J. Math. Sci. Univ. Tokyo
9 (2002), 637–658.

Twining Character Formula of Borel-Weil-Bott Type

By Satoshi NAITO

Abstract. We prove a twining character formula of Borel-Weil-Bott type for a connected, simply connected, simple affine algebraic group G over $\mathbb{C}$, by combining a Lefschetz type fixed point formula for the flag variety G/B and a Kostant type twining character formula for the complex simple Lie algebra $\mathfrak{g}$ of G. Our proof is analogous to the well-known "geometric" proof of the Weyl character formula for finite-dimensional irreducible G-modules.

1. Introduction

The geometric representation theory of a semi-simple affine algebraic group has been extensively studied, and nowadays it plays a central role in the representation theory. However, until recently, the main object of the study is a connected one, though there are various phenomena peculiar to the representation theory of a non-connected one (for example, see [M]).

The Borel-Weil-Bott theorem can surely be regarded as one of the most fundamental results in the geometric representation theory of a semi-simple affine algebraic group. In this paper, we prove a theorem of Borel-Weil-Bott type for a typical non-connected, simple affine algebraic group $\langle\omega\rangle \ltimes G$, which is the semi-direct product of a connected, simply connected, simple affine algebraic group G and the cyclic subgroup $\langle\omega\rangle$ of $\mathrm{Aut}(G)$ generated by (a lift of) a Dynkin diagram automorphism ω.

Let us explain our result more precisely. Let G be a connected, simply connected, simple affine algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$. We denote by $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{b}$ the Lie algebras of G, T, and B, respectively. The (rational) character group $X(T) := \mathrm{Hom}(T, \mathbb{C}^*)$ of T can be identified with the (additive) integral weight lattice $\mathfrak{h}^*_{\mathbb{Z}} \subset \mathfrak{h}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$.

1991 *Mathematics Subject Classification.* Primary 20G05; Secondary 17B10, 20G10, 14M15.

Key words: Twining character, Borel-Weil-Bott theorem.

We choose a set of positive roots $\Delta_+ \subset X(T) \simeq \mathfrak{h}^*_{\mathbb{Z}}$ in such a way that the roots of B (hence of $\mathfrak{b}$) are the negative roots $\Delta_- = -\Delta_+$.

The graph automorphism ω of the Dynkin diagram of $\mathfrak{g}$ induces (non-canonically) a certain automorphism ω of the Lie algebra $\mathfrak{g}$, which further lifts to an automorphism ω of the algebraic group G stabilizing both the subgroups T and B. We call these automorphisms ω (Dynkin) diagram automorphisms. Denote the naturally induced action of $\omega \in \mathrm{Aut}(G)$ on $X(T)$ again by ω, and set

$$X(T)^{\omega} := \{\lambda \in X(T) \mid \omega \cdot \lambda = \lambda\}.$$

Notice that, under the identification $X(T) \simeq \mathfrak{h}^*_{\mathbb{Z}}$, $X(T)^{\omega}$ is identified with $(\mathfrak{h}^*_{\mathbb{Z}})^0 := \{\lambda \in \mathfrak{h}^*_{\mathbb{Z}} \mid \omega^*(\lambda) = \lambda\}$, where $\omega^* \in GL(\mathfrak{h}^*)$ is the transposed map of the restriction of ω to $\mathfrak{h}$. Then the subgroup W^{ω} of the Weyl group $W \simeq N_G(T)/T$ consisting of all the elements fixed by the naturally induced action of $\langle\omega\rangle$ is given by:

$$\begin{aligned} W^{\omega} &= \{w \in W \mid (\omega^*)^{-1} w \omega^* = w\} \\ &= \{nT/T \in N_G(T)/T \mid n^{-1}\omega(n) \in T\}. \end{aligned}$$

It is known that W^{ω} is a Coxeter group. Furthermore, we form the semi-direct product $\langle\omega\rangle \ltimes G$ of G and the cyclic subgroup $\langle\omega\rangle$ of $\mathrm{Aut}(G)$ generated by ω, and then form its closed subgroups $\langle\omega\rangle \ltimes T$ and $\langle\omega\rangle \ltimes B$. It is clear that all these groups are closed subgroups of the affine algebraic group $\mathrm{Aut}(G) \ltimes G$, and hence are affine algebraic groups.

Let $\mathcal{B} := G/B$ be the flag variety, which is an $\langle\omega\rangle \ltimes G$-variety as well as a G-variety since $\omega \in \mathrm{Aut}(G)$ stabilizes B. For a weight $\lambda \in X(T)$, we denote by $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda})$ the (G-equivariant) sheaf of $\mathcal{O}_{\mathcal{B}}$-modules on $\mathcal{B}$ associated to the one-dimensional B-module $\mathbb{C}_{\lambda}$ on which B acts by the weight λ through the quotient $B \twoheadrightarrow T$. Then, for each $j \in \mathbb{Z}_{\geq 0}$, the cohomology group $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda}))$ of the associated sheaf $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda})$ on X is naturally equipped with a structure of rational G-module. If $\Lambda \in X(T)$ is a dominant weight and $v \in W \simeq N_G(T)/T$ is an element of the Weyl group, then the Borel-Weil-Bott theorem gives an identity of ordinary characters:

$$\mathrm{ch}\, H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho})) = \mathrm{ch}\, L(\Lambda),$$

where $L(\Lambda)$ is the irreducible highest weight G-module of highest weight Λ, $\rho := (1/2)\cdot\sum_{\alpha\in\Delta_+}\alpha \in X(T)^{\omega} \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0$ is the Weyl vector, and $\ell\colon W \to \mathbb{Z}_{\geq 0}$ is the length function of the Coxeter group W.

Now assume that $\lambda \in X(T)^{\omega}$ and fix an $\mathrm{ord}(\omega)$-th root of unity $\zeta \in \mathbb{C}^*$. We denote by $\mathbb{C}_{\lambda,\zeta}$ the one-dimensional $\langle\omega\rangle \ltimes B$-module on which B acts by the weight λ through the quotient $B \twoheadrightarrow T$ and ω by the scalar ζ. Then the sheaf $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi})$ of $\mathcal{O}_{\mathcal{B}}$-modules on $\mathcal{B}$ associated to $\mathbb{C}_{\lambda,\xi}$ becomes $\langle\omega\rangle \ltimes G$-equivariant, and hence each cohomology group $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi}))$, $j \geq 0$, naturally comes equipped with a structure of rational $\langle\omega\rangle \ltimes G$-module. Here we note that $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi})$ is the (locally free) sheaf of local sections of the algebraic line bundle

$$p\colon G \times^B \mathbb{C}_{\lambda,\xi} \to G/B = \mathcal{B},$$

and that the following isomorphism of line bundles gives $\mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi})$ a natural structure of $\langle\omega\rangle \ltimes G$-equivariant sheaf on $\mathcal{B}$ (see §4.1 for details):

$$\begin{array}{ccc} G \times^B \mathbb{C}_{\lambda,\xi} & \xrightarrow{\ \Phi\ } & (\langle\omega\rangle \ltimes G) \times^{(\langle\omega\rangle\ltimes B)} \mathbb{C}_{\lambda,\xi} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ G/B & \xrightarrow[\ \Psi\ ]{} & (\langle\omega\rangle \ltimes G)/(\langle\omega\rangle \ltimes B), \end{array}$$

where $\Phi([g,z]) := [(1,g),z]$ and $\Psi(gB) := (1,g)(\langle\omega\rangle \ltimes B)$ for $g \in G$ and $z \in \mathbb{C}_{\lambda,\xi}$. Thus, establishing a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle\omega\rangle \ltimes G$ is equivalent to determining the $\langle\omega\rangle \ltimes G$-module structure of $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi}))$ for each $j \geq 0$.

