathfrak{p}}}$. For an arbitrary modulus $M := \sum_{\mathfrak{q} \in P_{\widetilde{A/\mathfrak{p}}}} n_{\mathfrak{q}}\,\overline{(\mathfrak{q})}$ of $\widetilde{A/\mathfrak{p}}$ ($n_{\mathfrak{q}}$ is 0 except for finitely many primes) and the integer $i \geq 0$, we introduce the group $C_{K_{\mathfrak{p}}}(M,\, i)$ by

$$(4.3)\qquad C_{K_{\mathfrak{p}}}(M,\, i) := \mathrm{Coker}\left(K_3^M(K_{\mathfrak{p}}) \xrightarrow{diagonal} \bigoplus_{\mathfrak{q} \in P^1_{\widetilde{A/\mathfrak{p}}}} \left(\frac{K_3^M(K_{\mathfrak{p},\mathfrak{q}})}{F^{n_{\mathfrak{q}}} K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \cap U^i K_3^M(K_{\mathfrak{p},\mathfrak{q}})}\right)\right),$$

where we consider $F^{n_{\mathfrak{q}}} K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \subset K_3^M(K_{\mathfrak{p},\mathfrak{q}})$ by taking its image induced from the natural map

$$(4.4)\qquad (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]] \hookrightarrow \kappa(\mathfrak{p})_{\mathfrak{q}}[[u_{\mathfrak{p}}]] \hookrightarrow \kappa(\mathfrak{p})_{\mathfrak{q}}((u_{\mathfrak{p}})) \cong K_{\mathfrak{p},\mathfrak{q}}.$$

Now we can give

DEFINITION 6. Our topological idele class group is defined by

$$C_{K_{\mathfrak{p}}} = \varprojlim_{M,\, i\geq 0} C_{K_{\mathfrak{p}}}(M,\, i), \tag{4.5}$$

where M runs over all moduli of $\widetilde{A/\mathfrak{p}}$. For the topology, we put the discrete topology on each $C_{K_{\mathfrak{p}}}(M,\, i)$ and endow the induced inverse limit topology on $C_{K_{\mathfrak{p}}}$.

REMARK 4. This idele class group $C_{K_{\mathfrak{p}}}$ resembles that of complete discrete valuation fields having global residue fields. In fact, for such c.d.v.f. as has $\mathbb{F}_p(X)$ as its residue field, Kato gave the definition of its idele class group in [Ka1] III and proved its class field theory. It is found that our way of construction is essentially equivalent to Kato's one.

The next task is to construct the reciprocity pairing

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) \times C_{K_{\mathfrak{p}}}/p \to \mathbb{Z}/p. \tag{4.6}$$

First, choose isomorphisms

$$K_{\mathfrak{p}} \cong \kappa(\mathfrak{p})((u_{\mathfrak{p}})), \qquad \kappa(\mathfrak{p})_{\mathfrak{q}} = \mathrm{Frac}((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}) \cong \mathbb{F}_q((s_{\mathfrak{q}}))((t_{\mathfrak{q}})). \tag{4.7}$$

Then $K_{\mathfrak{p},\mathfrak{q}}$ is explicitly rewritten as

$$K_{\mathfrak{p},\mathfrak{q}} \cong \mathbb{F}_q((s_{\mathfrak{q}}))((t_{\mathfrak{q}}))((u_{\mathfrak{p}})). \tag{4.8}$$

On the other hand, for each three-dimensional local field $K_{\mathfrak{p},\mathfrak{q}}$, there exists the following residue pairing by Kato-Parshin:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{p},\mathfrak{q}},\, \mathbb{Z}/p) \times K^M_3(K_{\mathfrak{p},\mathfrak{q}})/p \to \mathbb{Z}/p \tag{4.9}$$

defined by $\left(\chi_{\mathfrak{p},\mathfrak{q}}, \overline{(a_{\mathfrak{p},\mathfrak{q}}, b_{\mathfrak{p},\mathfrak{q}}, c_{\mathfrak{p},\mathfrak{q}})}\right) \mapsto \mathrm{Res}_{\frac{ds_{\mathfrak{q}}}{s_{\mathfrak{q}}}\wedge\frac{dt_{\mathfrak{q}}}{t_{\mathfrak{q}}}\wedge\frac{du_{\mathfrak{p}}}{u_{\mathfrak{p}}}}(\chi_{\mathfrak{p},\mathfrak{q}}\frac{da_{\mathfrak{p},\mathfrak{q}}}{a_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{db_{\mathfrak{p},\mathfrak{q}}}{b_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{dc_{\mathfrak{p},\mathfrak{q}}}{c_{\mathfrak{p},\mathfrak{q}}}) \in \mathbb{Z}/p$, where $\mathrm{Res}_{\frac{ds_{\mathfrak{p}}}{s_{\mathfrak{p}}}\wedge\frac{dt_{\mathfrak{q}}}{t_{\mathfrak{q}}}\wedge\frac{du_{\mathfrak{q}}}{u_{\mathfrak{q}}}}$ denotes $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(c_{-1,-1,-1}) \in \mathbb{F}_p$. Here, $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$ is the trace operator and $c_{-1,-1,-1}$ is the coefficient of $\frac{ds_{\mathfrak{p}}}{s_{\mathfrak{p}}}\wedge\frac{dt_{\mathfrak{q}}}{t_{\mathfrak{q}}}\wedge\frac{du_{\mathfrak{q}}}{u_{\mathfrak{q}}}$ in the three-form $(\chi_{\mathfrak{p},\mathfrak{q}}\frac{da_{\mathfrak{p},\mathfrak{q}}}{a_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{db_{\mathfrak{p},\mathfrak{q}}}{b_{\mathfrak{p},\mathfrak{q}}} \wedge \frac{dc_{\mathfrak{p},\mathfrak{q}}}{c_{\mathfrak{p},\mathfrak{q}}}) \in \Omega^3_{K_{\mathfrak{p},\mathfrak{q}}}$. Fix $\chi \in H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)$. Then, by using the restriction homomorphism

$$r_{\mathfrak{p},\mathfrak{q}} \colon H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) \to H^1_{\mathrm{Gal}}(K_{\mathfrak{p},\mathfrak{q}},\, \mathbb{Z}/p) \tag{4.10}$$

in Galois cohomology, we can consider the above pairing by Kato-Parshin for each $\mathfrak{q}$. After gathering over all $\mathfrak{q}$, except for the reciprocity law which we will prove just below, it is found that the gathered pairings factor through $C_{K_\mathfrak{p}}(M,\, i)$ for some M and i which depend on χ. viz;

$$\chi \times C_{K_\mathfrak{p}}(M,\, i)/p \to \mathbb{Z}/p. \tag{4.11}$$

By considering all χ in the left hand side which is equivalent to consider the inductive limit for all χ, the corresponding right hand side in the pairing (4.11) turns out to be the inverse limit over all $M,\, i$, that is, $\varprojlim_{M,\, i\geq 0} C_{K_\mathfrak{p}}(M,\, i)/p = C_{K_\mathfrak{p}}/p$. Thus, we get the desired pairing (4.6). As mentioned above, we will prove the following reciprocity law:

Proposition 4.2. *For an arbitrary elements $a \in K_3^M(K_\mathfrak{p})$, $\chi \in H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p)$, the sum $\sum_{\mathfrak{q}\in P_1(\widetilde{A/\mathfrak{p}})}(r_{\mathfrak{p},\mathfrak{q}}(\chi),\, a)_\mathfrak{q} = 0$, where $r_{\mathfrak{p},\mathfrak{q}}$ is defined in* (4.10) *and* $(\ ,\ )_\mathfrak{q}$ *denotes the Kato-Parshin pairing* (4.9).

Proof. First, we remark that we can take an integer $i \geq 0$ such that $r_{\mathfrak{p},\mathfrak{q}}(\chi)$ annihilates the subgroup $\overline{U^i K_3^M(K_{\mathfrak{p},\mathfrak{q}})}$ for all $\mathfrak{q}$. We can see Theorem 2.7 shows that any element $\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} \in K_3^M(K_\mathfrak{p})/p$ can be written as

$$\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} = \overline{(1+\alpha\, u_\mathfrak{p}^j,\, \beta,\, \gamma)} \tag{4.12}$$

with $\alpha,\, \beta,\, \gamma \in \kappa(\mathfrak{p})$ modulo $U^i K_3^M(K_{\mathfrak{p},\mathfrak{q}})$. On the other hand, the representation of $K_\mathfrak{p}$ in (4.7) allows us to interpret each $\chi_\mathfrak{p} \in H^1_{\mathrm{Gal}}(K_\mathfrak{p},\, \mathbb{Z}/p) \cong K_\mathfrak{p}/(\mathfrak{P}-1)K_\mathfrak{p}$ $\big((\mathfrak{P}-1)x := x^p - x\big)$ as

$$\chi_\mathfrak{p} = \sum_{n\gg -\infty} \overline{\delta_n\, u_\mathfrak{p}^n},$$

where each $\delta_n \in \kappa(\mathfrak{p})$ and $\overline{\delta_n\, u_\mathfrak{p}^n}$ denotes the image of $\delta_n\, u_\mathfrak{p}^n$ in $K_\mathfrak{p}/(\mathfrak{P}-1)K_\mathfrak{p}$. But for $n \geq 1$, it is easily seen that the pair $\Big(\overline{\delta_n\, u_\mathfrak{p}^n},\, \overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})}\Big)$ goes to 0 under the pairing (4.11). Thus, we have only to check the above proposition in the case

$$\chi_\mathfrak{p} = \overline{(\frac{\delta_n}{u_\mathfrak{p}^n})}$$

for an arbitrary $n \geq 0$. Using the representation (4.12), we find the residue pairing $\left(\overline{(\frac{\delta_n}{u_{\mathfrak{p}}^n})}, \overline{(1+\alpha\, u_{\mathfrak{p}}^j,\, \beta,\, \gamma)}\right) = 0$ if $j \nmid n$. Otherwise, $n = kj$ and then it becomes

$$(4.13)\qquad \left(\overline{(\frac{\delta_{kj}}{u_{\mathfrak{p}}^{kj}})}, \overline{(1+\alpha\, u_{\mathfrak{p}}^j,\, \beta,\, \gamma)}\right) \mapsto \sum_{\mathfrak{q}\in P^1_{\widetilde{A}/\mathfrak{p}}} \mathrm{Res}_{\frac{ds_{\mathfrak{q}}}{s_{\mathfrak{q}}}\wedge\frac{dt_{\mathfrak{q}}}{t_{\mathfrak{q}}}} \left((-1)^{k-1} j\overline{\delta_{kj}}(\alpha^k\, \frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma})\right).$$

But if we consider $(-1)^{k-1}j\overline{\delta_{kj}} \in \kappa(\mathfrak{p})$ and $\alpha^k\,\frac{d\beta}{\beta} \wedge \frac{d\gamma}{\gamma} \in \Omega^2_{\kappa(\mathfrak{p})}$, the claim that the right hand side of (4.13) $= 0$ is nothing but the reciprocity law (in the class field theory) for $\kappa(\mathfrak{p})$, which is proved by Kato (cf. Proposition 7 in [Ka3]). Hence, we are done. □

Next, we give the definition of filtrations on $C_{K_{\mathfrak{p}}}$ and $H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)$.

Definition 7. For an arbitrary positive integer $i \geq 0$, we define the decreasing filtration $F^i C_{K_{\mathfrak{p}}}$ on $C_{K_{\mathfrak{p}}}$ and the increasing filtration $N_i H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)$ on $H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)$ by

(4.14) 1) $F^iC_{K_{\mathfrak{p}}} := \mathrm{Ker}\left(C_{K_{\mathfrak{p}}} \twoheadrightarrow \varprojlim_M C_{K_{\mathfrak{p}}}(M,\, i)\right)$

2) $N_iH^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) := \{\mathrm{Im}\colon x \mapsto (K_{\mathfrak{p}}/(\mathfrak{P}-1)K_{\mathfrak{p}}) \mid\ x \in K_{\mathfrak{p}}$ satisfies $v_{u_{\mathfrak{p}}}(x) \geq -i\}$, where $(\mathfrak{P}-1)$ is defined by $(\mathfrak{P}-1)\,(x) \;=\; x^p - x$.

For $N_iH^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)$, we see an isomorphism

$$(4.15)\qquad N_\infty H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p)/N_0H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) \cong H^2_{\mathfrak{p}}(A_{\mathfrak{p}},\, \mathbb{Z}/p)$$

$(N_\infty H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) = H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p))$ which immediately follows from the (exact) localization sequence

$$0 \to H^1_{\mathrm{Gal}}(A_{\mathfrak{p}},\, \mathbb{Z}/p) \to H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \mathbb{Z}/p) \to H^2_{\mathfrak{p}}(A_{\mathfrak{p}},\, \mathbb{Z}/p) \to 0.$$

We also consider the filtration $F^i(C_{K_\mathfrak{p}}/m)$ on $C_{K_\mathfrak{p}}/m$ for an arbitrary integer $m > 0$ by

$$F^i(C_{K_\mathfrak{p}}/m) := \mathrm{Image}\left(F^i C_{K_\mathfrak{p}} \to C_{K_\mathfrak{p}}/m\right) \tag{4.16}$$

and take the following graded-quotients:

$$Gr^i(C_{K_\mathfrak{p}}/p) := F^i(C_{K_\mathfrak{p}}/p)/F^{i+1}(C_{K_\mathfrak{p}}/p) \tag{4.17}$$

$$Gr_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) := N_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)/N_{i-1} H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p). \tag{4.18}$$

For these graded-quotients, we have

Lemma 4.3. *There hold the following isomorphisms*:

$$1.\ \ \textit{if } p \nmid i,\ \ \kappa(\mathfrak{p}) \xrightarrow{\sim} Gr_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \tag{4.19}$$

$$\text{by}\ \ \kappa(\mathfrak{p}) \ni a \mapsto \overline{(\frac{a}{u_\mathfrak{p}^i})} \in Gr_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$$

$$2.\ \ \textit{if } p \mid i,\ \ \kappa(\mathfrak{p})/\kappa(\mathfrak{p})^p \xrightarrow{\sim} Gr_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \tag{4.20}$$

$$\text{by}\ \ \kappa(\mathfrak{p}) \ni a \mapsto \overline{(\frac{a}{u_\mathfrak{p}^i})} \in Gr_i H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p).$$

This lemma is checked without any difficulty by explicit calculation.

