J. Math. Sci. Univ. Tokyo
9 (2002), 659–667.

L^2-estimates and Solvability of the $\bar\partial$ System on Weakly q-convex Dihedrons

By Luca Baracco and Pier Nicola Bettiol

Abstract. The weighted L^2 estimates are here exploited to solve the $\bar\partial$ problem on q–pseudoconvex dihedrons of $\mathbb{C}^n$.

Introduction

In this paper we prove local solvability of the $\bar\partial$ system in degree $\geq q$ on a weakly q-convex dihedron of $\mathbb{C}^n$. The basic method in the proof are the a priori L^2 estimates for $\bar\partial$ and $\bar\partial^*$ which provide both the existence and the regularity of the solutions. This method is very classic and goes back to the celebrated papers by KOHN [7] and HÖRMANDER [5]. In particular it yields solvability of $\bar\partial$ in any degree ≥ 1 on pseudoconvex domains Ω of $\mathbb{C}^n$. To this end one is lead to introduce a weight $e^{-\phi}$ in the L^2 norms for a strictly plurisubharmonic exhaustion function ϕ of Ω. In the present paper we do not require any more Ω to be pseudoconvex and allow $\bar\partial\partial\phi$, instead, to take negative eigenvalues. Our main contribution consists in finding a q-subharmonic exhaustion function for a non–smooth domain starting from the geometric assumption of q-convexity of its boundary. We owe to ZAMPIERI [8] a part of the techniques exploited in this contest. If, instead of a dihedron, we consider a C^2 half space, our results on solvability of the $\bar\partial$ system are already stated by HO in [3]. However our approach is original both for a new way of stating the relation of q-convexity of a domain and its boundary and for a simplified use of the L^2-estimates.

The authors wish to thank the referee for many remarks which improved the general expository quality of the paper.

§1. Basic Definitions

Let $\rho_1,\ \rho_2 : \mathbb{C}^n \to \mathbb{R}$, be functions of class C^2 with $\mathbb{C}$-transversal differentials that is $\partial\rho_1 \wedge \partial\rho_2 \neq 0$.

2000 *Mathematics Subject Classification.* 32F10, 32F20.

DEFINITION 1. A domain W of $\mathbb{C}^n$ defined by $W = \{z \in \mathbb{C}^n : \rho_1(z) < 0,\, \rho_2(z) < 0\}$ is said to be a *dihedron* with *edge* $S = \{z \in \mathbb{C}^n : \rho(z) = 0,\, \rho_2(z) = 0\}$.

Note that the edge S is generic. We will often focus our attention at a point $z_o \in S$ and consider wedges which are only defined in a neighborhood B of z_o (that is we require $z \in B$ in Definition 1). In this case we talk about *local wedges* at z_o. We denote by M_h $h = 1, 2$ the manifolds defined by $M_h = \{z \in \mathbb{C}^n : \rho_h(z) = 0\}$, and by M_h^+ the closed manifolds with boundary $M_h^+ = \{z \in M_h : \rho_k \leq 0 \text{ for } k \neq h\}$. The latter are the *faces* of W.

We will consider vectors $w = (w_J)_{|J|=q}$ for multiindices $J = (j_1, \dots, j_q)$ of total length q, with alternate coefficients. Often we will take, instead, vectors with coefficients having only ordered indices J that is verifying $j_1 < j_2 < \cdots < j_q$. When taking sums over ordered indices we will use the standard notation $\sum_J{}'$. We often play with the alternate and ordered notations. In particular we rewrite vectors of the form $(w_J)_J$ for J ordered of length q, in the form $\frac{1}{q}(w_{i\,K})_{i\,K}$, K ordered of length $q-1$. (Recall that if J and $i\,K$ are related by a permutation σ, then $w_{i\,K} = \operatorname{sgn}(\sigma) w_J$.) Let L_{M_h} be the Levi form of M_h that is the Hermitian form $\bar\partial\partial\rho_h|_{T^{\mathbb{C}}M}$ where $T^{\mathbb{C}}M$ is the complex tangent bundle to M_h. (For $z \in M_h$ this is defined by $T_z^{\mathbb{C}}M_h = \{u \in \mathbb{C}^n : \langle \partial\rho_h(z), u\rangle = 0\}$.) Let us denote by $\lambda_1^h \leq \lambda_2^h \leq \cdots \leq \lambda_{n-1}^h$ the ordered eigenvalues of L_{M_h}.

