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Formal Symbol Type Solutions of Fuchsian Microdifferential Equations

By Kiyoomi Kataoka and Yoshiaki Satoh

Abstract. We construct a basis of solutions for a micro-differential equation with Fuchsian singularities in microfunctions with one holomorphic parameter. More precisely, we construct solutions written by formal symbols with one holomorphic parameter. For such an equation of order m, we get $(m-1)$-regular formal symbols and one singular formal symbol; the latter is not holomorphic along the Fuchsian singularities but has boundary values in the sense of microfunctions.

1. Introduction

Let X be a complex manifold $\mathbb{C}_w \times \mathbb{C}^n_z$ and Z, M be its submanifolds

$$Z = \{(w, z) \in X; \operatorname{Im} z = 0\} \simeq Z^{\mathbb{R}} \supset M = \{\operatorname{Im} w = 0, \operatorname{Im} z = 0\}.$$

Here $Z^{\mathbb{R}}$ is the underlying real manifold of Z. We denote by $(w, z; \tau, \zeta)$ the coordinates of T^*X;

$$w = u + iv \in \mathbb{C},\ z = x + iy \in \mathbb{C}^n,\ \tau \in \mathbb{C},\ \zeta = \xi + i\eta \in \mathbb{C}^n.$$

Then the sheaf $\mathcal{CO}_Z$ on

$$T^*_Z X = \{(w, z; \tau, \zeta) \in T^*X; \tau = 0, \operatorname{Im} z = 0, \operatorname{Re} \zeta = 0\}$$

of microfunctions with a holomorphic parameter w is defined by

$$\mathcal{CO}_Z := \{f(u, v, x) \in \mathcal{C}_{Z^{\mathbb{R}}}; \bar{\partial}_w f = 0\}.$$

Here $\mathcal{C}_{Z^{\mathbb{R}}}$ is the sheaf of usual microfunctions on $Z^{\mathbb{R}}$, and it is well-known that $\mathcal{CO}_Z$ is identified with the sheaf $\mathcal{C}_{Z|X}$ of relative microfunctions as $\mathcal{E}_X$-modules. Setting $N = \mathbb{R}^n_x \subset \mathbb{C}^n_z = Y$, we denote by ρ a projection:

$$\rho : T^*_Z X \ni (w, x; i\eta) \mapsto (x; i\eta) \in T^*_N Y. \tag{1.1}$$

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Let us consider the following microdifferential equation for a section $f(w,x)$ of $\mathcal{CO}_Z$ around $\overset{\circ}{p}=(0,\overset{\circ}{x};i\overset{\circ}{\eta})\in T^*_Z X$ with $\overset{\circ}{\eta}\neq 0$:

$$P(w,x,D_w,D_x)f(w,x) := \Big(\sum_{k=0}^{m} A_k(w,x,D_w,D_x)D_w^{m-k}\Big)f=0, \tag{1.2}$$

where $D_w=\partial/\partial w$ and $D_{x_j}=\partial/\partial x_j$ $(j=1,...,n)$. We suppose that $P(w,z,D_w,D_z)$ has Fuchsian singularities along $\{w=\varphi(x,i\eta)\}$; that is, $A_k(w,z,D_w,D_z)$'s are microdifferential operators defined at $\overset{\circ}{p}$ satisfying the conditions:

$$\begin{cases}\mathrm{ord}(A_k)\leq 0\ (k=0,...,m),\\ \sigma_0(A_0)(\overset{\circ}{p})=0,\ \partial_w\sigma_0(A_0)(\overset{\circ}{p})\neq 0,\\ \sigma_0(A_1)(\overset{\circ}{p})/\partial_w\sigma_0(A_0)(\overset{\circ}{p})\notin\mathbb{Z}.\end{cases} \tag{1.3}$$

Therefore by the Späth type theorem for $\mathcal{E}_X$ in Sato-Kawai-Kashiwara [13] (hereafter, referred to as S-K-K [13]) we can write

$$A_0(w,z,D_w,D_z)=\alpha(w,z,D_w,D_z)(w-\Phi(z,D_w,D_z)). \tag{1.4}$$

Here $\alpha(w,z,D_w,D_z),\Phi(z,D_w,D_z)\in\mathcal{E}_X|_{\overset{\circ}{p}}$ are operators of order 0 with

$$\begin{cases}[\Phi,D_w]=0,\ \ \sigma_0(\Phi)(z,0,\zeta)\equiv\varphi(z,\zeta),\\ \sigma_0(\alpha)(\overset{\circ}{p})\neq 0,\ \varphi(\overset{\circ}{x},i\overset{\circ}{\eta})=0.\end{cases} \tag{1.5}$$

Since the equation (1.2) is microlocally equivalent to $D_w^m f(w,x)=0$ in $\{w-\varphi(x,i\eta)\neq 0\}$, the solution sheaf in $\mathcal{CO}_Z$ of the equation (1.2) is isomorphic to $\rho^{-1}\mathcal{C}_N^m$ on $\{w-\varphi(x,i\eta)\neq 0\}$. The subject of this article is to study the behavior of solutions in $\mathcal{CO}_Z$ around the singular locus $K=\{w-\varphi(x,i\eta)=0\}$. Precisely, we have the following (Theorem 5.8):

THEOREM. *We can construct a system* $\{U^{(\ell)}(w,z,D_z);\ell=1,...,m\}$ *of formal symbols of microdifferential operators defined around* $\overset{\circ}{p}$ *satisfying the following conditions:*

(1) $U^{(1)}$ *is a multivalued section of* $\mathcal{E}_X$ *over* $\{(w,x;i\eta)\in T^*_Z X; 0<|w-\varphi(x,i\eta)|<r, |x-\overset{\circ}{x}|<r, |\eta-\overset{\circ}{\eta}|<r\}$ *for some small* $r>0$, *and* $U^{(\ell)}\in\mathcal{E}_X|_{\overset{\circ}{p}}$ *for* $\ell=2,...,m$.

(2) $U^{(\ell)}(w, z, D_z)$ $(\ell = 1, ..., m)$ *commute with* w.

(3) $P(w, z, D_w, D_z)U^{(\ell)}(w, z, D_z) = 0$ $(mod\, \mathcal{E}_X \cdot D_w)$ *for* $\ell = 1, ..., m$.

(4) *ord*$(U^{(\ell)}) = 0$ *for* $\ell = 1, ..., m$, *and holomorphic functions* $\{\sigma_0(U^{(\ell)})(w, x, i\eta); \ell = 1, ..., m\}$ *give a complete system of solutions in* $\{w - \varphi(x, i\eta) \neq 0\}$ *of the following linear ordinary differential equation :*

$$LU := \Big(\sum_{k=0}^{m} \sigma_0(A_k)(w, x, 0, i\eta)\frac{\partial^{m-k}}{\partial w^{m-k}}\Big)U = 0 \tag{1.6}$$

(5) *For any microfunction* $f(x) \in \mathcal{C}_N|_{\rho(\overset{\circ}{p})}$, $U^{(1)}(w, x, D_x)f(x)$ *has a microfunction boundary value at* $w = \varphi(x, i\eta)$*; that is, a microfunction boundary value from any side of any* $\mathbb{R}$*-conic and real analytic hypersurface* H *of* $T^*_Z X$ *passing through* $K = \{w - \varphi(x, i\eta) = 0\}$.

The precise meaning of the condition (5) will be given in Section 3. As a direct consequence, we have a unique decomposition of a solution $f(w, x) \in \mathcal{CO}_Z$ around $\overset{\circ}{p}$ of (1.2) into a sum:

$$f(w, x) = \sum_{\ell=1}^{m} U^{(\ell)}(w, x, D_x)f_\ell(x), \tag{1.7}$$

where $f_\ell(x) \in \mathcal{C}_N|_{\rho(\overset{\circ}{p})}$ $(\ell = 1, ..., m)$ are uniquely determined by $f(w, x)$. Further, we conclude from the condition 5 that any solution $f(w, x)$ has a microfunction boundary value at $w = \varphi(x, i\eta)$; this fact will be applied to a construction of microlocal solutions for some differential equations with variable multiplicities discussed in Yamane [16], Kataoka [11].

This article consists of 5 sections as follows:

In Section 2, after giving a brief survey on formal symbols, we transform our P into the normal form under some quantized contact transformation preserving $T^*_Z X$. Further we prepare some estimates for holomorphic solutions of Fuchsian ordinary differential equations, which will be used in Section 4.

Section 3 is devoted to give an elementary proof of the invariance of $\mathcal{CO}_Z$ under quantized contact transformations preserving $T^*_Z X$. The key theorem is Theorem 3.9 on the structure of holomorphic contact transformations preserving $T^*_Z X$: That is, *holomorphic contact transformations preserving $T^*_Z X$ are essentially generated by the holomorphic functions of the following type:*

$$z_n^* = h_0(z, z^{*\prime}) + (\Psi(w, z, w^*, z^{*\prime}))^2,$$

where $z^{*\prime} = (z_1^*, ..., z_{n-1}^*)$, and holomorphic functions $\Psi(w, z, w^*, z^{*\prime})$, $h_0(z, z^{*\prime})$ satisfy the following conditions:

(1) $\Psi(\overset{\circ}{w}, \overset{\circ}{x}, \overset{\circ}{w}{}^*, \overset{\circ}{x}{}^{*\prime}) = 0, \partial_w \Psi \neq 0, \partial_{w^*} \Psi \neq 0.$

(2) *h_0 is real-valued for real $(z, z^{*\prime})$, and $\{x_n^* - h_0(x, x^{*\prime}) = 0\}$ gives a real analytic contact transformation $S' : (x; \eta) \mapsto (x^*; \eta^*)$.*

At the same time, we justify our definition of microfunction boundary values of sections of $\mathcal{CO}_Z$ from one side of an $\mathbb{R}$-conic and real analytic hypersurface H of $T^*_Z X$. As direct consequences, we can reduce our P to the normalized operator obtained in Section 2 for the equation $Pf(w, x) = 0$ in $\mathcal{CO}_Z$.

In Section 4, we construct formal symbols $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ satisfying $PU = 0$ (mod $\mathcal{E}_X \cdot D_w$) by successive approximation. The key idea here is in applying some suitable formal norms of Boutet de Monvel and Krée's type to prove the convergence. Once we establish some inequalities on those formal norms, we easily obtain a priori estimates for U. The difficulty appears only when we construct the non-regular type formal symbols related to Fuchsian singularities. To deal with this case, we introduce weighted sup-norms for holomorphic functions with Fuchsian singularities and modifications of the formal norms by these weighted sup-norms.

In Section 5, before we deal with our main theorem, we prove under some growth order conditions near a boundary that a classical formal symbol of pseudo-differential operators has a microfunction boundary value. That is, the following is another main result of this article (Theorem 5.5):

THEOREM. *Let $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ be a classical formal symbol of a pseudo-differential operator with order ≤ 0 defined in an $\mathbb{R}$-conic open set*

$$W_r \equiv \Big\{ (w, z; *, \zeta) \in T^*X; \operatorname{Im} w > 0, |w| < r, |z| < \kappa,$$
$$|\zeta_j| < \rho|\zeta_n| \ (1 \leq \forall j \leq n-1), |\operatorname{Re} \zeta_n| < \delta \operatorname{Im} \zeta_n \Big\}$$

for some $r, \kappa, \rho, \delta > 0\,(\delta < 1)$. *We suppose that* $U_j \in \mathcal{O}(W_r)$ $(\forall j \leq 0)$ *and that there exists some constants* $C, \mu > 0$ *satisfying the following inequalities:*

$$|U_{-p}(w, z, \zeta)| \leq C^{p+1} p! |\operatorname{Im} w|^{-p-\mu} |\zeta|^{-p} \quad \text{on } W_r \ (\forall p \geq 0).$$

Then for any microfunction $f(x) \in \mathcal{C}_N|_{(0;idx_n)}$, *a section* $U(w, x, D_x)f(x) \in \Gamma(\{w \in \mathbb{C}; \operatorname{Im} w > 0, |w| < r\} \times \{(0; idx_n)\}; \mathcal{CO}_Z)$ *has a microfunction boundary value at* $(0, 0; idx_n)$ *from* $\operatorname{Im} w > 0$.

Further, we show by a counter-example that the growth condition above is the best possible in some sense.