Lemma 4.4. *For $i \geq 0$, there exist the following surjections*:

$$1.\ \ Gr^0(C_{K_\mathfrak{p}}/p) = 0. \tag{4.21}$$

$$2.\ \ \textit{if}\ \ p \nmid i,\ \ \Big(\prod_{\mathfrak{q} \in P^1_{\widehat{A}/\mathfrak{p}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}}\Big)/\Omega^2_{\kappa(\mathfrak{p})} \twoheadrightarrow Gr^i(C_{K_\mathfrak{p}}/p). \tag{4.22}$$

$$3.\ \ \textit{if}\ \ p \mid i\ (>0), \tag{4.23}$$

$$\Big(\prod_{\mathfrak{q} \in P^1_{\widehat{A}/\mathfrak{p}}} (\Omega^1_{\kappa(\mathfrak{p})_\mathfrak{q}}/\Omega^1_{\kappa(\mathfrak{p})_\mathfrak{q},\, d=0})\Big)/(\Omega^1_{\kappa(\mathfrak{p})}/\Omega^1_{\kappa(\mathfrak{p}),\, d=0}) \twoheadrightarrow Gr^i(C_{K_\mathfrak{p}}/p).$$

As an important fact, each left hand side of (4.22), (4.23) *coincides with the differential ideles defined at* (3.15), (3.16), *respectively.*

Proof. By Kato's Theorem 2.7, (4.22) and (4.23) follow by considering the definition of $C_{K_\mathfrak{p}}$ in Definition 6 (one should pay our attention to the fact that in the definition of $C_{K_{\mathfrak{p}}}$, the filtration $F^\bullet C_{K_{\mathfrak{p}}}$ is taken into account). Thus, only the case 1 requires the proof. It suffices to prove $Gr^0(K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p) = 0$ for each component of $C_{K_{\mathfrak{p}}}/p$. According to Theorem 2.7 again, we have $Gr^0(K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p) \cong K_3^M(\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}})/p$, where $\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ denotes the maximal ideal of the valuation ring $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}$. But the residual field $\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}}$ is, by definition, $\kappa(\mathfrak{p})_{\mathfrak{q}}$ which is two-dimensional local field of characteristic $p > 0$. Thus we have isomorphisms $Gr^0(K_3^M(K_{\mathfrak{p},\mathfrak{q}})/p) \cong K_3^M(\mathcal{O}_{K_{\mathfrak{p},\mathfrak{q}}}/\mathfrak{m}_{K_{\mathfrak{p},\mathfrak{q}}})/p \cong K_3^M(\kappa(\mathfrak{p})_{\mathfrak{q}})/p$. But thanks to Bloch-Gabber-Kato Theorem 2.5, we have $K_3^M(\kappa(\mathfrak{p})_{\mathfrak{q}})/p \cong \Omega^3_{\kappa(\mathfrak{p})_{\mathfrak{q}},\log}$ which is zero considering that fact that the number of p-bases of $\kappa(\mathfrak{p})_{\mathfrak{q}}$, which is two-dimensional local field, is two. $\square$

Now, we are in the stage to prove Theorem 4.1.

Proof of Theorem 4.1. First, we see that the pairing (4.6) induces the pairing

$$H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \times F^0\left(C_{K_{\mathfrak{p}}}/p\right) \to \mathbb{Z}/p \tag{4.24}$$

from which we obtain the dual reciprocity homomorphism

$$H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \to \mathrm{Hom}_{\mathrm{cont}}(F^0\left(C_{K_{\mathfrak{p}}}/p\right), \mathbb{Z}/p). \tag{4.25}$$

We will prove that (4.25) is an isomorphism. For the proof, we will define the filtration on $H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p)$ induced from those on $H^2_{\mathrm{Gal}}(K_{\mathfrak{p}}, \mathbb{Z}/p)$ by using (4.15). We denote by the same symbol $N_iH^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p)$ these filtrations. Thus, we have

(4.26)
$$Gr_0H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) = 0, \quad Gr_iH^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \cong Gr_iH^1_{\mathrm{Gal}}(K_{\mathfrak{p}}, \mathbb{Z}/p) \quad (i > 0).$$

We consider the following pairing between each gr-quotients:

$$Gr_nH^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \times Gr^n(C_{K_{\mathfrak{p}}}/p) \to \mathbb{Z}/p \tag{4.27}$$

which is induced from the pairing (4.24). The well-definedness of this pairing is easily checked. It is also found that the pairing (4.27) induces the homomorphism

$$Gr_nH^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \to \mathrm{Hom}_{\mathrm{cont}}(Gr^n(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p), \tag{4.28}$$

where we put the induced topology on $Gr^n(C_{K_\mathfrak{p}}/p)$ from that on $C_{K_\mathfrak{p}}$. The key result is

Claim 4.5. The homomorphism (4.28) is an isomorphism.

Proof. We treat the case $p \nmid n$ (the case $p \mid n$ is also proved without any change). Let us consider the commutative diagram

$$\begin{array}{ccc} Gr_n H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p) & \longrightarrow & \mathrm{Hom}_{\mathrm{cont}}(Gr^n(C_{K_\mathfrak{p}}/p), \mathbb{Z}/p) \\ \big\downarrow\cong & & \big\downarrow\cap \\ \kappa(\mathfrak{p}) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{cont}}((\prod_{\mathfrak{q}\in P^1_{\widetilde{A}/\mathfrak{p}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}})/\Omega^2_{\kappa(\mathfrak{p})}, \mathbb{Z}/p), \end{array} \tag{4.29}$$

where the left vertical isomorphism follows from Lemma 4.3 together with the fact that $Gr_n H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p) \cong Gr_n H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ and the lower horizontal isomorphism follows from Theorem 3.7. Finally, the right vertical inclusion follows from Lemma 4.4 together with the fact that for an arbitrary continuous homomorphism $\chi \in \mathrm{Hom}_{\mathrm{cont}}(Gr^n(C_{K_\mathfrak{p}}/p), \mathbb{Z}/p)$, the composition $(\prod_{\mathfrak{q}\in P^1_{\widetilde{A}/\mathfrak{p}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}})/\Omega^2_{\kappa(\mathfrak{p})} \twoheadrightarrow Gr^n(C_{K_\mathfrak{p}}/p) \xrightarrow{\chi} \mathbb{Z}/p$ becomes the continuous homomorphism of $(\prod_{\mathfrak{q}\in P^1_{\widetilde{A}/\mathfrak{p}}} \Omega^2_{\kappa(\mathfrak{p})_\mathfrak{q}})/\Omega^2_{\kappa(\mathfrak{p})}$ in the sense of Theorem 3.7. Now, the commutativity of the diagram (4.29) shows the desired bijectivity of (4.28). □

We prove Theorem 4.1 by downward induction, which is the original method by Kato in his proof of higher dimensional local class field theory [Ka1]. We begin the proof. First, we see that the injectivity of (4.25) follows immediately from Claim 4.5. So, we will show the surjectivity of (4.25). Let us take an arbitrary element $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^0\left(C_{K_\mathfrak{p}}/p\right), \mathbb{Z}/p)$. By the continuity of χ, χ annihilates some filtration $F^j\left(C_{K_\mathfrak{p}}/p\right)$ on $C_{K_\mathfrak{p}}/p$. So, if we will restrict χ to the subgroup $F^{j-1}\left(C_{K_\mathfrak{p}}/p\right)$, we have the induced homomorphism $\chi\colon Gr^{j-1}\left(C_{K_\mathfrak{p}}/p\right) \to \mathbb{Z}/p$. But Claim 4.5 shows that such continuous homomorphism always comes from $Gr_{j-1}H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p)$. Thus we may take some element $\chi_{j-1} \in N_j H^2_\mathfrak{p}(A_\mathfrak{p}, \mathbb{Z}/p)$ such that $\chi_{j-1} = \chi$ on $F^{j-1}(C_{K_\mathfrak{p}}/p)$. We proceed this way taking the new character $(\chi - \chi_{j-1})$ on

$F^{j-2}(C_{K_\mathfrak{p}}/p)$. In the same way, we have $(\chi - \chi_{j-1})\colon Gr^{j-2}(C_{K_\mathfrak{p}}/p) \to \mathbb{Z}/p$ and this comes from $N_{j-1}H^2_\mathfrak{p}(A_\mathfrak{p},\, \mathbb{Z}/p)$. That is, we have $\chi_{j-2} = (\chi - \chi_{j-1})$ on $F^{j-2}C_{K_\mathfrak{p}}/p$, where $\chi_{j-2} \in N_{j-1}H^2_\mathfrak{p}(A_\mathfrak{p},\, \mathbb{Z}/p)$. By the inductive procedure, it is easily found that there exist elements $\chi_{j-1}, \chi_{j-2}, \ldots, \chi_1 \in N_j H^2_\mathfrak{p}(A_\mathfrak{p},\, \mathbb{Z}/p) \subset H^2_\mathfrak{p}(A_\mathfrak{p},\, \mathbb{Z}/p)$ (N_j is the increasing filtration) such that

$$\chi = \chi_{j-1} + \chi_{j-2} + \ldots + \chi_1 \quad \text{on} \quad F^1\left(C_{K_\mathfrak{p}}/p\right). \tag{4.30}$$

But as proved in Lemma 4.4, $Gr^0(C_{K_\mathfrak{p}}/p) = 0$, which shows that $F^0\left(C_{K_\mathfrak{p}}/p\right) = F^1\left(C_{K_\mathfrak{p}}/p\right)$. So from (4.30), it follows that $\chi \in H^2_\mathfrak{p}(A_\mathfrak{p},\, \mathbb{Z}/p)$, which shows the desired surjectivity of (4.25). Thus the bijectivity of (4.1) is established. □

5. Proof of the Existence Theorem (p-Primary Parts)

The following existence theorem for p primary parts is our aim:

Theorem 5.1. *Let $A := \mathbb{F}_q[[X_1, X_2, X_3]]$ and K be the fractional field of A. Then there exists the canonical dual reciprocity isomorphism*

$$\rho_K^*\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Q}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(C_K,\, \mathbb{Q}_p/\mathbb{Z}_p), \tag{5.1}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ *denotes the set of all continuous homomorphisms from C_K to $\mathbb{Q}_p/\mathbb{Z}_p$ of finite order.*

The proof of Theorem 5.1 is given at the end of this section by using the key commutative diagram (5.71). For the dual reciprocity homomorphism ρ_K^*, its definition mod p, i.e. ρ_K^*/p is given at (5.14). The idea of constructing general ρ_K^*/p^m goes same way, and the reason why we focus our consideration to ρ_K^*/p is explained by the following lemma

Lemma 5.2. *The isomorphism* (5.1) *is deduced from the isomorphism*

$$\rho_K^*/p\colon H^1_{\mathrm{Gal}}(K,\, \mathbb{Z}/p) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(C_K,\, \mathbb{Z}/p). \tag{5.2}$$

Proof. From the short exact sequence

$$0 \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p \to 0,$$

we get the long exact sequence of the Galois cohomology

$$0 \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^n) \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^{n+1}) \to H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \to 0,$$

where the vanishing $H^2_{\mathrm{Gal}}(K, \mathbb{Z}/p) = 0$ comes from the well-known fact that Galois cohomology with p-torsion coefficients vanishes for fields of characteristic $p > 0$. On the other hand, we have the exact sequence

$$C_K/p \to C_K/p^{n+1} \to C_K/p^n \to 0.$$

So, by taking the Pontryagin dual of this exact sequence, we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathrm{cont}}(C_K/p^n, \mathbb{Z}/p^n) &\to \mathrm{Hom}_{\mathrm{cont}}(C_K/p^{n+1}, \mathbb{Z}/p^{n+1}) \\ &\to \mathrm{Hom}_{\mathrm{cont}}(C_K/p, \mathbb{Z}/p). \end{aligned}$$

Now, let us consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^n) \arrow[r] \arrow[d, dashed, "{\rho_K}^*/p^n"] & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^{n+1}) \arrow[r] \arrow[d, dashed, "{\rho_K}^*/p^{n+1}"] & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \arrow[r] \arrow[d, dashed, "{\rho_K}^*/p"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}((C_K/p^n), \mathbb{Z}/p^n) \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}((C_K/p^{n+1}), \mathbb{Z}/p^{n+1}) \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}((C_K/p), \mathbb{Z}/p). & \end{tikzcd}$$

Applying the snake lemma to this diagram, we see that the bijectivity of ${\rho_K}^*/p^{n+1}$ is deduced from those of ${\rho_K}^*/p^n$ and ${\rho_K}^*/p$. □

By this lemma, we have only to prove the bijectivity of (5.2). We will begin to construct the (dual) reciprocity map ρ_K^*/p, which is the key to analyze the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. First, we construct the reciprocity pairing

$$H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \times C_K/p \to \mathbb{Z}/p \tag{5.3}$$

which is the special case of the reciprocity pairing

$$H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^m) \times C_K/p^m \to \mathbb{Z}/p^m$$

for an arbitrary $m \geq 1$. The idea of constructing this pairing is to identify the group of characters of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ with the quotient $W_m(K)/(1-\gamma)W_m(K)$, by which we mean the isomorphism $H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p^m) \cong W_m(K)/(1-\gamma)W_m(K)$. The next task is to construct the pairing

$W_m(K)/(1-\gamma)W_m(K) \times C_K/p^m \to \mathbb{Z}/p^m$. We also want to put the stress on the fact that these pairings were first introduced by A. N. Parshin and K. Kato in their celebrated papers (loc.cit.) completely independently. Especially, Parshin's geometric exposition in [Pa1] is highly understandable. For details on the reciprocity pairing, we refer the paper of Parshin. As stated above, we have only to construct the pairing (5.3). Take an arbitrary $\chi \in H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p)$. Then, by the restriction map $r_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$: $H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \to H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \mathbb{Z}/p)$ in Galois cohomology, we can send χ into $H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \mathbb{Z}/p)$. Recall the following canonical reciprocity pairing by Kato-Parshin in three-dimensional local field $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$:

$$H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \mathbb{Z}/p) \times K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p \to \mathbb{Z}/p \tag{5.4}$$

by which for $\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} \in H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \mathbb{Z}/p)$ and $\overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})} \in K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$, the pair $(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})})$ goes as

$$\begin{aligned}&(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})})\\ &\quad\mapsto \mathrm{Res}_{\frac{ds_{\mathfrak{p}_\mathfrak{m}}}{s_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dt_{\mathfrak{p}_\mathfrak{m}}}{t_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{du_{\mathfrak{p}_\mathfrak{m}}}{u_{\mathfrak{p}_\mathfrak{m}}}}(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} \frac{da_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}} \wedge \frac{db_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}} \wedge \frac{dc_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}).\end{aligned}$$

Gathering the pairing (5.4) for all $K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}$, we get the pairing

$$H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) \times \left(\prod\nolimits'_{\mathfrak{m}\in P_A^2,\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right) \to \mathbb{Z}/p, \tag{5.5}$$

which can be explicitly written as

$$\begin{aligned}&\left(\chi, \overline{(a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})}_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}\right)\\ &\quad\mapsto \sum\nolimits_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} \mathrm{Res}_{\frac{ds_{\mathfrak{p}_\mathfrak{m}}}{s_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{dt_{\mathfrak{p}_\mathfrak{m}}}{t_{\mathfrak{p}_\mathfrak{m}}}\wedge\frac{du_{\mathfrak{p}_\mathfrak{m}}}{u_{\mathfrak{p}_\mathfrak{m}}}}(\chi_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}} \frac{da_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{a_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}} \wedge \frac{db_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{b_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}} \wedge \frac{dc_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}{c_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}}).\end{aligned}$$

By conditions 1), 2) of $\left(\prod'_{\mathfrak{m}\in P_A^2,\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right)$ stated in Lemma 2.4, the above pairing (5.5) is maybe well defined. Now, we prove the very important reciprocity property.