DEFINITION 2. A dihedron W is said to be *weakly q-convex* if for $h = 1$ and $h = 2$ we have

$$\sum_{j=1}^{q} \lambda_j^h(z) \geq 0 \quad \forall z \in M_h^+. \tag{1}$$

It is obvious that condition (1) is invariant under complex orthonormal transformations in $\mathbb{C}^n$. One can also check (cf. [3. Lemma 2.2]) that condition (1) is equivalent to

$$(2)\quad \sum_{|K|=q-1}{}' \sum_{i,j=1}^{n} \bar\partial_{z_j}\partial_{z_i}\rho_h(z) w_{iK}\bar w_{jK} \geq 0 \quad \forall z \in M_h^+,$$

$$\forall (w_{i\,K})_{i\,K} \text{ with } |K| = q-1 \text{ and } \sum_{i=1}^{n} \frac{\partial \rho_h}{\partial z_i} w_{iK} = 0.$$

Indeed the equivalence of the two above condition is immediate once one proves that the second is invariant under orthonormal complex transformation (cf. [3]).) We similarly define the notion of *q–subharmonicity* for a real function ϕ on $\mathbb{C}^n$. We denote by $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ the ordered eigenvalues of $\bar\partial\partial\phi$.

Definition 3. ϕ is said to be *q-subharmonic* if and only if

$$(3)\qquad \sum_{j=1}^{q} \mu_j \geq 0.$$

Again, it is possible to check (cf. [3, Lemma 1.3]), that (3) is equivalent to

$$(4)\qquad \sum_{|K|=q-1}{}' \sum_{i,j=1}^{n} \bar\partial_{z_j}\partial_{z_i}\phi w_{iK}\bar w_{jK} \geq 0.$$

We point out the main diffence between the two above Definitions 2 and 3. In the first case only *restricted* Levi forms such as $\bar\partial\partial\rho_h|_{T^{\mathbb{C}}M}$, of size $(n-1)\times(n-1)$ appear, whereas in the second case, the entire Levi form $\bar\partial\partial\phi$ of size $n\times n$ enters into play. The main content of subsequent Theorem 5 will consist in relating positivity of forms of such different size.

Definition 4. A real function ϕ defined on W is said an *exhaustion* function for W when $\phi^{-1}([0,c]) \subset\subset W$ for any positive c.

§2. q–convexity of the Boundary and q–subharmonicity of a Defining Function

Let W be a dihedron of $\mathbb{C}^n$ in a neighborhood of a point z_o of its edge S.

Theorem 5. *Let W be weakly q-convex at z_o. Then there exists an exhaustion function ϕ defined on $B \cap W$ (for a suitable spheric neighborhood of z_o) which is q-subharmonic.*

Proof. We shall find two q-subharmonic functions ϕ_1, ϕ_2 defined on $B \cap W$, such that $\phi_h(z) \to +\infty$ for $z \to M_h, h = 1, 2$. If we then take a subharmonic exhaustion function φ for B, it is clear that

$$\phi := \phi_1 + \phi_2 + \varphi \tag{5}$$

will serve the purpose. Let us construct ϕ_1 and ϕ_2. Let $T_{z_o}M_1^+$ and $T_{z_o}M_2^+$ be the tangent spaces to the manifolds with boundary M_1^+ and M_2^+ respectively. Since $S \subset M_h^+$, $h = 1, 2$, we have, for suitable v_1, v_2 that $T_{z_o}M_1^+ = T_{z_o}S \oplus \mathbb{R}^+ v_1$ and $T_{z_o}M_2^+ = T_{z_o}S \oplus \mathbb{R}^+ v_2$ with $\Re e\langle \partial\rho_2(z_o), v_1\rangle < 0$ and $\Re e\langle \partial\rho_1(z_o), v_2\rangle < 0$. Set $w = v_2 - \varepsilon v_1$ with $\varepsilon > 0$ small, and consider the map

$$\begin{array}{rccc} G: & M_1^+ \times \mathbb{R}^+ & \to & \mathbb{C}^n \\ & (z, \lambda) & \mapsto & z + \lambda w. \end{array}$$

We show that for a convenient neighborhood V of z_o over M_1^+ and for $\delta > 0$

$$G(V \times (0, \delta]) \supset B(z_o, r) \cap W.$$

To this end it is enough to show that $G'(T_{z_o}M_1^+ \times T_o\mathbb{R}^+) \supset T_{z_o}W$. In fact

$$\begin{aligned} G'(T_{z_o}M_1^+ \times T_o\mathbb{R}^+) &= \{v + \lambda w \text{ such that } \lambda > 0, v \in T_{z_o}M_1^+\} \\ &= \{s + \lambda w + \mu v_1 \text{ such that } \lambda > 0, \mu > 0, s \in T_{z_o}S\}. \end{aligned}$$

On the other hand

$$T_{z_o}W = \{s + \mu_1 v_1 + \mu_2 v_2 \text{ such that } \mu_1, \mu_2 > 0, s \in T_{z_o}S\}.$$

Clearly $T_{z_o}W \subset G'(T_{z_o}M_1^+ \times T_o\mathbb{R}^+)$. It follows that the image of G contains $B \cap W$ for B small.