2. Preliminaries

2.1. Formal symbols and quantized contact transformations

Definition 2.1. A microdifferential operator $Q(w, z, D_w, D_z) \in \mathcal{E}_X$ at $\mathring{q} = (\mathring{w}, \mathring{z}; \mathring{\tau}, \mathring{\zeta}) \in T^*X$ of order $\leq m (\in \mathbb{Z})$ is identified with a formal sum

$$Q(w, z, D_w, D_z) = \sum_{j=-\infty}^{m} Q_j(w, z, D_w, D_z).$$

of holomorphic functions $\{Q_j(w, z, \tau, \zeta)\}_{j=-\infty}^{m}$ satisfying the following: There exist an $\mathbb{R}$-conic neighborhood W of $\mathring{q}$ in T^*X and a positive constant C such that each $Q_j(z, x, \zeta, \xi)$ is holomorphic in W, and homogeneous of degree j with respect to $(\tau, \zeta) \in \mathbb{C} \times \mathbb{C}^n$, and that we have the following estimates on W:

$$(2.1) \qquad |Q_j(w, z, \tau, \zeta)| \leq (m - j)! C^{1+m-j} (|\tau| + |\zeta|)^j \quad (\forall j \leq m).$$

The formal sum $\sum_{j=-\infty}^{m} Q_j(w, z, \tau, \zeta)$ is called the formal symbol of $Q(w, z, D_w, D_z)$. The composition of two formal symbols $\sum_{j=-\infty}^{m'} Q'_j$, $\sum_{j=-\infty}^{m''} Q''_j$ is defined by a formal symbol $\sum_{j=-\infty}^{m'+m''} Q'''_j$ with

$$(2.2) \qquad Q'''_j(w, z, \tau, \zeta) = \sum_{j'+j''-k-|\alpha|=j} \frac{1}{k!\alpha!} \frac{\partial^{k+|\alpha|} Q'_{j'}}{\partial \tau^k \partial \zeta^\alpha} \frac{\partial^{k+|\alpha|} Q''_{j''}}{\partial w^k \partial z^\alpha},$$

where $k \in \mathbb{N} \cup \{0\}, \alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$. Note that the summation is performed on some finite terms for each j and that $\{Q'''_j\}_j$ satisfy some estimates like (2.1). It is well-known that this composition rule is associative and gives the operator product $Q'(w, z, D_w, D_z) \times Q''(w, z, D_w, D_z)$ ([6, 5], [13]).

REMARK 2.2. This definition naturally extends to the classical definition of formal symbols of pseudo-differential operators $Q \in \mathcal{E}_X^{\mathbb{R}}$ due to Boutet de Monvel and Krée [6, 5]. That is, a formal sum $\sum_{j=-\infty}^{m} Q_j$ is said to be a classical formal symbol at $\mathring{q} = (\mathring{w}, \mathring{z}; \mathring{\tau}, \mathring{\zeta}) \in T^*X$ of pseudo-differential operators of order $\leq m (\in \mathbb{Z})$ if there exist an $\mathbb{R}$-conic neighborhood W of $\mathring{q}$ in T^*X and a positive constant C such that each $Q_j(z, x, \zeta, \xi)$ is a holomorphic function in W satisfying (2.1) (not necessarily of homogeneous degree j with respect to (ζ, ξ)). Then the most of arguments for formal symbols of $\mathcal{E}_X$ extend to the arguments for classical formal symbols of $\mathcal{E}_X^{\mathbb{R}}$. Only one difference is the non-uniqueness of expressions; we cannot determine the j-th order term for a pseudo-differential operator with finite order in general.

REMARK 2.3. We mean by *a classical formal symbol* of a pseudo-differential operator the following: The definition domain of j-th term of the formal symbol does not depend on j, and that they satisfy some inequalities like (2.1) there. On the other hand, in Aoki's modern definition [2] of formal symbols such domains may decrease when $j \to -\infty$. Indeed, their intersection may be void. Aoki's definition of formal symbols of pseudo-differential operators is much simpler, and easy to handle. Further a classical formal symbol canonically induces a formal symbol of Aoki's type, and for micro-differential operators these definitions coincide with each other. However, one cannot define the formal norms for Aoki's formal symbols, which are successfully introduced by Boutet de Monvel and Krée [6] for the theory of (classical) formal symbols. In our construction of formal symbol type solutions we essentially use some variations of formal norms. Hence we employ the classical definition of formal symbols for pseudo-differential operators.

Before constructing the solutions of (1.2) we reduce P to a simpler microdifferential operator by using some quantized contact transformation preserving $T^*_Z X$.

Proposition 2.4. *Let $W^*(w,z,\tau,\zeta)$ be a holomorphic function defined at $\overset{\circ}{p}$ of homogeneous degree* 0 *with respect to (τ,ζ) satisfying*

$$W^*(\overset{\circ}{p})=0,\ \partial_w W^*(\overset{\circ}{p})\neq 0.$$

Then there exists a holomorphic contact transformation:

$$S:\begin{cases} w^*=W^*(w,z,\tau,\zeta), & \tau^*=\tau^*(w,z,\tau,\zeta) \\ z_k^*=z_k^*(w,z,\tau,\zeta), & \zeta_k^*=\zeta_k^*(w,z,\tau,\zeta)\ (k=1,\dots,n) \end{cases} \tag{2.3}$$

defined in a neighborhood of $\overset{\circ}{p}$ satisfying

$$\begin{cases} \tau^*|_{\tau=0}=0, & \partial_\tau \tau^*(\overset{\circ}{p})\neq 0, \\ z_k^*|_{\tau=0}=z_k, & \zeta_k^*|_{\tau=0}=\zeta_k\ \ (k=1,\dots,n). \end{cases} \tag{2.4}$$

Proof. Solve the following Cauchy problem for $\chi=\chi(w,z,\tau^*,\zeta^*)$

$$\begin{cases} \partial_{\tau^*}\chi=W^*(w,z,\partial_w\chi,\partial_z\chi), \\ \chi|_{\tau^*=0}=z\cdot\zeta^*. \end{cases}$$

Then the unique holomorphic solution χ at $\overset{\circ}{p}$ is of homogeneous degree 0 with respect to (τ^*,ζ^*), and generates the desired contact transformation:

$$S:\begin{cases} w^*=\partial_{\tau^*}\chi=W^*, & \tau=\partial_w\chi, \\ z_k^*=\partial_{\zeta_k^*}\chi, & \zeta_k=\partial_{z_k}\chi\ \ (k=1,\dots,n). \ \square \end{cases}$$

Applying Proposition 2.4 to $W^*=w-\sigma_0(\Phi)(z,\tau,\zeta)$ for Φ at (1.4), we get a holomorphic contact transformation S satisfying (2.3). Since the solution χ has the form $\chi=w\tau^*+z\cdot\zeta^*+\psi(z,\tau^*,\zeta^*)$ in this case, we know that $\tau^*=\tau$. Therefore by the theory of quantizations of contact transformations due to S-K-K [13] we have an isomorphism between sheaves of rings

$$\mathcal{S}:S^{-1}\mathcal{E}_X\xrightarrow{\sim}\mathcal{E}_X$$

such that

$$\mathcal{S}(D_{w^*}) = D_w, \ \ \mathcal{S}(w^*) = w - \Phi(z, D_w, D_z).$$

Thus we obtain

$$\mathcal{S}^{-1}(P) = \alpha^*(w^*, z^*, D_{w^*}, D_{z^*})w^* D_{w^*}^m + \sum_{k=1}^{m} A_k^*(w^*, z^*, D_{w^*}, D_{z^*}) D_{w^*}^{m-k}.$$

Here, $\alpha^* = \mathcal{S}^{-1}(\alpha), A_k^* = \mathcal{S}^{-1}(A_k) \in \mathcal{E}_X|_{S(\mathring{p})}$ $(k = 1, \ldots, m)$ are operators of order ≤ 0, and α^* is an elliptic operator at $S(\mathring{p})$ of order 0. Note that S preserves $T_Z^*X = \{(w, z; \tau, \zeta) \in T^*X; \tau = 0, \operatorname{Im} z = 0, \operatorname{Re}\zeta = 0\}$; that is, in a neighborhood of $\mathring{p}$ we have $S(T_Z^*X) \subset T_Z^*X$. Further the ordinary differential operator L at (1.6) associated with P is transformed into

$$L^* = \sum_{k=0}^{m} \sigma_0(A_k^*) \Big(\frac{\partial}{\partial w^*}\Big)^{m-k} = \sum_{k=0}^{m} \Big(\sigma_0(A_k) \circ S^{-1}\Big) \Big(\frac{\partial}{\partial w^*}\Big)^{m-k}.$$

Therefore the conditions (1.3) are also satisfied for $\mathcal{S}^{-1}(P)$ because

$$\begin{aligned}\Big(\sigma_0(A_1^*)/\partial_{w^*}\sigma_0(A_0^*)\Big)(S(\mathring{p})) &= \sigma_0(A_1)(\mathring{p})/\sigma_0(\alpha^*)(S(\mathring{p})) \\ &= \sigma_0(A_1)(\mathring{p})/\sigma_0(\alpha)(\mathring{p}) = \Big(\sigma_0(A_1)/\partial_w\sigma_0(A_0)\Big)(\mathring{p}).\end{aligned}$$

We strengthen this reduction as follows:

LEMMA 2.5. *Set $K = \{\sigma_0(A_0) = 0\} \cap T_Z^*X = \{(w, x; i\eta) \in T_Z^*X; w = \varphi(x, i\eta)\}$. Let H be any $\mathbb{R}$-conic and real analytic hypersurface in T_Z^*X passing through K. Then there exist a holomorphic contact transformation S and a quantization $\mathcal{S}$ of S defined in a neighborhood $\mathring{p}$ such that*

$$\begin{cases} S(K) = \{w^* = 0\} \cap T_Z^*X \ \subset \ S(H) = \{\operatorname{Im} w^* = 0\} \cap T_Z^*X, \\ \qquad\qquad\qquad\qquad S(T_Z^*X) \ \subset \ T_Z^*X \end{cases} \tag{2.5}$$

and that

$$\mathcal{S}^{-1}(D_w) \in \mathcal{E}_X \cdot D_{w^*}, \tag{2.6}$$

$$\mathcal{S}^{-1}(P) = \alpha^* w^* D_{w^*}^m + \sum_{k=1}^{m} A_k^* D_{w^*}^{m-k}. \tag{2.7}$$

Here, $\alpha^*, A_k^* \in \mathcal{E}_X|_{S(\overset{\circ}{p})}$ $(k = 1, \dots, m)$ *are operators of order* ≤ 0*, and* α^* *is an elliptic operator at* $S(\overset{\circ}{p})$ *of order* 0*. Further* $\mathcal{S}^{-1}(P)$ *also satisfies the conditions* (1.3) *at* $S(\overset{\circ}{p})$*.*

Remark 2.6. As a direct consequence of (2.6) we have the following equivalence for a microdifferential operator $U \in \mathcal{E}_X|_{\overset{\circ}{p}}$ of order ≤ 0:

$$PU = 0 \ (\mathrm{mod}\ \mathcal{E}_X \cdot D_w) \Longleftrightarrow \mathcal{S}^{-1}(P)U^* = 0 \ (\mathrm{mod}\ \mathcal{E}_X \cdot D_{w^*})$$

with $U^* = \mathcal{S}^{-1}(U)$. In particular, we get

$$\sigma_0(U)(w, x, 0, i\eta) = \sigma_0(U^*)(w^*, x^*, 0, i\eta^*)$$

under the correspondence $S : (w, x; 0, i\eta) \mapsto (w^*, x^*, 0, i\eta^*)$ and the ordinary differential equations:

$$L(\sigma_0(U)|_{\tau=0}) = 0, \ \ L^*(\sigma_0(U^*)|_{\tau^*=0}) = 0.$$

Proof. By the arguments above we may suppose that $A_0 = \alpha \cdot w$ with an elliptic operator $\alpha(w, z, D_w, D_z) \in \mathcal{E}_X|_{\overset{\circ}{p}}$ of order 0. Hence $K = \{w = 0\} \cap T_Z^*X$ and H is written locally as

$$H = \{w = T(t, x, \eta); t \in \mathbb{R}\}$$

in T_Z^*X. Here $T(t, x, \eta)$ is some $\mathbb{C}$-valued real analytic function defined at $(0, \overset{\circ}{x}, \overset{\circ}{\eta})$ of homogeneous degree 0 with respect to η such that

$$T(0, x, \eta) \equiv 0 \ \text{ and } \ \partial_t T(0, \overset{\circ}{x}, \overset{\circ}{\eta}) \neq 0.$$