Reciprocity Proposition 5.3. *Both $K_3^M(K_\mathfrak{m})/p$ and $K_3^M(K_\mathfrak{p})/p$, if embedded diagonally into $\prod'_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$, $\prod'_{\mathfrak{p}_\mathfrak{m}\to\mathfrak{p}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$*

$\left(\subset \prod'_{\mathfrak{m}\in P_A^2, \mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p\right)$, *respectively, are annihilated by an arbitrary element* χ *of* $H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p)$ *in the pairing* (5.5).

Proof. We begin with $K_3^M(K_\mathfrak{m})/p$. In this case, we have to prove that in the pairing

$$(5.6) \qquad H^1_{\mathrm{Gal}}(K_\mathfrak{m}, \mathbb{Z}/p) \times \underline{\prod}_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p \to \mathbb{Z}/p,$$

where $\underline{\prod}$ is defined by $\mathfrak{p}_\mathfrak{m}$-component $a_{\mathfrak{p}_\mathfrak{m}}$ lies in $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})$ for almost all $\mathfrak{p}_\mathfrak{m} \in P_\mathfrak{m}^1$, the diagonal image of $K_3^M(K_\mathfrak{m})/p$ into $\underline{\prod}_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/p$ is annihilated by an arbitrary $\chi_\mathfrak{m} \in H^1_{\mathrm{Gal}}(K_\mathfrak{m}, \mathbb{Z}/p)$. But this is nothing but Kato's reciprocity law for two-dimensional complete normal local rings whose residue fields are higher dimensional local fields which was already established as Proposition 7 in [Ka3].

Next, we prove the reciprocity law for $K_\mathfrak{p}$. In this case, we have to prove that any pair $\left(\chi_\mathfrak{p}, \overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})}\right)$ with $\chi_\mathfrak{p} \in H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p)$ and $\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} \in K_3^M(K_\mathfrak{p})/p$ goes to zero under the pairing (5.5). By using Bloch-Gabber-Kato Theorem 2.5, we can consider $\overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})} \in \Omega^3_{K_\mathfrak{p},\log}$. So, by the cup product

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \mathbb{Z}/p) \times H^0_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}) \to H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}),$$

we can consider $\left(\chi_\mathfrak{p}, \overline{(a_\mathfrak{p}, b_\mathfrak{p}, c_\mathfrak{p})}\right) \in H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log})$. Now we can see that the reciprocity law for $K_\mathfrak{p}$ is equivalent to the existence of the following complex:

Claim 5.4. There exists a natural complex

$$H^1_{\mathrm{Gal}}(K_\mathfrak{p}, \Omega^3_{K_\mathfrak{p},\log}) \to \bigoplus_{\substack{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1, \mathfrak{p}_\mathfrak{m}\mapsto\mathfrak{p} \\ \mathfrak{m}\in P_A^2}} H^1_{\mathrm{Gal}}(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}},\log}) \xrightarrow{\textit{addition}} \mathbb{Z}/p,$$

where the first map is the restriction homomorphism in Galois cohomology.

PROOF. Consider the coniveau-spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in (\mathrm{Spec} A)^{(p)}} H_x^{p+q}(\mathrm{Spec} A,\, \Omega^3_{A,\log}[-3]) \Longrightarrow H_{\text{ét}}^{p+q}(\mathrm{Spec} A,\, \Omega^3_{A,\log}[-3]),$$

where $(\mathrm{Spec} A)^{(p)}$ denotes the set of primes of codimension p. By the E_1-term sequence

$$E_1^{1,4} \xrightarrow{d_1^{1,4}} E_1^{2,4} \xrightarrow{d_1^{2,4}} E_1^{3,4} \to 0,$$

we get

$$\begin{aligned}(5.7)\qquad \bigoplus_{\mathfrak{p} \in P_A^1} H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{A,\log}) \to \bigoplus_{\mathfrak{m} \in P_A^2} & H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\log}) \\ & \to H^4_{\mathfrak{m}_A}(\mathrm{Spec} A,\, \Omega^3_{\mathrm{Spec} A,\log}) \to 0.\end{aligned}$$

Purity Theorem 2.6 shows $H^4_{\mathfrak{m}_A}(\mathrm{Spec} A,\, \Omega^3_{\mathrm{Spec} A,\log}) \cong H^1_{\text{ét}}(\mathrm{Spec}\, \mathbb{F}_q,\, \mathbb{Z}/p) \cong \mathrm{Hom}(\widehat{\mathbb{Z}},\, \mathbb{Z}/p) \cong \mathbb{Z}/p$. Next we analyze the group $H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{A,\log})$. From the localization sequences for $A^{\mathrm{h}}_{\mathfrak{p}}$, we see an isomorphism $H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{p}},\log}) \cong H^1_{\mathrm{Gal}}(K^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{K^{\mathrm{h}}_{\mathfrak{p}},\log})$. In the same way, $H^2_{\mathfrak{p}}(A_{\mathfrak{p}},\, \Omega^3_{A_{\mathfrak{p}},\log}) \cong H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log})$, where $A_{\mathfrak{p}}$, $K_{\mathfrak{p}}$ denote the completion of $A^{\mathrm{h}}_{\mathfrak{p}}$, $K^{\mathrm{h}}_{\mathfrak{p}}$, respectively. But Theorem 2.6 shows $H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{p}},\log}) \cong H^1(\kappa(\mathfrak{p}),\, \Omega^2_{\kappa(\mathfrak{p}),\log}) \cong H^2_{\mathfrak{p}}(A_{\mathfrak{p}},\, \Omega^3_{A_{\mathfrak{p}},\log})$. So putting all together, we have an isomorphism

$$H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{p}},\log}) \cong H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}),$$

by which we can rewrite (5.7) as

$$(5.8)\qquad \bigoplus_{\mathfrak{p} \in P_A^1} H^1_{\mathrm{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}) \to \bigoplus_{\mathfrak{m} \in P_A^2} H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\log}) \to \mathbb{Z}/p \to 0.$$

We will analyze the group $H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\log})$. By excision, $H^3_{\mathfrak{m}}(A,\, \Omega^3_{A,\log}) \cong H^3_{\mathfrak{m}}(A^{\mathrm{h}}_{\mathfrak{m}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{m}},\log})$ and by Theorem 2.6, it holds $H^3_{\mathfrak{m}}(A^{\mathrm{h}}_{\mathfrak{m}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{m}},\log}) \cong H^1_{\text{ét}}(\kappa(\mathfrak{m}),\, \Omega^1_{\kappa(\mathfrak{m}),\log}) \cong H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \Omega^3_{A_{\mathfrak{m}},\log})$, where $A_{\mathfrak{m}}$ denotes the completion of $A^{\mathrm{h}}_{\mathfrak{m}}$ at $\mathfrak{m}$. So we can change the henselian ring $A^{\mathrm{h}}_{\mathfrak{m}}$ to the complete local ring $A_{\mathfrak{m}}$. By considering the localization sequence on $A_{\mathfrak{m}}$, we see an isomorphism

$$(5.9) \qquad H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \Omega^3_{A_{\mathfrak{m}},\log}) \cong H^2_{\text{ét}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}),$$

where $T_{\mathfrak{m}} := \operatorname{Spec} A_{\mathfrak{m}} \setminus \{\mathfrak{m}\}$. We will calculate the group $H^2_{\text{ét}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log})$. Consider the localization sequence

(5.10)

$$H^1_{\text{Gal}}(K_{\mathfrak{m}},\, \Omega^3_{K_{\mathfrak{m}},\log}) \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} H^2_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \to H^2_{\text{ét}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \to 0.$$

In this sequence, we can replace each $H^2_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log})$ by $H^2_{\mathfrak{p}_{\mathfrak{m}}}(T^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{T^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log})$, where $T^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} \cong \operatorname{Spec} A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$. So, $H^2_{\mathfrak{p}_{\mathfrak{m}}}(T^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{T^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \cong H^2_{\mathfrak{p}_{\mathfrak{m}}}(\operatorname{Spec} A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log})$. It also follows from Theorem 2.6 that $H^2_{\mathfrak{p}_{\mathfrak{m}}}(\operatorname{Spec} A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \cong H^2_{\mathfrak{p}_{\mathfrak{m}}}(\operatorname{Spec} A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log})$, where $A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ is the completion of $A^{\mathrm{h}}_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ at $\mathfrak{p}_{\mathfrak{m}}$. Moreover, the localization sequence for $A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ shows an isomorphism $H^2_{\mathfrak{p}_{\mathfrak{m}}}(\operatorname{Spec} A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \cong H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log})$. Putting all together, we have

$$(5.11) \qquad H^2_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \cong H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}),$$

where $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ denotes the fractional field of the complete discrete local ring $A_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$.

From (5.9), (5.10), (5.11), the complex (5.8) is rewritten as

$$(5.12) \qquad \bigoplus_{\mathfrak{p} \in P^1_A} H^1_{\text{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}) \to \bigoplus_{\mathfrak{m} \in P^2_A} \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \xrightarrow{\textit{addition}} \mathbb{Z}/p \to 0.$$

From this, we can take out the desired complex in Claim 5.4 by focusing our attention to the component $H^1_{\text{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log})$ in (5.12). That is, we can deduce

$$H^1_{\text{Gal}}(K_{\mathfrak{p}},\, \Omega^3_{K_{\mathfrak{p}},\log}) \to \bigoplus_{\substack{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}},\ \mathfrak{p}_{\mathfrak{m}} \mapsto \mathfrak{p} \\ \mathfrak{m} \in P^2_A}} H^1_{\text{ét}}(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\, \Omega^3_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}},\log}) \xrightarrow{\textit{addition}} \mathbb{Z}/p. \ \square$$

By Reciprocity Proposition 5.3, the above pairing (5.5) factors as

$$(5.13)\qquad H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)\times\Bigl({\prod}'_{\mathfrak{m}\in P^2_A,\,\mathfrak{p}_{\mathfrak{m}}\in P^1_{\mathfrak{m}}}K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/p\Bigr)\Big/ \prod\nolimits_{\mathfrak{m}\in P^2_A}K_3^M(K_{\mathfrak{m}})/p\prod\nolimits_{\mathfrak{p}\in P^1_A}K_3^M(K_{\mathfrak{p}})/p \to \mathbb{Z}/p,$$

where the right hand side of the pairing (5.13) is D_K/p defined in (2.). Further, it is found that each element $\chi\in H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)$ annihilates F^MD_K for some modulus M. So, by taking the limit on M, we get the reciprocity pairing

$$H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)\times \varprojlim{}_M(D_K/F^MD_K)/p\to\mathbb{Z}/p.$$

The isomorphism in (2.7) rewrites this as

$$H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)\times C_K/p\to\mathbb{Z}/p.$$

By considering the dual, we at last get the dual reciprocity homomorphism

$$(5.14)\qquad \rho^*_K/p\colon H^1_{\mathrm{Gal}}(K,\,\mathbb{Z}/p)\to\mathrm{Hom}_{\mathrm{cont}}(C_K,\,\mathbb{Z}/p).$$

We prove the bijectivity of (5.2). First, we explain briefly our approach. We consider the scheme $X=\operatorname{Spec}A\smallsetminus\{\mathfrak{m}_A\}$ which is the regular excellent scheme of Krull-dimension two. Then, we consider the closed subscheme $Z=\cup_{i=1\dots m}\mathfrak{m}_i$ of X where each $\mathfrak{m}_i$ is a closed point of X (hence Z is codimension two). For the pair (X,Z), we consider the localization sequence ((5.16) below) in the étale cohomology with $\mathbb{Z}/p$-coefficient. This is the first step.

