J. Math. Sci. Univ. Tokyo
9 (2002), 669–687.

Special Solutions of the Hamiltonian System on an Elliptic Curve

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Abstract. Special solutions of Hamiltonian systems defined on an elliptic curve are studied; those solutions are reduced to the ordinary differential equation of the first order. The first integral of the equation is given in terms of a ratio of elliptic theta functions. Two cases of degeneration of the elliptic curve are also investigated. The functions describing the first integrals are similar to those given as solutions of Bruschi-Calogero equation.

Introduction

The present article concerns a Hamiltonian system on an elliptic curve. This system is obtained in [O2] by means of holonomic deformation of a linear differential equation defined on an elliptic curve.

The theory of holonomic deformation, or monodromy preserving deformation, has been considered mainly on the projective line $\mathbb{CP}^1$. It is well-known that the Painlevé equations are obtained by holonomic deformation; cf. [O1]. In particular, a Hamiltonian structure associated with the Painlevé equations is defined in a natural manner; the sixth Painlevé equation P_{VI} is written in the form :

$$t(t-1)\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}, \quad t(t-1)\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H}{\partial q},$$

with the Hamiltonian function

$$\begin{aligned} H &= q(q-1)(q-t)p^2 \\ &\quad -\{\kappa_0(q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)\}p \\ &\quad +\kappa(q-t), \end{aligned}$$

$$\kappa = \frac{1}{4}[(\kappa_0+\kappa_1+\theta-1)^2 - \kappa_\infty^2].$$

2000 *Mathematics Subject Classification.* Primary 58F05; Secondary 33E05, 34A20.

When $\kappa = 0$, we obtain special solutions of P_{VI}, given by:

$$\begin{aligned} t(t-1)\frac{\mathrm{d}q}{\mathrm{d}t} &= \frac{\partial H}{\partial p}, \\ p &= 0, \end{aligned}$$

which is reduced to the Riccati equation:

$$t(t-1)\frac{\mathrm{d}q}{\mathrm{d}t} = -\kappa_0(q-1)(q-t) - \kappa_1 q(q-t) - (\theta-1)q(q-1). \tag{1}$$

Moreover, (1) can be linearized by the use of Gauß Hypergeometric Functions.

The holonomic deformation of a linear ordinary differential equation of the second order, defined on an elliptic curve $\mathbb{E}$, is also governed by a Hamiltonian system. We begin with recalling results obtained by [O2].

Let Ω be the lattice generated by two complex numbers $2\omega_1, 2\omega_3$ such that $\mathrm{Im}(\omega_3/\omega_1) > 0$. We denote by $\wp(x)$ Weierstraß $\wp$-function with the fundamental periods $2\omega_1, 2\omega_3$, and by $\zeta(x)$ Weierstraß ζ-function. By identifying an elliptic curve $\mathbb{E}$ with $\mathbb{C}/\Omega$, we represent a linear ordinary differential equation defined on $\mathbb{E}$ as an equation whose coefficients are elliptic functions.

Consider the linear differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = p(z;t)y \tag{2}$$

defined on $\mathbb{E}$, such that

$$\begin{aligned} p(z;t) = \nu &+ a_0\wp(z) + a_1\wp(z-t) \\ &+ \frac{3}{4}\wp(z-\lambda_1) + \frac{3}{4}\wp(z-\lambda_2) \\ &+ H\mathfrak{z}(z;t) - \mu_1\mathfrak{z}(z;\lambda_1) - \mu_2\mathfrak{z}(z;\lambda_2) \end{aligned} \tag{3}$$

where $\mathfrak{z}(\lambda;\mu)$ is the function :

$$\begin{aligned} \mathfrak{z}(\lambda;\mu) &= \zeta(\lambda-\mu) - \zeta(\lambda) + \zeta(\mu) \\ &= \frac{1}{2}\frac{\wp'(\lambda)+\wp'(\mu)}{\wp(\lambda)-\wp(\mu)}. \end{aligned}$$

Equation (2) is of the Fuchsian type and the Riemannian scheme of (2) reads:

$$\left\{ \begin{array}{llll} z \equiv 0 & z \equiv t & z \equiv \lambda_k & (k = 1, 2) \quad \mathrm{mod}.\Omega \\ \frac{1}{2}(1 + c_0) & \frac{1}{2}(1 + c_1) & \frac{3}{2} & \\ \frac{1}{2}(1 - c_0) & \frac{1}{2}(1 - c_1) & -\frac{1}{2} & \end{array} \right\},$$

where $a_0 = \frac{1}{4}({c_0}^2 - 1), a_1 = \frac{1}{4}({c_1}^2 - 1)$.

We make the following assumption :

(H) none of $\lambda_k + \Omega$ $(k = 1, 2)$ is a logalithmic singularity.