Find a holomorphic function $F(w, z, \zeta)$ of homogeneous degree 0 with respect to ζ satisfying $T(F(w, x, i\eta), x, \eta) \equiv w$ by the implicit function theorem. Then we apply Proposition 2.4 to $W^* = F(w, z, \zeta)$. Hence we get a holomorphic contact transformation S satisfying (2.5) and

$$w^* = F(w, z, \zeta) = \beta(w, z, \zeta)w, \ \tau^* = \gamma(w, z, \tau, \zeta)\tau$$

with some non-vanishing holomorphic functions β, γ of homogeneous degree 0 with respect to (τ, ζ). Choose a quantization $\mathcal{S} : S^{-1}\mathcal{E}_X \xrightarrow{\sim} \mathcal{E}_X$ of S such that

$$\begin{cases} \mathcal{S}(w^*) = w \cdot \beta(w, z, D_z) + \delta(w, z, D_w, D_z), \\ \mathcal{S}(D_{w^*}) = \gamma(w, z, D_w, D_z) D_w, \end{cases} \tag{2.8}$$

where $\delta \in \mathcal{E}_X|_{\overset{\circ}{p}}$ is an operator of order ≤ -1. Then, we have

$$\mathcal{S}^{-1}(P) = \alpha^* \lambda^* (G^* D_{w^*})^m + \sum_{k=1}^{m} A_k^* (G^* D_{w^*})^{m-k},$$

where $\alpha^* = \mathcal{S}^{-1}(\alpha), \lambda^* = \mathcal{S}^{-1}(w), G^* = \mathcal{S}^{-1}(\gamma^{-1}), A_k^* = \mathcal{S}^{-1}(A_k)$ $(k = 1, \ldots, m)$ are operators of order ≤ 0. Write

$$(G^* D_{w^*})^j = (G^*)^j D_{w^*}^j + \sum_{\ell=0}^{j-1} G_{j\ell} D_{w^*}^{\ell}$$

with some $G_{j\ell} \in \mathcal{E}_X|_{S(\overset{\circ}{p})}$ of order ≤ 0. Therefore we have

$$\mathcal{S}^{-1}(P) = \sum_{k=0}^{m} A_k^{*\prime} D_{w^*}^{m-k}$$

with

$$\begin{aligned} A_0^{*\prime} &= \alpha^* \lambda^* (G^*)^m, \\ A_k^{*\prime} &= A_k^* (G^*)^{m-k} + \sum_{j=1}^{k-1} A_j^* G_{m-j,m-k} + \alpha^* \lambda^* G_{m,m-k} \end{aligned}$$

for $k = 1, \ldots, m$. Since $\sigma_0(A_0^{*\prime}) = w^* \sigma_0(\alpha^*) \sigma_0(\mathcal{S}^{-1}(\beta))^{-1} \sigma_0(G^*)^m$, we can write $A_0^{*\prime}$ as follows:

$$A_0^{*\prime} = \alpha^{*\prime}(w^*, z^*, D_{w^*}, D_{z^*}) \Big(w^* + \Psi(z^*, D_{w^*}, D_{z^*}) \Big),$$

where $\alpha^{*\prime}, \Psi \in \mathcal{E}_X|_{S(\overset{\circ}{p})}$, $\mathrm{ord}(\alpha^{*\prime}) = 0$, $\mathrm{ord}(\Psi) \leq -1$ and $\alpha^{*\prime}$ is an elliptic operator. Find an elliptic operator $\kappa(z^*, D_{w^*}, D_{z^*}) \in \mathcal{E}_X|_{S(\overset{\circ}{p})}$ of order 0 satisfying

$$\kappa^{-1} \Big(w^* + \Psi(z^*, D_{w^*}, D_{z^*}) \Big) \kappa = w^*,$$

and define a modification $\mathcal{S}'$ of $\mathcal{S}$:

$$\mathcal{S}'(Q) := \mathcal{S}(\kappa Q \kappa^{-1}). \tag{2.9}$$

Then, $\mathcal{S}'$ is also a quantization of S and ${\mathcal{S}'}^{-1}(P)$ gives the normalized form (2.7) of P. Further, since $\sigma_0(A_1^{*\prime}) = \sigma_0(A_1^*)\sigma_0(G^*)^{m-1} + w^*\sigma(\alpha^*)\sigma_0(\mathcal{S}^{-1}(\beta))^{-1}\sigma_0(G_{m,m-1})$, we have

$$\begin{aligned}
&\Big(\sigma_0(A_1^{*\prime})/\partial_{w^*}\sigma_0(A_0^{*\prime})\Big)(S(\mathring{p}))\\
&= \Big(\sigma_0(A_1^*)/(\sigma_0(\alpha^*)\sigma_0(\mathcal{S}^{-1}(\beta))^{-1}\sigma_0(G^*))\Big)(S(\mathring{p}))\\
&= \Big(\sigma_0(A_1)\sigma_0(\beta)\sigma_0(\gamma)/\sigma_0(\alpha)\Big)(\mathring{p}) = \Big(\sigma_0(A_1)/\partial_w\sigma_0(A_0)\Big)(\mathring{p}).
\end{aligned}$$

We used at the last step that $\Big(\sigma_0(\beta)\sigma_0(\gamma)\Big)(\mathring{p}) = 1$, which is a conclusion from the commutation relation $[D_{w^*}, w^*] = 1$ for the equations (2.8). Therefore ${\mathcal{S}'}^{-1}(P)$ also satisfies the conditions (1.3). This completes the proof. □

2.2. Fuchsian ordinary differential operators

For an $\varepsilon > 0$ we set D, $\Omega \subset \mathbb{C}$ as follows:

$$D = \{w \in \mathbb{C}; |w| \leq 1\}, \tag{2.10}$$

$$\Omega = \{z \in \mathbb{C}; 0 < |w| \leq 1, |\arg w| \leq \pi - \varepsilon\}. \tag{2.11}$$

Let L be an m-th order ordinary differential operator of the form

$$L = \sum_{k=0}^{m} a_k(w)\partial_w^{m-k},$$

where $a_0(w) = w$ and each $a_k(w)$ is holomorphic in a neighborhood of D. Then, we obtain estimations for solutions of

$$LU = f$$

for two cases: Holomorphic functions $f(w)$ on D and also on Ω.

Definition 2.7. For a holomorphic function $U(w)$ in a neighborhood of D, we define two norms as follows:

$$\|U\| = \sup_D |U(w)|, \quad \|U\|' = \sup_{w\in D, 0\leq j\leq m} |U^{(j)}(w)| \tag{2.12}$$

and define another two norms with weight $\mu \in \mathbb{R}$ by

$$\|U\|_\mu = \sup_{w \in \Omega} |w|^\mu |U(w)|, \tag{2.13}$$

$$\|U\|'_\mu = \sup_{w \in \Omega, 0 \le j \le m} |w|^{\mu - m + 1 + j} |U^{(j)}(w)| \tag{2.14}$$

for a holomorphic function $U(w)$ defined in a neighborhood of Ω .

LEMMA 2.8. *We suppose that $a_1(0) \neq 0, -1, -2, \ldots$. Set*

$$M = 1 + \sup_{w \in D} \sum_{k=1}^{m} |a_k(w)| < +\infty, \tag{2.15}$$

$$\delta = \min\{|p + a_1(0)|; p = 0, 1, 2, \ldots\} > 0. \tag{2.16}$$

Then we have a positive constant C depending only on M and δ, which satisfies the following estimations:

(1) **Regular case**: *For a $f(w) \in \mathcal{O}(D)$, any solution $U(w) \in \mathcal{O}(D)$ of $LU = f$ satisfies*

$$\|U\|' \le C\{\|f\| + |U(0)| + \cdots + |U^{(m-2)}(0)|\}. \tag{2.17}$$

(2) **Non-regular case**: *For a $f(w) \in \mathcal{O}(\Omega)$, any solution $U(w) \in \mathcal{O}(\Omega)$ of $LU = f$ satisfies*

$$\|U\|'_\mu \le C\{\|f\|_\mu + |U(1)| + \cdots + |U^{(m-1)}(1)|\} \tag{2.18}$$

with $\forall \mu \ge M + m + 1$.

REMARK 2.9. It is well known by the theory of Fuchsian differential equations that under the assumption $a_1(0) \neq 0, -1, -2, \ldots$, there exists a unique solution for any given $(U(0), \ldots, U^{(m-2)}(0))$ or $(U(1), ..., U^{(m-1)}(1))$ for both cases.

PROOF. Put an $m \times m$-matrix

$$A(w) = \begin{pmatrix} 0 & w & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & w \\ -a_m(w) & \cdots & \cdots & \cdots & -a_1(w) \end{pmatrix},$$

and two m-dimensional vectors

$$X(w) = \begin{pmatrix} U(w) \\ U'(w) \\ \vdots \\ U^{(m-1)}(w) \end{pmatrix}, \quad B(w) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f(w) \end{pmatrix}.$$

Then, $LU = f$ reduces to

$$\frac{dX(w)}{dw} = \frac{1}{w}A(w)X(w) + \frac{1}{w}B(w).$$

Hence,

$$X(w) = X(w_0) + \int_{w_0}^{w} \frac{1}{s}A(s)X(s)ds + \int_{w_0}^{w} \frac{1}{s}B(s)ds. \tag{2.19}$$

Firstly we consider the non-regular case. We introduce the following norms for $m \times m$ matrix $X = (X_{jk})_{j,k=1}^{m}$ and m-vector $B = (B_j)_{j=1}^{m}$:

$$|X| \equiv \max_{j=1,\dots,m} \Big(\sum_{k=1}^{m} |X_{jk}| \Big), \quad |B| \equiv \max_{j=1,\dots,m} |B_j|.$$

Therefore we have $|A(w)| \leq M$ on D. We put $w = e^{i\theta}$ and $w_0 = 1$ in (2.19) and we get the following integral inequality for $\theta \in [0, \pi - \varepsilon]$:

$$\begin{aligned} |X(e^{i\theta})| &\leq |X(1)| + \int_0^{\theta} M|X(e^{i\varphi})|d\varphi + \int_0^{\theta} |f(e^{i\varphi})|d\varphi \\ &\leq |X(1)| + \pi\|f\|_{\mu} + \int_0^{\theta} M|X(e^{i\varphi})|d\varphi. \end{aligned} \tag{2.20}$$

Further putting $w = re^{i\theta}$ and $w_0 = e^{i\theta}$ we get the following for $|\theta| \leq \pi - \varepsilon$ and $r \in (0, 1]$:

$$\begin{aligned} |X(re^{i\theta})| &\leq |X(e^{i\theta})| + \int_r^1 \frac{M}{s}|X(se^{i\theta})|ds + \int_r^1 \frac{|f(se^{i\theta})|}{s}ds \\ &\leq |X(e^{i\theta})| + \frac{r^{-\mu} - 1}{\mu}\|f\|_{\mu} + \int_r^1 \frac{M}{s}|X(se^{i\theta})|ds. \end{aligned} \tag{2.21}$$

Now we apply Gronwall's lemma to (2.21) for $\mu \geq M+m+1$:

$$\begin{aligned}|X(re^{i\theta})| &\leq |X(e^{i\theta})| + \frac{r^{-\mu}-1}{\mu}\|f\|_\mu \\ &\quad + \int_r^1 \{|X(e^{i\theta})| + \frac{t^{-\mu}-1}{\mu}\|f\|_\mu\}\frac{M}{t}\exp\Big(\int_r^t \frac{M}{s}ds\Big)dt \\ &\leq r^{-M}|X(e^{i\theta})| + \frac{r^{-\mu}}{\mu-M}\|f\|_\mu \leq r^{-M}|X(e^{i\theta})| + r^{-\mu}\|f\|_\mu.\end{aligned}$$

In the same way, we obtain from (2.20) that

$$\begin{aligned}|X(e^{i\theta})| &\leq |X(1)| + \pi\|f\|_\mu + \int_0^\theta \{|X(1)| + \pi\|f\|_\mu\}Me^{M(\theta-\varphi)}d\varphi \\ &\leq e^{M\pi}\{|X(1)| + \pi\|f\|_\mu\}.\end{aligned}$$

The last inequality holds for $|\theta| \leq \pi - \varepsilon$. Therefore we have

$$\begin{aligned}|X(re^{i\theta})| &\leq r^{-M}e^{M\pi}|X(1)| + (r^{-M}\pi e^{M\pi} + r^{-\mu})\|f\|_\mu \\ &\leq r^{-\mu}(1+\pi e^{M\pi})(\|f\|_\mu + |X(1)|).\end{aligned}$$

Thus we obtain the inequalities:

$$\begin{aligned}|U^{(m)}(w)| &= |w|^{-1}\cdot\big| - a_1(w)U^{(m-1)}(w) - \cdots - a_m(w)U(w) + f(w)\big| \\ &\leq |w|^{-\mu-1}M(1+\pi e^{M\pi})\big(\|f\|_\mu + |X(1)|\big),\end{aligned}$$

and

$$|U^{(j)}(w)| \leq |X(w)| \leq |w|^{-\mu}\big(1+\pi e^{M\pi}\big)\big(\|f\|_\mu + |X(1)|\big) \tag{2.22}$$

for $j = 0, 1, ..., m-1$. Hence for $j = m, m-1$ we have the uniform estimates of $|w|^{\mu+j-m+1}U^{(j)}(w)$. Further by using (2.22) for $j = m-1, m-2$ and an integral expression

$$U^{(m-2)}(re^{i\theta}) = U^{(m-2)}(e^{i\theta}) - \int_r^1 U^{(m-1)}(se^{i\theta})e^{i\theta}ds,$$

we get a similar uniform estimate for $j = m-2$. Hence by repetitive arguments we have the estimate (2.18).