In the second step, we consider the localization sequence obtained from the pair $(X\setminus Z,\,W\setminus Z)$, where $W=\cup_{j=1\dots n}\overline{(\mathfrak{p}_j)}$ is the union of finite codimension one closed sub-schemes of X ($\overline{(\mathfrak{p}_j)}$ denotes the closure of $\mathfrak{p}_j$ in X). This is (5.17) below. Under these settings, we consider the limit $(\cup\,\mathfrak{m}_i)\to P^2_A$ and $(\cup\,\mathfrak{p}_j)\to P^1_A$ set-theoretically, where $\mathfrak{m}_i$ and $\mathfrak{p}_j$ run over all height two primes of A and all height one primes of A, respectively. As an important fact, under the limit of the above procedure, the localization sequence (5.17) below turns out to involve the very important Galois cohomology group $H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p)$ which is nothing but the Pontryagin dual of $\mathrm{Gal}(K^{\mathrm{ab}}/K)/p$. The following is the key theorem in this section:

THEOREM 5.5. *There exists an exact complex*

$$0 \to \mathbb{Z}/p \to H^1_{et}(K, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P^1_A} H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m}\in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p). \tag{5.15}$$

PROOF. We will prove this theorem by several steps and it is completed by Lemma 5.12. Consider the following localization sequence:

$$\begin{aligned}\cdots \longrightarrow & \bigoplus_{i=1\,\ldots n} H^1_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) \longrightarrow \\ & H^1_{et}(X, \mathbb{Z}/p) \longrightarrow H^1_{et}(X \setminus \cup_i \mathfrak{m}_i, \mathbb{Z}/p) \longrightarrow \\ & \bigoplus_{i=1\,\ldots n} H^2_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) \longrightarrow \\ & H^2_{et}(X, \mathbb{Z}/p) \longrightarrow H^2_{et}(X \setminus \cup_i \mathfrak{m}_i, \mathbb{Z}/p) \longrightarrow \\ & \bigoplus_{i=1\,\ldots n} H^3_{\mathfrak{m}_i}(X_{et}, \mathbb{Z}/p) \longrightarrow \\ & H^3_{et}(X, \mathbb{Z}/p) \longrightarrow H^3_{et}(X \setminus \cup_i \mathfrak{m}_i, \mathbb{Z}/p) \longrightarrow \cdots .\end{aligned} \tag{5.16}$$

Also consider the following second localization sequence:

$$\begin{aligned}\to & \bigoplus_{j=1\,\ldots m} H^1_{\overline{\mathfrak{p}_j}\setminus(\cup_i \mathfrak{m}_i)}(X \setminus (\cup_i \mathfrak{m}_i)) \to H^1_{et}(X \setminus (\cup_i \mathfrak{m}_i)) \to \\ & H^1_{et}(X \setminus ((\cup_i \mathfrak{m}_i) \cup (\cup_j \overline{\mathfrak{p}_j}))) \to \\ & \bigoplus_{j=1\,\ldots m} H^2_{\overline{\mathfrak{p}_j}\setminus(\cup_i \mathfrak{m}_i)}(X \setminus (\cup_i \mathfrak{m}_i)) \to H^2_{et}(X \setminus (\cup_i \mathfrak{m}_i)) \to \\ & H^2_{et}(X \setminus ((\cup_i \mathfrak{m}_i) \cup (\cup_j \overline{\mathfrak{p}_j}))) \to,\end{aligned} \tag{5.17}$$

where $H^1_{et}(X \setminus (\cup_i \mathfrak{m}_i)) := H^1_{et}(X \setminus (\underset{i=1,..,n}{\cup} \mathfrak{m}_i), \mathbb{Z}/p)$ and $H^1_{et}(X \setminus ((\cup_i \mathfrak{m}_i) \cup (\cup_j \overline{\mathfrak{p}_j}))) := H^1_{et}(X \setminus ((\underset{i=1,..,n}{\cup} \mathfrak{m}_i) \cup (\underset{j=1,..,m}{\cup} \overline{\mathfrak{p}_j})), \mathbb{Z}/p)$, respectively. We put

$$L^k := \varinjlim_{U_\theta} H^k_{\text{ét}}(U_\theta, \mathbb{Z}/p), \tag{5.18}$$

where U_θ runs over all open subschemes of X such that each complement $X \setminus U_\theta$ is a closed subscheme of X of codimension two. Then, under the increasing limit of $(\cup\, \mathfrak{m}_i) \to P_A^2,\ (\cup\, \mathfrak{p}_j) \to P_A^1$, we get the following exact sequence from (5.17):

(5.19)
$$\bigoplus_{\mathfrak{p}\in P_A^1} H^1_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \mathbb{Z}/p) \to L^1 \to H^1_{et}(K,\, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P_A^1} H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \mathbb{Z}/p) \to L^2 \to 0,$$

where the final 0 is obtained by replacing $H^2_{et}(K,\, \mathbb{Z}/p)$ with 0 (cf. [SGA4], X), and we define $H^i_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \mathbb{Z}/p)\colon = \varinjlim_{U\supset\mathfrak{p}} H^i_{\mathfrak{p}}(U,\, \mathbb{Z}/p)$, where U runs over all open subschemes of X containing $\mathfrak{p}$ and $A^{\mathrm{h}}_{\mathfrak{p}}$ denotes the henselization of A at $\mathfrak{p}$. We remark $H^1_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}},\, \mathbb{Z}/p) = 0$.

LEMMA 5.6. $L^1 \cong \mathbb{Z}/p$.

PROOF. First, we see $H^j_{\mathfrak{m}_i}(X_{et},\, \mathbb{Z}/p) \cong H^j_{\mathfrak{m}_i}(A^{\mathrm{h}}_{\mathfrak{m}_i},\, \mathbb{Z}/p)$ by excision. We will now check that this group vanishes for $i = 1, 2$. Consider the exact sequence

$$\begin{aligned} H^0_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}) &\to H^1_{\mathfrak{m}_i}(\mathbb{Z}/p) \to H^1_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}) \to H^1_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}) \\ &\to H^2_{\mathfrak{m}_i}(\mathbb{Z}/p) \to H^2_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}) \xrightarrow{\iota_{\mathfrak{m}_i}} H^2_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}), \end{aligned}$$

where we abbreviate the cohomology group $H^j_{\mathfrak{m}_i}(A^{\mathrm{h}}_{\mathfrak{m}_i},\, \mathcal{O}_{A^{\mathrm{h}}_{\mathfrak{m}_i}})$ as $H^j_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i})$ in the above. We can see, using Corollary 3.3 with $M = \mathcal{O}_{A^{\mathrm{h}}_{\mathfrak{m}_i}}$ and $n = 2$, that $H^j_{\mathfrak{m}_i}(\mathcal{O}^{\mathrm{h}}_{\mathfrak{m}_i}) = 0$ for $j = 0, 1$. Further, we see that the map $\iota_{\mathfrak{m}_i}$ is injective. This follows from the surjectivity of the homomorphism $(1 - C)\colon \Omega^2_{A^{\mathrm{h}}_{\mathfrak{m}_i}} \twoheadrightarrow \Omega^2_{A^{\mathrm{h}}_{\mathfrak{m}_i}}$ (cf. proof of Lemma 5.8 below). Consequently, it holds that $H^1_{\mathfrak{m}_i}(X_{et},\, \mathbb{Z}/p) = H^2_{\mathfrak{m}_i}(X_{et},\, \mathbb{Z}/p) = 0$ for an arbitrary height two prime $\mathfrak{m}_i$. Hence by (5.16), we have

(5.20)
$$L^1 \cong H^1_{\text{ét}}(X, \mathbb{Z}/p).$$

But we have the localization sequence

(5.21)
$$H^1_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p) \to H^1_{et}(A,\, \mathbb{Z}/p) \to H^1_{et}(X,\, \mathbb{Z}/p) \to H^2_{\mathfrak{m}_A}(A,\, \mathbb{Z}/p)$$

and in the same way, it holds that $H^1_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = H^2_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = 0$ (this can be seen, for example, by considering the long exact sequence

$$\begin{aligned} 0 &\to H^1_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to H^1_{\mathfrak{m}_A}(A, \mathcal{O}_A) \to H^1_{\mathfrak{m}_A}(A, \mathcal{O}_A) \\ &\to H^2_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \to H^2_{\mathfrak{m}_A}(A, \mathcal{O}_A) \end{aligned}$$

together with the vanishing $H^i_{\mathfrak{m}_A}(A, \mathcal{O}_A) = 0$ $(i = 0, 1, 2)$ obtained by Corollary 3.3). By (5.21), we get $H^1_{et}(X, \mathbb{Z}/p) \cong H^1_{et}(A, \mathbb{Z}/p) \cong \mathbb{Z}/p$, which together with an isomorphism (5.20) furnishes lemma. $\square$

Next, we analyze L^2. For this, we have the following result:

PROPOSITION 5.7. *There exists a canonical injection*

$$L^2 \hookrightarrow \bigoplus_{\mathfrak{m} \in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p),$$

where we consider $K^M_3(A_{\mathfrak{m}})$ as an abstract discrete module.

PROOF. We use the localization sequence (5.16). First, we state a lemma.

LEMMA 5.8. *We have $H^2_{et}(X, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = 0$.*

PROOF. The isomorphism in the statement of lemma follows from (5.16) together with the theorem by M. Artin on the cohomological dimension of affine schemes of characteristic p for p-torsion sheaves (cf. loc.cit.). We begin the proof. Consider the commutative diagram

(5.22)
$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & H^3_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \arrow[r] \arrow[d, dashed] & H^3_{\mathfrak{m}_A}(\mathrm{Spec}\, A, \mathcal{O}_A) \arrow[r] \arrow[d, dashed, "\cong"] & H^3_{\mathfrak{m}_A}(\mathrm{Spec}\, A, \mathcal{O}_A) \arrow[r] \arrow[d, dashed, "\cong"] & H^4_{\mathfrak{m}_A}(A, \mathbb{Z}/p) \arrow[r] \arrow[d, dashed] & 0 \\ & 0 \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_A, \mathbb{Z}/p) \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_A, \mathbb{Z}/p) \arrow[r] & \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A,\log}, \mathbb{Z}/p) \arrow[r] & 0, \end{tikzcd}$$

where the extreme left and right zeros in the upper row are obtained from the vanishings $H^2_{\mathfrak{m}_A}(\mathrm{Spec} A, \mathcal{O}_A) = H^4_{\mathfrak{m}_A}(\mathrm{Spec} A, \mathcal{O}_A) = 0$ which follows directly from Corollary 3.3. The vertical isomorphisms also come from Corollary 3.3 noticing the fact that the residue field of $\kappa(\mathfrak{m}_A)$ is a finite field. Further, the bottom exact sequence is obtained by considering the Pontryagin dual of the short exact sequence

$$0 \to \Omega^3_{A,\log} \to \Omega^3_A \xrightarrow{1-C} \Omega^3_A \to 0,$$

where the surjection $\Omega^3_A \xrightarrow{1-C} \Omega^3_A$ is proved by the explicit calculation (we put the $\mathfrak{m}_A$-adic topology on Ω^3_A). The desired vanishing $H^3_{\mathfrak{m}_A}(A, \mathbb{Z}/p) = 0$ follows from the diagram (5.22) immediately. □

By Lemma 5.8, the exact sequence (5.16) provides the injection $L^2 \hookrightarrow \bigoplus_{\mathfrak{m}} H^3_{\mathfrak{m}}(X, \mathbb{Z}/p)$. Further, the excision theorem in étale cohomology provides isomorphisms

$$H^3_{\mathfrak{m}}(X, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}}(A^{\mathrm{h}}_{\mathfrak{m}}, \mathbb{Z}/p) \cong H^3_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathbb{Z}/p),$$

where the final isomorphism between henselian local ring $A^{\mathrm{h}}_{\mathfrak{m}}$ and the complete local ring $A_{\mathfrak{m}}$ is assured by Hartshorne (cf. [Ha]). So, Proposition 5.7 follows from

Lemma 5.9. *There exists the following injective homomorphism*:

$$H^3_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(K^M_3(A_{\mathfrak{m}})/p, \mathbb{Z}/p).$$

Proof. From the Artin-Schreier sequence $0 \to \mathbb{Z}/p \to \mathcal{O}_{A_{\mathfrak{m}}} \xrightarrow{x^p - x} \mathcal{O}_{A_{\mathfrak{m}}} \to 0$, we deduce the long exact sequence

(5.23)

$$H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathbb{Z}/p) \to H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) \xrightarrow{x^p - x} H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) \to H^3_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathbb{Z}/p) \to 0$$

of the local cohomology, where the final 0 is obtained by the vanishing $H^3_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}})$ which follows from Corollary 3.3. We need a Sub-lemma.

Sub-lemma 5.10. *We put the inverse limit topology on $A_{\mathfrak{m}}$ induced from each locally compact group $A_{\mathfrak{m}}/\mathfrak{m}^n$ (notice that each $A_{\mathfrak{m}}/\mathfrak{m}^n$ $(n \geq 1)$ is a finite vector space over one dimensional local field $\kappa(\mathfrak{m})$, hence has the natural induced topology). We can also induce the topology on $\Omega^3_{A_{\mathfrak{m}}}$ from the above mentioned topology on $A_{\mathfrak{m}}$. Then, we have*

$$H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p). \tag{5.24}$$

PROOF. By Corollary 3.3, we have an isomorphism

$$H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})}, \kappa(\mathfrak{m})), \tag{5.25}$$

where $\mathrm{Hom}_{\mathrm{cont}}$ considers the set of all continuous homomorphisms with respect to "$\mathfrak{m}_A$-adic topology" on $A_{\mathfrak{m}}$. As $\kappa(\mathfrak{m})$ is the usual one-dimensional local field, the residue pairing shows an isomorphism $\kappa(\mathfrak{m}) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^1_{\kappa(\mathfrak{m})}, \mathbb{Z}/p)$, where we put the usual locally compact topology on $\Omega^1_{\kappa(\mathfrak{m})}$ induced from that of $\kappa(\mathfrak{m})$. Inserting this isomorphism in (5.25), we have

$$\begin{aligned} H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) &\cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})}, \kappa(\mathfrak{m})) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})}, \mathrm{Hom}_{\mathrm{cont}}(\Omega^1_{\kappa(\mathfrak{m})}, \mathbb{Z}/p)) \\ &\cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})} \otimes_{\kappa(\mathfrak{m})} \Omega^1_{\kappa(\mathfrak{m})}, \mathbb{Z}/p), \end{aligned} \tag{5.26}$$

where we put the product topology on $\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})} \otimes_{\kappa(\mathfrak{m})} \Omega^1_{\kappa(\mathfrak{m})}$. But it is not difficult to see an isomorphism

$$\Omega^2_{A_{\mathfrak{m}}/\kappa(\mathfrak{m})} \otimes_{\kappa(\mathfrak{m})} \Omega^1_{\kappa(\mathfrak{m})} \cong \Omega^3_{A_{\mathfrak{m}}} \tag{5.27}$$

between topological groups, where $\Omega^3_{A_{\mathfrak{m}}}$ has the topology mentioned in Sublemma 5.10 (we can also see that $\Omega^3_{A_{\mathfrak{m}}}$ is the absolute differential module over $\mathbb{Z}$). Now the isomorphism (5.24) follows from (5.26) and (5.27). □

We return to the proof of Lemma 5.9. By considering the exact sequence

$$0 \to \Omega^3_{A_{\mathfrak{m}},\log} \to \Omega^3_{A_{\mathfrak{m}}} \xrightarrow{1-C} \Omega^3_{A_{\mathfrak{m}}},$$

we get the following commutative diagram:

$$\begin{array}{ccccccc} H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) & \longrightarrow & H^2_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathcal{O}_{A_{\mathfrak{m}}}) & \longrightarrow & H^3_{\mathfrak{m}}(A_{\mathfrak{m}}, \mathbb{Z}/p) & \longrightarrow & 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p) & \to & \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}}}, \mathbb{Z}/p) & \to & \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}},\log}, \mathbb{Z}/p) & \to & 0, \end{array}$$

where we endow on $\Omega^3_{A_{\mathfrak{m}}}$ the topology stated in Sub-lemma 5.10. The top horizontal exact sequence comes from (5.23) and the vertical isomorphism is (5.24) stated in Sub-lemma 5.10.