Considering the holonomic deformation of (2)-(3) under Assumption (H), we obtain the following result:

Proposition 0.1 ([O2]). *The holonomic deformation of* (2)-(3) *is governed by the Hamiltonian system :*

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = \frac{\partial H}{\partial \mu_k}, \quad \frac{\mathrm{d}\mu_k}{\mathrm{d}t} = -\frac{\partial H}{\partial \lambda_k} \quad ; k = 1, 2\ , \tag{4}$$

with the Hamiltonian H :

$$H = M\{({\mu_1}^2 - {\mu_2}^2) + (\mu_1 + \mu_2)N - P\}.$$

Here we put :

$$\begin{aligned} M &= \{\zeta(\lambda_1 - t) - \zeta(\lambda_2 - t) - \zeta(\lambda_1) + \zeta(\lambda_2)\}^{-1} \\ N &= \mathfrak{z}(\lambda_1; \lambda_2) = \zeta(\lambda_1 - \lambda_2) - \zeta(\lambda_1) + \zeta(\lambda_2) \\ P &= a_0\{\wp(\lambda_1) - \wp(\lambda_2)\} + a_1\{\wp(\lambda_1 - t) - \wp(\lambda_2 - t)\}. \end{aligned}$$

Moreover, consider the linear equation :

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = q(z; t)y, \tag{5}$$

such that

$$
\begin{aligned}
(6) \qquad q(z;t) = \rho + {r_0}^2 t^2 \wp(z)^2 + r_1 t \wp'(z) \\
+ \frac{3}{4}\wp(z-\lambda_1) + \frac{3}{4}\wp(z-\lambda_2) \\
+ K\wp(z) - \mu_1 \mathfrak{z}(z;\lambda_1) - \mu_2 \mathfrak{z}(z;\lambda_2).
\end{aligned}
$$

(5)-(6) is the linear differential equation with an irregular singularity; we obtain this by means of coalescence of two singularities of (2)-(3). It is known that :

PROPOSITION 0.2 ([O2]). *The holonomic deformation of* (5)-(6) *is governed by the Hamiltonian system as follows:*

$$
(7) \qquad D\lambda_k = \frac{\partial K}{\partial \mu_k}, \quad D\mu_k = -\frac{\partial K}{\partial \lambda_k} \quad ; k = 1, 2\ ,\ D = t\frac{\mathrm{d}}{\mathrm{d}t},
$$

with

$$
K = L\{({\mu_1}^2 - {\mu_2}^2) + (\mu_1 + \mu_2)N - Q\}.
$$

Here we put:

$$
\begin{aligned}
L &= \{\wp(\lambda_1) - \wp(\lambda_2)\}^{-1} \\
N &= \mathfrak{z}(\lambda_1;\lambda_2)(= \zeta(\lambda_1 - \lambda_2) - \zeta(\lambda_1) + \zeta(\lambda_2)) \\
Q &= {r_0}^2 t^2 \{\wp(\lambda_1)^2 - \wp(\lambda_2)^2\} + r_1 t\{\wp'(\lambda_1) - \wp'(\lambda_2)\}.
\end{aligned}
$$

The aim of the present article is to consider special solutions of (4) in the case: $a_0 = a_1 = 0$, and those of (7) with $r_0 = r_1 = 0$. For Hamiltonian system (4), we have from Proposition 0.1 the following expressions:

$$\begin{cases} \dfrac{\partial H}{\partial \mu_1} = M(2\mu_1 + N) \\ \dfrac{\partial H}{\partial \mu_2} = M(-2\mu_2 + N) \\ \dfrac{\partial H}{\partial \lambda_1} = M\Big\{(-\dfrac{\partial}{\partial \lambda_1}\dfrac{1}{M})H + (\dfrac{\partial}{\partial \lambda_1}N)(\mu_1 + \mu_2) \\ \qquad\qquad - a_0\wp'(\lambda_1) - a_1\wp'(\lambda_1 - t)\Big\} \\ \dfrac{\partial H}{\partial \lambda_2} = M\Big\{(-\dfrac{\partial}{\partial \lambda_2}\dfrac{1}{M})H + (\dfrac{\partial}{\partial \lambda_2}N)(\mu_1 + \mu_2) \\ \qquad\qquad + a_0\wp'(\lambda_2) + a_1\wp'(\lambda_2 - t)\Big\} \end{cases}$$

Then, when $a_0 = a_1 = 0$, (4) admits particular solutions given by :

$$\mu_1 + \mu_2 = 0, \quad \frac{\mathrm{d}\mu_1}{\mathrm{d}t} = \frac{\mathrm{d}\mu_2}{\mathrm{d}t} = 0,$$

and then

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = M(N + h), \tag{8}$$

where $h(= 2\mu_1 = -2\mu_2)$ being a constant.

We will show below (Theorem 1) that (8) admits a first integral $F_1(\lambda_1, \lambda_2, t)$. Here we say that F is a first integral of (8) if the function takes a constant value along a particular solution of (8). That is, the general solution of (8) is given by:

$$\lambda_1 - \lambda_2 = 2a, \tag{9}$$

$$F_1(\lambda_1, \lambda_2, t) = b, \tag{10}$$

where a, b are arbitrary constants.