To deal with the regular case we expand $A(w), B(w), X(w)$ into power series:

$$A(w) = \sum_{p=0}^{\infty} A_p w^p, \quad B(w) = \sum_{p=0}^{\infty} B_p w^p, \quad X(w) = \sum_{p=0}^{\infty} X_p w^p.$$

Hence we have the following equations for the coefficients:

$$\begin{cases} a_1(0)U^{(m-1)}(0) + \cdots + a_m(0)U^{(0)}(0) = f(0), \\ (p - A_0)X_p = \sum_{q=1}^{p} A_q X_{p-q} + B_p \quad (p \geq 1). \end{cases} \tag{2.23}$$

By the Cauchy estimates we obtain $|A_p| \leq M, |B_p| \leq \|f\|$ $(p \geq 0)$. Further $(p - A_0)^{-1}$ is given by

$$(p - A_0)^{-1} = \frac{1}{p(p + a_1(0))} \begin{pmatrix} p + a_1(0) & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p + a_1(0) & 0 \\ -a_m(0) & \cdots & \cdots & -a_2(0) & p \end{pmatrix}.$$

for $p \geq 1$. Therefore we have

$$\begin{aligned} |X_p| &\leq |(p - A_0)^{-1}| \Bigl(\sum_{q=1}^{p} |A_q||X_{p-q}| + |B_p| \Bigr) \\ &\leq \max \left\{ \frac{1}{p}, \frac{p + |a_2(0)| + \cdots + |a_m(0)|}{p|p + a_1(0)|} \right\} \Bigl(\sum_{q=1}^{p} M|X_{p-q}| + \|f\| \Bigr) \\ &\leq K \Bigl(M \sum_{q=0}^{p-1} |X_q| + \|f\| \Bigr) \end{aligned}$$

with $K = \max\{1, M/\delta\}$. Thus, adding $\sum_{q=0}^{p-1} |X_q|$ to both sides we get

$$\begin{aligned} \sum_{q=0}^{p} |X_q| &\leq (KM + 1) \sum_{q=0}^{p-1} |X_q| + K\|f\| \\ &\leq \frac{(KM + 1)^p - 1}{M} \|f\| + (KM + 1)^p |X_0|. \end{aligned}$$

Since we obtain from the first equation of (2.23) that $|U^{(m-1)}(0)| \leq (1/\delta)(M|U^{(m-2)}(0)| + \cdots + M|U^{(0)}(0)| + \|f\|)$, we have

$$|X_0| \leq K(|U^{(m-2)}(0)| + \cdots + |U^{(0)}(0)| + \|f\|).$$

Consequently

$$|X_p| \leq (KM+1)^p \Big\{ \big(\frac{1}{M} + K\big)\|f\| + K\big(|U(0)| + \cdots + |U^{(m-2)}(0)|\big) \Big\}$$

for $\forall p \geq 0$, and so we have

$$\sup \left\{ |X(z)|; |z| \leq \frac{1}{2(KM+1)} \right\} \leq 2\Big(K + \frac{1}{M}\Big)\Big(\|f\| + \sum_{j=0}^{m-2} |U^{(j)}(0)|\Big).$$

Putting $r_0 = 1/\{2(MK+1)\} < 1$, we get an integral inequality similar to (2.21) for $r \geq r_0$:

$$|X(re^{i\theta})| \leq |X(r_0 e^{i\theta})| + \|f\| \log \frac{1}{r_0} + \int_{r_0}^{r} \frac{M}{s} |X(se^{i\theta})| ds.$$

Hence by Gronwall's lemma we obtain for $r \in [r_0, 1]$ that

$$|X(re^{i\theta})| \leq \Big(\frac{r}{r_0}\Big)^M \Big\{ 2(K + \frac{1}{M}) + \log \frac{1}{r_0} \Big\} \Big(\|f\| + \sum_{j=0}^{m-2} |U^{(j)}(0)|\Big)$$

Therefore,

$$\sup_D |X(w)| \leq \Big(\frac{1}{r_0}\Big)^M \Big\{ 2(K + \frac{1}{M}) + \log \frac{1}{r_0} \Big\} \Big(\|f\| + \sum_{j=0}^{m-2} |U^{(j)}(0)|\Big)$$

Note that

$$\begin{aligned} \sup_{w \in D} |U^{(m)}(w)| &= \sup_{|w|=1} \left| \frac{-\sum_{j=1}^{m} a_j(w) U^{(m-j)}(w) + f(w)}{w} \right| \\ &\leq M \sup_{w \in D} |X(w)| + \|f\|. \end{aligned}$$

Therefore since $M \geq 1$,

$$\|U\|' \leq M \sup_{w \in D} |X(w)| + \|f\| \leq C\Big(\|f\| + \sum_{j=0}^{m-2} |U^{(j)}(0)|\Big)$$

with

$$C = M\Big\{2(KM+1)\Big\}^M\Big[2\big(K+\frac{1}{M}\big)+\log\big\{2(KM+1)\big\}\Big]+1$$

and

$$K = \max\{1, M/\delta\}.$$

This completes the proof of Lemma 2.8. □

3. Quantized Contact Transformations for Sheaf $\mathcal{CO}_Z$

Before introducing quantized contact transformations for sheaf $\mathcal{CO}_Z$, we investigate the structure of holomorphic contact transformations preserving $T^*_Z X$.

DEFINITION 3.1. Let $S : T^*X \xrightarrow{\sim} T^*X$ be a holomorphic contact transformation defined in a neighborhood of $\overset{\circ}{p} \in T^*X \setminus X$. Then *the anti-graph of* S is defined by the following G:

$$G = \{(w, z, w^*, z^*; \tau, \zeta, -\tau^*, -\zeta^*) \in T^*(X \times X)\},$$
$$\pi : T^*(X \times X) \supset G \to \pi(G) \subset X \times X.$$

Here $(w^*, z^*; \tau^*, \zeta^*) = S((w, z; \tau, \zeta))$ and $(w, z; \tau, \zeta)$ moves over a neighborhood of $\overset{\circ}{p}$, and π denotes the natural projection. It is clear that G becomes an $\mathbb{R}$-conic and complex Lagrangian submanifold of $T^*(X \times X)$.

S is said to be *of generic type* if and only if the projection $\pi(G)$ becomes a complex hypersurface of $X \times X$; more precisely, there exists either one of coordinates $z^*_1, \dots, z^*_n$, for example z^*_n, such that we have

$$\pi(G) = \{g \equiv z^*_n - h(w, z, w^*, z^{*\prime}) = 0\} \tag{3.1}$$

with some holomorphic function h. Here $z^{*\prime} = (z^*_1, \dots, z^*_{n-1})$.

REMARK 3.2. We can get such an expression (3.1) if the complex submanifold G with dimension $2n+2$ has $(w, z, w^*, z^{*\prime}, \zeta^*_n)$ as a local coordinate system in a neighborhood of $(\overset{\circ}{w}, \overset{\circ}{z}, \overset{\circ}{w}{}^*, \overset{\circ}{z}{}^*; \overset{\circ}{\tau}, \overset{\circ}{\zeta}, -\overset{\circ}{\tau}{}^*, -\overset{\circ}{\zeta}{}^*)$. It is the most important in this case that G coincides with the conormal (line) bundle of $\pi(G)$:

$$G = T^*_{\pi(G)}(X \times X).$$

That is, we have the equations:

$$(3.2)\qquad \begin{cases} z_n^* = h(w,z,w^*,z^{*\prime}) \\ \tau^* = -\zeta_n^*\partial_{w^*}h, & \zeta_k^* = -\zeta_n^*\partial_{z_k^*}h \ (k=1,\dots,n-1), \\ \tau = \zeta_n^*\partial_w h, & \zeta_j = \zeta_n^*\partial_{z_j}h \quad (j=1,\dots,n). \end{cases}$$

Further, a holomorphic function $h(w,z,w^*,z^{*\prime})$ in (3.1) induces a local contact transformation if and only if

$$(3.3)\qquad \det\begin{pmatrix} \partial_w\partial_{w^*}h, & \partial_w\partial_{z^{*\prime}}h, & \partial_w h \\ \partial_z\partial_{w^*}h, & \partial_z\partial_{z^{*\prime}}h, & \partial_z h \end{pmatrix} \neq 0.$$

A contact transformation preserving T_Z^*X also preserves $\{\tau = 0\}$. As for these transformations we have the following lemma:

LEMMA 3.3. *Let* $S : T^*X \ni (w,z;\tau,\zeta) \mapsto (w^*,z^*;\tau^*,\zeta^*) \in T^*X$ *be a holomorphic contact transformation defined in a neighborhood* $\mathring{p} = (\mathring{w},\mathring{z};0,\mathring{\zeta})$ *with* $\mathring{\zeta} \neq 0$. *We suppose that* S *preserves* $\{\tau = 0\}$*; that is,* $S(\{\tau = 0\}) \subset \{\tau^* = 0\}$. *Let* S *be given by the holomorphic functions*

$$\begin{cases} w^* = W(w,z,\tau,\zeta), & \tau^* = T(w,z,\tau,\zeta), \\ z_j^* = Z_j(w,z,\tau,\zeta), & \zeta_j = \Xi_j(w,z,\tau,\zeta) \ (j=1,...,n). \end{cases}$$

Then, $Z_j(w,z,0,\zeta), \Xi_j(w,z,0,\zeta)$ *do not depend on* w *for* $j = 1,...,n$. *Hence* S *induces a holomorphic contact transformation*

$$(3.4)\qquad S' : T^*Y \ni (z;\zeta) \mapsto (Z(*,z,0,\zeta);\Xi(*,z,0,\zeta)) \in T^*Y.$$

PROOF. By the assumption we have

$$\sum_{j=1}^n d\Xi_j(w,z,0,\zeta) \wedge dZ_j(w,z,0,\zeta) = \sum_{j=1}^n d\zeta_j \wedge dz_j.$$

Therefore we obtain a system of equations for $k = 1,...,n$:

$$\sum_{j=1}^n \Big(\frac{\partial Z_j}{\partial z_k}\frac{\partial \Xi_j}{\partial w} - \frac{\partial \Xi_j}{\partial z_k}\frac{\partial Z_j}{\partial w}\Big)\Big|_{\tau=0} = 0,$$

$$\sum_{j=1}^n \Big(\frac{\partial Z_j}{\partial \zeta_k}\frac{\partial \Xi_j}{\partial w} - \frac{\partial \Xi_j}{\partial \zeta_k}\frac{\partial Z_j}{\partial w}\Big)\Big|_{\tau=0} = 0.$$

Since the $(2n) \times (2n)$-matrix

$$\begin{pmatrix} \partial Z_j/\partial z_k, & \partial \Xi_{j'}/\partial z_k \\ \partial Z_j/\partial \zeta_{k'}, & \partial \Xi_{j'}/\partial \zeta_{k'} \end{pmatrix}_{kk',jj'}$$

is non-singular, we have that

$$\frac{\partial \Xi_k}{\partial w}(w, z, 0, \zeta) = 0, \quad \frac{\partial Z_k}{\partial w}(w, z, 0, \zeta) = 0 \ \ (k = 1, ..., n). \ \square$$

Example 3.4. Let $\alpha \in \mathbb{C}$ be a non-zero constant. Set 2 generating functions g_0, g_1 by

$$g_0 \equiv z_n^* - z_n + \sum_{j=1}^{n-1} z_j z_j^* + \frac{1}{2\alpha}(w^* - w)^2 \qquad (n \geq 1), \tag{3.5}$$

$$g_1 \equiv z_n^* - z_n + \sum_{j=2}^{n-1} z_j z_j^* + (z_1^* - z_1)(w^* - w) \qquad (n \geq 2). \tag{3.6}$$

Then the contact transformations S_0, S_1 corresponding to $\{g_0 = 0\}$, $\{g_1 = 0\}$ respectively are given as follows:

$$S_0 : \begin{cases} \tau^* = \tau, & w^* = w + \alpha(\tau/\zeta_n), \\ \zeta_j^* = \zeta_n z_j, & z_j^* = -\zeta_j/\zeta_n \qquad (j = 1, \dots, n-1), \\ \zeta_n^* = \zeta_n, & z_n^* = z_n + (\sum_{k=1}^{n-1} \zeta_k z_k)/\zeta_n - \frac{\alpha}{2}(\tau/\zeta_n)^2. \end{cases} \tag{3.7}$$

$$S_1 : \begin{cases} \tau^* = \tau, & w^* = w + (\zeta_1/\zeta_n), \\ \zeta_1^* = \zeta_1, & z_1^* = z_1 + (\tau/\zeta_n), \\ \zeta_j^* = \zeta_n z_j, & z_j^* = -\zeta_j/\zeta_n \qquad (j = 2, \dots, n-1), \\ \zeta_n^* = \zeta_n, & z_n^* = z_n + (\sum_{k=2}^{n-1} \zeta_k z_k)/\zeta_n - (\tau\zeta_1/\zeta_n^2). \end{cases} \tag{3.8}$$

It is clear that S_0, S_1 are holomorphic contact transformations of generic type preserving T_Z^*X. Further if $\alpha \in \mathbb{R} \setminus \{0\}$, S_0, S_1 also preserve T_M^*X; that is, real contact transformations.