From this diagram, we get the isomorphism

$$H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}},\log},\, \mathbb{Z}/p).$$

As $\mathrm{Hom}_{\mathrm{cont}}(\Omega^3_{A_{\mathfrak{m}},\log},\, \mathbb{Z}/p) \subset \mathrm{Hom}(\Omega^3_{A_{\mathfrak{m}},\log},\, \mathbb{Z}/p)$ (Hom denotes the set of all homomorphisms between discrete abelian groups), we have the injection

$$H^3_{\mathfrak{m}}(A_{\mathfrak{m}},\, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(\Omega^3_{A_{\mathfrak{m}},\log},\, \mathbb{Z}/p).$$

Thus, Lemma 5.9 follows from this injection together with the following claim:

Claim 5.11. There exists a canonical injective homomorphism

$$\mathrm{Hom}(\Omega^3_{A_{\mathfrak{m}},\log},\, \mathbb{Z}/p) \hookrightarrow \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}),\, \mathbb{Z}/p).$$

Proof. By considering dual, we prove that there exists a surjective homomorphism

$$K^M_3(A_{\mathfrak{m}})/p \twoheadrightarrow \Omega^3_{A_{\mathfrak{m}},\log} \tag{5.28}$$

between discrete modules. Consider the localization sequence on $T_{\mathfrak{m}} = \operatorname{Spec} A_{\mathfrak{m}} \setminus \{\mathfrak{m}\}$

$$\begin{aligned} 0 \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} H^0_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) &\to H^0_{et}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \to H^0_{et}(K_{\mathfrak{m}},\, \Omega^3_{K_{\mathfrak{m}},\log}) \to \\ \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} H^1_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) &\to H^1_{et}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}) \to H^1_{et}(K_{\mathfrak{m}},\, \Omega^3_{K_{\mathfrak{m}},\log}) \to \dots \end{aligned}$$

From the vanishing $H^0_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}}, \Omega^3_{T_{\mathfrak{m}},\log}) = 0$ and the equality $H^0_{et}(T_{\mathfrak{m}}, \Omega^3_{T_{\mathfrak{m}},\log}) = \Omega^3_{A_{\mathfrak{m}},\log}$, we deduce the exact sequence

$$0 \to \Omega^3_{A_{\mathfrak{m}},\log} \to \Omega^3_{K_{\mathfrak{m}},\log} \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} H^1_{\mathfrak{p}_{\mathfrak{m}}}(T_{\mathfrak{m}},\, \Omega^3_{T_{\mathfrak{m}},\log}).$$

But Purity Theorem 2.6 rewrites this sequence as

$$0 \to \Omega^3_{A_{\mathfrak{m}},\log} \to \Omega^3_{K_{\mathfrak{m}},\log} \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} \Omega^2_{\kappa(\mathfrak{p}_{\mathfrak{m}}),\log}. \tag{5.29}$$

On the other hand by works of [Pan], [Gra], [Po1], [So], the following Gersten-Quillen complex is known to be exact:

(5.30)

$$K^M_3(A_{\mathfrak{m}})/p \to K^M_3(K_{\mathfrak{m}})/p \to \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} K^M_2(\kappa(\mathfrak{p}_{\mathfrak{m}}))/p \to K^M_1(\kappa(\mathfrak{m}))/p \to 0.$$

Combining (5.29) and (5.30), we get the commutative diagram

$$\begin{array}{ccccccc}
 & & K^M_3(A_{\mathfrak{m}})/p & \longrightarrow & K^M_3(K_{\mathfrak{m}})/p & \to & \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} K^M_2(\kappa(\mathfrak{p}_{\mathfrak{m}}))/p \\
 & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \Omega^3_{A_{\mathfrak{m}},\log} & \longrightarrow & \Omega^3_{K_{\mathfrak{m}},\log} & \longrightarrow & \bigoplus_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} \Omega^2_{\kappa(\mathfrak{p}_{\mathfrak{m}}),\log},
\end{array}$$

where the vertical isomorphisms follow from Theorem 2.5. From this diagram, it is easily seen that the extreme left vertical arrow is surjective, which is nothing but our desired surjectivity (5.28). Thus, we finished the proof of Claim 5.11, Lemma 5.9. Consequently, Proposition 5.7 is proved completely. □

By Lemma 5.6 and Proposition 5.7, the exact sequence (5.19) can be rewritten as

$$\begin{aligned}
0 &\to \mathbb{Z}/p \to H^1_{et}(K, \mathbb{Z}/p) \\
&\to \bigoplus_{\mathfrak{p} \in P^1_A} H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m} \in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p).
\end{aligned}$$

From this, the existence of the exact complex (5.15) immediately follows once we show

LEMMA 5.12. *For each henselian discrete valuation ring $A^{\mathrm{h}}_{\mathfrak{p}}$, we have an isomorphism*

$$H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \cong H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p),$$

where $A_{\mathfrak{p}}$ is the completion of $A^{\mathrm{h}}_{\mathfrak{p}}$ at $\mathfrak{p}$.

Proof. By using the localization sequence

$$0 \to H^1_{et}(A^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \to H^1_{et}(K^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \to H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \to 0,$$

we get the isomorphism

$$H^2_{\mathfrak{p}}(A^{\mathrm{h}}_{\mathfrak{p}}, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}((K^{\mathrm{h}}_{\mathfrak{p}})^{\mathrm{ab}}/(K^{\mathrm{h}}_{\mathfrak{p}})^{\mathrm{ur}}), \mathbb{Z}/p), \tag{5.31}$$

where $(K^{\mathrm{h}}_{\mathfrak{p}})^{\mathrm{ur}}$ denotes the maximal unramified extension of $K^{\mathrm{h}}_{\mathfrak{p}}$. In the same way, we get the following isomorphism:

$$H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K^{\mathrm{ab}}_{\mathfrak{p}}/K^{\mathrm{ur}}_{\mathfrak{p}}), \mathbb{Z}/p), \tag{5.32}$$

where $K^{\mathrm{ur}}_{\mathfrak{p}}$ denotes the maximal unramified extension of $K_{\mathfrak{p}}$. But, Artin's approximation theorem in [A] provides the isomorphism

$$\mathrm{Gal}(\overline{K^{\mathrm{h}}_{\mathfrak{p}}}/K^{\mathrm{h}}_{\mathfrak{p}}) \cong \mathrm{Gal}(\overline{K_{\mathfrak{p}}}/K_{\mathfrak{p}}). \tag{5.33}$$

Now, the isomorphism in Lemma 5.12 immediately follows from (5.31) and (5.32) by (5.33). □

Now, we finish the proof of Theorem 5.5. Next, we state another key complex.

Theorem 5.13. *There exists a complex*

$$\begin{aligned} 0 &\to \mathbb{Z}/p \to \mathrm{Hom}_{\mathrm{cont}}(C_K/p, \mathbb{Z}/p) \\ &\to \bigoplus_{\mathfrak{p}\in P^1_A} \mathrm{Hom}_{\mathrm{cont}}(F^0(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p) \to \bigoplus_{\mathfrak{m}\in P^2_A} \mathrm{Hom}(K^M_3(A_{\mathfrak{m}}), \mathbb{Z}/p) \end{aligned} \tag{5.34}$$

which is exact at $\mathbb{Z}/p$ and $\mathrm{Hom}_{\mathrm{cont}}(C_K/p, \mathbb{Z}/p)$.

Proof. The existence of the complex (5.34) is proved by the following two steps:

Step 1. There exists a complex

(5.35)

$$0 \to \mathbb{Z}/p \to \mathrm{Hom}_{\mathrm{cont}}(C_K/p, \mathbb{Z}/p) \to \bigoplus_{\mathfrak{p}\in P^1_A} \mathrm{Hom}_{\mathrm{cont}}(F^0(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p)$$

which is exact at $\mathbb{Z}/p$ and $\mathrm{Hom}_{\mathrm{cont}}(C_K/p,\, \mathbb{Z}/p)$.

Step 2. There exists a complex

$$\begin{aligned}(5.36)\qquad \mathrm{Hom}_{\mathrm{cont}}(C_K/p,\, \mathbb{Z}/p) &\to \bigoplus_{\mathfrak{p}\in P_A^1} \mathrm{Hom}_{\mathrm{cont}}(F^0(C_{K_\mathfrak{p}}/p),\, \mathbb{Z}/p)\\ &\to \bigoplus_{\mathfrak{m}\in P_A^2} \mathrm{Hom}(K_3^M(A_\mathfrak{m}),\, \mathbb{Z}/p).\end{aligned}$$

Proof of Step 1. We first prove the exactness of

$$(5.37)\qquad F^0C_K \to C_K \to \mathbb{Z} \to 0.$$

By definition of F^0C_K in (2.10), it is sufficient to prove the exactness

$$(5.38)\qquad F^0C_K(M) \to C_K(M) \to \mathbb{Z} \to 0$$

for each modulus M. But this is rewritten as

$$\begin{aligned}(5.39)\qquad &\mathrm{Coker}\left(\bigoplus_{\mathfrak{p}\in P_A^1} U^0K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m}\in P_A^2} F^0C_\mathfrak{m}(M)\right)\\ &\to \mathrm{Coker}\left(\bigoplus_{\mathfrak{p}\in P_A^1} K_3^M(K_\mathfrak{p}) \to \bigoplus_{\mathfrak{m}\in P_A^2} C_\mathfrak{m}(M)\right) \to \mathbb{Z}.\end{aligned}$$

For each $\mathfrak{m} \in P_A^2$, we have the canonical isomorphism

$$(5.40)\qquad \mathrm{Coker}\,\big(F^0C_\mathfrak{m}(M) \to C_\mathfrak{m}(M)\big) \cong \kappa(\mathfrak{m})^*,$$

which immediately follows from the isomorphism $\frac{K_3^M(K_{\mathfrak{m},\mathfrak{m}_\mathfrak{p}})}{U^0K_3^M(K_{\mathfrak{m},\mathfrak{m}_\mathfrak{p}})} \cong K_2^M(\kappa(\mathfrak{p}_\mathfrak{m}))$ (cf. Theorem 2.7 (1)) together with the following exact Gersten-Quillen complex for $A_\mathfrak{m}$ established by Panin (loc.cit.):

$$(5.41)\qquad K_3^M(K_\mathfrak{m}) \to \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P_\mathfrak{m}^1} K_2^M(\kappa(\mathfrak{p}_\mathfrak{m})) \to \kappa(\mathfrak{m})^* \to 0.$$

Hence, the proof of the exactness of (5.39) is reduced to that of

$$(5.42)\qquad \bigoplus_{\mathfrak{p}\in P_A^1} K_3^M(K_{\mathfrak{p}}) \to \bigoplus_{\mathfrak{m}\in P_A^2} \kappa(\mathfrak{m})^* \to \mathbb{Z}.$$

As $\operatorname{Coker}\left(K_3^M(K_{\mathfrak{p}}) \overset{\text{diagonal}}{\longrightarrow} \bigoplus \frac{K_3^M(K_{\mathfrak{m},\mathfrak{m}_{\mathfrak{p}}})}{U^0K_3^M(K_{\mathfrak{m},\mathfrak{m}_{\mathfrak{p}}})}\right) \cong \operatorname{Coker}\left(\frac{K_3^M(K_{\mathfrak{p}})}{U^0K_3^M(K_{\mathfrak{p}})} \overset{\text{diagonal}}{\longrightarrow} \bigoplus \frac{K_3^M(K_{\mathfrak{m},\mathfrak{m}_{\mathfrak{p}}})}{U^0K_3^M(K_{\mathfrak{m},\mathfrak{m}_{\mathfrak{p}}})}\right)$ together with an isomorphism $K_3^M(K_{\mathfrak{p}})/U^0K_3^M(K_{\mathfrak{p}}) \cong K_2^M(\kappa(\mathfrak{p}))$ in Theorem 2.7 (1), we can replace $K_3^M(K_{\mathfrak{p}})$ with $K_2^M(\kappa(\mathfrak{p}))$. Thus, the aiming sequence in (5.42) becomes

$$(5.43)\qquad \bigoplus_{\mathfrak{p}\in P_A^1} K_2^M(\kappa(\mathfrak{p})) \to \bigoplus_{\mathfrak{m}\in P_A^2} \kappa(\mathfrak{m})^* \to \mathbb{Z}.$$

But this is nothing but the Gersten-Quillen complex of A proved to be exact by Panin (cf. loc.cit.), which shows the exactness of (5.37). By putting $\bigotimes_{\mathbb{Z}} \mathbb{Z}/p$ to (5.37) and taking the Pontryagin dual, we get the exact sequence

$$(5.44)\qquad 0 \to \mathbb{Z}/p \to \operatorname{Hom}_{\text{cont}}(C_K/p, \mathbb{Z}/p) \to \operatorname{Hom}_{\text{cont}}(F^0C_K/p, \mathbb{Z}/p).$$

The proof of the existence of the complex (5.35) is obtained immediately from (5.44) and the following proposition:

Proposition 5.14. *For each height one prime $\mathfrak{p} \in P_A^1$, we have the homomorphism*

$$(5.45)\qquad \Psi_{\mathfrak{p}}\colon F^0C_{K_{\mathfrak{p}}} \to F^0C_K,$$

which, being gathered over all elements in P_A^1, gives the surjective homomorphism

$$(5.46)\qquad \prod_{\mathfrak{p}\in P_A^1} \Psi_{\mathfrak{p}}\colon \prod F^0C_{K_{\mathfrak{p}}} \twoheadrightarrow F^0C_K.$$

Further, in the dual homomorphism $\Psi_{\mathfrak{p}}^*\colon \operatorname{Hom}(F^0C_K/p, \mathbb{Z}/p) \to \operatorname{Hom}(F^0C_{K_{\mathfrak{p}}}/p, \mathbb{Z}/p)$ of (5.45), it holds $\Psi_{\mathfrak{p}}^*(\operatorname{Hom}_{\text{cont}}(F^0C_K/p, \mathbb{Z}/p)) \subset \operatorname{Hom}_{\text{cont}}(F^0C_{K_{\mathfrak{p}}}/p, \mathbb{Z}/p)$.