To give an explicit form of F_1, we fix a moduli of $\mathbb{E}$ as follows:

a) (elliptic case) $2\omega_1 = 1$, $2\omega_3 = \tau$, $\mathrm{Im}\,\tau > 0$.

Moreover, we will consider equations of the form of (8), also on a rational curve obtained from the elliptic curve $\mathbb{E}$ by the degenerations:

b) (trigonometric case) $\quad \mathrm{Im}\,\tau \to \infty$,

c) (rational case) $\quad \omega_1 \to \infty,\ \omega_3 \to \infty$.

Then we have the

THEOREM 1. (8) *admits a first integrals given as follows:*

a).Elliptic case.

$$F_1(\lambda_1, \lambda_2, t) = e^{\kappa_1 t}\frac{\vartheta_0(x-c)\vartheta_0(u-c)}{\vartheta_0(x+c)\vartheta_0(u+c)}$$

b).Trigonometric case.

$$F_2(\lambda_1, \lambda_2, t) = e^{\kappa_2 t}\frac{(e^{2\pi\sqrt{-1}x}-\gamma)(e^{2\pi\sqrt{-1}u}-\gamma)}{(e^{2\pi\sqrt{-1}x}-\delta)(e^{2\pi\sqrt{-1}u}-\delta)}$$

c).Rational case.

$$F_3(\lambda_1, \lambda_2, t) = e^{\kappa_3 t}\frac{(x-d)(u-d)}{(x+d)(u+d)}$$

Here we introduce variables x, u as follows:

$$x = \frac{\lambda_1+\lambda_2}{2},$$
$$u = t - \frac{\lambda_1+\lambda_2}{2},$$

and ϑ_0 is the elliptic theta function, $c, d, \gamma, \delta, \kappa_1, \kappa_2, \kappa_3$ being constants.

On the other hand, consider a class of integrable dynamical systems characterized by the equations:

$$\ddot{q}_j = \sum_{\substack{k=1\\k\neq j}}^{n} \dot{q}_j\dot{q}_k v(q_j-q_k), \quad q_j = q_j(t), \quad j = 1, 2, \cdots, n,$$

v being a function. This is called Ruijsenaars-Schneider system ([RS]), which can be written in the Lax form ([BC1]) :

$$\dot{\mathbf{L}} = [\mathbf{L}, \mathbf{M}],$$

where $\mathbf{L}$ and $\mathbf{M}$ are the $n \times n$ matrices such that:

$$L_{jk} = \delta_{jk}\dot{q}_j + (1-\delta_{jk})(\dot{q}_j\dot{q}_k)^{1/2}\alpha(q_j - q_k),$$
$$M_{jk} = \delta_{jk}\sum_{\substack{m=1\\ m\neq j}}^{n} \dot{q}_m\beta(q_j - q_m) + (1-\delta_{jk})(\dot{q}_j\dot{q}_k)^{1/2}\gamma(q_j - q_k).$$

Here the function $\alpha(z)$ satisfies the equation:

$$\alpha(z)\alpha'(w) - \alpha'(z)\alpha(w) = (\alpha(z+w) - \alpha(z)\alpha(w))(\eta(z) - \eta(w)), \tag{11}$$

which is called Bruschi-Calogero equation.

Let α and η be holomorphic functions defined on a punctured disk $\{z \in \mathbb{C}; 0 < |z| < r\}$ for some $r > 0$. It is shown recently by [KS] that if they satisfy Bruschi-Calgero equation, then they are equal to one of the following functions.

0-1) $\alpha(z) = 0$ or $e^{\rho z}$ $\quad (\rho \in \mathbb{C})$,
η :arbitrary,

0-2) $\alpha(z) = Ce^{\rho z}$ $\quad (C, \rho \in \mathbb{C}, C \neq 0, 1)$,
η :constant,

1) $$\alpha(z) = e^{\rho z}\frac{\sigma(\mu)\sigma(\nu+\lambda z)}{\sigma(\nu)\sigma(\mu+\lambda z)}, \quad \begin{pmatrix} \rho, \mu, \nu, A \in \mathbb{C}, \lambda, \omega_1, \omega_3 \in \mathbb{C}\backslash 0, \\ \mathrm{Im}\omega_3/\omega_1 > 0, \mu, \nu \notin \mathbb{Z}(2\omega_1) + \mathbb{Z}(2\omega_3) \end{pmatrix}$$

$$\eta(z) = \lambda\zeta(\lambda z) - \lambda\zeta(\lambda z + \mu) + A,$$

$\sigma(w) = \sigma(w; \omega_1, \omega_3)$ being the σ-function of Weierstraß;