For a rational $\langle\omega\rangle \ltimes T$-module V with the T-weight space decomposition $V = \bigoplus_{\mu\in X(T)} V_{\mu}$, the twining character $\mathrm{ch}^{\omega}(V)$ of V is defined by

$$\mathrm{ch}^{\omega}(V) := \sum_{\mu\in X(T)^{\omega}} \mathrm{Tr}(\omega|_{V_{\mu}})\, e(\mu)$$

as an element of the group algebra $\mathbb{C}[X(T)^{\omega}]$ of $X(T)^{\omega}$ over $\mathbb{C}$ with the canonical basis $e(\mu)$, $\mu \in X(T)^{\omega} \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0$. Note that the twining character $\mathrm{ch}^{\omega}(V) \in \mathbb{C}[X(T)^{\omega}]$ can be viewed as the trace function:

$$T \ni t \mapsto \mathrm{Tr}((\omega,t)\ ;\ V) \in \mathbb{C}.$$

In fact, we have for each $t \in T$,

$$\mathrm{Tr}((\omega, t)\ ;\ V) = \sum_{\mu \in X(T)^{\omega}} \mathrm{Tr}(\omega|_{V_{\mu}})\, e(\mu)(t) \in \mathbb{C},$$

where $e(\mu)(t) := \mu(t) \in \mathbb{C}^{*}$ for $\mu \in X(T)^{\omega}$ and $t \in T$. We know from [M, Ch. 2] that the character (i.e., the trace function) of a rational $\langle\omega\rangle \ltimes G$-module is completely determined by its values on $\langle\omega\rangle \ltimes T^{\omega} = \langle\omega\rangle \times T^{\omega}$, where $T^{\omega} := \{t \in T \mid \omega(t) = t\}$. Therefore, the study of the $\langle\omega\rangle \ltimes G$-module structure of $H^{j}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi}))$ is reduced to the study of the twining character $\mathrm{ch}^{\omega}(H^{j}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\xi})))$ for $j \geq 0$. In fact, we obtain the following theorem, which may be thought of as a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle\omega\rangle \ltimes G$.

THEOREM. *Let $\Lambda \in X(T)^{\omega}$ be a dominant weight fixed by ω and $v \in W^{\omega}$ an element of the Weyl group $W \simeq N_{G}(T)/T$ fixed by ω. Then we have an identity of twining characters*

$$\mathrm{ch}^{\omega}(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho,\zeta}))) = (-1)^{\ell(v)-\widehat{\ell}(v)} \times \zeta \times \mathrm{ch}^{\omega}(L(\Lambda)),$$

where $\widehat{\ell} \colon W^{\omega} \to \mathbb{Z}_{\geq 0}$ is the length function of the Coxeter group W^{ω}. Hence we have an isomorphism of $\langle\omega\rangle \ltimes G$-modules

$$H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{v(\Lambda+\rho)-\rho,\zeta})) \simeq \mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta} \otimes_{\mathbb{C}} L(\Lambda),$$

where $\mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta}$ denotes a one-dimensional rational $\langle\omega\rangle \ltimes G$-module on which ω acts by the scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta$ and G trivially.

REMARK. The scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ in the theorem above, which does not appear in the ordinary Borel-Weil-Bott theorem, seems to come from the character of the cyclic group $\langle\omega\rangle$.

Our proof of this theorem is analogous to the well-known "geometric" proof in [AB, §5] (see also [CG, Ch. 6.1]) of the Weyl character formula for the irreducible highest weight G-module $L(\Lambda)$ of dominant highest weight

$\Lambda \in X(T)$. But, unlike the case of an ordinary character, it is not so easy to determine the alternating sum

$$\sum_{j \ge 0} (-1)^j \operatorname{ch}^{\omega}(\bigwedge^j (\mathfrak{g}/\mathfrak{b})^*)$$

of the twining characters $\operatorname{ch}^{\omega}(\bigwedge^j (\mathfrak{g}/\mathfrak{b})^*)$, $j \ge 0$, whose description can be given by a Kostant type twining character formula obtained in [Na3].

This paper is organized as follows. In Section 2, following [FSS], [FRS], and [Na3], we recall the definition of a twining character, the twining character formula for $L(\Lambda)$ with $\Lambda \in X(T)$ dominant, and a Kostant type twining character formula. In Section 3, following [Ni] (and also [CG]), we briefly review a Lefschetz type fixed point formula. In Section 4, we prove our main theorem above, by combining a Lefschetz type fixed point formula and a Kostant type twining character formula.

Acknowledgments. I am very grateful to the referee for some invaluable comments on an earlier version of this paper.

2. Twining Characters

2.1. Diagram automorphisms

Let $\mathfrak{g}$ be a (finite-dimensional) complex simple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. We choose a set of positive roots $\Delta_+ \subset \mathfrak{h}^* := \operatorname{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ such that the roots of $\mathfrak{b}$ are the negative roots $\Delta_- = -\Delta_+$. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots in Δ_+, $\{h_i \mid i \in I\}$ the set of simple coroots in $\mathfrak{h}$, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix with $a_{ij} = \alpha_j(h_i)$, and $W = \langle r_i \mid i \in I \rangle \subset GL(\mathfrak{h}^*)$ the Weyl group with r_i a simple reflection. We take and fix a Chevalley basis $\{e_\alpha, f_\alpha \mid \alpha \in \Delta_+\} \cup \{h_i \mid i \in I\}$ of $\mathfrak{g}$ so that $\mathfrak{g}_{\alpha_i} = \mathbb{C}\, e_{\alpha_i}$, $\mathfrak{g}_{-\alpha_i} = \mathbb{C}\, f_{\alpha_i}$, and $[e_{\alpha_i}, f_{\alpha_i}] = h_i$ for each $i \in I$.

A graph automorphism of the Dynkin diagram of $\mathfrak{g}$ is a bijection $\omega\colon I \to I$ of the index set I of the simple roots α_i such that

$$a_{\omega(i),\omega(j)} = a_{ij} \quad \text{for all } i, j \in I.$$

Let N be the order of ω, and N_i the number of elements of the ω-orbit I_i of $i \in I$:

$$I_i := \{\omega^k(i) \mid 0 \le k \le N_i - 1\}.$$

This ω can be extended (non-canonically) to an automorphism ω, called a diagram automorphism, of order N of the Lie algebra $\mathfrak{g}$ in such a way that

$$\begin{cases} \omega(e_{\alpha_i}) := e_{\alpha_{\omega(i)}} & \text{for } i \in I, \\ \omega(f_{\alpha_i}) := f_{\alpha_{\omega(i)}} & \text{for } i \in I, \\ \omega(h_i) := h_{\omega(i)} & \text{for } i \in I. \end{cases}$$

Note that we have

$$(\omega(x)|\omega(y)) = (x|y) \quad \text{for all } x, y \in \mathfrak{g},$$

where $(\cdot|\cdot)$ is the Killing form on $\mathfrak{g}$.