As we see in the next theorem, S_0 is a typical example of the generic and normal case, and S_1 is a typical example of the generic and non-normal case.

THEOREM 3.5. *Let $S : T^*X \xrightarrow{\sim} T^*X$ be a holomorphic contact transformation defined in a neighborhood of $\mathring{p} = (\mathring{w}, \mathring{x}; 0, i\mathring{\eta}) \in T^*_Z X$ with $\mathring{\eta} \neq 0$ preserving $T^*_Z X$. We assume that S is of generic type and that the antigraph is given by the conormal bundle of $\{z^*_n = h(w, z, w^*, z^{*\prime})\}$ like (3.1). Set $S(\mathring{p}) = \mathring{p}^* = (\mathring{w}^*, \mathring{x}^*; 0, i\mathring{\eta}^*)$. We suppose $\mathring{\eta}^*_n \neq 0$ and the following condition (the normal case condition):*

$$\partial^2_{w^*} h(\mathring{w}, \mathring{x}, \mathring{w}^*, \mathring{x}^{*\prime}) \neq 0.$$

Then h has a form:

$$h = h_0(z, z^{*\prime}) + \Psi(w, z, w^*, z^{*\prime})^2, \tag{3.9}$$

where holomorphic functions $\Psi(w, z, w^*, z^{*\prime}), h_0(z, z^{*\prime})$ satisfy the following conditions:

(1) $\Psi(\mathring{w}, \mathring{x}, \mathring{w}^*, \mathring{x}^{*\prime}) = 0, \partial_w \Psi \neq 0, \partial_{w^*} \Psi \neq 0.$

(2) *h_0 is real-valued for real $(z, z^{*\prime})$, and $\{x^*_n - h_0(x, x^{*\prime}) = 0\}$ gives a real analytic contact transformation $S' : (x; \eta) \mapsto (x^*; \eta^*)$, which is the induced transformation in the sense of Lemma 3.3.*

Conversely, if Ψ, h_0 satisfy these conditions, then the hypersurface $\{z^*_n = h_0(z, z^{*\prime}) + \Psi(w, z, w^*, z^{*\prime})^2\}$ generates a holomorphic contact transformation preserving $T^*_Z X$.

PROOF. Since S preserves $T^*_Z X$, we have a nowhere-vanishing holomorphic function $\phi(w, z, w^*, z^{*\prime})$ satisfying

$$\partial_w h = \phi \partial_{w^*} h. \tag{3.10}$$

Here we note that $\partial_w h = \partial_{w^*} h = 0$ at $(\mathring{w}, \mathring{x}, \mathring{w}^*, \mathring{x}^{*\prime})$. By the assumption $\partial^2_{w^*} h(\mathring{w}, \mathring{x}, \mathring{w}^*, \mathring{x}^{*\prime}) \neq 0$ we can find a holomorphic function $w^* = \psi(w, z, z^{*\prime})$ satisfying

$$\partial_{w^*} h|_{w^* = \psi} = 0, \ \ \psi(\mathring{w}, \mathring{z}, \mathring{z}^{*\prime}) = \mathring{w}^*.$$

Thus, by expanding h into a power series of $w^* - \psi(w, z, z^{*\prime})$, we know that any branch

$$\Psi(w, z, w^*, z^{*\prime}) \equiv \pm\sqrt{h - (h|_{w^* = \psi})}$$

is a holomorphic function satisfying $\partial_{w^*}\Psi \neq 0$ at $(\overset{\circ}{w}, \overset{\circ}{z}, \overset{\circ}{w}{}^*, \overset{\circ}{z}{}^{*\prime})$. We note here that the critical value

$$h_0 \equiv h(w, z, \Psi(w, z, z^{*\prime}), z^{*\prime})$$

does not depend on w. Because

$$\begin{aligned}\partial_w h_0 &= (\partial_w h)|_{w^*=\Psi} + (\partial_{w^*} h)|_{w^*=\Psi} \cdot \partial_w \Psi \\ &= (\phi|_{w^*=\Psi} + \partial_w \Psi)(\partial_{w^*} h)|_{w^*=\Psi} = 0.\end{aligned}$$

Therefore we can write h in the form (3.9). Further, on $\{\Psi = 0\}$ we know that the determinant of the matrix (3.3) is equal to

$$\Psi_w \Psi_{w^*} \det\left(\partial_z \partial_{z^{*\prime}} h_0, \partial_z h_0\right).$$

Hence we directly obtain the conditions (1), (2) and the converse statement. □

LEMMA 3.6. *Let $S : T^*X \to T^*X$ be any holomorphic contact transformation defined in a neighborhood of $\overset{\circ}{p} = (\overset{\circ}{w}, \overset{\circ}{x}; 0, i\overset{\circ}{\eta}) \in T^*_Z X$ with $\overset{\circ}{\eta} \neq 0$ preserving $T^*_Z X$. Suppose that the induced transformation $S' : T^*Y \to T^*Y$ for S is equal to the identity map. Then for the contact transformation S_0 in* (3.7) *with a sufficiently large positive number α, the composition $S_0 \circ S$ becomes a holomorphic contact transformation preserving $T^*_Z X$ of the generic and normal case type in the sense of Theorem 3.5.*

PROOF. We may assume that $\overset{\circ}{\eta}_n \neq 0$. We write

$$\begin{aligned} S &: (w, z; \tau, \zeta) \mapsto (w^*, z^*; \tau^*, \zeta^*), \\ S_0 &: (w^*, z^*; \tau^*, \zeta^*) \mapsto (w^{**}, z^{**}; \tau^{**}, \zeta^{**}).\end{aligned}$$

In order to get an expression $\{z_n^{**} - h(w, z, w^{**}, z^{**\prime}) = 0\}$ for the projection of the anti-graph of $S_0 \circ S$, It is sufficient to show that

$$I \equiv \det \begin{pmatrix} \partial_\tau w^{**}, & \partial_\tau z^{**\prime}, & \partial_\tau \zeta_n^{**} \\ \partial_\zeta w^{**}, & \partial_\zeta z^{**\prime}, & \partial_\zeta \zeta_n^{**} \end{pmatrix} \neq 0 \tag{3.11}$$

at $\overset{\circ}{p}$. Here $z^{**\prime} = (z_1^{**}, ..., z_{n-1}^{**})$ is a row vector, and ∂_ζ denotes the column vector of the gradient here. For any function $F(w^*, z^*, \tau^*, \zeta^*)$ we have that

$$\partial_{\zeta_j}(F \circ S) = \partial_{\zeta_j} w^* \cdot \partial_{w^*} F + \partial_{\zeta_j^*} F \quad (j = 1, ..., n)$$

on $\{\tau = 0\}$. Therefore at $\overset{\circ}{p}$ we have

$$\begin{aligned} I &= \det\begin{pmatrix} \partial_\tau w^* + (\alpha\partial_\tau\tau^*)/\zeta_n^*, & \partial_\tau\zeta^*\partial_{\zeta^*}(-\zeta^{*\prime}/\zeta_n^*), & \partial_\tau\zeta_n^* \\ \partial_\zeta w^*, & \partial_{\zeta^*}(-\zeta^{*\prime}/\zeta_n^*), & \partial_{\zeta^*}\zeta_n^* \end{pmatrix} \\ &= \det\begin{pmatrix} \partial_\tau w^* + (\alpha\partial_\tau\tau^*)/\zeta_n^* - \partial_{\zeta_n} w^*\partial_\tau\zeta_n^*, & -(\partial_\tau\zeta^{*\prime})/\zeta_n^* \\ \partial_{\zeta'} w^*, & -E_{n-1}/\zeta_n^* \end{pmatrix} \\ &= (-\zeta_n^*)^{1-n}\Big(\partial_\tau w^* + (\alpha\partial_\tau\tau^*)/\zeta_n^* - \sum_{j=1}^{n}\partial_{\zeta_j} w^*\partial_\tau\zeta_j^*\Big). \end{aligned}$$

Here E_{n-1} is the identity matrix of size $n-1$. Since $\partial_\tau\tau^*(\overset{\circ}{p}) \neq 0$, we have $I \neq 0$ for any sufficiently large $\alpha > 0$. Note that $\partial^2_{w^{**}} h(\overset{\circ}{w}, \overset{\circ}{x}, \overset{\circ}{w}{}^{**}, \overset{\circ}{x}{}^{**\prime}) \neq 0$ is equivalent to $\tilde{\partial}_{w^{**}}\tau^{**} \neq 0$, where $\tilde{\partial}$ means the differentiation in the coordinates $(w, z, w^{**}, z^{**\prime}, \zeta_n^{**})$. On the other hand we have

$$\partial_\tau\tau^{**} = \partial_\tau w^{**}\tilde{\partial}_{w^{**}}\tau^{**} + \sum_{j=1}^{n-1}\partial_\tau z_j^{**}\tilde{\partial}_{z_j^{**}}\tau^{**} + \partial_\tau\zeta_n^{**}\tilde{\partial}_{\zeta_n^{**}}\tau^{**},$$

$$\partial_\zeta\tau^{**} = \partial_\zeta w^{**}\tilde{\partial}_{w^{**}}\tau^{**} + \sum_{j=1}^{n-1}\partial_\zeta z_j^{**}\tilde{\partial}_{z_j^{**}}\tau^{**} + \partial_\zeta\zeta_n^{**}\tilde{\partial}_{\zeta_n^{**}}\tau^{**}.$$

Hence we obtain

$$\tilde{\partial}_{w^{**}}\tau^{**} = \frac{1}{I}\det\begin{pmatrix} \partial_\tau\tau^{**}, & \partial_\tau z^{**\prime}, & \partial_\tau\zeta_n^{**} \\ \partial_\zeta\tau^{**}, & \partial_\zeta z^{**\prime}, & \partial_\zeta\zeta_n^{**} \end{pmatrix}.$$

Since $\partial_\zeta\tau^{**} = 0$ on $\{\tau = 0\}$, we get

$$\tilde{\partial}_{w^{**}}\tau^{**} = \frac{\partial_\tau\tau^{**}}{I}\det\big(\partial_{\zeta^*}(-\zeta^{*\prime}/\zeta_n^*), \quad \partial_{\zeta^*}\zeta_n^*\big) = \frac{\partial_\tau\tau^{**}}{I(-\zeta_n^*)^{n-1}} \neq 0$$

at $\overset{\circ}{p}$. This completes the proof. □

THEOREM 3.7. *Let S be a holomorphic contact transformation defined in a neighborhood of $\overset{\circ}{p} \in T_Z^*X\backslash Z$ preserving T_Z^*X, and $\mathcal{S}$ be any quantization of S. Then, there exists a sheaf isomorphism*

$$\mathcal{T}_{\mathcal{S}} : S^{-1}\mathcal{CO}_Z \overset{\sim}{\longrightarrow} \mathcal{CO}_Z$$

satisfying

$$\mathcal{T}_{\mathcal{S}}(Qf) = \mathcal{S}(Q)\mathcal{T}_{\mathcal{S}}(f)$$

at any point q near $\overset{\circ}{p}$ for any germs $f \in \mathcal{CO}_Z|_{S(q)}$, $Q \in \mathcal{E}_X|_{S(q)}$. Further such a $\mathcal{T}_{\mathcal{S}}$ is determined up to a constant by $\mathcal{S}$.

Remark 3.8. This result is proven in a general situation in [8, 9]. We introduce here a more elementary proof based on Theorem 4.2.17 of [10] for clarifying a definition given later.

Proof. We have only to prove this theorem for some quantization $\mathcal{S}$ of S because other quantizations are written as the composition of $\mathcal{S}$ and some inner automorphisms similar to (2.9). Further by the preceding lemma we can reduce S to the following 3 cases:

(1) S is of the generic and normal case type.

(2) $S = S_0^{-1}$, where S_0 is the one at (3.7).

(3) S is induced by a (tangential) real analytic contact transformation $S' : T_N^*Y \to T_N^*Y$; that is, $w^* \equiv w, \tau^* \equiv \tau$.