2) $$\alpha(z) = e^{\rho z}\frac{a(e^{2z/\lambda} - 1) + b}{c(e^{2z/\lambda} - 1) + b},$$
$$\begin{pmatrix} \lambda, \rho, a, b, c, A \in \mathbb{C}, \lambda \neq 0, \\ b(a-c) \neq 0, a \neq 0 \,\mathrm{or}\, b \neq c, c \neq 0 \,\mathrm{or}\, a \neq b \end{pmatrix}$$
$$\eta(z) = \frac{2\lambda^{-1}e^{2z/\lambda}}{e^{2z/\lambda} - 1} - 2\lambda^{-1}ce^{2z/\lambda}(e^{2z/\lambda} - 1) + b + A,$$

3) $\alpha(z)=e^{\rho z}\dfrac{az+b}{cz+b}, \quad (\rho,a,b,c,A\in\mathbb{C}, b(a-c)\neq 0)$

$$\eta(z)=\frac{b}{z(cz+b)}+A.$$

Let $F_1=F_1(\lambda_1,\lambda_2,t)$ be the function given in Theorem 1. Since

$$\begin{aligned}\vartheta_0(v)&=\exp(-2\eta_1\omega_1v^2)\vartheta_0(0)\sigma_3(2\omega_1v),\\ \sigma_3(z)&=\exp(-\eta_3z)\sigma(z+\omega_3)/\sigma(\omega_3),\end{aligned}$$

we have

$$F_1=e^{\kappa_1t}e^{8c\omega_1(\eta_1t+\eta_3)}\frac{\sigma(2\omega_1x+(\omega_3-2c\omega_1))\sigma(2\omega_1u+(\omega_3-2c\omega_1))}{\sigma(2\omega_1x+(\omega_3+2c\omega_1))\sigma(2\omega_1u+(\omega_3+2c\omega_1))},$$

where σ_3 is the Weierstraß co-σ-function. By putting

$$\begin{aligned}\nu&=-2\omega_1x-\omega_3-2c\omega_1,\\ \mu&=-2\omega_1x-\omega_3+2c\omega_1,\end{aligned}$$

and then using the pseudo-periodicity of σ-function, we obtain the following expression:

$$F_1=e^{(\kappa_1+8c\omega_1\eta_1)t}\frac{\sigma(\mu)\sigma(\nu+2\omega_1t)}{\sigma(\nu)\sigma(\mu+2\omega_1t)}.$$

This shows that a first integral of (8) solves Bruschi-Calogero equation (11). A mathematical meaning of this fact is not yet clear.

When considering (7) which is the Hamiltonian system of confluent type with $r_0=r_1=0$, we have a special solution of the form:

$$\mu_1+\mu_2=0,\quad \frac{\mathrm{d}\mu_1}{\mathrm{d}t}=\frac{\mathrm{d}\mu_2}{\mathrm{d}t}=0,$$

$$t\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=t\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=L(N+h), \tag{12}$$

where $h(=2\mu_1=-2\mu_2)$ being a constant. We can give an explicit form of a first integral of (12); in fact we have the

Theorem 2. (12) *admits first integrals as follows:*

a).Elliptic case.

$$G_1(\lambda_1,\lambda_2,t)=t\frac{\sigma(x-a)\sigma(x+a)}{\sigma(x-b)\sigma(x+b)}$$

b).Trigonometric case.

$$\begin{aligned}G_2(\lambda_1,\lambda_2,t) &= t^{\varkappa_2}\frac{e^{2\pi\sqrt{-1}x}-C^{-1}}{e^{2\pi\sqrt{-1}x}-C}\\ &\quad\times\exp\left\{\frac{c_1}{e^{2\pi\sqrt{-1}x}-C^{-1}}+\frac{c_2}{e^{2\pi\sqrt{-1}x}-C}\right.\\ &\qquad\left.+\frac{c_3}{(e^{2\pi\sqrt{-1}x}-C^{-1})^2}+\frac{c_4}{(e^{2\pi\sqrt{-1}x}-C)^2}\right\},\end{aligned}$$

c).Rational case.

$$G_3(\lambda_1,\lambda_2,t)=t\frac{(x-\bar{a})(x+\bar{a})}{(x-\sqrt{-3}\bar{a})(x+\sqrt{-3}\bar{a})},$$

Here $x=\frac{\lambda_1+\lambda_2}{2}$, *and* σ *denotes the* σ*-function of Weierstraß,* $\varkappa_2$, a, b, C, $\bar{a}$, $c_i (i=1,2,3,4)$ *being constants.*

We will verify the theorems in the following two sections. In Section 1 we consider (8) and (12) in the elliptic case; Section 2 is devoted to an investigation of the degeneration cases.

1. First Integrals

In this section, we study differential equations (8) and (12) in the elliptic case; the degeneration cases are investigated in the next section. Consider the equation:

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=M(N+h), \tag{8}$$

where

$$M=\{\zeta(\lambda_1-t)-\zeta(\lambda_2-t)-\zeta(\lambda_1)+\zeta(\lambda_2)\}^{-1},$$
$$N=\zeta(\lambda_1-\lambda_2)-\zeta(\lambda_1)+\zeta(\lambda_2).$$

It is clear that

$$a = \frac{\lambda_1 - \lambda_2}{2}$$

is independent of t.