The restriction of ω to the Cartan subalgebra $\mathfrak{h}$ induces a transposed map $\omega^*\colon \mathfrak{h}^* \to \mathfrak{h}^*$ by $\omega^*(\lambda)(h) := \lambda(\omega(h))$ for $\lambda \in \mathfrak{h}^*$ and $h \in \mathfrak{h}$. We set

$$(\mathfrak{h}^*)^0 := \{\lambda \in \mathfrak{h}^* \mid \omega^*(\lambda) = \lambda\},$$

and call an element of $(\mathfrak{h}^*)^0$ a symmetric weight. Notice that the Weyl vector $\rho := (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha$ is a symmetric weight.

2.2. Fixed point Weyl group

We define the fixed point subgroup W^ω of the Weyl group W by

$$W^\omega := \{w \in W \mid (\omega^*)^{-1} w \omega^* = w\},$$

and call it the fixed point Weyl group. Notice that W^ω stabilizes the subspace $(\mathfrak{h}^*)^0$ of $\mathfrak{h}^*$. Choose and fix a complete set $\widehat{I}$ of representatives of the ω-orbits in I, and set for each $i \in \widehat{I}$,

$$w_i := \begin{cases} \displaystyle\prod_{k=0}^{N_i/2-1} (r_{\omega^k(i)}\, r_{\omega^{k+N_i/2}(i)}\, r_{\omega^k(i)}) & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1, \\ \displaystyle\prod_{k=0}^{N_i-1} r_{\omega^k(i)} & \text{if } \displaystyle\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 2. \end{cases}$$

Note that for each $i \in \widehat{I}$ we have $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$ or 2, and that if $\sum_{k=0}^{N_i-1} a_{i,\omega^k(i)} = 1$ then N_i is an even integer. It is also known (see, for example, [FRS]) that the fixed point Weyl group W^ω is a Coxeter group with the canonical generator system $\{w_i \mid i \in \widehat{I}\}$. We denote the length function of the Coxeter system $(W, \{r_i \mid i \in I\})$ (resp. $(W^\omega, \{w_i \mid i \in \widehat{I}\})$) by $\ell\colon W \to \mathbb{Z}_{\geq 0}$ (resp. $\widehat{\ell}\colon W^\omega \to \mathbb{Z}_{\geq 0}$). Recall from [KN, Lemma 1.3.1] that if $w = w_{i_1} w_{i_2} \cdots w_{i_n} \in W^\omega$ is a reduced expression of $w \in W^\omega$ in the Coxeter system $(W^\omega, \{w_i \mid i \in \widehat{I}\})$, i.e., $\widehat{\ell}(w) = n$, then

$$\ell(w) = \ell(w_{i_1}) + \ell(w_{i_2}) + \cdots + \ell(w_{i_n}). \tag{2.2.1}$$

2.3. Twining character formulas

We denote by $\langle\omega\rangle$ the cyclic subgroup of $\mathrm{Aut}(\mathfrak{g})$ generated by the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$. A finite-dimensional vector space V over $\mathbb{C}$ is called an $\mathfrak{h}$-module if V admits a weight space decomposition with respect to $\mathfrak{h}$:

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu.$$

Further, an $\mathfrak{h}$-module V is called an $(\mathfrak{h}, \langle\omega\rangle)$-module if V admits a $\mathbb{C}$-linear $\langle\omega\rangle$-action such that

$$\omega \cdot (xv) = \omega(x)(\omega \cdot v) \quad \text{for all } x \in \mathfrak{g} \text{ and } v \in V.$$

Notice that $\omega \cdot V_\mu = V_{(\omega^*)^{-1}(\mu)}$ for $\mu \in \mathfrak{h}^*$. In view of this fact, we define the twining character $\mathrm{ch}^\omega(V)$ of V by

$$\mathrm{ch}^\omega(V) := \sum_{\mu \in (\mathfrak{h}^*)^0} \mathrm{Tr}(\omega|_{V_\mu})\, e(\mu)$$

as an element of the group algebra $\mathbb{C}[(\mathfrak{h}^*)^0]$ of $(\mathfrak{h}^*)^0$ over $\mathbb{C}$ with the canonical basis $\{e(\mu) \mid \mu \in (\mathfrak{h}^*)^0\}$.

Here we recall two kinds of formulas for twining characters, which have been proved more generally for Kac-Moody algebras.

Formula 1. Let $\Lambda \in \mathfrak{h}^*$ be a symmetric dominant integral weight. Then, as in [FSS], [FRS], and [Na3], the irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ of highest weight Λ is equipped with a structure of $(\mathfrak{h}, \langle\omega\rangle)$-module such that $\omega \cdot v_\Lambda = v_\Lambda$, where v_Λ is a (nonzero) highest weight vector of $L(\Lambda)$. It is shown in [FRS] that

$$\mathrm{ch}^\omega(L(\Lambda)) = \frac{\displaystyle\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)}\, e(w(\Lambda+\rho))}{\displaystyle\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)}\, e(w(\rho))}.$$

Formula 2. Let $\mathfrak{n}_-$ (resp. $\mathfrak{n}_+$) be the sum of all negative (resp. positive) root spaces $\mathfrak{g}_\alpha$, $\alpha \in \Delta_-$ (resp. $\alpha \in \Delta_+$). Then $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_+$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_-$, and $\mathfrak{n}_+$ all become $(\mathfrak{h}, \langle\omega\rangle)$-modules under the adjoint action of $\mathfrak{h}$ and the natural action of $\omega \in \mathrm{Aut}(\mathfrak{g})$ since ω stabilizes $\mathfrak{n}_-$, $\mathfrak{h}$, and $\mathfrak{n}_+$. Notice that the quotient module $\mathfrak{g}/\mathfrak{b}$ is isomorphic to $\mathfrak{n}_-$ as an $(\mathfrak{h}, \langle\omega\rangle)$-module. Moreover, since the Killing form $(\cdot|\cdot)$ on $\mathfrak{g}$ is nondegenerate and $\langle\omega\rangle$-invariant, we have an isomorphism of $(\mathfrak{h}, \langle\omega\rangle)$-modules

$$(\mathfrak{g}/\mathfrak{b})^* \simeq \mathfrak{n}_-,$$

where $(\mathfrak{g}/\mathfrak{b})^*$ is the dual $(\mathfrak{h}, \langle\omega\rangle)$-module of $\mathfrak{g}/\mathfrak{b}$. Hence, by taking the exterior power, we obtain that

$$\mathrm{ch}^\omega\Big(\bigwedge\nolimits^j (\mathfrak{g}/\mathfrak{b})^*\Big) = \mathrm{ch}^\omega\Big(\bigwedge\nolimits^j \mathfrak{n}_-\Big)$$

for each $j \geq 0$. It is shown in [Na3] that in $\mathbb{C}[(\mathfrak{h}^*)^0]$,

$$\sum_{j \geq 0} (-1)^j\, \mathrm{ch}^\omega\Big(\bigwedge\nolimits^j \mathfrak{n}_-\Big) = \sum_{j \geq 0} (-1)^j\, \mathrm{ch}^\omega(H_j(\mathfrak{n}_-, \mathbb{C}))$$

and

$$\mathrm{ch}^\omega(H_j(\mathfrak{n}_-, \mathbb{C})) = \sum_{\substack{w \in W^\omega \\ \ell(w) = j}} (-1)^{\widehat{\ell}(w) - j}\, e(w(\rho) - \rho)$$

for each $j \geq 0$, where $H_j(\mathfrak{n}_-, \mathbb{C})$ is the usual Lie algebra homology module of $\mathfrak{n}_-$ with coefficients in the trivial module $\mathbb{C}$.