We can assume, under a unitary change of coordinates, that the direction of w is ∂_{y_1}. Under this assumption we can choose an equation for M_1 of the form $y_1 - g(x_1, z') = 0$; the manifold $G(M_1 \times \{\lambda\})$ will then have equation $y_1 - g(x_1, z') = \lambda$. We set $\rho_1 = -y_1 + g$ and $\phi_1 = -\log(-\rho_1) + C|z|^2$, where C is a big positive constant. We claim that ϕ_1 is q-subharmonic. To see this, we decompose each vector $w = (w_{i\,K})_{i\,K}$, with K ordered, as a sum $w = w^t + w^n$ of a tangential and normal component to $\rho_1 = \lambda$ respectively. (This simply means that for each fixed K we take the decomposition in tangential and normal components for the vectors in $\mathbb{C}^n$: $(w_{i\,K})_i = (w_{i\,K})_i^t + (w_{i\,K})_i^n$, and then take the collection over K ordered for these decompositions.) We also write $w_{\cdot\,K}$ instead of $(w_{i\,K})_i$ and so on. We then have for each K

$$
\begin{aligned}
\partial\bar\partial\phi_1(w_{\cdot\,K}, \bar w_{\cdot\,K}) &= \frac{1}{\rho_1^2}\partial\rho_1 \otimes \bar\partial\rho_1(w_{\cdot\,K}, \bar w_{\cdot\,K}) \\
&\quad - \frac{1}{\rho_1}\partial\bar\partial\rho_1(w_{\cdot\,K}, \bar w_{\cdot\,K}) + C|w_{\cdot\,K}|^2 \\
&\geq \frac{1}{2\rho_1^2}|w^n_{\cdot\,K}|^2 - 2\Re e\frac{1}{\rho_1}\partial\bar\partial\rho_1(w^n_{\cdot\,K}, \bar w^t_{\cdot\,K}) \\
&\quad - \frac{1}{\rho_1}\partial\bar\partial\rho_1(w^t_{\cdot\,K}, \bar w^t_{\cdot\,K}) + C|w_{\cdot\,K}|^2
\end{aligned}
$$

Since the Levi form at the point $z = (x_1 + iy_1, z')$ coincides with the Levi form at the boundary point $(x_1 + ig(x_1, z'), z')$ (and since $\rho_1 < 0$ on W), we have that

$$
\sum_{|K|=q-1}{}' \left(-\frac{1}{\rho_1}\partial\bar\partial\rho_1(w^t_{\cdot\,K}, \bar w^t_{\cdot\,K}\right) \geqslant 0
$$

(as a consequence of the definition of q-pseudoconvexity). Note now that $\Re e\partial\bar\partial\rho_1(w^n_{\cdot K}, \bar w^t_{\cdot K}) \geq -D|w^t_{\cdot K}||w^n_{\cdot K}|$ for some constant $D \geqslant 0$. We then choose our constant C such that $C \geq 2D^2$. Hence the remaining terms in the above expression for $\partial\bar\partial\phi_1(w_{\cdot\,K}, \bar w_{\cdot\,K})$ are bounded from below by $C|w_{\cdot\,K}|^2 - \frac{2D}{|\rho_1|}|w^t_{\cdot K}||w^n_{\cdot K}| + \frac{1}{2\rho_1^2}|w^n_{\cdot K}|^2 \geq |\frac{1}{\sqrt{2}|\rho_1|}w^n_{\cdot K} - \sqrt{2}Dw^t_{\cdot K}|^2 > 0$. This implies that ϕ_1 is q-subharmonic. In the same way we construct ϕ_2. We finally take φ, strictly subharmonic exhaustion function for the sphere, and define ϕ by means of (5). $\square$

§3. Solvability of the $\bar\partial$ Operator on q–convex Dihedrons of $\mathbb{C}^n$