We show:

LEMMA 1.1. (8) *has a first integral of this form:*

$$F_1(\lambda_1, \lambda_2, t) = e^{\kappa_1 t} \frac{\vartheta_0(x-c)\vartheta_0(u-c)}{\vartheta_0(x+c)\vartheta_0(u+c)}$$

where $\vartheta_0(\mu)$ *is the elliptic theta function. Here we put:*

$$x = \frac{\lambda_1 + \lambda_2}{2},$$
$$u = t - \frac{\lambda_1 + \lambda_2}{2},$$

κ_1, c *being constants.*

PROOF OF LEMMA.

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t} = M(N+h),$$

that is, $M^{-1}\mathrm{d}x = (N+h)\mathrm{d}t$, we have:

$$M^{-1}\mathrm{d}x = (N+h)\mathrm{d}x + (N+h)\mathrm{d}u.$$

On the other hand, we compute:

$$\begin{aligned} N &= \zeta(\lambda_1 - \lambda_2) - \zeta(\lambda_1) + \zeta(\lambda_2) \\ &= \zeta(2a) - \zeta(x+a) + \zeta(x-a) \\ &= -\mathfrak{z}(x-a; x+a), \end{aligned}$$

$\mathfrak{z}(u; v)$ being the function defined in Introduction, and then

$$\begin{aligned} M^{-1} - N &= \zeta(\lambda_1 - t) - \zeta(\lambda_2 - t) - \zeta(\lambda_1 - \lambda_2) \\ &= \zeta(a-u) - \zeta(-a-u) - \zeta(2a) \\ &= -\zeta(2a) + \zeta(u+a) - \zeta(u-a) \\ &= \mathfrak{z}(u-a; u+a). \end{aligned}$$

It follows that:

$$\frac{\mathrm{d}x}{\mathfrak{z}(x)-h}+\frac{\mathrm{d}u}{\mathfrak{z}(u)-h}=0, \tag{13}$$

where we put:

$$\mathfrak{z}(w) \underset{\text{def.}}{=} \mathfrak{z}(w-a; w+a).$$

Using the addition formulae of elliptic functions, we can show

$$\mathfrak{z}(z)=-\frac{1}{2}\frac{\wp''(a)}{\wp'(a)}-\frac{\wp'(a)}{\wp(z)-\wp(a)},$$

then we have:

$$\frac{1}{\mathfrak{z}(z)-h}=\frac{\alpha}{\wp'(a)}\left\{1+\frac{\alpha}{\wp(z)-\beta}\right\},$$

where

$$\alpha=-\frac{2\wp'(a)^2}{2h\wp'(a)+\wp''(a)},$$
$$\beta=\wp'(a)-\frac{2\wp'(a)^2}{2h\wp'(a)+\wp''(a)}=\alpha+\wp'(a).$$

Now we need some constants, related to the elliptic curve $\mathbb{E}$:

$$\omega_2=-(\omega_1+\omega_3),$$
$$e_\nu=\wp(\omega_\nu),\ \eta_\nu=\zeta(\omega_\nu),\quad \nu=1,2,3$$
$$\rho=\sqrt{e_1-e_3},\ k=\sqrt{\frac{e_2-e_3}{e_1-e_3}}.$$

Then we define a constant q by:

$$-k^2\,\mathrm{sn}^2 q=\frac{e_3-\beta}{\rho^2}\ .$$

Here we denote by $\mathrm{sn}q$ Jacobi elliptic function, which is related to the Weierstraß $\wp$ as follows:

$$\wp(z)=e_3+\frac{e_1-e_3}{\mathrm{sn}^2(\rho z)}\ .\ \square$$

PROPOSITION 1.2. *By using the notations given above, we have:*

$$\frac{\mathrm{d}w}{\mathfrak{z}(w)-h}=\mathrm{d}\left(\left\{1+2\varphi\frac{\sigma_3'(c)}{\sigma_3(c)}\right\}w+\varphi\log\frac{\sigma_3(w-c)}{\sigma_3(w+c)}\right),$$

where $\varphi=\dfrac{1}{2}\dfrac{\alpha}{\rho^3k^2\,\mathrm{sn}q\,\mathrm{cn}q\,\mathrm{dn}q}$, $c=\dfrac{q}{\rho}$, $\sigma_3(w)$ *being the Weierstraß co-σ-function.*

Lemma 1.1 follows from the Proposition; in fact, we obtain:

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathfrak{z}(x)-h}+\frac{\mathrm{d}u}{\mathfrak{z}(u)-h}\\&\qquad=\mathrm{d}\left(\left\{1+2\varphi\frac{\sigma_3'(c)}{\sigma_3(c)}\right\}(x+u)+\varphi\log\frac{\sigma_3(x-c)\sigma_3(u-c)}{\sigma_3(x+c)\sigma_3(u+c)}\right).\end{aligned}$$