3. Lefschetz Type Fixed Point Formula

In this section, following [Ni, §4] (see also [CG, Ch. 5]), we review briefly a special case of (a K-theoretic version of) the Lefschetz fixed point formula that suffices for our purpose. Here we often identify an algebraic vector bundle over a smooth algebraic variety with the locally free coherent sheaf of local sections of it.

Let A be a (not necessarily connected) diagonalizable algebraic group over $\mathbb{C}$. We denote by $X(A)$ the (rational) character group $\mathrm{Hom}(A, \mathbb{C}^*)$ of A, by $R(A)$ the group ring $\mathbb{Z}[X(A)]$ of $X(A)$ with the canonical basis $\{e(\chi) \mid \chi \in X(A)\}$, and by $S^{-1}R(A)$ the localization of $R(A)$ with respect to the multiplicative subset of $R(A)$ generated by elements of the form $1-e(\chi)$ for nontrivial $\chi \in X(A)$. Note that the group algebra $\mathbb{C}[X(A)] = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[X(A)]$ is identified with the algebra $\mathrm{Mor}(A, \mathbb{C})$ of regular algebraic functions on A by: $e(\chi)(a) = \chi(a) \in \mathbb{C}^*$ for $\chi \in X(A)$ and $a \in A$.

For a (finite-dimensional) rational A-module V, we set

$$\mathrm{Tr}(V) := \sum_{\chi \in X(A)} (\dim_{\mathbb{C}} V_\chi)\, e(\chi) \in R(A),$$

where $V_\chi := \{v \in V \mid a\,v = \chi(a)v \ \text{ for all } a \in A\}$ for $\chi \in X(A)$. Note that the evaluation map (at $a \in A$) $\mathrm{ev}_a \colon X(A) \to \mathbb{C}$ given by $\chi \mapsto \chi(a)$ gives rise to a ring homomorphism $\mathrm{ev}_a \colon R(A) \to \mathbb{C}$ such that $\mathrm{ev}_a(\mathrm{Tr}(V))$ is equal to the usual trace $\mathrm{Tr}(a\,; V) \in \mathbb{C}$ for the operation of $a \in A$ on V. Also, for an algebraic vector bundle E over Z and a point $z \in Z$, we denote by $E|_{\{z\}}$ the fiber of E at the point z.

The following is a special case of the Lefschetz fixed point formula (see [Ni, §4], and also [CG, Ch. 5]).

Theorem 3.1. *Let Z be a smooth projective A-variety (i.e., a variety with an algebraic action of A) such that all the A-fixed points Z^A of Z form a variety of finite set. For an A-equivariant algebraic vector bundle E over Z (viewed as a locally free coherent sheaf on Z), we have in $S^{-1}R(A)$,*

$$(\heartsuit) \qquad \sum_{j \geq 0} (-1)^j\, \mathrm{Tr}(H^j(Z, E)) = \sum_{z \in Z^A} \mathrm{Tr}(E|_{\{z\}}) \times \Bigl(\mathrm{Tr}(\lambda_A|_{\{z\}})\Bigr)^{-1}.$$

Here

$$\mathrm{Tr}(\lambda_A|_{\{z\}}) := \sum_{j\geq 0} (-1)^j \,\mathrm{Tr}(\textstyle\bigwedge^j T_z^* Z),$$

where $\bigwedge^j T_z^* Z$ *is the* j*-th exterior power of the* (*Zariski*) *cotangent space* $T_z^* Z$ *to* Z *at the point* z*.*

4. Twining Character Formula of Borel-Weil-Bott Type

4.1. Geometric setting

Let G be a connected, simply connected, simple affine algebraic group over $\mathbb{C}$ with maximal torus T and Borel subgroup $B \supset T$, so that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G is a complex simple Lie algebra with Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T)$ and Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B) \supset \mathfrak{h}$. Thus we can use the (algebraic) setting in Section 2. Recall that the (rational) character group $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$ of T is identified with the (additive) integral weight lattice $\mathfrak{h}_{\mathbb{Z}}^* \subset \mathfrak{h}^*$ by taking the differential at the identity element, i.e., by the map $\lambda \mapsto \mathrm{d}\lambda$. Recall also that the Weyl group $W \subset GL(\mathfrak{h}^*)$ is identified with the quotient group $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. For each $w \in W \simeq N_G(T)/T$, we denote by $\dot{w} \in N_G(T)$ a right coset representative of w. Notice that if V is a rational G-module and V_μ is a T-weight space corresponding to $\mu \in X(T)$, then we have $\dot{w}V_\mu = V_{w(\mu)}$.

Since the affine algebraic group G is simply connected, there exists an automorphism (of order N) of G whose differential at the identity element coincides with the diagram automorphism $\omega \in \mathrm{Aut}(\mathfrak{g})$ in §2.1. By abuse of notation, we denote by ω this automorphism of G and by $\langle\omega\rangle$ the cyclic subgroup of $\mathrm{Aut}(G)$ generated by the $\omega \in \mathrm{Aut}(G)$. Note that, under the isomorphism of algebraic groups $\mathrm{Aut}(G) \tilde{\to} \mathrm{Aut}(\mathfrak{g})$ given by taking the differential at the identity element, the two cyclic groups $\langle\omega\rangle$ are isomorphic. We define the fixed point subgroup

$$X(T)^\omega := \{\lambda \in X(T) \mid \omega \cdot \lambda = \lambda\}$$

of $X(T)$ by the naturally induced action of ω, which coincides with the restriction of $(\omega^{-1})^* = (\omega^*)^{-1} \in GL(\mathfrak{h}^*)$ under the identification $X(T) \simeq \mathfrak{h}_{\mathbb{Z}}^*$. Hence we have an identification:

$$X(T)^\omega \simeq (\mathfrak{h}_{\mathbb{Z}}^*)^0 := \{\lambda \in \mathfrak{h}_{\mathbb{Z}}^* \mid \omega^*(\lambda) = \lambda\}.$$

Furthermore, it is easy to check that for $w \in W$, the element $\omega(\dot{w}) \in N_G(T)$ is a right coset representative of $(\omega^*)^{-1} w \omega^* \in W \simeq N_G(T)/T$. Hence we see that, under the identification $W \simeq N_G(T)/T = \{\dot{w}T/T \mid w \in W\}$, W^ω is identified with $\{\dot{w}T/T \mid \dot{w}^{-1}\omega(\dot{w}) \in T\}$. In fact, for each $w \in W^\omega$, we can (and will) take a right coset representative $\dot{w} \in N_G(T)$ of w such that $\omega(\dot{w}) = \dot{w}$ (see [Sp, Ch. 9.3]).

We form the semi-direct product $\langle\omega\rangle \ltimes G$ of G and $\langle\omega\rangle \subset \mathrm{Aut}(G)$, and then its closed subgroups $\langle\omega\rangle \ltimes B$ and $\langle\omega\rangle \ltimes T$, all of which are closed subgroups of the affine algebraic group $\mathrm{Aut}(G) \ltimes G$, and hence are affine algebraic groups. Notice that a (finite-dimensional) rational $\langle\omega\rangle \ltimes T$-module obviously becomes an $(\mathfrak{h}, \langle\omega\rangle)$-module in §2.3.