The third case is a trivial case. Further the second case belongs to the first case because S_0^{-1} is generated by

$$g_1 = z_n^* - z_n - \sum_{j=1}^{n-1} z_j z_j^* - \frac{1}{2\alpha}(w^* - w)^2. \tag{3.12}$$

Hence we have only to deal with the first case. Using the form (3.9) for h, we consider the following integral transformation

$$(\mathcal{T}f)(w, x) = \int \delta(x_n^* - h(w, x, w^*, x^{*\prime}))f(w^*, x^*)dw^*dx^* \tag{3.13}$$

for a section $f(w^*, x^*)$ of $\mathcal{CO}_Z$. This integral has no meaning because the values of h are not limited to real numbers when $w, x, w^*, x_1^*, \ldots, x_{n-1}^*$ move.

However we can formally modify this integral as follows:

$$
\begin{aligned}
(3.14)\qquad \mathcal{T}f &= \int f(w^*, x^{*\prime}, h(w, x, w^*, x^{*\prime}))dw^*dx^{*\prime} \\
&= \int f(w^*, x^{*\prime}, h_0(x, x^{*\prime}) + \Psi(w, x, w^*, x^{*\prime})^2)dw^*dx^{*\prime} \\
&= \int f(W^*(w, x, t, x^{*\prime}), x^{*\prime}, h_0(x, x^{*\prime}) + t^2)\partial_t W^* dtdx^{*\prime},
\end{aligned}
$$

where $W^*(w, z, t, z^{*\prime})$ is a holomorphic function satisfying

$$t = \Psi(w, z, w^*, z^{*\prime})|_{w^*=W^*}, \quad W^*(\mathring{w}, \mathring{x}, 0, \mathring{x}^{*\prime}) = \mathring{w}^*.$$

Then the last integral of (3.14) has a meaning as an integral for microfunctions of $(\operatorname{Re} w, \operatorname{Im} w, x, t, x^{*\prime})$ with respect to $(t, x^{*\prime})$. Further it is clear that this integral becomes a section of $\mathcal{CO}_Z$ as a microfunction of $(\operatorname{Re} w, \operatorname{Im} w, x)$. Then, by the well-known method in [13], we can show the following: There exists a quantization $\mathcal{S}$ of S such that this integral transformation $\mathcal{T}$ induces a desired sheaf isomorphism $\mathcal{T}_S : S^{-1}\mathcal{CO}_Z \xrightarrow{\sim} \mathcal{CO}_Z$ for $\mathcal{S}$. □

We give here the precise meaning concerning boundary values of sections of $\mathcal{CO}_Z$. Let K be a real analytic submanifold of T_Z^*X with codimension 2, and H be a real analytic hypersurface in T_Z^*X passing through K given as follows:

$$
\begin{aligned}
&K = \{(w, x; i\eta) \in T_Z^*X; w = \psi(x, \eta)\} \\
&\subset H = \{(u + iv, x; i\eta) \in T_Z^*X; \Phi(u, v, x, \eta) = 0\}.
\end{aligned}
$$

Here $\psi(x, \eta)$ is a complex valued analytic function of (x, η) with homogeneous degree 0 with respect to η, and $\Phi(u, v, x, \eta)$ is a real-valued analytic function of (u, v, x, η) of homogeneous degree 0 with respect to η satisfying the following:

$$\nabla\Phi \neq 0 \ \text{ on } \Phi = 0, \quad \Phi \circ \psi = 0.$$

By Lemma 2.5, we can choose a holomorphic contact transformation S defined in a neighborhood $\mathring{p} \in K$ such that

$$
(3.15)\qquad \begin{cases} S(K) = \{w^* = 0\} \cap T_Z^*X & \subset\ S(H) = \{\operatorname{Im} w^* = 0\} \cap T_Z^*X, \\ \qquad\qquad S(T_Z^*X) & \subset\ T_Z^*X. \end{cases}
$$

Set $\sigma =$ the signature of $S^*(d\operatorname{Im} w^*)/d\Phi$, where $S^*(\omega)$ denotes the pull-back of a differential form ω by S. We denote by $\pi : T^*X \to X$ the canonical projection, and by $\mathcal{BO}_Z = \mathcal{CO}_Z|_Z$ the sheaf on Z of hyperfunctions with a holomorphic parameter w.

Definition 3.9. Let $\mathring{p} = (\mathring{w}, \mathring{x}; i\mathring{\eta})$ be a point of K, and $f(w,x)$ be a section of $\mathcal{CO}_Z$ on $\{\Phi > 0\} \cap U$ with an $\mathbb{R}$-conic neighborhood $U \subset T^*_Z X$ of $\mathring{p}$. Then, $f(w,x)$ is said to have a boundary value at $\mathring{p}$ from $\Phi > 0$ if there exist a small neighborhood U' of $\mathring{p}$ and a section $F(w^*, x^*) \in \Gamma(\{\sigma \operatorname{Im} w^* > 0\} \cap \pi(S(U')); \mathcal{BO}_Z)$ satisfying

$$(\mathcal{T}_{\mathcal{S}}^{-1} f)(w^*, x^*) = [F(w^*, x^*)]$$

as sections of $\Gamma(\{\sigma \operatorname{Im} w^* > 0\} \cap S(U'); \mathcal{CO}_Z)$. Here $\mathcal{T}_{\mathcal{S}}$ is a quantization of S introduced in Theorem 3.7.

Though the boundary value $[F(u^* + i\sigma 0, x^*)]$ itself depends on a choice of $\mathcal{T}_{\mathcal{S}}$, this definition neither depends on a choice of S nor $\mathcal{T}_{\mathcal{S}}$ (shown as below).

Remark 3.10. A germ of $\mathcal{CO}_Z$ is represented by a germ of $\mathcal{BO}_Z$. However it is well-known that a section of $\mathcal{CO}_Z$ cannot be represented globally by a section of $\mathcal{BO}_Z$ in general. Indeed, the cohomological boundary value $(\mathcal{T}_{\mathcal{S}}^{-1} f)(u^* + i\sigma 0, x^*)$ defines a second hyperfunction on $\Sigma = \{(w^*, x^*; i\eta^*) \in T^*_Z X; \operatorname{Im} w^* = 0\}$. On the other hand the sheaf $\mathcal{B}^2_\Sigma$ of second hyperfunctions is essentially larger than the sheaf $\mathcal{C}_M|_\Sigma$. Here $M = \{(w,z) \in X; \operatorname{Im} w = 0, \operatorname{Im} z = 0\}$. Hence the definition above is equivalent to the following:

$$(\mathcal{T}_{\mathcal{S}}^{-1} f)(u^* + i\sigma 0, x^*) \in \mathcal{C}_M|_{S(\mathring{p})}.$$

Further this boundary value is equal to $[F(u^* + i\sigma 0, x^*)]$ as a microfunction of (u^*, x^*) at $S(\mathring{p})$. The uniqueness of this boundary value $[F(u^* + i\sigma 0, x^*)] \in \mathcal{C}_M|_{S(\mathring{p})}$ for a section $(\mathcal{T}_{\mathcal{S}}^{-1} f)(w^*, x^*)$ is justified by Schapira's N-regularity property of $\overline{\partial}_w$-operator. We refer to [3, 4] as for the second microlocal analysis, and to [14] as for the N-regularity of $\overline{\partial}_w$-operator. Further as for a self-contained proof of the equivalent fact, see Proposition 4.1.11 of [10].

A holomorphic contact transformation S generated by (3.9) preserves $H = \{\operatorname{Im} w = 0\}$ if and only if the holomorphic function $\Psi(w, z, w^*, z^{*\prime})$ is real-valued on $\{\mathrm{Im}\, w = \mathrm{Im}\, w^* = 0, \mathrm{Im}\, z = 0, \mathrm{Im}\, z^{*\prime} = 0\}$. Hence the explicit formula in (3.14) for $\mathcal{T}$ together with the partial flabbiness of $\mathcal{BO}_Z$ leads to the following lemma, which is also due to Theorem 4.2.17 of [10].

LEMMA 3.11. *Let $H = \{(w, x; i\eta) \in T^*_Z X; \mathrm{Im}\, w = 0\}$, and $S : T^*X \xrightarrow{\sim} T^*X$ be a holomorphic contact transformation defined in a neighborhood of $\mathring{p} \in H$. We assume that S preserves $T^*_Z X, H$ respectively. Let $F_\pm(w^*, x^*)$ be sections of $\mathcal{BO}_Z$ on $\{\pm \mathrm{Im}\, w^* > 0\} \cap \pi(U)$ for a neighborhood U in $T^*_Z X$ of $\mathring{p}$, respectively. Then for any quantization $\mathcal{T}_S : S^{-1}\mathcal{CO}_Z \xrightarrow{\sim} \mathcal{CO}_Z$ of S, each $(\mathcal{T}_S[F_\pm])(w, x)$ has a boundary value at $S^{-1}(\mathring{p})$ from $\pm \mathrm{Im}\, w > 0$. That is, there exist a neighborhood U' of $S^{-1}(\mathring{p})$ and sections $G_\pm(w, x) \in \Gamma(\{\pm \mathrm{Im}\, w > 0\} \cap \pi(U'); \mathcal{BO}_Z)$ such that*

$$(\mathcal{T}_S[F_\pm])(w, x) = [G_\pm(w, x)] \text{ on } \{\pm \mathrm{Im}\, w > 0\} \cap U'.$$

By Lemma 2.5 and this lemma, we can reduce our equation $Pf = 0$ to the case

$$P = wD_w^m + \sum_{k=1}^{m} A_k(w, z, D_w, D_z) D_w^{m-k} \tag{3.16}$$

with some operators $A_k \in \mathcal{E}_X|_{\mathring{p}}$ $(k = 1, \ldots, m)$ of order ≤ 0.

4. Construction of Solutions of Formal Symbol Type

4.1. An iteration scheme

As seen in Section 3 we can assume P has the form (3.16); that is, $A_0(w, z, D_w, D_z) \equiv w$. Write

$$A_k(w, z, D_w, D_z) = \sum_{j=-\infty}^{0} A_{jk}(w, z, D_w, D_z),$$

where $A_{jk}(w, z, \tau, \zeta)$ is the j-th order part of A_k with homogeneous degree j in (τ, ζ) for each k. Hence the ordinary differential operator defined at (1.6) is given by

$$L = w\partial_w^m + \sum_{k=1}^{m} A_k^0(w, z, \zeta)\partial_w^{m-k}.$$

Here $A_k^0(w,z,\zeta) \equiv A_{0,k}(w,z,0,\zeta)$ satisfying

$$A_1^0(0,\overset{\circ}{x},i\overset{\circ}{\eta}) \notin \mathbb{Z}.$$

Definition 4.1. For a formal symbol $U=\sum_{j=-\infty}^{0} U_j(w,z,\zeta)$ of order ≤ 0 at $\overset{\circ}{p}$, we define linear operators $L, \mathcal{L}$ by

$$LU = \sum_{j=-\infty}^{0} (LU_j)(w,z,\zeta), \tag{4.1}$$

$$\mathcal{L}U = \sum_{j=-\infty}^{0} \Bigg(\sum_{\substack{-|\alpha|+q=j \\ 0\leq k\leq m}} \frac{1}{\alpha!} \partial_\zeta^\alpha A_k^0(w,z,\zeta) \partial_w^{m-k} \partial_z^\alpha U_q(w,z,\zeta) \Bigg). \tag{4.2}$$

Here L operates on each holomorphic function U_j as an ordinary differential operator with parameters (z,ζ). It is easy to see that the results of these operations also become formal symbols of type $V=\sum_{j=-\infty}^{0} V_j(w,z,\zeta)$. Indeed, $\mathcal{L}U$ coincides with the operator composition (mod. $\mathcal{E}_X D_w$):

$$\Big(\sum_{k=0}^{m} A_k^0(w,z,D_z) D_w^{m-k}\Big) U(w,z,D_z) \equiv (\mathcal{L}U)(w,z,D_z).$$

Further let R be a microdifferential operator of the form

$$\begin{cases} R(w,z,D_w,D_z) &= \sum_{k=0}^{m} R_k(w,z,D_w,D_z) D_w^{m-k}, \\ R_k(w,z,\tau,\zeta) &= \sum_{j=-\infty}^{-1} R_{jk}(w,z,\tau,\zeta). \end{cases} \tag{4.3}$$

Here each R_k is a formal symbol of order ≤ -1 defined at $\overset{\circ}{p}$. Then we define an operator $R\circ$ by

$$R\circ U = \sum_{j=-\infty}^{-1} \Bigg(\sum_{\substack{-|\alpha|-s+\ell+q=j \\ 0\leq k\leq m}} \frac{(\partial_\tau^s \partial_\zeta^\alpha R_{\ell k})(w,z,0,\zeta)}{s!\alpha!} \partial_w^{m-k+s} \partial_z^\alpha U_q(w,z,\zeta) \Bigg).$$

Indeed $R\circ U$ becomes a formal symbol of type $V=\sum_{j=-\infty}^{-1} V_j(w,z,\zeta)$ of order ≤ -1 satisfying

$$R\circ U \equiv R(w,z,D_w,D_z) U(w,z,D_z) \pmod{\mathcal{E}_X D_w}.$$

Then, our successive approximation process for formal symbols $U_k = \sum_{j=-\infty}^{0} U_{jk}(w,z,\zeta)$ $(k=0,1,2,\dots)$ is formulated as follows:

$$\begin{cases} LU_0 = 0, \\ LU_{k+1} = \{(L-\mathcal{L}) - R\circ\}U_k \quad (k=0,1,2,\dots). \end{cases} \tag{4.4}$$

Indeed, if $\sum_{k=0}^{\infty} U_k$ converges as a formal symbol, the sum $U(w,z,\zeta)$ satisfies the following equation (mod. $\mathcal{E}_X D_w$):

$$\Big(\sum_{k=0}^{m} A_k^0(w,z,D_z)D_w^{m-k} + R(w,z,D_w,D_z)\Big)U(w,z,D_z) \equiv 0. \tag{4.5}$$

Further since we have

$$\operatorname{ord}((\mathcal{L}-L)U) \le \operatorname{ord}(U)-1,\ \operatorname{ord}(R\circ U) \le \operatorname{ord}(U)-1, \tag{4.6}$$

we can choose U_k's satisfying $\operatorname{ord}(U_k) \le -k$ $(\forall k \ge 0)$. That is, the j-degree component of $\sum_{k=0}^{\infty} U_k$ is determined only by $U_0,\dots,U_{|j|}$.