It follows that, if we put;

$$F_1(\lambda_1,\lambda_2,t)=e^{\psi t}\frac{\sigma_3(x-c)\sigma_3(z-c)}{\sigma_3(x+c)\sigma_3(z+c)}\ ,$$
$$\psi=\frac{1}{\varphi}+2\frac{\sigma_3'(c)}{\sigma_3(c)}\ ,$$

then $\mathrm{d}F_1=0$, along a solution of (8). the Weierstraß co-σ-functions are related to theta functions as follows:

$$\sigma_j(\lambda)=e^{2\eta_1\omega_1\mu^2}\frac{\vartheta_{j+1}(\mu)}{\vartheta_{j+1}(0)}\quad ;j=1,2,3\ ,\quad(\vartheta_0(\mu)=\vartheta_4(\mu))$$
$$\mu=\frac{\lambda}{2\omega_1}.$$

Then, by normalizing as $2\omega_1=1$, we obtain the following expression for the first integral:

$$F_1(\lambda_1,\lambda_2,t)=e^{\kappa_1 t}\frac{\vartheta_0(x-c)\vartheta_0(u-c)}{\vartheta_0(x+c)\vartheta_0(u+c)},$$

where $\kappa_1=\psi-4\eta_1c(=2\omega_1\psi-4\eta_1c)$.

Proposition follows from the following formula on elliptic integrals of the third kind:

$$k^2\,\mathrm{sn}q\,\mathrm{cn}q\,\mathrm{dn}q\int_0^w \frac{\mathrm{sn}^2w}{1-k^2\,\mathrm{sn}^2q\,\mathrm{sn}^2w}\mathrm{d}w=\frac{1}{2}\frac{\sigma_3(x-c)}{\sigma_3(x+c)}+\frac{\sigma_3'(c)}{\sigma_3(c)}x,$$

with $w=\rho x$, $q=\rho c$. The left hand side of this formula is often denoted as $\Pi(w,q)$, called the Jacobi-Π-function ([E], [HC]). Lemma 1.1 has been established.

Next we investigate the equation

$$t\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=t\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=L(N+h),$$

$$\begin{aligned}&L=\{\wp(\lambda_1)-\wp(\lambda_2)\}^{-1},\\&N=\mathfrak{z}(\lambda_1;\lambda_2)(=\zeta(\lambda_1-\lambda_2)-\zeta(\lambda_1)+\zeta(\lambda_2)).\end{aligned}$$

We deduce from (12) the equation

$$\frac{\mathrm{d}t}{t}+\frac{\wp(x+a)-\wp(x-a)}{\mathfrak{z}(x)-h}\mathrm{d}x=0,$$

where we put:

$$x=\frac{\lambda_1+\lambda_2}{2},\ a=\frac{\lambda_1-\lambda_2}{2}.$$

Note that a is constant and recall the identity:

$$\frac{1}{\mathfrak{z}(z)-h}=\frac{\alpha}{\wp'(a)}\left\{1+\frac{\alpha}{\wp(z)-\beta}\right\},$$

$$\begin{aligned}&\alpha=-\frac{2\wp'(a)^2}{2h\wp'(a)+\wp''(a)},\\&\beta=\wp'(a)-\frac{2\wp'(a)^2}{2h\wp'(a)+\wp''(a)}=\alpha+\wp'(a).\end{aligned}$$

By the use of the addition formulae, we have

$$\wp(x+a)-\wp(x-a)=-\frac{\wp'(a)\wp'(x)}{\{\wp(x)-\wp(a)\}^2},$$

and then

$$\begin{aligned}
-\frac{1}{L(N+h)} &= \frac{\wp(x+a)-\wp(x-a)}{\mathfrak{z}(x)-h} \\
&= \frac{\alpha}{\wp'(a)}\left\{1+\frac{\alpha}{\wp(x)-\beta}\right\}\frac{-\wp'(a)\wp'(x)}{\{\wp(x)-\wp(a)\}^2} \\
&= -\wp'(x)\left\{\frac{1}{\wp(x)-\beta}-\frac{1}{\wp(x)-\wp(a)}\right\}.
\end{aligned}$$

Then it follows that:

$$\begin{aligned}
&\frac{\mathrm{d}t}{t}-\frac{\mathrm{d}x}{L(N+h)} \\
=&\frac{\mathrm{d}t}{t}-\left\{\frac{\wp'(x)\mathrm{d}x}{\wp(x)-\beta}-\frac{\wp'(x)\mathrm{d}x}{\wp(x)-\wp(a)}\right\} \\
=&\mathrm{d}\left(\log t-\log\{\wp(x)-\beta\}+\log\{\wp(x)-\wp(a)\}\right).
\end{aligned}$$