Remark 4.1.1. Let V be a rational $\langle\omega\rangle \ltimes T$-module with the T-weight space decomposition $V = \bigoplus_{\mu \in X(T)} V_\mu$. Then, for each $t \in T$, we have

$$\mathrm{Tr}((\omega, t)\ ;\ V) = \sum_{\mu \in X(T)^\omega} \mathrm{Tr}(\omega|_{V_\mu})\, e(\mu)(t) \in \mathbb{C},$$

where $e(\mu)(t) := \mu(t) \in \mathbb{C}^*$ for $\mu \in X(T)^\omega$ and $t \in T$. Hence the twining character $\mathrm{ch}^\omega(V) \in \mathbb{C}[(\mathfrak{h}^*_{\mathbb{Z}})^0] \simeq \mathbb{C}[X(T)^\omega]$ can be thought of as the trace function:

$$T \ni t \mapsto \mathrm{Tr}((\omega, t)\ ;\ V) \in \mathbb{C}.$$

Let $\mathcal{B} := G/B$ be the flag variety and $\pi\colon G \to G/B = \mathcal{B}$ the quotient morphism. It is well-known that $\mathcal{B}$ is a smooth projective G-variety and $\pi\colon G \to \mathcal{B}$ has local sections. To each (finite-dimensional) rational B-module M, we can associate a G-equivariant vector bundle $p\colon L_{\mathcal{B}}(M) \to \mathcal{B}$ over $\mathcal{B}$ by setting

$$L_{\mathcal{B}}(M) := G \times^B M = (G \times M)/B \quad \text{and} \quad p([g, m]) := gB/B \in \mathcal{B},$$

where B acts from the right on the direct product $G \times M$ by

$$(g, m)b := (gb, b^{-1}m) \quad \text{for } g \in G,\ m \in M, \text{ and } b \in B,$$

and where G acts from the left on $G \times^B M$ via left multiplication on the first factor. (The quotient $(G \times M)/B$ actually is a smooth algebraic variety since $\pi\colon G \to \mathcal{B}$ has local sections.) Note that the sheaf $\mathcal{L}_{\mathcal{B}}(M)$, called the associated sheaf to M on $\mathcal{B}$, of local sections of the algebraic vector bundle $L_{\mathcal{B}}(M)$ is a locally free G-equivariant (or G-linearized) sheaf of $\mathcal{O}_{\mathcal{B}}$-modules (see [J, Part I. Ch. 5] and [CG, Ch. 5]).

Furthermore, if M is a (finite-dimensional) rational $\langle\omega\rangle \ltimes B$-module, then the vector bundle $p\colon L_{\mathcal{B}}(M) \to \mathcal{B}$ comes equipped with a structure of $\langle\omega\rangle \ltimes G$-equivariant algebraic vector bundle by the rational $\langle\omega\rangle$-action:

$$\omega \cdot ([g, m]) := [\omega(g), \omega \cdot m] \quad \text{for } g \in G \text{ and } m \in M$$

on $L_{\mathcal{B}}(M)$ and the natural algebraic action of $\langle\omega\rangle$ on $\mathcal{B}$. Hence the associated sheaf $\mathcal{L}_{\mathcal{B}}(M)$ of $\mathcal{O}_{\mathcal{B}}$-modules becomes $\langle\omega\rangle \ltimes G$-equivariant (cf. [KN, §2.3]).

Remark 4.1.2. To each (finite-dimensional) rational $\langle\omega\rangle \ltimes B$-module M, we can associate an $\langle\omega\rangle \ltimes G$-equivariant algebraic vector bundle

$$q\colon (\langle\omega\rangle \ltimes G) \times^{(\langle\omega\rangle \ltimes B)} M \to (\langle\omega\rangle \ltimes G)/(\langle\omega\rangle \ltimes B)$$

by replacing G and B in the definition of $L_{\mathcal{B}}(M)$ above with $\langle\omega\rangle \ltimes G$ and $\langle\omega\rangle \ltimes B$, respectively. (Note that the canonical morphism $\langle\omega\rangle \ltimes G \to (\langle\omega\rangle \ltimes G)/(\langle\omega\rangle \ltimes B)$ has local sections since $\pi\colon G \to G/B$ does.) Then we have the following isomorphism of $\langle\omega\rangle \ltimes G$-equivariant vector bundles:

$$\begin{array}{ccc} G \times^{B} M & \xrightarrow{\Phi} & (\langle\omega\rangle \ltimes G) \times^{(\langle\omega\rangle \ltimes B)} M \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ G/B & \xrightarrow[\Psi]{} & (\langle\omega\rangle \ltimes G)/(\langle\omega\rangle \ltimes B), \end{array}$$

where $\Phi([g, m]) := [(1, g), m]$ and $\Psi(gB) := (1, g)(\langle\omega\rangle \ltimes B)$ for $g \in G$ and $m \in M$.

The quotient $\mathfrak{g}/\mathfrak{b}$ is viewed as a rational B-module by the adjoint representation $\mathrm{Ad}\colon B \to GL(\mathfrak{g}/\mathfrak{b})$. In addition, since $\omega \in \mathrm{Aut}(\mathfrak{g})$ and hence $\omega \in \mathrm{Aut}(G)$ stabilize $\mathfrak{b} \subset \mathfrak{g}$, the quotient $\mathfrak{g}/\mathfrak{b}$ can be made into a rational $\langle\omega\rangle \ltimes B$-module. Moreover, we can check the following lemma.

Lemma 4.1.3. *The cotangent bundle $T^*\mathcal{B}$ over the flag variety $\mathcal{B}$ is isomorphic to the vector bundle $L_{\mathcal{B}}((\mathfrak{g}/\mathfrak{b})^*)$ over $\mathcal{B}$ associated to the dual*

$\langle\omega\rangle \ltimes B$-module $(\mathfrak{g}/\mathfrak{b})^*$ of $\mathfrak{g}/\mathfrak{b}$ as $\langle\omega\rangle \ltimes G$-equivariant vector bundles over $\mathcal{B}$.

PROOF. Cf. the proof of [CG, Lemma 1.4.9]. □

4.2. Proof of the formula

Let $\lambda \in X(T)^\omega$ and $\zeta \in \mathbb{C}^*$ an N-th root of unity. We denote by $\mathbb{C}_{\lambda,\zeta}$ the one-dimensional rational $\langle\omega\rangle \ltimes B$-module on which B acts by the weight λ through the quotient $B \twoheadrightarrow T$ and ω by the scalar ζ. Then, for each $j \geq 0$, the cohomology group $H^j(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}))$ of the associated sheaf $\mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})$ on $\mathcal{B}$ naturally comes equipped with a rational $\langle\omega\rangle \ltimes G$-module structure, since $\mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})$ is a locally free $\langle\omega\rangle \ltimes G$-equivariant sheaf of $\mathcal{O}_\mathcal{B}$-modules (cf. [J, Part I. Ch. 5] and also [KN, §2.3]).