We set

$$R_{jk}^A = \begin{cases} A_{-1,k}(w,z,\tau,\zeta) + \Big(A_{0,k+1}(w,z,\tau,\zeta) - A_{0,k+1}(w,z,0,\zeta)\Big)/\tau \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=-1), \\ A_{jk}(w,z,\tau,\zeta) \qquad (j \le -2). \end{cases} \tag{4.7}$$

Then $R^A = \sum_{k=0}^{m}\sum_{j=-\infty}^{-1} R_{jk}^A(w,z,D_w,D_z)D_w^{m-k}$ is a microdifferential operator at $\mathring{p}$ satisfying the conditions in (4.3). Further if we set $R = R^A$ in our successive approximation process (4.4), the corresponding equation (4.5) is just equal to $P(w,z,D_w,D_z)U(w,z,D_z) \equiv 0$. Consequently our program reduces to a construction of 'convergent series' $\sum_{k=0}^{\infty} U_k$ of formal symbols satisfying (4.4).

4.2. Formal norms

Boutet de Monvel and Krée introduced so-called a formal norm $N(Q;t)$ for a formal symbol Q of analytic pseudo-differential operators [6]. $N(Q;t)$ is a formal power series of a variable t with real non-negative coefficients depending Q. In particular the following properties are the most important for $Q_1, Q_2 \in \mathcal{E}_X$ of order ≤ 0:

$$N(Q_1 + Q_2; t) \ll N(Q_1; t) + N(Q_2; t),$$
$$N(Q_1 Q_2; t) \ll N(Q_1; t) N(Q_2; t),$$

where $F_1(t) \ll F_2(t)$ means that $F_2(t)$ is a majorant series for $F_1(t)$. To show the convergence of our formal symbols, we introduce some variants of formal norms for $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$. We may assume that each $A_k(w, z, D_w, D_z)$ is defined in a neighborhood of

$$D_\nu = \{(w, z; \tau, \zeta) \in T^*X; |w| \leq 1 + \nu, |z - \mathring{x}| \leq \nu, \frac{|\tau|}{|\zeta|} + \Big|\frac{\zeta}{|\zeta|} - \frac{i\mathring{\eta}}{|\mathring{\eta}|}\Big| \leq \nu\}$$

with some small $\nu > 0$.

Definition 4.2 (Regular type). When each component $U_j(w, z, \zeta)$ of $U = \sum_{j=-\infty}^{0} U_j$ is holomorphic in D_ν, we define a formal power series $N_{m'}(U; t)$ in t with parameters z, ζ for each $m' = 0, 1, 2, \ldots$ by

$$N_{m'}(U; t) \equiv \sum_{p,\alpha,\beta,\ell} \frac{p! t^{2p+\ell+|\alpha+\beta|} |\zeta|^{p+|\beta|}}{(p + \ell + |\alpha|)!(p + |\beta|)!} \max_{0 \leq k \leq m'} \|\partial_w^{k+\ell} \partial_z^\alpha \partial_\zeta^\beta U_{-p}\|.$$

Here $\|\cdot\|$ is the sup-norm in $w \in D$ introduced at (2.12). Indeed, if U is a section of $\mathit{\Gamma}(D \times \{(\mathring{x}; i\mathring{\eta})\}; \mathcal{E}_X)$, $N_{m'}(U; t)$ has a convergent majorant series independent of (z, ζ). Conversely, if the formal norm $N_{m'}(U; t)$ for a set of homogeneous holomorphic functions $U_j(w, z, \zeta) \in \mathcal{O}(D_\nu)$ has a convergent majorant series independent of (z, ζ), then $U = \sum_{j-\infty}^{0} U_j(w, z, D_z)$ becomes a section of $\mathit{\Gamma}(D \times \{(\mathring{x}; i\mathring{\eta})\}; \mathcal{E}_X)$.

Definition 4.3 (Non-regular type). When each component $U_j(w, z, \zeta)$ of U is holomorphic in a neighborhood of $\{w \in \Omega\} \cap D_\nu$ with

$$\Omega = \{w \in \mathbb{C}; 0 < |w| \leq 1, |\arg w| \leq \pi - \varepsilon\},$$

we define a formal power series $N^{\mu}_{m'}(U;t)$ in t with parameters z,ζ for each $m'=0,1,2,\dots$ and a positive constant μ by

$$(4.8)\qquad N^{\mu}_{m'}(U;t) \equiv \sum_{p,\alpha,\beta,\ell} \frac{p!t^{2p+\ell+|\alpha+\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!} \times |\zeta|^{p+|\beta|} \max_{0\le k\le m'} \|\partial_w^{k+\ell}\partial_z^{\alpha}\partial_\zeta^{\beta}U_{-p}\|_{\mu+k+\ell+|\alpha+\beta|+p-\kappa(m')}$$

with $\kappa(0)=0, \kappa(m')=m'-1$ $(\forall\, m'\ge 1)$. Here $\|\cdot\|_\mu$ is the sup-norm in $w\in\Omega$ with some weight introduced at (2.13). Further, when each component $U_j(w,z,\zeta)\equiv U_j(z,\zeta)$ is not depending on w, we define

$$(4.9)\qquad K(U;t)\equiv \sum_{p,\alpha,\beta} \frac{p!t^{2p+|\alpha+\beta|}}{(p+|\alpha|)!(p+|\beta|)!}|\zeta|^{p+|\beta|}|\partial_z^{\alpha}\partial_\zeta^{\beta}U_{-p}|.$$

In the approximation process, we need some a priori estimates for $N_m(U_k;t)$ or $N^{\mu}_m(U_k;t)$. In the next subsections we get our main estimates by these formal norms.

4.3. Estimates for L

Let us consider the following equation for formal symbols $U=\sum_{j=-\infty}^{0}U_j(w,z,\zeta), F=\sum_{j=-\infty}^{0}F_j(w,z,\zeta)$:

$$LU=F \iff LU_j=F_j\ (j=0,-1,-2,\dots).$$

We estimate $N_m(U;t)$ by $N_0(F;t)$ and $\sum_{k=0}^{m-2}K(\partial_w^kU(0,z,\zeta);t)$ for regular type formal symbols. Further, we estimate $N^{\mu}_m(U;t)$ by $N^{\mu}_0(F;t)$ and $\sum_{k=0}^{m-1}K(\partial_w^kU(1,z,\zeta);t)$ for non-regular type formal symbols.

To derive such estimates we apply $\partial_w^{\ell}\partial_z^{\alpha}\partial_\zeta^{\beta}$ to both sides of $LU_{-p}=F_{-p}$. Then we obtain

$$\begin{aligned} L(\partial_w^{\ell}\partial_z^{\alpha}\partial_\zeta^{\beta}U_{-p}) &= \partial_w^{\ell}\partial_z^{\alpha}\partial_\zeta^{\beta}F_{-p} \\ &- \sum_{\substack{\ell',\alpha',\beta'\\ \ell'',\alpha'',\beta''}}\sum_{k=0}^{m}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\partial_w^{\ell'}\partial_z^{\alpha'}\partial_\zeta^{\beta'}A_k^0\cdot\partial_w^{\ell''+m-k}\partial_z^{\alpha''}\partial_\zeta^{\beta''}U_{-p} \\ &(\ell=\ell'+\ell'',\alpha=\alpha'+\alpha'',\beta=\beta'+\beta'',(\ell',\alpha',\beta')\neq 0). \end{aligned}$$

Here we employ Lemma 2.8. For a sufficiently small $\nu > 0$ we set

$$M_\nu = 1 + \sup_{(w,z;0,\zeta)\in D_\nu} \sum_{k=1}^{m} |A_k^0(w,z,\zeta)| < +\infty$$

and

$$\delta_\nu = \inf\{|p + A_1^0(0,z,\zeta)|; p = 0,1,2,\dots,(z;\zeta) \in V_\nu\} > 0$$

with

$$V_\nu = \{(z;\zeta) \in \mathbb{C}^n \times \mathbb{C}^n; |z - \mathring{x}| \le \nu, \big|\zeta/|\zeta| - i\mathring{\eta}/|\mathring{\eta}|\big| \le \nu\}.$$

Then there exists a positive constant C_0 depending only on M_ν and δ_ν, which satisfies the estimates (2.17), (2.18) for

$$L = \sum_{k=0}^{m} A_k^0(w,z,\zeta)\partial_w^{m-k}.$$

In particular we have the following estimates:

$$|\partial_w^\ell \partial_z^\alpha \partial_\zeta^\beta A_k^0(w,z,\zeta)| \le \ell!\alpha!\beta!(2n/\nu)^{\ell+|\alpha|+|\beta|}|\zeta|^{-|\beta|}M_\nu$$

for $|w| \le 1, (z,\zeta) \in V_{\nu/2}$. Hereafter we fix a $(z,\zeta) \in V_{\nu/2}$ and set

$$C_1 = \max\{M_\nu, 2n/\nu\}.$$

(1) Regular type case: Firstly for $m' = m$ we consider

$$\begin{aligned}
&\max_{0\le k\le m} \|\partial_w^{k+\ell}\partial_z^\alpha\partial_\zeta^\beta U_{-p}\| \\
&\le C_0\biggl(\|\partial_w^\ell\partial_z^\alpha\partial_\zeta^\beta F_{-p}\| + \sum_{k=0}^{m-2} |\partial_w^{k+\ell}\partial_z^\alpha\partial_\zeta^\beta U_{-p}(0,z,\zeta)| \\
&+ \sum_{(\ell',\alpha',\beta')\neq 0} \frac{(m+1)\ell!\alpha!\beta! C_1^{\ell'+|\alpha'|+|\beta'|+1}|\zeta|^{-|\beta'|}}{\ell''!\alpha''!\beta''!} \max_{0\le k\le m} \|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''} U_{-p}\|\biggr).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
N_m(U;t) &\ll C_0\Big\{N_0(F;t)+\sum_{k=0}^{m-2}K(\partial_w^k U(0,z,\zeta);t)\\
&+\sum_{\substack{p,\alpha,\beta\\ \ell>0}}\frac{p!t^{2p+\ell+|\alpha+\beta|}|\zeta|^{p+|\beta|}(m-1)}{(p+\ell+|\alpha|)!(p+|\beta|)!}\max_{0\le k\le m-2}\left|\partial_w^{k+\ell}\partial_z^{\alpha}\partial_\zeta^{\beta}U_{-p}|_{w=0}\right|\\
&+\sum_{\substack{p,\ell,\alpha,\beta\\ (\ell',\alpha',\beta')\neq 0}}\frac{p!t^{2p+\ell+|\alpha+\beta|}|\zeta|^{p+|\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!}\\
&\times\frac{(m+1)\ell!\alpha!\beta!C_1^{\ell'+|\alpha'|+|\beta'|+1}|\zeta|^{-|\beta'|}}{\ell''!\alpha''!\beta''!}\max_{0\le k\le m}\|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''}U_{-p}\|\Big\}\\
&\ll C_0\Big\{N_0(F;t)+\sum_{k=0}^{m-2}K(\partial_w^k U(0,z,\zeta);t)+(m-1)t\cdot N_m(U;t)\\
&+\sum_{\substack{p,\ell,\alpha,\beta\\ (\ell',\alpha',\beta')\neq 0}}\frac{(m+1)C_1\ell!\alpha!\beta!(p+\ell''+|\alpha''|)!(p+|\beta''|)!}{\ell''!\alpha''!\beta''!(p+\ell+|\alpha|)!(p+|\beta|)!}(C_1t)^{\ell'+|\alpha'+\beta'|}\\
&\times\frac{p!t^{2p+\ell''+|\alpha''+\beta''|}|\zeta|^{p+|\beta''|}}{(p+\ell''+|\alpha''|)!(p+|\beta''|)!}\max_{0\le k\le m}\|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''}U_{-p}\|\Big\}.
\end{aligned}$$