Let W be a dihedron of $\mathbb{C}^n$, and z_o a point of its edge. We will deal with the $\bar\partial$–complex over forms with coefficients in $C^\infty(W)_{z_o}$ the space of germs at z_o of infinitely differentiable functions on W. We denote by $C^\infty(W)^k_{z_o}$ the space of forms $f = \sum'_{|J|=k} f_J d\bar z_J$ with $f_J \in C^\infty(W)_{z_o}$. (It will be understood all through the paper that the f_J's are alternate in J whereas the sum $\sum_J{}'$ is taken over ordered indices.) We then consider the $\bar\partial$–complex

$$\cdots \to C^\infty(W)^{k-1}_{z_o} \overset{\bar\partial}{\to} C^\infty(W)^k_{z_o} \overset{\bar\partial}{\to} C^\infty(W)^{k+1}_{z_o} \to \dots \tag{6}$$

THEOREM 6. *Let W be a dihedron which is weakly q–convex in a neighborhood of a point z_o of its edge. Then for any form $f \in C^\infty(W)^k_{z_o}$ of degree $k \geq q$ which verifies $\bar\partial(f) = 0$ there is a form $u \in C^\infty(W)^{k-1}_{z_o}$ which solves the equation $\bar\partial u = f$.*

PROOF. If B is a small sphere with center at z_o, we still denote by W the intersection $B \cap W$. According to Theorem 5, there exists a global q–subharmonic exhaustion function for this new W, that we will still denote by ϕ. We will then prove global solvability of $\bar\partial$ on W. Along with forms with $C^\infty(W)$ coefficients, we will also consider forms whose coefficients f_J belong to the space $L^2_\phi(W)$, that is which have finite integrals $\int_W e^{-\phi}|f_J|^2 dV$. We will adopt the above integrals as norm in the space $L^2_\phi(W)$ and also denote it by $||f_J||_\phi$. We then switch from the complex (6) to the new complex of closed densely defined operators

$$\cdots \to L^2_{\phi-2\psi}(W)^{k-1} \overset{\bar\partial}{\to} L^2_{\phi-\psi}(W)^k \overset{\bar\partial}{\to} L^2_\phi(W)^{k+1} \to \dots, \tag{7}$$

where ψ is a new function which will be chosen later on. We write ∂_j and $\bar\partial_j$ instead of ∂_{z_j} and $\bar\partial_{z_j}$ respectively. We denote by $\bar\partial^*$ (resp. δ_j) the adjoint of $\bar\partial$ (resp. $\bar\partial_j$). The following inequality holds (cf. [3, pages 83-84]):

$$\begin{aligned} &\sum_{|K|=k-1}{}' \sum_{i,j=1..n} \int e^{-\phi}(\delta_i f_{iK} \delta_j f_{jK} - \bar\partial_j f_{iK} \overline{\bar\partial_i f_{jK}}) dV \\ &\quad + \sum_{|J|=k}{}' \sum_{j=1..n} \int e^{-\phi} (\bar\partial_j f_J)^2 dV \\ &\leq 2\|\bar\partial^* f\|_{\phi-2\psi} + \|\bar\partial f\|^2_\phi + 2\|\partial\psi f\|^2_\phi \quad \forall f \in C^\infty_c(W). \end{aligned} \tag{8}$$

We also have the commutation relations :

$$\delta_i\bar\partial_j - \bar\partial_j\delta_i = \bar\partial_j\partial_i\phi. \tag{9}$$

Let us denote by (I) the two first double sums in (8). By discarding the second sums which are positive, we get by the aid of (9): $\mathrm{I} \geq \sum_K{}' \sum_{i,j} \cdot \int e^{-\phi}\bar\partial_j\partial_i\phi\bar f_{jK} f_{iK} dV$. On the other hand, since ϕ is q–subharmonic and therefore satisfies (4), we then have

$$\sum_K{}' \sum_{i,j} \int e^{-\phi}\bar\partial_j\partial_i\phi\bar f_{jK} f_{iK} dV \geq \lambda\|f\|^2_\phi, \tag{10}$$

for some $\lambda \geq 0$. Thus in conclusion: $\mathrm{I} \geq \lambda\|f\|^2_\phi$. This yields, in combination with (8):

$$\lambda\|f\|^2_\phi \leq 2\|\bar\partial^* f\|^2_{\phi-2\psi} + \|\bar\partial f\|^2_\phi + 2\|\partial\psi f\|^2_\phi \quad \forall f \in C^\infty_c(W)^k. \tag{11}$$