Let $\Lambda \in X(T)^\omega$ be dominant and $v \in W^\omega$. We set

$$\lambda := v \circ \Lambda = v(\Lambda + \rho) - \rho \in X(T)^\omega,$$

where $\rho = (1/2) \cdot \sum_{\alpha \in \Delta_+} \alpha \in X(T)^\omega$ is the Weyl vector. The Borel-Weil-Bott theorem tells us that $H^j(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})) = 0$ unless $j \neq \ell(v)$, and that $H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}))$ is isomorphic, as a G-module, to the irreducible highest weight module $L(\Lambda)$ of highest weight Λ.

Now we are ready to state our main result, which, in view of Remarks 4.1.1 and 4.1.2, may be thought of as a Borel-Weil-Bott theorem for the non-connected, simple affine algebraic group $\langle\omega\rangle \ltimes G$.

THEOREM 4.2.1. *Let $\Lambda \in X(T)^\omega$ be a dominant weight fixed by the diagram automorphism ω, $v \in W^\omega$, and $\zeta \in \mathbb{C}^*$ an N-th root of unity. Set $\lambda := v \circ \Lambda = v(\Lambda + \rho) - \rho$. Then we have the following identity of twining characters*:

$$\mathrm{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}))) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \mathrm{ch}^\omega(L(\Lambda))$$

in the group algebra $\mathbb{C}[X(T)^\omega]$ of $X(T)^\omega$ over $\mathbb{C}$ with the canonical basis $\{e(\mu) \mid \mu \in X(T)^\omega\}$. Here $L(\Lambda)$ is the irreducible highest weight G-module of highest weight Λ. Hence we have an isomorphism of $\langle\omega\rangle \ltimes G$-modules

$$H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})) \simeq \mathbb{C}_{(-1)^{\ell(v) - \widehat{\ell}(v)}\zeta} \otimes_\mathbb{C} L(\Lambda),$$

where $\mathbb{C}_{(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta}$ *denotes a one-dimensional rational* $\langle\omega\rangle \ltimes G$*-module on which* ω *acts by the scalar* $(-1)^{\ell(v)-\widehat{\ell}(v)}\zeta$ *and* G *trivially.*

REMARK 4.2.2. We know from the Borel-Weil-Bott theorem (see, for example, [J, Part II. Ch. 5]) that each $\lambda \in X(T)$ such that $H^j(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_\lambda)) \neq 0$ for some $j \in \mathbb{Z}_{\geq 0}$ can be written uniquely in the form $\lambda = v \circ \Lambda$ for some $v \in W$ and dominant $\Lambda \in X(T)$. It is obvious that if this $\lambda = v \circ \Lambda$ is an element of $X(T)^\omega$, then $v \in W^\omega$.

REMARK 4.2.3. It follows from Equality (2.2.1) that for $v \in W^\omega$, the scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ is not equal to 1 in general. Hence Theorem 4.2.1 implies that, as $\langle\omega\rangle \ltimes G$-modules, $H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,1}))$ and $L(\Lambda)$ are not necessarily isomorphic. This scalar $(-1)^{\ell(v)-\widehat{\ell}(v)}$ seems to come from the character of the cyclic group $\langle\omega\rangle$. In fact, when G is of type D_4 and ω is of order 3, we see by (2.2.1) that $(-1)^{\ell(v)-\widehat{\ell}(v)} = 1$ for all $v \in W^\omega$.

The rest of this subsection is devoted to the proof of Theorem 4.2.1. Now we set $V := H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta}))$ and compute the twining character

$$\mathrm{ch}^\omega(V) = \sum_{\mu \in X(T)^\omega} \mathrm{Tr}(\omega|_{V_\mu})\, e(\mu)$$

of the rational $\langle\omega\rangle \ltimes T$-module V with the T-weight space decomposition $V = \bigoplus_{\mu \in X(T)} V_\mu$. Because we already know from the Borel-Weil-Bott theorem that $V = H^{\ell(v)}(\mathcal{B}, \mathcal{L}_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta}))$ is a simple module of highest weight Λ as a G-module (and hence simple as an $\langle\omega\rangle \ltimes G$-module), it suffices to prove the equality

$$\mathrm{ch}^\omega(V) = (-1)^{\ell(v)-\widehat{\ell}(v)} \times \zeta \times \mathrm{ch}^\omega(L(\Lambda)) \tag{4.2.1}$$

in the group algebra $\mathbb{C}[X(T)^\omega] \subset \mathbb{C}[X(T)]$. (Here we recall that $\mathbb{C}[X(T)] = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[X(T)]$ is identified with the algebra $\mathrm{Mor}(T, \mathbb{C})$ of regular algebraic functions on T by: $e(\mu)(t) = \mu(t) \in \mathbb{C}^*$ for $\mu \in X(T)$ and $t \in T$.)

We define the fixed point torus $T^\omega \subset T$ by

$$T^\omega := \{t \in T \mid \omega(t) = t\}.$$

The Lie algebra $\mathrm{Lie}(T^\omega)$ of $T^\omega \subset T$ is

$$\mathfrak{h}^0 := \{h \in \mathfrak{h} \mid \omega(h) = h\}.$$

It is known (see [St, §8]) that T^ω is, in fact, connected. Hence T^ω is a subtorus of T. Furthermore, it easily follows from the canonical isomorphism $(\mathfrak{h}^*)^0 \simeq (\mathfrak{h}^0)^*$ that the group homomorphism

$$\mathrm{Res}\colon X(T)^\omega \to X(T^\omega) := \mathrm{Hom}(T^\omega, \mathbb{C}^*)$$

given by the restriction to $T^\omega \subset T$ is injective. So the induced algebra homomorphism

$$\mathrm{Res}\colon \mathbb{C}[X(T)^\omega] \to \mathbb{C}[X(T^\omega)] \simeq \mathrm{Mor}(T^\omega, \mathbb{C})$$

is also injective. Thus it suffices for us to show Equality (4.2.1) in the group algebra $\mathbb{C}[X(T^\omega)]$.

Here we remark that the semi-direct product $\langle\omega\rangle \ltimes T$ itself is not abelian, but its closed subgroup $\langle\omega\rangle \ltimes T^\omega$ splits into the direct product $\langle\omega\rangle \times T^\omega$, and hence that $\langle\omega\rangle \ltimes T^\omega = \langle\omega\rangle \times T^\omega$ is a (not necessarily connected) diagonalizable algebraic group. Thus the (closed) embedding $T^\omega \tilde{\to} \{\omega\} \times T^\omega \subset \langle\omega\rangle \times T^\omega$ induces the evaluation map (at $\omega \in \langle\omega\rangle$)

$$\mathrm{ev}_\omega\colon \mathbb{Z}[X(\langle\omega\rangle \times T^\omega)] \to \mathbb{C}[X(T^\omega)].$$

By Remark 4.1.1, we see that for a (finite-dimensional) rational $\langle\omega\rangle \ltimes T$-module U,

$$\mathrm{ev}_\omega(\mathrm{Tr}(U)) = \mathrm{ch}^\omega(U) \tag{4.2.2}$$

in the group algebra $\mathbb{C}[X(T^\omega)] \simeq \mathrm{Mor}(T^\omega, \mathbb{C})$, where the rational $\langle\omega\rangle \ltimes T$-module U is regarded as a rational $\langle\omega\rangle \times T^\omega$-module by restriction. Note that $\mathrm{ch}^\omega(U)$ is, in fact, an element of the group algebra $\mathbb{C}[X(T)^\omega] \hookrightarrow \mathbb{C}[X(T^\omega)]$, since U is a rational $\langle\omega\rangle \ltimes T$-module.