Proof. We begin with constructing each homomorphism $\Psi_{\mathfrak{p}}\colon F^0C_{K_{\mathfrak{p}}} \to F^0C_K$. By Definition 6, it suffices to construct the homomorphism for each

$\mathfrak{q}$-component of $F^0 C_{K_\mathfrak{p}}$, by which we mean to construct the homomorphism $\Psi_{\mathfrak{p},\mathfrak{q}}\colon U^0 K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \to F^0 C_K$. We recall that $K_{\mathfrak{p},\mathfrak{q}}$ is the complete discrete valuation field having $\kappa(\mathfrak{p})_\mathfrak{q}$ as its residue field. We need the following result by Nagata:

THEOREM (Nagata, cf. [Na], Cor.37.6, 37.9, 37.10). *For an arbitrary complete (henselian) integral local ring R, there exists a one-to-one correspondence between maximal ideals of the normalization $\widetilde{R}$ of R and prime ideals of zero of the completion $\widehat{R}$ of R at its maximal ideal $\mathfrak{m}_R$.*

We will use this theorem of Nagata to $(A/\mathfrak{p})_{(\overline{\mathfrak{m}})}$ defined now. By taking a height two prime ideal $\mathfrak{m}$ of A which contains $\mathfrak{p}$, we consider the localization of the ring $A/\mathfrak{p}$ at its prime ideal $\overline{\mathfrak{m}}$, where $\overline{\mathfrak{m}}$ is the image of $\mathfrak{m}$ in $A/\mathfrak{p}$. Note that all elements outside the maximal prime ideal $\overline{\mathfrak{m}}$ are made to be invertible in $(A/\mathfrak{p})_{(\overline{\mathfrak{m}})}$. This is our ring. We will complete this local ring $(A/\mathfrak{p})_{(\overline{\mathfrak{m}})}$ at $\overline{\mathfrak{m}}$ obtaining $(A/\mathfrak{p})_{\overline{\mathfrak{m}}}$. I.e.,

$(A/\mathfrak{p})_{(\overline{\mathfrak{m}})} :=$ localization of $A/\mathfrak{p}$ at $\overline{\mathfrak{m}}$ ($\overline{\mathfrak{m}}$ denotes the image of $\mathfrak{m}$ in $A/\mathfrak{p}$).
$(A/\mathfrak{p})_{\overline{\mathfrak{m}}} :=$ completion of $(A/\mathfrak{p})_{(\overline{\mathfrak{m}})}$ at $\overline{\mathfrak{m}}$.

If we denote by $\widetilde{(A/\mathfrak{p})}_{(\overline{\mathfrak{m}})}$ the normalization of $(A/\mathfrak{p})_{(\overline{\mathfrak{m}})}$, then the above theorem of Nagata asserts the following one to one correspondence:

$$\{\text{maximal ideals of } \widetilde{(A/\mathfrak{p})}_{(\overline{\mathfrak{m}})}\}$$
$$\overset{1:1}{\Longleftrightarrow} \{\text{prime divisors of } 0 \text{ in } (A/\mathfrak{p})_{\overline{\mathfrak{m}}}\} \cdots\cdots (\blacklozenge)$$

It is found that the normalization $\widetilde{(A/\mathfrak{p})}_{(\overline{\mathfrak{m}})}$ is the direct sum of localizations of $\widetilde{A/\mathfrak{p}}$ at $\mathfrak{q}_1, \dots, \mathfrak{q}_i$, respectively, where each $\mathfrak{q}_i$ is a height one prime ideal of $\widetilde{A/\mathfrak{p}}$ lying over $\overline{\mathfrak{m}}$. Moreover, we can check that each prime divisor of 0 in the complete local ring $(A/\mathfrak{p})_{\overline{\mathfrak{m}}}$ corresponds to each height one prime ideal $\mathfrak{p}_\mathfrak{m}$ of $A_\mathfrak{m}$, which lies over $\mathfrak{p}$. By considering $(\blacklozenge)$ for all $\mathfrak{m} \in P_A^2$ containing $\mathfrak{p}$, we get

(5.47)
$$\{\text{ height one prime } \mathfrak{q}\text{s of } \widetilde{A/\mathfrak{p}}\} \overset{1:1}{\Longleftrightarrow} \{(\mathfrak{m}, \mathfrak{p}_\mathfrak{m}) | \mathfrak{m} \in P_A^2,\ \mathfrak{p}_\mathfrak{m} \in P_\mathfrak{m}^1 \text{s.t.} \mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p}\}$$

Moreover by the correspondence (5.47), we have an isomorphism

$$\zeta_{\mathfrak{q}}\colon \kappa(\mathfrak{p})_{\mathfrak{q}} \cong \kappa(\mathfrak{p}_{\mathfrak{m}}), \tag{5.48}$$

which consequently shows that both $K_{\mathfrak{p},\mathfrak{q}}$ and $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$ are complete discrete valuation fields of positive characteristic with isomorphic residue fields. Thus, we can choose isomorphisms

$$\phi_{\mathfrak{q}}\colon K_{\mathfrak{p},\mathfrak{q}} \cong K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}} \tag{5.49}$$

$$\psi_{\mathfrak{q}}\colon K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \cong K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \tag{5.50}$$

$$\text{(hence } U^iK_3^M(K_{\mathfrak{p},\mathfrak{q}}) \cong U^iK_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \text{ for } i \geq 0), \tag{5.51}$$

where (5.50) is induced from (5.49). By combining this isomorphism with the natural map $U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \to F^0C_K$ which follows from Definition 5, we get

$$\Psi_{\mathfrak{p},\mathfrak{q}}\colon U^0K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \to F^0C_K. \tag{5.52}$$

This, we will define, is the $\mathfrak{q}$-component of $\Psi_{\mathfrak{p}}$ in (5.45). As for the surjectivity of (5.46), it is obvious by considering each isomorphism $U^0K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \cong U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ in (5.51) and the definition of $F^0C_{K_{\mathfrak{p}}}$ and F^0C_K. We remark that the well-definedness of the product homomorphism in (5.46) is assured by Lemma 2.4.

Now, we will show the most essential part in Step 1. That is, each induced homomorphism

$$\Psi_{\mathfrak{p}} \circ \chi\colon F^0(C_{K_{\mathfrak{p}}}/p) \to \mathbb{Z}/p \tag{5.53}$$

associated to an arbitrary continuous character $\chi \in \mathrm{Hom}_{\mathrm{cont}}(F^0C_K/p,\, \mathbb{Z}/p)$ is also a continuous character of $F^0(C_{K_{\mathfrak{p}}}/p)$. By the continuity of χ, we can assume that χ factors as $\chi\colon F^0C_K \twoheadrightarrow F^0C_K(M_\chi) \to \mathbb{Z}/p$ for some modulus M_χ. Let us denote it explicitly by

$$M_\chi\colon= \sum_{\mathfrak{p}\in P_A^1} n_{\mathfrak{p}}\,\overline{(\mathfrak{p})}. \tag{5.54}$$

By the above mentioned isomorphism $K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \cong K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$, it is immediately found that $\chi \circ \Psi_{\mathfrak{p}}$ annihilates each subgroup $U^{n_{\mathfrak{p}}}K_3^M(K_{\mathfrak{p},\mathfrak{q}})$ for *all* $\mathfrak{q}$. We consider the composite map

$$\lambda_{\mathfrak{q}}\colon K_3^M(\widetilde{(A/\mathfrak{p})}_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \to F^0C_{K_{\mathfrak{p}}} \xrightarrow{\chi\circ\Psi_{\mathfrak{p}}} \mathbb{Z}/p, \tag{5.55}$$

where $K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \stackrel{natural}{\longrightarrow} F^0 C_{K_{\mathfrak{p}}}$ is obtained by (4.4). We will show that $\lambda_{\mathfrak{q}}$ annihilates the subgroup $F^{n_{\mathfrak{q}}} K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$ of $K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$ for some integer $n_{\mathfrak{q}} \geq 0$, and $n_{\mathfrak{q}} = 0$ for almost all $\mathfrak{q}$. This should prove $\chi \circ \Psi_{\mathfrak{p}}$ is continuous. For the convenience of the proof, let us introduce an invariant $N_{\mathfrak{q}}$ as follows: by the correspondence of Nagata in (5.47), we can associate the couple $(\mathfrak{m}, \mathfrak{p}_0)$ to $\mathfrak{q}$. Choose an arbitrary height one prime ideal $\mathfrak{p}_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$, and associate a height one prime $\mathfrak{p}$ of A by considering the image of $\mathfrak{p}_{\mathfrak{m}}$ in $\operatorname{Spec} A_{\mathfrak{m}} \to \operatorname{Spec} A$. We will take the coefficient $n_{\mathfrak{p}}$ of $\overline{(\mathfrak{p})}$ in the modulus M_χ, and define $n_{\mathfrak{p}_{\mathfrak{m}}} := n_{\mathfrak{p}}$. Then, $N_{\mathfrak{q}}$ is given by

$$N_{\mathfrak{q}} := \sum_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{A_{\mathfrak{m}}}, \mathfrak{p}_{\mathfrak{m}} \neq \mathfrak{p}_0} n_{\mathfrak{p}_{\mathfrak{m}}}. \tag{5.56}$$

The well defined-ness of (5.56) is understood from $n_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$. We will prove

Theorem 5.15. *If $N \gg N_{\mathfrak{q}}$ is an integer sufficiently bigger than $N_{\mathfrak{q}}$, the character $\chi \circ \Psi_{\mathfrak{p}}$ in* (5.53) *annihilates $\psi_{\mathfrak{q}}\left(F^N K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])\right)$ ($\psi_{\mathfrak{q}}$ is defined in* (5.50)*).*

Proof. The actual size of N will be understood in the course of the proof of Claim 5.16 below. Let us focus our attention to the homomorphism $\lambda_{\mathfrak{q}}$ (5.55). By assumption, χ factors as $\chi\colon C_K \twoheadrightarrow C_K(M_\chi) \to \mathbb{Z}/p$ and it suffices to show that the kernel

$$\lambda_{\mathfrak{q},M}\colon K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \to F^0 C_{K_{\mathfrak{p}}} \to C_K \twoheadrightarrow C_K(M_\chi)/p \tag{5.57}$$

contains $F^N K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$ for sufficiently large $N \gg N_{\mathfrak{q}}$. We will examine the kernel of $\lambda_{\mathfrak{q},M}$. The homomorphism $K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \to C_K$ factors as $K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) \hookrightarrow K_3^M(K_{\mathfrak{p},\mathfrak{q}}) \overset{\psi_{\mathfrak{q}}}{\xrightarrow{\sim}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \to C_K$. So, it suffices to show that the kernel of

$$\xi_{\mathfrak{p}_0,M}\colon K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \to C_K \twoheadrightarrow C_K(M_\chi)/p \tag{5.58}$$

contains $\psi_{\mathfrak{q}}(F^N K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]))$. We show this by using the fact that the homomorphism $K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \to C_K$ factors through the idele class group

$C_{K_{\mathfrak{m}}}$ for $K_{\mathfrak{m}}$. That is, $\xi_{\mathfrak{p}_0,M}$ factorizes as

(5.59)
$$\xi_{\mathfrak{p}_0,M}\colon K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \xrightarrow{\iota_{\mathfrak{p}_0}} \frac{\coprod_{\mathfrak{p}_{\mathfrak{m}}\in P^1_{A_{\mathfrak{m}}}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}{K_3^M(K_{\mathfrak{m}})} \to C_K \twoheadrightarrow C_K(M_\chi)/p,$$

where $\iota_{\mathfrak{p}_0}$ denotes the embedding of $K_3^M(K_{\mathfrak{m},\mathfrak{p}_0})$ to $\mathfrak{p}_0$-component of $\coprod_{\mathfrak{p}_{\mathfrak{m}}\in P^1_{A_{\mathfrak{m}}}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$. Now, the definition of $C_K(M)$ in (2.3) shows that each subgroup $U^{n_{\mathfrak{p}_{\mathfrak{m}}}}K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \subset K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ is annihilated in $C_K(M_\chi)$. So (5.59) is rewritten as

$$\xi_{\mathfrak{p}_0,M}\colon K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \xrightarrow{\iota_{\mathfrak{p}_0,M}} \frac{\coprod_{\mathfrak{p}_{\mathfrak{m}}\in P^1_{A_{\mathfrak{m}}}} \left(\frac{K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}{U^{n_{\mathfrak{p}_{\mathfrak{m}}}}K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}\right)}{K_3^M(K_{\mathfrak{m}})} \to C_K(M_\chi)/p.$$

We are interested in the kernel of

$$\iota_{\mathfrak{p}_{0,M}}\colon K_3^M(K_{\mathfrak{m},\mathfrak{p}_0}) \to \frac{\coprod_{\mathfrak{p}_{\mathfrak{m}}\in P^1_{A_{\mathfrak{m}}}} \left(\frac{K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}{U^{n_{\mathfrak{p}_{\mathfrak{m}}}}K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}\right)}{K_3^M(K_{\mathfrak{m}})}. \tag{5.60}$$

Recall that Kato-filtration is given by $U^{n_{\mathfrak{p}_{\mathfrak{m}}}}K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) := \{1+\mathfrak{p}_{\mathfrak{p}_{\mathfrak{m}}}^{n_{\mathfrak{p}_{\mathfrak{m}}}}\mathcal{O}_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}, K^*_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}, K^*_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}\}$. We promise as in Theorem 2.7 that $U^{n_{\mathfrak{p}_{\mathfrak{m}}}}k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) := \{1+\mathfrak{p}_{\mathfrak{p}_{\mathfrak{m}}}^{n_{\mathfrak{p}_{\mathfrak{m}}}}\mathcal{O}_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}, K^*_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}, K^*_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}\}/p$. We can take mod p in (5.60) and put

$$V := \{1 + \prod_{\mathfrak{p}_{\mathfrak{m}}\neq\mathfrak{p}_0} \mathfrak{p}_{\mathfrak{m}}^{n_{\mathfrak{p}_{\mathfrak{m}}}} A_{\mathfrak{m}}, K^*_{\mathfrak{m}}, K^*_{\mathfrak{m}}\}/p \subset k_3^M(K_{\mathfrak{m},\mathfrak{p}_0}). \tag{5.61}$$