We arrived at the first integral of (12):

$$G_1(\lambda_1,\lambda_2,t)=t\frac{\wp(x)-\wp(a)}{\wp(x)-\beta}.$$

Since

$$\wp(x)-\wp(a)=-\frac{\sigma(x-a)\sigma(x+a)}{\sigma(x)^2\sigma(a)^2},$$

by defining b by $\wp(b)=\beta$, we have the

LEMMA 1.3. (12) *has the first integral*

$$G_1(\lambda_1,\lambda_2,t)=t\frac{\sigma(x-a)\sigma(x+a)}{\sigma(x-b)\sigma(x+b)}.$$

2. Degeneration of the Elliptic Curve

Firstly we consider the trigonometric case: $2\omega_1=1, \mathrm{Im}\,\tau\to\infty$. Put:

$$\begin{aligned}
&x=\frac{\lambda_1+\lambda_2}{2},\ u=t-\frac{\lambda_1+\lambda_2}{2},\ a=\frac{\lambda_1-\lambda_2}{2}, \\
&X=e^{2\pi\sqrt{-1}x}, \qquad U=e^{2\pi\sqrt{-1}u}, \qquad A=e^{2\pi\sqrt{-1}a}.
\end{aligned}$$

For the trigonometric case, the Weierstraß $\wp$-functions reduce to the function:

$$\wp(x) = \pi^2 \left\{ \frac{1}{\sin^2 \pi x} - \frac{1}{3} \right\},$$
$$\wp'(x) = -2\pi^3 \frac{\cos \pi x}{\sin^3 \pi x}.$$

Then we have:

$$\wp(x-a) = \pi^2 \left\{ \frac{-4e^{2\pi\sqrt{-1}(x-a)}}{(e^{2\pi\sqrt{-1}(x-a)}-1)^2} - \frac{1}{3} \right\} = \pi^2 \left\{ \frac{-4AX}{(X-A)^2} - \frac{1}{3} \right\},$$
$$\wp(x+a) = \pi^2 \left\{ \frac{-4e^{2\pi\sqrt{-1}(x+a)}}{(e^{2\pi\sqrt{-1}(x+a)}-1)^2} - \frac{1}{3} \right\} = \pi^2 \left\{ \frac{-4AX}{(AX-1)^2} - \frac{1}{3} \right\},$$
$$\begin{aligned} \wp'(x-a) &= 8\pi^3\sqrt{-1} \frac{e^{\pi\sqrt{-1}(x-a)} + e^{-\pi\sqrt{-1}(x-a)}}{(e^{\pi\sqrt{-1}(x-a)} - e^{-\pi\sqrt{-1}(x-a)})^3} \\ &= 8\pi^3\sqrt{-1} \frac{AX(X+A)}{(X-A)^3}, \\ \wp'(x+a) &= 8\pi^3\sqrt{-1} \frac{e^{\pi\sqrt{-1}(x+a)} + e^{-\pi\sqrt{-1}(x+a)}}{(e^{\pi\sqrt{-1}(x+a)} - e^{-\pi\sqrt{-1}(x+a)})^3} \\ &= 8\pi^3\sqrt{-1} \frac{AX(AX+1)}{(AX-1)^3}. \end{aligned}$$

(See [S].)

LEMMA 2.1. *For the trigonometric case, the equation*

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = M(N+h)$$

has the first integral

$$F_2(\lambda_1, \lambda_2, t) = e^{\kappa_2 t} \frac{(X-\gamma)(U-\gamma)}{(X-\delta)(U-\delta)} \ ,$$

κ_2, γ, δ *being constants.*

PROOF OF LEMMA 2.1. Note that

$$A = e^{2\pi\sqrt{-1}a} = e^{2\pi\sqrt{-1}\frac{\lambda_1-\lambda_2}{2}}$$

is not depending on t. Since $\mathrm{d}X = 2\pi\sqrt{-1}X\mathrm{d}x$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathfrak{z}(x)-h} &= \frac{\mathrm{d}x}{\zeta(x+a)-\zeta(x-a)-\zeta(2a)-h} \\
&= \left\{\frac{1}{2}\frac{\wp'(x-a)+\wp'(x+a)}{\wp(x-a)-\wp(x+a)} - h\right\}^{-1}\mathrm{d}x \\
&= \frac{m}{2\pi\sqrt{-1}}\frac{\mathrm{d}X}{X}\frac{1+X^2-\xi X}{1+X^2-\eta X},
\end{aligned}$$

where we put:

$$\begin{aligned}
m &= -\frac{A(1-A^2)}{2hA(1-A^2)-2\pi\sqrt{-1}A(1+A^2)}, \\
\xi &= -\frac{1+A^2}{A(1-A^2)}, \\
\eta &= \frac{2h(1-A^2)+2\pi\sqrt{-1}(1-6A+A^4)}{2hA(1-A^2)-2\pi\sqrt{-1}A(1+A^2)}.
\end{aligned}$$