We first want to remove the restrain "$f \in C^\infty_c(W)$" in (11). To this end let $D_{\bar\partial},\ D_{\bar\partial^*}$ be the domains in (7) of $\bar\partial$ and $\bar\partial^*$ resp. If we choose ψ according to the density Lemma 4.1.3 by [4], then (8) will be true for any $f \in D_\partial \cap D_{\bar\partial^*}$. Note that for all $K \subset\subset W$ the above ψ can be chosen with the property $\psi_{|K} = 0$. Finally we want to put in better form (11). Define $K(= K_t) = \{z \in W; \phi(z) \leq t\}$. We replace ϕ by $\chi \circ \phi + 2|z|^2$ where χ has the following properties

$$\begin{cases} \chi(t) \geq 0,\ \chi'(t) \geq 0,\ \chi(t) = 0\ \forall t \leq c \\ \chi'' \geq 0,\ \chi'(t) \geq \frac{\sup_{K_t} 2(|\partial\psi| + e^\psi)}{\lambda_{K_t}} \end{cases}$$

By this new ϕ we then conclude

$$\|f\|^2_{\phi-\psi} \leq \|\bar\partial^* f\|^2_{\phi-2\psi} + \|\bar\partial f\|^2_\phi \quad \forall f \in D_{\bar\partial} \cap D_{\bar\partial^*}. \tag{12}$$

Once the estimate (12) is established the proof of Theorem 6 follows from classical lines. For reader convenience we give its outline.

Sketch of end of proof of Theorem 6. Let $f \in L^2_{\mathrm{loc}}(W)^k$ with $\bar\partial f = 0$. We wish to find a form $u \in L^2_{\mathrm{loc}}(W)^{k-1}$ such that $\bar\partial u = f$ or equivalently:

$$(\bar\partial u, g) = (u, \bar\partial^* g) = (f, g) \quad \forall g \in D(\bar\partial^*). \tag{13}$$

We define an antilinear functional on $R(\bar\partial^*)$, the range of $\bar\partial^*$, by:

$$\bar\partial^* g \mapsto (f,g). \tag{14}$$

(14) is well defined and, besides, we have

$$|(f,g)| \leq C\|f\|\|\bar\partial^* g\|,$$

whenever g belongs to $\ker\bar\partial$ (this follows from (12)). If g lies in the orthogonal of $\ker\bar\partial$, then the inequality trivially holds. Hence we can extend (14) to an antilinear functional on L^2 and thus we get that there exists u such that (13) is satisfied. Note that u can be chosen up to an element of $\ker\bar\partial$. In particular, we can choose u in the orthogonal complement of $\ker\bar\partial$ namely $\overline{R(\bar\partial^*)}$. As $\bar\partial^*\circ\bar\partial^* = 0$ we can find a solution u such that $\bar\partial^* u = 0$. This equation together with $\bar\partial u = f$ will yield regularity. It can be proved that if $\bar\partial u$ and $\bar\partial^* u$ are both L^2-forms, then $u \in W^1$ the Sobolev space of index 1. By an induction argument we can say that for each $f \in W^s_{\mathrm{loc}}(W)$ there exists $u \in W^{s+1}_{\mathrm{loc}}(W)$ which solves the Cauchy Riemann system. To get the C^∞ solution, we have just to observe that $\bigcap_{s\geq 1} W^s_{\mathrm{loc}}(W) = C^\infty(W)$. $\square$

References

[1] Andreotti, A. and C. D. Hill, E. E. Levi convexity and the Hans Lewy problem. Part II: Vanishing Theorems, Ann. Sc. Norm. Sup. Pisa (1972), 747–806.

[2] Dufresnoy, A., Sur l'operateur $\bar\partial$ et les fonctions différentiables au sens de Whitney, Ann. Inst. Fourier **29** (1979), 229–238.

[3] Ho, L. H., $\bar\partial$ problem on weakly q-convex domains, Math. Ann. **290** (1991), 3–18.

[4] Hörmander, L., An introduction to complex analysis in several complex variables, Van Nostrand, Princeton N.J. (1966).

[5] Hörmander, L., L^2 estimates and existence theorems for the $\bar\partial$ operator, Acta Math. **113** (1965), 89–152.

[6] Hörmander, L., Sur la regularité C^∞ du $\bar\partial$ au bord d'un domaine de C^n dont la forme de Levi a exactement s valeurs propres strictement negatives, Math. Ann. **195** (1993), 131–165.

[7] Kohn, J. J., Regularity at the boundary of the $\bar\partial$–Neumann problem, Proc. Nat. Acad. Sci. U.S.A. **49** (1963), 206–213.

[8] Zampieri, G., Solvability of the $\bar\partial$ problem with C^∞ regularity up to the boundary on wedges of $\mathbb{C}^N$, Israel J. Math. **115** (2000), 321–331.

(Received May 21, 2001)
(Revised March 1, 2002)

Luca Baracco
Dipartimento di Matematica
Universita di Padova
via Belzoni, 7 Padova
Italy
E-mail: baracco@math.unipd.it

Pier Nicola Bettiol
E-mail: bettiol@math.unipd.it