We claim that V goes to zero by $\iota_{\mathfrak{p}_{0,M}}$. This is because we can use elements of the group $k_3^M(K_{\mathfrak{m}})$, which is diagonally embedded into $\coprod_{\mathfrak{p}_{\mathfrak{m}}\in P^1_{A_{\mathfrak{m}}}} \left(\frac{k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}{U^{n_{\mathfrak{p}_{\mathfrak{m}}}}k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})}\right)$. For example, take an element $a \in V$. Then a is sent to the restricted product as an element $(1,..,a,..,1)$, where a sits in $\mathfrak{p}_0$-component and all other component is 1. This latter element is changed to $(a^{-1},..,1,..,a^{-1})$, where this time only $\mathfrak{p}_0$-component has 1. But a^{-1} lies in the group $U^{n_{\mathfrak{p}_{\mathfrak{m}}}}k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ for all $\mathfrak{p}_{\mathfrak{m}}(\neq \mathfrak{p}_0)$ by definition of V. Thus it goes zero in $C_K(M_\chi)$. Now, we have the following claim which furnishes our proof:

CLAIM 5.16. For sufficiently large $N \gg N_{\mathfrak{q}}$, V satisfies

$$V \supset \psi_{\mathfrak{q}}\left(F^N k_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])\right) \mod U^{n_{\mathfrak{p}}} k_3^M(K_{\mathfrak{m},\mathfrak{p}_0}). \tag{5.62}$$

PROOF. The idea is to use Theorem 2.7. For the 3-dimensional local field $K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}$, what Theorem 2.7 tells us is that any element $a \in k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$ is written as

$$a = \sum_{i=1,\ldots,n_{\mathfrak{p}}-1} \sum_{j=1,\ldots,m_i} \{1 + \widetilde{\alpha_{ij}}\,\pi^i\,, \widetilde{\beta_{ij}}\,, \widetilde{\gamma_{ij}}\} \tag{5.63}$$

modulo $U^{n_{\mathfrak{p}}} k_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$, where $\alpha_{ij}\,, \beta_{ij}\,, \gamma_{ij} \in \kappa(\mathfrak{p}_{\mathfrak{m}})$ and $\widetilde{\alpha_{ij}}\,, \widetilde{\beta_{ij}}\,, \widetilde{\gamma_{ij}}$ denote their liftings to $\mathcal{O}_{K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}}$, respectively. We will prove our claim using the explicit representation (5.63). So let us take an arbitrary element $a \in F^N k_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]])$. The explicit representation $F^N K_3^M((\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]) = \{(1 + \mathfrak{q}^N(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]), (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]^*, (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}[[u_{\mathfrak{p}}]]^*\}$ in (4.2) shows

$$\alpha_{ij} \in 1 + \mathfrak{q}^N(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}} \quad \text{and} \quad \beta_{ij},\, \gamma_{ij} \in (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}^{*}. \tag{5.64}$$

We will show that elements α_{ij} and β_{ij}, $\gamma_{ij} \in (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}^{*}$ in (5.64) are also expressed as in (5.61). First, we check β_{ij}, γ_{ij}. Let us recall that the complete discrete valuation field $\kappa(\mathfrak{p})_{\mathfrak{q}}$ is the fractional field of $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$, which is the completion of $(\widetilde{A/\mathfrak{p}})$ at $\mathfrak{q}$. As is stated in (5.48), Nagata's correspondence provides us with an isomorphism $\kappa(\mathfrak{p})_{\mathfrak{q}} \cong \kappa(\mathfrak{p}_0) = A_{\mathfrak{m},\mathfrak{p}_0}/\mathfrak{p}_0$. So, it holds $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}^{*} \subset \kappa(\mathfrak{p})_{\mathfrak{q}}^{*} \cong \kappa(\mathfrak{p}_0)^* = (A_{\mathfrak{m},\mathfrak{p}_0}/\mathfrak{p}_0)^*$. But $A_{\mathfrak{m},\mathfrak{p}_0}/\mathfrak{p}_0 =$\{fractional field of $A_{\mathfrak{m}}/\mathfrak{p}_0$\}, so we can find some lifting $\widetilde{\beta_{ij}}$ of $\beta_{ij} \in \kappa(\mathfrak{p})_{\mathfrak{q}}$ such that $\widetilde{\beta_{ij}} \in K_{\mathfrak{m}}(\subset K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})$. By the same reason, this is true for γ_{ij}. Thus we get the desired assertion for β_{ij}, γ_{ij}.

Next, we check that any element $\alpha_{ij} \in 1 + \mathfrak{q}^N(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$ in (5.64) is also written in the form (5.61). That is, for sufficiently large $N \gg N_{\mathfrak{q}}$, it holds

$$\prod_{\mathfrak{p}_{\mathfrak{m}} \neq \mathfrak{p}_0} (\overline{\mathfrak{p}}_{\mathfrak{m}})^{n_{\mathfrak{p}_{\mathfrak{m}}}} \overline{A_{\mathfrak{m}}} \supset \mathfrak{q}^N(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}, \tag{5.65}$$

where $\overline{A_{\mathfrak{m}}} := A_{\mathfrak{m}}/\mathfrak{p}_0$ which is the image of $A_{\mathfrak{m}}$ in the residue field $\kappa(\mathfrak{p}_0)$ and $\overline{\mathfrak{p}}_{\mathfrak{m}}$ also denotes the image of $\mathfrak{p}_{\mathfrak{m}}$ in $\overline{A_{\mathfrak{m}}}$. First it holds $\overline{A_{\mathfrak{m}}} \subset (\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$ and each image $\overline{\mathfrak{p}}_{\mathfrak{m}}$ of prime ideal $\mathfrak{p}_{\mathfrak{m}}$ in $\overline{A_{\mathfrak{m}}}$ generates an ideal $\overline{\mathfrak{p}}_{\mathfrak{m}}(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$ by extending to $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$. If it generates whole ring, the inclusion (5.65) easily follows. Otherwise it is a proper ideal, and we may assume $\prod_{\mathfrak{p}_{\mathfrak{m}}\neq\mathfrak{p}_0}(\overline{\mathfrak{p}}_{\mathfrak{m}})^{n_{\mathfrak{p}_{\mathfrak{m}}}} \subset \mathfrak{q}^{\sum n_{\mathfrak{p}_{\mathfrak{m}}}} = \mathfrak{q}^{N_{\mathfrak{q}}}$. This shows $\prod_{\mathfrak{p}_{\mathfrak{m}}\neq\mathfrak{p}_0}(\overline{\mathfrak{p}}_{\mathfrak{m}})^{n_{\mathfrak{p}_{\mathfrak{m}}}} = \mathfrak{q}^{m}$ for some $m \geq N_{\mathfrak{q}}$. Consequently for $N_1 \gg m \geq N_{\mathfrak{q}}$, it holds

$$\prod_{\mathfrak{p}_{\mathfrak{m}}\neq\mathfrak{p}_0}(\overline{\mathfrak{p}}_{\mathfrak{m}})^{n_{\mathfrak{p}_{\mathfrak{m}}}} \supset \mathfrak{q}^{N_1}. \tag{5.66}$$

Next we observe that $(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}$ coincides with the normalization $\widetilde{\overline{A_{\mathfrak{m}}}}$ of $\overline{A_{\mathfrak{m}}}$ which follows from Nagata's correspondence (5.47). As the normalization is finite, it holds

$$\overline{A_{\mathfrak{m}}} \supset \mathfrak{q}^{N_2}(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}} \tag{5.67}$$

for sufficiently large integer N_2. Putting (5.66) and (5.67) together, we have finally

$$\prod_{\mathfrak{p}_{\mathfrak{m}}\neq\mathfrak{p}_0}(\overline{\mathfrak{p}}_{\mathfrak{m}})^{n_{\mathfrak{p}_{\mathfrak{m}}}}\overline{A_{\mathfrak{m}}} \supset \mathfrak{q}^{N_1+N_2}(\widetilde{A/\mathfrak{p}})_{\mathfrak{q}}. \tag{5.68}$$

So if we take $N \geq N_1 + N_2$, we have the desired inclusion (5.65). The procedure of the proof also tells us that we can take $N = 0$ for almost all $\mathfrak{q}$. Thus Theorem 5.16 is established, which proves Theorem 5.15, hence Proposition 5.14.

PROOF OF STEP 2. For the proof, we use the explicit representation of C_K in Lemma 2.4. According to (2.9), F^0C_K/p is explicitly given by $F^0C_K/p \xrightarrow{\sim} \varprojlim_M D_{K,p}/F^M D_{K,p}$ $\left(D_{K,p} := \left(\prod'_{\mathfrak{m}\in P_A^2,\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/p\right)/\prod_{\mathfrak{m}\in P_A^2}K_3^M(K_{\mathfrak{m}})\prod_{\mathfrak{p}\in P_A^1}K_3^M(K_{\mathfrak{p}})\right)$. We introduce the auxiliary group $E_{K,p} := \left(\prod'_{\mathfrak{m}\in P_A^2,\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/p\right)/\prod_{\mathfrak{p}\in P_A^1}K_3^M(K_{\mathfrak{p}})$, where $\prod'$ has the same meaning as in $D_{K,p}$. Moreover, $\prod_{\mathfrak{p}\in P_A^1}\Psi_{\mathfrak{p}} \colon \prod F^0C_{K_{\mathfrak{p}}}/p \twoheadrightarrow F^0C_K/p$ factors as

$$\prod F^0C_{K_{\mathfrak{p}}}/p \to \varprojlim_M E_{K,p}/F^M E_{K,p} \twoheadrightarrow \varprojlim_M D_{K,p}/F^M D_{K,p} = F^0C_K/p. \tag{5.69}$$

Also by the explicit construction of $\Psi_{\mathfrak{p}}$, each embedded group $K_3^M(A_{\mathfrak{m}}) \subset \prod F^0 C_{K_{\mathfrak{p}}}$, after being sent in $\varprojlim_M E_{K,p}/F^M E_{K,p}$ in (5.69), coincides with the diagonal embedding

$$K_3^M(A_{\mathfrak{m}}) \xrightarrow{\textit{diagonal}} \prod_{\mathfrak{p}_{\mathfrak{m}} \in P^1_{\mathfrak{m}}} \left(U^0 K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) / U^{M(\mathfrak{p}_{\mathfrak{m}})} K_3^M(K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}}) \right). \tag{5.70}$$

But such diagonally embedded group $K_3^M(A_{\mathfrak{m}})$ vanishes automatically after being sent into $F^0 C_K/p$ which is immediately understood from the reciprocity for $K_{\mathfrak{m}}$ in (2.2). This completes the proof of Step 2. □

By using Theorem 5.5 and Theorem 5.13, we can prove our main Theorem.

Proof of Theorem 5.1. By (5.15) and (5.34), we have the commutative diagram

(5.71)
$$\begin{array}{ccccccccc}
0 \to \mathbb{Z}/p \to & H^1_{\mathrm{Gal}}(K, \mathbb{Z}/p) & \longrightarrow & \bigoplus_{\mathfrak{p} \in P^1_A} H^2_{\mathfrak{p}}(A_{\mathfrak{p}}, \mathbb{Z}/p) & \longrightarrow & \bigoplus_{\mathfrak{m} \in P^2_A} \mathrm{Hom}(K_3^M(A_{\mathfrak{m}}), \mathbb{Z}/p) \\
\| & \downarrow \rho_K{}^*/p & & \downarrow \cong & & \| \\
0 \to \mathbb{Z}/p \to & \mathrm{Hom_c}(C_K/p, \mathbb{Z}/p) & \to & \bigoplus_{\mathfrak{p} \in P^1_A} \mathrm{Hom_c}(F^0(C_{K_{\mathfrak{p}}}/p), \mathbb{Z}/p) & \to & \bigoplus_{\mathfrak{m} \in P^2_A} \mathrm{Hom}(K_3^M(A_{\mathfrak{m}}), \mathbb{Z}/p),
\end{array}$$

where the vertical isomorphism comes from Theorem 4.1. The top row is exact by Theorem 5.5 and the bottom row is exact at $\mathbb{Z}/p$ and at $\mathrm{Hom_c}(C_K/p, \mathbb{Z}/p)$ by Theorem 5.13. The desired bijectivity of $\rho_K{}^*/p$ follows immediately by the diagram chase in (5.71). □

6. Proof of the Existence Theorem (l-Primary Parts)

In this final section, we will prove the bijectivity of ρ_K^* modulo arbitrary natural number m which is prime to the characteristic of K. Let us state our purpose in this section.

Theorem 6.1. *Let $A := \mathbb{F}_q[[X, Y, Z]]$ and K be its fractional field. Then for an arbitrary prime $l \neq p$ under the Bloch-Milnor-Kato conjecture for K (see Conjecture in page 8), it holds the dual reciprocity isomorphism*

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\sim} \mathrm{Hom_c}(C_K, \mathbb{Q}_l/\mathbb{Z}_l). \tag{6.1}$$

As stated in the introduction, we need the holding of Bloch-Milnor-Kato conjecture in the shape $K_3^M(K)/l^m \cong H^3_{\text{ét}}(K, \mu_{l^m}^{\otimes 3})$. By combining Theorem 5.1 and Theorem 6.1, we get the class field theory for K.

THEOREM 6.2. *Let A, K be as above. Then under the Bloch-Milnor-Kato conjecture for K, we have the following dual reciprocity isomorphism*:

$$\rho_K^* \colon H^1_{\mathrm{Gal}}(K, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Q}/\mathbb{Z}), \tag{6.2}$$

where $\mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Q}/\mathbb{Z})$ *means the set of all continuous homomorphisms of finite order from C_K to $\mathbb{Q}/\mathbb{Z}$.*

Here, we give some corollaries of Theorem 6.2.

COROLLARY 6.3. *Let A and K be as above. Then under the Bloch-Milnor-Kato conjecture, the canonical reciprocity map*

$$\rho_K \colon C_K \to \mathrm{Gal}(K^{ab}/K) \tag{6.3}$$

has its dense image in $\mathrm{Gal}(K^{ab}/K)$ *by the Krull topology.*

PROOF. This follows perhaps from the injectivity of (6.2) by considering dual. □

Next, we give explicit isomorphisms for certain finite abelian extensions.