We define constants $\gamma, \gamma', \delta, \delta'$ by:

$$\begin{aligned}
&\xi = \gamma' + \delta', \\
&\eta = \gamma + \delta, \\
&\gamma\delta = \gamma'\delta' = 1;
\end{aligned}$$

hence we obtain:

$$\begin{aligned}
\frac{(1+X^2)-\xi X}{(1+X^2)+\eta X} &= \frac{(X-\gamma')(X-\delta')}{(X-\gamma)(X-\delta)} \\
&= 1 + \frac{(\gamma+\delta)-(\gamma'+\delta')}{\gamma-\delta}\left(\frac{X}{X-\gamma} - \frac{X}{X-\delta}\right).
\end{aligned}$$

By putting:

$$\kappa_2 = \frac{\gamma-\delta}{(\gamma+\delta)-(\gamma'+\delta')},$$

we rewrite the equation as follows:

$$\begin{aligned}\frac{2\pi\sqrt{-1}}{m}\left\{\frac{\mathrm{d}x}{\mathfrak{z}(x)-h}+\frac{\mathrm{d}u}{\mathfrak{z}(u)-h}\right\} &= \frac{\mathrm{d}X}{X}+\frac{1}{\kappa_2}\left(\frac{\mathrm{d}X}{X-\gamma}-\frac{\mathrm{d}X}{X-\delta}\right)\\ &\quad+\frac{\mathrm{d}U}{U}+\frac{1}{\kappa_2}\left(\frac{\mathrm{d}U}{U-\gamma}-\frac{\mathrm{d}U}{U-\delta}\right)\\ &= \mathrm{d}\left(\log XU+\frac{1}{\kappa_2}\log\frac{(X-\gamma)(U-\gamma)}{(X-\delta)(U-\delta)}\right).\end{aligned}$$

We thus obtain the integral:

$$F_2(\lambda_1,\lambda_2,t)=e^{\kappa_2 t}\frac{(X-\gamma)(U-\gamma)}{(X-\delta)(U-\delta)}.$$

For the rational case: $2\omega_1\to\infty$, $2\omega_3\to\infty$, a first integral of (8) is given by the following Lemma:

LEMMA 2.2. (8) *has the first integral*

$$F_3(\lambda_1,\lambda_2,t)=e^{\kappa_3 t}\frac{(x-d)(u-d)}{(x+d)(u+d)}$$

Here we put

$$\begin{aligned}x&=\frac{\lambda_1+\lambda_2}{2},\\ u&=t-\frac{\lambda_1+\lambda_2}{2},\end{aligned}$$

and define the constants κ_3, d *as follows:*

$$\kappa_3=-\frac{\sqrt{(2ah+1)(2ah-3)}}{2a},\quad d=a\sqrt{\frac{2ah-3}{2ah+1}}\qquad \textit{for}\quad a=\frac{\lambda_1-\lambda_2}{2}.$$

PROOF OF LEMMA 2.2. In the rational case, we have:

$$\zeta(z)=\frac{1}{z}.$$

Then

$$\begin{aligned}\mathfrak{z}(z) &= \zeta(z+a) - \zeta(z-a) - \zeta(2a) \\ &= \frac{1}{z+a} - \frac{1}{z-a} - \frac{1}{2a} \\ &= -\frac{z^2+3a^2}{2a(z^2-a^2)}\ ;\end{aligned}$$

therefore

$$\begin{aligned}\frac{\mathrm{d}z}{\mathfrak{z}(z)-h} &= \left(-\frac{2a}{1+2ah} + \frac{8a^3}{(1+2ah)^2}\frac{1}{z^2+\frac{3-2ah}{1+2ah}a^2}\right)\mathrm{d}z \\ &= \mathrm{d}\left(-\frac{2az}{1+2ah} + \frac{4a^3d^{-1}}{(1+2ah)^2}\log\frac{1-d^{-1}z}{1+d^{-1}z}\right),\end{aligned}$$

where

$$d = a\sqrt{\frac{2ah-3}{2ah+1}}.$$

By putting

$$\kappa_3 = -\frac{1+2ah}{2a^2d^{-1}},$$

we have

$$\begin{aligned}&\frac{\mathrm{d}x}{\mathfrak{z}(x)-h} + \frac{\mathrm{d}u}{\mathfrak{z}(u)-h} \\ &\qquad = \mathrm{d}\left(-\frac{2a(x+u)}{1+2ah} + \frac{2a}{\kappa_3(1+2ah)}\log\frac{(1-\frac{x}{d})(1-\frac{u}{d})}{(1+\frac{x}{d})(1+\frac{u}{d})}\right),\end{aligned}$$

from which we have the integral:

$$F_3(\lambda_1,\lambda_2,t) = e^{\kappa_3 t}\frac{(x-d)(u-d)}{(x+d)(u+d)}.$$

We have computed the proof of Theorem. For confluent type equation (12), Theorem 2 can be established in a similar manner; we do not enter into detail of verification. □

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(Received April 27, 2001)

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