In order to compute $\mathrm{ev}_\omega(\mathrm{Tr}(V)) \in \mathbb{C}[X(T^\omega)]$, we want to apply Theorem 3.1 to the flag variety $\mathcal{B} = G/B$, which is a smooth projective $\langle\omega\rangle \ltimes T$-variety.

Lemma 4.2.4. *We have*

$$\mathcal{B}^{\langle\omega\rangle\times T^{\omega}} = (\mathcal{B}^{T})^{\langle\omega\rangle}.$$

Proof. Take a regular semi-simple element $t \in T^{\omega}$ such that $t^{N} \in T^{\omega}$ is also regular. Let $x \in \mathcal{B}$ be a point fixed by $(\omega, t) \in \langle\omega\rangle \times T^{\omega}$. Then we have

$$x = (\omega, t)^{N}\, x = (\omega^{N}, t^{N})\, x = (1, t^{N})\, x = t^{N}\, x.$$

Thus it suffices to show that if $t \in T$ is a regular semi-simple element, then we have $\mathcal{B}^{t} = \mathcal{B}^{T}$. Let $t \in T$ be a regular semi-simple element and $gB/B \in \mathcal{B}^{t}$ with $g \in G$, i.e., $tgB = gB$. Then we have $g^{-1}tg \in B$, so that $t \in gBg^{-1} \cap T$. Since gBg^{-1} is also a Borel (i.e., maximal closed connected solvable) subgroup of G, there exists a maximal torus (i.e., closed connected diagonalizable subgroup) T' of gBg^{-1} such that $t \in T' \subset gBg^{-1}$. So we get $t \in T \cap T'$. Note that T' is also a maximal torus of G. Since $t \in T$ is a regular semi-simple element, it belongs to a unique maximal torus. Therefore, it follows that $T' = T$, and hence $T \subset gBg^{-1}$, i.e., $g^{-1}Tg \subset B$. Consequently, we see that $(g^{-1}Tg)B = B$, so that $TgB = gB$. Hence we have shown that $gB/B \in \mathcal{B}^{T}$, which proves the lemma. □

Remark 4.2.5. The set of regular elements t such that t^{N} is also regular are clearly dense in $T^{\omega} \subset T$, since the restriction of any positive root $\alpha \in \Delta_{+} \subset X(T)$ to $T^{\omega} \subset T$ is not identically equal to 1.

Put $A := \langle\omega\rangle \times T^{\omega}$ and $E := L_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta})$, where $\lambda = v \circ \Lambda$ with $\Lambda \in X(T)^{\omega}$ dominant and $v \in W^{\omega}$. We recall the well-known fact that $\mathcal{B}^{T} = \{\dot{w}B/B \mid w \in W\}$. Hence we see by Lemma 4.2.4 that

$$\begin{aligned}\mathcal{B}^{A} = (\mathcal{B}^{T})^{\langle\omega\rangle} &= \{\dot{w}B/B \mid \dot{\omega}^{-1}\omega(\dot{\omega}) \in T,\ w \in W\}\\ &= \{\dot{w}B/B \mid w \in W^{\omega}\}.\end{aligned}$$

In particular, $\mathcal{B}^{A}$ is a variety of finite set. Therefore, we can apply Theorem 3.1 to our setting.

Let S be the multiplicative subset of $\mathbb{Z}[X(\langle\omega\rangle \times T^{\omega})]$ generated by elements of the form $1 - e(\chi)$ for nontrivial $\chi \in X(\langle\omega\rangle \times T^{\omega})$ as in §3. Then an easy consideration shows that $0 \notin \mathrm{ev}_\omega(S) \subset \mathbb{C}[X(T^\omega)]$. Hence the evaluation map $\mathrm{ev}_\omega \colon \mathbb{Z}[X(\langle\omega\rangle \times T^\omega)] \to \mathbb{C}[X(T^\omega)]$ lifts to a ring homomorphism

$$S^{-1}\mathrm{ev}_\omega \colon S^{-1}\mathbb{Z}[X(\langle\omega\rangle \times T^\omega)] \to \mathrm{Frac}(\mathbb{C}[X(T^\omega)]),$$

where $\mathrm{Frac}(\mathbb{C}[X(T^\omega)])$ is the fraction field of $\mathbb{C}[X(T^\omega)]$. Therefore, by applying $S^{-1}\mathrm{ev}_\omega$ to both sides of ($\heartsuit$) in Theorem 3.1, we obtain the following formula in the fraction field $\mathrm{Frac}(\mathbb{C}[X(T)^\omega]) \hookrightarrow \mathrm{Frac}(\mathbb{C}[X(T^\omega)])$,

(4.2.3)

$$\sum_{j\geq 0}(-1)^j \,\mathrm{ch}^\omega\left(H^j(\mathcal{B},\mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}))\right) = \sum_{w\in W^\omega}\mathrm{ch}^\omega\left(L_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})|_{\{\dot{w}B/B\}}\right)\times \times\left(\sum_{j\geq 0}(-1)^j\,\mathrm{ch}^\omega\big(\textstyle\bigwedge^j T^*_{\dot{w}B/B}\mathcal{B}\big)\right)^{-1}.$$

Since $H^j(\mathcal{B},\mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})) = 0$ unless $j = \ell(v)$ by the Borel-Weil-Bott theorem, the left-hand side of (4.2.3) becomes

$$(-1)^{\ell(v)} \times \mathrm{ch}^\omega(H^{\ell(v)}(\mathcal{B},\mathcal{L}_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}))).$$

We now compute the right-hand side of (4.2.3).

LEMMA 4.2.6. *Let $\lambda \in X(T)^\omega$ and $w \in W^\omega$. Then the fiber $L_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})|_{\{\dot{w}B/B\}}$ of the line bundle $p\colon L_\mathcal{B}(\mathbb{C}_{\lambda,\zeta}) \to \mathcal{B}$ at the point $\dot{w}B/B \in \mathcal{B}$ is isomorphic to $\mathbb{C}_{w(\lambda),\zeta}$ as an $\langle\omega\rangle \ltimes T$-module.*

PROOF. Let $[\dot{w}, z] \in G \times^B \mathbb{C}_{\lambda,\zeta}$ with $z \in \mathbb{C}_{\lambda,\zeta}$ be an element of the fiber $L_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})|_{\{\dot{w}B/B\}}$ of the line bundle $L_\mathcal{B}(\mathbb{C}_{\lambda,\zeta})$ at the point $\dot{w}B/B \in \mathcal{B}$. For $t \in T$, we have

$$\begin{aligned} t\cdot[\dot{w},z] &= [t\dot{w},z] \\ &= [\dot{w}(\dot{w}^{-1}t\dot{w}),z] = [\dot{w},(\dot{w}^{-1}t\dot{w})z] \quad \text{since } \dot{w}^{-1}t\dot{w} \in T \\ &= [\dot{w},\lambda(\dot{w}^{-1}t\dot{w})\times z] = [\dot{w},(w(\lambda))(t)\times z]. \end{aligned}$$