COROLLARY 6.4. *Let A, K be as above. We assume the Bloch-Milnor-Kato conjecture for K. Then for an arbitrary finite abelian extension L/K such that the integral closure of A in L is regular, there exists a canonical reciprocity isomorphism*

$$\rho_K \colon C_K/N_{L/K}(C_L) \xrightarrow{\sim} \mathrm{Gal}(L/K). \tag{6.4}$$

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Hom}_{\mathrm{c}}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) & \to & H^1_{\mathrm{Gal}}(K, \mathbb{Q}/\mathbb{Z}) & \to & H^1_{\mathrm{Gal}}(L, \mathbb{Q}/\mathbb{Z}) & \to 0 \\
& \downarrow & & \downarrow \cong & & \downarrow \cong & \\
0 \to & \mathrm{Hom}_{\mathrm{c}}(C_K/N_{L/K}(C_L), \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Q}/\mathbb{Z}) & \to & \mathrm{Hom}_{\mathrm{c}}(C_L, \mathbb{Q}/\mathbb{Z}). &
\end{array}$$

From this, we get the bijectivity of the extreme left vertical arrow. Corollary follows by taking Pontryagin dual of $\mathrm{Hom}_{\mathrm{c}}(\mathrm{Gal}(L/K), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{c}}(C_K/N_{L/K}(C_L), \mathbb{Q}/\mathbb{Z})$ proved just now noticing that the group $C_K/N_{L/K}(C_L)$ is discrete because $N_{L/K}(C_L)$ contains $\mathrm{Ker}\Big(C_K \to C_K(M)\Big)$ for some modulus M (this follows from [Ka1], II). $\square$

Remark 5. For a general abelian extension L/K that does not satisfy the above condition in Corollary 6.4, there may occur the inequality $|C_K/N_{L/K}(C_L)| > [L : K]$. Indeed, in the case of class field theory for two-dimensional complete regular local rings (i.e. $n = 2$), such examples were given and studied by Shiho in [Sh1].

Proof of Theorem 6.1. For verifying the isomorphism (6.1), we have only to prove

$$\rho_K^*/l\colon H^1_{\mathrm{Gal}}(K, \mathbb{Z}/l) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{c}}(C_K, \mathbb{Z}/l\ ) \tag{6.5}$$

for an arbitrary prime $l \neq p$ (the isomorphism of ρ_K^*/l^m for $m > 1$ is proved in the same way to the case $m = 1$ without any essential change). As l is prime to the characteristic K, the group C_K/l becomes discrete. By norm arguments, we may assume $\mu_l \in K$, where μ_l denotes the group of l-th power of unity. By Kummer theory, we have the isomorphism

$$H^1_{\mathrm{Gal}}(K, \mu_l) \cong K^*/K^{*l}. \tag{6.6}$$

Moreover, as A is a unique factorial domain, we have the factorization

$$K^*/K^{*l} \cong \mathbb{F}_q{}^*/\mathbb{F}_q{}^{*l} \times \bigoplus_{\mathfrak{p}\in P^1_A} u_{\mathfrak{p}}^{\mathbb{Z}/l}, \tag{6.7}$$

where $u_{\mathfrak{p}}$ denotes the regular parameter of the prime ideal $\mathfrak{p}$. We consider the following exact sequences:

$$F^0C_K/l \to C_K/l \to \mathbb{Z}/l \to 0, \tag{6.8}$$

$$0 \to \mathbb{F}_q{}^*/\mathbb{F}_q{}^{*l} \to H^1_{\mathrm{Gal}}(K, \mu_l) \to \bigoplus_{\mathfrak{p}\in P^1_A} \mathbb{Z}/l \to 0, \tag{6.9}$$

where the exactness of (6.8) is obtained by putting $\bigotimes_{\mathbb{Z}} \mathbb{Z}/l$ to the exact sequence (5.37). The exactness of (6.9) is obvious from (6.6) and (6.7). Now, we have the key theorem.

Theorem 6.5. *There exists a canonical isomorphism*

$$F^0C_K/l \cong \prod\nolimits_{\mathfrak{p}\in P_A^1} \mu_l. \tag{6.10}$$

The proof of this theorem is given below, and we check that Theorem 6.1 is easily deduced from Theorem 6.5. In fact, ${\mathbb{F}_q}^*/{\mathbb{F}_q}^{*l} \times \mathbb{Z}/l \to \mu_l$ is the perfect pairing considering the duality of finite fields. Next for each height one prime $\mathfrak{p}$, we have the duality $\mu_l \times u_{\mathfrak{p}}^{\mathbb{Z}/l} \to \mu_l$. This also follows from the isomorphism $\mathrm{Gal}(K(u_{\mathfrak{p}}^{1/l})/K) \cong \mu_l$ which is an easy corollary of Kummer Theory. Now substituting F^0C_K/l in (6.8) by (6.10), and comparing two short exact sequences (6.8) and (6.9), the above mentioned duality results yields the desired bijectivity (6.5). □

Now, we begin to prove Theorem 6.5.

Proof of Theorem 6.5. Recall the definition in (2.10) that $F^0C_K\colon = \varprojlim_M F^0C_K(M)$, where $F^0C_K(M) \cong \mathrm{Coker}\left(\bigoplus_{\mathfrak{p}\in P_A^1} U^0K_3^M(K_{\mathfrak{p}}) \to \bigoplus_{\mathfrak{m}\in P_A^2} F^0C_{\mathfrak{m}}(M)\right)$. So for the proof of Theorem 6.5, it suffices to show the following isomorphism for each modulus M:

$$\begin{aligned} F^0C_K(M)/l &\cong \mathrm{Coker}\left(\bigoplus_{\mathfrak{p}\in P_A^1} U^0K_3^M(K_{\mathfrak{p}})/l \to \bigoplus_{\mathfrak{m}\in P_A^2} F^0C_{\mathfrak{m}}(M)/l\right) \\ &\cong \prod\nolimits_{\mathfrak{p}\in P_A^1} \mu_l. \end{aligned} \tag{6.11}$$

We begin with the following result:

Proposition 6.6. *We have an isomorphism*

$$F^0C_{\mathfrak{m}}(M)/l \cong \bigoplus\nolimits_{\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} K_3^M(\kappa(\mathfrak{p}_{\mathfrak{m}}))/l \tag{6.12}$$

for an arbitrary modulus M.

Proof. By (2.9), $F^0C_{\mathfrak{m}}(M)/l\colon = \mathrm{Image}\Big(\Delta_{\mathfrak{m}}\colon \bigoplus_{\mathfrak{p}_{\mathfrak{m}}\in P_{\mathfrak{m}}^1} U^0K_3^M \times (K_{\mathfrak{m},\mathfrak{p}_{\mathfrak{m}}})/l \longrightarrow C_{\mathfrak{m}}(M)/l\Big)$, where $C_{\mathfrak{m}}(M)\colon = \mathrm{Coker}\Big(K_3^M(K_{\mathfrak{m}}) \xrightarrow{diagonal}$

$\bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} \left(K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/U^{M(\mathfrak{p}_\mathfrak{m})}K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)\Bigg)$ as in (2.2). From this representation, it is found

$$\text{(6.13)}\quad \operatorname{Ker}\Delta_\mathfrak{m} \cong \operatorname{Ker}\left(K_3^M(K_\mathfrak{m}) \xrightarrow{\textit{diagonal}} K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})\right)/l.$$

On the other hand, the Gersten-Quillen complex

$$\text{(6.14)}\qquad K_3^M(A_\mathfrak{m}) \to K_3^M(K_\mathfrak{m}) \to \bigoplus_{\mathfrak{p}_\mathfrak{m}\in P^1_{A_\mathfrak{m}}} K_2^M(\kappa(\mathfrak{p}))$$

together with an isomorphism $K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \cong K_2^M(\kappa(\mathfrak{p}))$ (cf.Theorem 2.7) shows that the right hand side of (6.13) comes from $K_3^M(A_\mathfrak{m})/l$. But it holds

Lemma 5.7. *We have the vanishing* $K_3^M(A_\mathfrak{m})/l = 0$.

Proof. We will prove by the explicit calculation. As $A_\mathfrak{m}$ has positive characteristic and also complete, we may assume that the residue field $\kappa(\mathfrak{m})$ of $A_\mathfrak{m}$ is (non-canonically) contained in $A_\mathfrak{m}$. Any element $a \in A_\mathfrak{m}^*$ is written as $a = \overline{a}\,\delta$ where $\overline{a}$ is the image of a in the residue field $\kappa(\mathfrak{m})$ and $\delta \in (1+\mathfrak{m}A_\mathfrak{m})$, which implies an isomorphism $A_\mathfrak{m}^* \cong \kappa(\mathfrak{m})^* \times (1+\mathfrak{m}A_\mathfrak{m})$. But as is easily seen, we have $(1+\mathfrak{m}A_\mathfrak{m})^l = 1+\mathfrak{m}A_\mathfrak{m}$, so modulo l, we may assume that $K_3^M(A_\mathfrak{m})/l \cong K_3^M(\kappa(\mathfrak{m}))/l$. But $\kappa(\mathfrak{m})$ is one-dimensional local field, so it is well-known that $K_3^M(\kappa(\mathfrak{m}))/l = 0$ (cf. proof of Lemma 6.8 below). $\square$

By Lemma 6.7, we see that the kernel of $\Delta_\mathfrak{m}$ is zero, by which we get

$$\text{(6.15)}\qquad F^0C_\mathfrak{m}(M)/l \cong \bigoplus\nolimits_{\mathfrak{p}_\mathfrak{m}\in P^1_\mathfrak{m}} U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/l.$$

By using (6.15), the proof of Proposition 6.6 will be completed by

Lemma 6.8. *We have an isomorphism*

$$U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/l \cong K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l.$$

Proof. By Theorem 2.7 (1), we have the exact sequence

$$U^1K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \to U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}}) \to K_3^M(\kappa(\mathfrak{p}_\mathfrak{m})) \to 0.$$

By putting $\bigotimes_\mathbb{Z} \mathbb{Z}/l$ to this, we have

$$U^1K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/l \to U^0K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/l \to K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \to 0. \tag{6.16}$$

But as $U^1K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})$ is l-divisible, we have $U^1K_3^M(K_{\mathfrak{m},\mathfrak{p}_\mathfrak{m}})/l = 0$. Thus by considering (6.16), we have the desired isomorphism in Lemma 6.8. □

By combining Proposition 6.6 together with the following theorem, we obtain the desired isomorphism (6.11):

Theorem 6.9. *For each $\mathfrak{p} \in P_A^1$, it holds an isomorphism*

$$\mathrm{Coker}\left(U^0K_3^M(K_\mathfrak{p})/l \to \bigoplus_{\substack{\mathfrak{p}_\mathfrak{m} \in P^1_{A_\mathfrak{m}}, \mathfrak{p}_\mathfrak{m} \mapsto \mathfrak{p} \\ \mathfrak{m} \in P_A^2}} K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \right) \cong \mu_l. \tag{6.17}$$

Proof. We again use the correspondence (5.47) by Nagata. From the isomorphism $\kappa(\mathfrak{p}_\mathfrak{m}) \cong \kappa(\mathfrak{p})_\mathfrak{q}$ stated in (5.48), we have $K_3^M(\kappa(\mathfrak{p}_\mathfrak{m}))/l \cong K_3^M(\kappa(\mathfrak{p})_\mathfrak{q})/l \cong K_2^M(\kappa(\mathfrak{q}))/l$ and it also holds $U^0K_3^M(K_\mathfrak{p})/l \cong K_3^M(\kappa(\mathfrak{p}))/l$. Thus, we can rewrite (6.17) as

$$\mathrm{Coker}\left(K_3^M(\kappa(\mathfrak{p}))/l \to \bigoplus_{\mathfrak{q} \in P^1_{\widehat{A/\mathfrak{p}}}} K_2^M(\kappa(\mathfrak{q}))/l \right) \cong \mu_l. \tag{6.18}$$

We can use the following cohomological Hasse principle by S. Saito:

Theorem (S. Saito, [Sa1]). *For an arbitrary two-dimensional excellent normal complete local ring R with finite residue field, the following sequence is exact*:

$$0 \to (\mathbb{Z}/m)^{r_R} \to H^3_{\mathrm{Gal}}(F, \mu_m^{\otimes 2}) \to \bigoplus_{\mathfrak{q} \in P_R^1} H^2_{\mathrm{Gal}}(\kappa(\mathfrak{q}), \mu_m) \to \mathbb{Z}/m \to 0$$

for an arbitrary natural number m prime to the characteristic of R, where F denotes the fractional field of R, P_R^1 denotes the set of all height one primes of R, and r_R is the rank of R (for details, we refer the original paper).

This theorem by Saito provides, when $m = l$ and $R \supset \mu_l$, the exact sequence

$$(6.19) \qquad H^3_{et}(F, \mu_l^{\otimes 3}) \to \bigoplus_{\mathfrak{q} \in P_R^1} H^2_{et}(\kappa(\mathfrak{q}), \mu_l^{\otimes 2}) \to \mu_l \to 0.$$

Now, we have the Bloch-Milnor-Kato isomorphism $K_3^M(\kappa(\mathfrak{p}))/l \cong H^3_{\mathrm{Gal}}(\kappa(\mathfrak{p}), \mu_l^{\otimes 3})$ by assumption, or Merkur'ev-Suslin isomorphism $K_2^M(\kappa(\mathfrak{q}))/l \cong H^2_{\mathrm{Gal}}(\kappa(\mathfrak{q}), \mu_l^{\otimes 2})$. So if we replace each term in (6.19) with these isomorphisms, we get the exact sequence

$$(6.20) \qquad K_3^M(F)/l \to \bigoplus_{\mathfrak{q} \in P_R^1} K_2^M(\kappa(\mathfrak{q}))/l \to \mu_l \to 0.$$

By putting $R = \widetilde{A/\mathfrak{p}}$ in (6.20) and noticing the isomorphism $F \cong \kappa(\mathfrak{p})$, we get the desired isomorphism (6.18). Thus, we finally established the existence theorem for prime to p parts. □

Acknowledgment. The author heartily thanks Professors Kazuya Kato, Yasuo Morita for their unceasing encouragement. He is also grateful to the referee for his keen advice. This paper is much refinement of author's thesis begun during his visit to Institute de Henri Poincaré and Cambridge University. Much hospitality of Director Joseph Oesterlé and of Professor John Coates is both heavily appreciated. The author will dedicate this paper to his late grandmothers and to the late Japanese mathematician Osamu Hyodo.

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(Received June 14, 2000)
(Revised September 6, 2001)

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