Also, we have

$$\begin{aligned} \omega([\dot{w},z]) &= [\omega(\dot{w}),\omega\cdot z] \\ &= [\dot{w},\zeta\times z] \end{aligned}$$

since the right coset representative $\dot{w} \in N_G(T)$ of $w \in W^\omega$ is chosen in such a way that $\omega(\dot{w}) = \dot{w}$. This proves the lemma. $\square$

By this lemma, we get

$$\mathrm{ch}^\omega \left(L_{\mathcal{B}}(\mathbb{C}_{\lambda,\zeta})|_{\{\dot{w}B/B\}} \right) = \zeta \times e(w(\lambda)) \in \mathbb{C}[X(T)^\omega].$$

On the other hand, we see by Lemma 4.1.3 that the (Zariski) cotangent space $T^*_{\dot{w}B/B}\mathcal{B}$ to $\mathcal{B}$ at the point $\dot{w}B/B$ for $w \in W^\omega$ is isomorphic to the dual $\langle\omega\rangle \ltimes T$-module $(\mathfrak{g}/\mathrm{Ad}(\dot{w})\mathfrak{b})^*$ of $\mathfrak{g}/\mathrm{Ad}(\dot{w})\mathfrak{b}$ as an $\langle\omega\rangle \ltimes T$-module, and hence is isomorphic to $\mathrm{Ad}(\dot{w})\mathfrak{n}_-$ as an $\langle\omega\rangle \ltimes T$-module. Therefore, we deduce that for each $w \in W^\omega$ and $j \geq 0$,

$$\mathrm{ch}^\omega(\textstyle\bigwedge^j T^*_{\dot{w}B/B}\mathcal{B}) = \mathrm{ch}^\omega(\textstyle\bigwedge^j (\mathrm{Ad}(\dot{w})\mathfrak{n}_-)).$$

Here we note that the following diagram commutes for each $w \in W^\omega$:

$$\begin{array}{ccc} \mathfrak{n}_- & \xrightarrow{\ \omega\ } & \mathfrak{n}_- \\ {\scriptstyle \mathrm{Ad}(\dot{w})}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Ad}(\dot{w})} \\ \mathrm{Ad}(\dot{w})\mathfrak{n}_- & \xrightarrow[\ \omega\ ]{} & \mathrm{Ad}(\dot{w})\mathfrak{n}_-, \end{array}$$

since we have

$$\omega(\mathrm{Ad}(\dot{y})x) = \mathrm{Ad}(\omega(\dot{y}))\omega(x)$$

for all $y \in W$ and $x \in \mathfrak{g}$. From this commutative diagram, we see that for each $j \geq 0$,

$$\mathrm{ch}^\omega(\textstyle\bigwedge^j (\mathrm{Ad}(\dot{w})\mathfrak{n}_-)) = w\Big(\mathrm{ch}^\omega(\textstyle\bigwedge^j \mathfrak{n}_-)\Big).$$

Consequently, we obtain in $\mathbb{C}[X(T)^\omega]$,

$$\begin{aligned} \sum_{j\geq 0} (-1)^j \, \mathrm{ch}^\omega(\textstyle\bigwedge^j T^*_{\dot{w}B/B}\mathcal{B}) &= \sum_{j\geq 0} (-1)^j \, \mathrm{ch}^\omega(\textstyle\bigwedge^j (\mathrm{Ad}(\dot{w})\mathfrak{n}_-)) \\ &= \sum_{j\geq 0} (-1)^j \left(w\Big(\mathrm{ch}^\omega(\textstyle\bigwedge^j \mathfrak{n}_-)\Big)\right) \\ &= w\Big(\sum_{j\geq 0} (-1)^j \, \mathrm{ch}^\omega(\textstyle\bigwedge^j \mathfrak{n}_-)\Big). \end{aligned}$$

We put

$$H := \sum_{j\geq 0} (-1)^j \operatorname{ch}^{\omega}(\bigwedge^j \mathfrak{n}_-)$$

as an element in $\mathbb{C}[X(T)^{\omega}] \simeq \mathbb{C}[(\mathfrak{h}_{\mathbb{Z}}^*)^0]$. It follows from Formula 2 that

$$H = e(-\rho) \cdot \left(\sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

So we have for each $w \in W^{\omega}$,

$$w(H) = e(-w(\rho)) \cdot w \left(\sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

Here, since for each $y \in W^{\omega}$

$$(-1)^{\widehat{\ell}(w)}(-1)^{\widehat{\ell}(y)} = (-1)^{\widehat{\ell}(w)+\widehat{\ell}(y)} = (-1)^{\widehat{\ell}(wy)},$$

we deduce that

(4.2.4)

$$\begin{aligned} w \left(\sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right) &= \sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(w)}(-1)^{\widehat{\ell}(y)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(wy)} e(wy(\rho)) \\ &= (-1)^{\widehat{\ell}(w)} \times \sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(y(\rho)). \end{aligned}$$

Hence we get

$$w(H) = (-1)^{\widehat{\ell}(w)} \times e(-w(\rho)) \cdot \left(\sum_{y \in W^{\omega}} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right).$$

To sum up, the right-hand side of (4.2.3) is

$$F := \sum_{w \in W^\omega} \frac{\zeta \times e(w(\lambda))}{(-1)^{\widehat{\ell}(w)} \times e(-w(\rho)) \cdot \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)}$$

$$\in \mathrm{Frac}(\mathbb{C}[X(T)^\omega]).$$

Since $\lambda = v \circ \Lambda$, we have $w(\lambda) = wv(\Lambda + \rho) - w(\rho)$ for each $w \in W^\omega$. Hence we get

$$F = \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)^{-1} \times$$

$$\times \left(\zeta \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\rho)) \cdot e(wv(\Lambda + \rho) - w(\rho)) \right)$$

$$= \left(\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho)) \right)^{-1} \cdot \left(\zeta \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(wv(\Lambda + \rho)) \right).$$

As in (4.2.4), we deduce that

$$\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(wv(\Lambda + \rho)) = (-1)^{\widehat{\ell}(v)} \times \sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\Lambda + \rho)).$$

Hence we have in $\mathrm{Frac}(\mathbb{C}[X(T)^\omega])$,

$$F = \zeta \times (-1)^{\widehat{\ell}(v)} \times \frac{\sum_{w \in W^\omega} (-1)^{\widehat{\ell}(w)} e(w(\Lambda + \rho))}{\sum_{y \in W^\omega} (-1)^{\widehat{\ell}(y)} e(y(\rho))}$$

$$= \zeta \times (-1)^{\widehat{\ell}(v)} \times \mathrm{ch}^\omega(L(\Lambda)) \quad \text{by Formula 1.}$$

Thus, by Equality (4.2.3), we obtain Equality (4.2.1):

$$\mathrm{ch}^\omega(V) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \mathrm{ch}^\omega(L(\Lambda)) \in \mathbb{C}[X(T)^\omega].$$

Recall that $V = H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{v \circ \Lambda, \zeta}))$. Thus we have proved that

$$\mathrm{ch}^\omega(H^{\ell(v)}(\mathcal{B}, \mathcal{L}_\mathcal{B}(\mathbb{C}_{v \circ \Lambda, \zeta}))) = (-1)^{\ell(v) - \widehat{\ell}(v)} \times \zeta \times \mathrm{ch}^\omega(L(\Lambda)),$$

as desired.

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(Received July 2, 2001)

Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571
Japan
E-mail: naito@math.tsukuba.ac.jp