We recall here an inequality for $r\ge r'\ge 0, s\ge s'\ge 0$:

$$\binom{r}{r'}\binom{s}{s'}\le\binom{r+s}{r'+s'}. \tag{4.10}$$

Therefore

$$\begin{aligned}
&\frac{\ell!\alpha!\beta!(p+\ell''+|\alpha''|)!(p+|\beta''|)!}{\ell''!\alpha''!\beta''!(p+\ell+|\alpha|)!(p+|\beta|)!}\\
&=\frac{\ell'!\alpha'!\beta'!}{(\ell'+|\alpha'|)!|\beta'|!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\binom{p+\ell+|\alpha|}{\ell'+|\alpha'|}^{-1}\binom{p+|\beta|}{|\beta'|}^{-1}\le 1.
\end{aligned}$$

Hence we have

$$N_m(U;t) \ll C_0\Big\{ N_0(F;t) + \sum_{k=0}^{m-2} K(\partial_w^k U(0,z,\zeta);t)$$
$$+ (m-1)tN_m(U;t) + (m+1)C_1 N_m(U;t) \sum_{(l',\alpha',\beta')\neq 0} (C_1 t)^{\ell'+|\alpha'+\beta'|}\Big\}.$$

Set

$$\psi(t) \equiv \sum_{(\ell',\alpha',\beta')\neq 0} (C_1 t)^{\ell'+|\alpha'+\beta'|},$$
$$\Phi(t) \equiv \frac{C_0}{1-(m+1)\{C_0C_1\psi(t)+C_0 t\}}.$$

Since $\psi(0)=0$, we get the following proposition:

Proposition 4.4. *If each component of F and U is holomorphic on a neighborhood of $\{|w|\leq 1\}$ with respect to w, we have*

$$N_m(U;t) \ll \Phi(t)\Big\{ N_0(F;t) + \sum_{k=0}^{m-2} K(\partial_w^k U(0,z,\zeta);t)\Big\}.$$

Here $\Phi(t)$ is a convergent power series of t with non-negative coefficients independent of F, U.

(2) Non-regular type case: For $m' = m \geq 1$ and $\mu \geq M_\nu + m + 1$, we obtain

$$\max_{0\leq k\leq m} \|\partial_w^{k+\ell}\partial_z^\alpha\partial_\zeta^\beta U_{-p}\|_{\mu+k+\ell+p+|\alpha+\beta|-m+1}$$
$$\leq C_0\Big\{ \|\partial_w^\ell\partial_z^\alpha\partial_\zeta^\beta F_{-p}\|_{\mu+\ell+|\alpha+\beta|+p} + \sum_{k=0}^{m-1} |\partial_w^{k+\ell}\partial_z^\alpha\partial_\zeta^\beta U_{-p}(1,z,\zeta)|$$
$$+ \sum_{(\ell',\alpha',\beta')\neq 0} \frac{(m+1)\ell!\alpha!\beta! C_1^{\ell'+|\alpha'|+|\beta'|+1}|\zeta|^{-|\beta'|}}{\ell''!\alpha''!\beta''!}$$
$$\times \max_{0\leq k\leq m} \|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''} U_{-p}\|_{\mu+\ell+|\alpha+\beta|+p}\Big\}.$$

Note that $\mu+\ell+|\alpha+\beta|+p \ge \mu+k+\ell''+|\alpha''+\beta''|+p-m+1$ in the last term because $\ell'+\alpha'+\beta' \ge 1$. Hence we obtain

$$\begin{aligned}&\max_{0\le k\le m} \|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''}U_{-p}\|_{\mu+\ell+|\alpha+\beta|+p}\\ &\le \max_{0\le k\le m} \|\partial_w^{k+\ell''}\partial_z^{\alpha''}\partial_\zeta^{\beta''}U_{-p}\|_{\mu+k+\ell''+|\alpha''+\beta''|+p-m+1}.\end{aligned}$$

In the same way as the regular type case (replace $\max_{0\le k\le m}\|\cdot\|$ by the norms above), we obtain the following proposition:

PROPOSITION 4.5. *If each component of F and U is holomorphic on a neighborhood of Ω with respect to w, for $\forall\mu \ge M_\nu+m+1$ we have*

$$N_m^\mu(U;t) \ll \Phi(t)\biggl\{N_0^\mu(F;t)+\sum_{k=0}^{m-1}K(\partial_w^k U(1,z,\zeta);t)\biggr\}.$$

Here $\Phi(t)$ is a convergent power series of t with non-negative coefficients independent of F,U.

4.4. Estimates for $\mathcal{L}-L$

We estimate $N_0((\mathcal{L}-L)U;t)$ by $N_m(U;t)$ and $N_0^\mu((\mathcal{L}-L)U;t)$ by $N_m^\mu(U;t)$ respectively.

PROPOSITION 4.6. *Set a convergent power series of t with positive coefficients with value 0 at $t=0$:*

$$\psi_1(t) \equiv mC_1\sum_{\ell'=0}^{\infty}(C_1t)^{\ell'}\sum_{\alpha'\ge 0}(C_1t)^{|\alpha'|}\sum_{\beta'\ge 0}(C_1t)^{|\beta'|}\sum_{|\gamma|\ge 1}(C_1t)^{|\gamma|}.$$

(1) *Regular type case: If each component of U is holomorphic on a neighborhood of $\{|w|\le 1\}$ with respect to w, we have*

$$N_0((\mathcal{L}-L)U;t) \ll \psi_1(t)N_m(U;t). \tag{4.11}$$

(2) *Non-regular type case: If each component of U is holomorphic on a neighborhood of Ω with respect to w, for $\forall\mu\ge 0$ we have*

$$N_0^\mu((\mathcal{L}-L)U;t) \ll \psi_1(t)N_m^\mu(U;t). \tag{4.12}$$

PROOF. (1) Regular type case:

$$
\begin{aligned}
N_0((\mathcal{L}-L)U;t) &= \sum_{p,\alpha,\beta,\ell} \frac{p!t^{2p+\ell+|\alpha+\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!}|\zeta|^{p+|\beta|} \\
&\times \left\| \partial_w^\ell \partial_z^\alpha \partial_\zeta^\beta \Bigl(\sum_{p=|\gamma|+q,|\gamma|>0,1\le k\le m} \frac{1}{\gamma!}\partial_\zeta^\gamma A_k^0 \cdot \partial_w^{m-k}\partial_z^\gamma U_{-q} \Bigr) \right\| \\
&\ll \sum_{\ell',\ell'',\alpha',\alpha'',\beta',\beta'',q,|\gamma|>0} \frac{p!t^{2p+\ell+|\alpha+\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!} \frac{1}{\gamma!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'} m \\
&\times \ell'!\alpha'!(\beta'+\gamma)!C_1^{|\alpha'+\beta'+\gamma|+\ell'+1}|\zeta|^{q+|\beta''|} \max_{0\le k\le m-1} \|\partial_w^{k+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\| \\
&\ll \sum_{\ell',\ell'',\alpha',\alpha'',\beta',\beta'',q,|\gamma|>0,\alpha''^*=\alpha''+\gamma} (*) mC_1(C_1t)^{\ell'+|\alpha'+\beta'+\gamma|} \\
&\times \frac{q!t^{2q+\ell''+|\alpha''^*+\beta''|}|\zeta|^{q+|\beta''|}}{(q+\ell''+|\alpha''^*|)!(q+|\beta''|)!} \max_{0\le k\le m} \|\partial_w^{k+\ell''}\partial_z^{\alpha''^*}\partial_\zeta^{\beta''}U_{-q}\|.
\end{aligned}
$$

Here $(*)$ is given by

$$
\begin{aligned}
&\frac{p!(p+\ell''+|\alpha''|)!(q+|\beta''|)!\ell!\alpha!\beta!(\beta'+\gamma)!}{(p+\ell+|\alpha|)!(p+|\beta|)!\gamma!q!\ell''!\alpha''!\beta'!\beta''!} \\
&= \frac{\ell'!\alpha'!\beta'!}{(\ell'+|\alpha'|)!|\beta'|!}\binom{\beta'+\gamma}{\beta'}\binom{p}{|\gamma|}\binom{\beta}{\beta'}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'} \\
&\times \binom{|\beta'+\gamma|}{|\beta'|}^{-1}\binom{p+|\beta|}{|\gamma+\beta'|}^{-1}\binom{p+\ell+|\alpha|}{\ell'+|\alpha'|}^{-1} \le 1,
\end{aligned}
$$

where we used the inequality (4.10). Therefore, we obtain the estimate (4.11) for the regular type case.
(2) Non-regular type case: In the same way as above, we have

$$
\begin{aligned}
N_0^\mu((\mathcal{L}-L)U;t) &\ll \sum_{\ell',\ell'',\alpha',\alpha'',\beta',\beta'',q,|\gamma|>0} \frac{p!t^{2p+\ell+|\alpha+\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!} \\
&\times \frac{1}{\gamma!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'} m\ell'!\alpha'!(\beta'+\gamma)!C_1^{|\alpha'+\beta'+\gamma|+\ell'+1} \\
&\times |\zeta|^{q+|\beta''|} \max_{0\le k\le m-1} \|\partial_w^{k+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\|_{\mu+p+\ell+|\alpha+\beta|}.
\end{aligned}
$$

Since $\mu+p+\ell+|\alpha+\beta| \geq \mu+k+\ell''+|\alpha''+\gamma|+|\beta''|+q-m+1$, we have

$$\max_{0\leq k\leq m-1} \|\cdot\|_{\mu+p+\ell+|\alpha+\beta|} \leq \max_{0\leq k\leq m} \|\cdot\|_{\mu+k+\ell''+|\alpha''+\gamma|+|\beta''|+q-(m-1)}.$$

Therefore the same argument as in the regular type case leads to the conclusion (4.12). □

4.5. Estimates for $R\circ *$

We estimate $N_0(R\circ U;t)$, $N_0^{\mu}(R\circ U;t)$ by $N_m(U;t)$, $N_m^{\mu}(U;t)$ respectively. Here $R=\sum_{k=0}^{m} R_k(w,z,D_w,D_z)D_w^{m-k}$ with $\mathrm{ord}(R_k)\leq -1 (k=0,1,\ldots,m)$ is supposed to be a microdifferential operator defined in D_ν for some $\nu>0$. Therefore we have a constant $C_2>0$ satisfying the following estimates for each j-th degree component R_{jk} of R_k on $D_{\nu/2}$:

$$|\partial_w^\ell \partial_z^\alpha \partial_\tau^s \partial_\zeta^\beta R_{-p,k}| \leq C_2^{1+p+\ell+s+|\alpha+\beta|}\ell!\alpha!s!\beta!p!|\zeta|^{-p-s-|\beta|}.$$

PROPOSITION 4.7. *Set a convergent power series of t with positive coefficients with value 0 at $t=0$:*

$$\psi_2(t)\equiv (m+1)\sum_{\substack{\ell',\alpha',\beta',\gamma\geq 0\\ r\geq 1, s\geq 0}} C_2^{1-r}(C_2 t)^{s+2r+\ell'+|\gamma+\alpha'+\beta'|}.$$

(1) *Regular type case: If each component of U is holomorphic on a neighborhood of $\{|w|\leq 1\}$ with respect to w, we have*

$$N_0((R\circ U;t) \ll \psi_2(t)N_m(U;t). \tag{4.13}$$

(2) *Non-regular type case: If each component of U is holomorphic on a neighborhood of Ω with respect to w, for $\forall\mu\geq 0$ we have*

$$N_0^{\mu}(R\circ U;t) \ll \psi_2(t)N_m^{\mu}(U;t). \tag{4.14}$$

PROOF. Firstly we consider the regular type case:

$$N_0(R\circ U;t)\ll \sum_{p,\alpha,\beta,\ell}\frac{p!t^{2p+\ell+|\alpha+\beta|}|\zeta|^{p+|\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}$$

$$\times\sum_{\substack{|\gamma|+s+r+q=p\\ 0\le k\le m}}\left\|\frac{(\partial_w^{\ell'}\partial_z^{\alpha'}\partial_\tau^s\partial_\zeta^{\beta'+\gamma}R_{-r,k})|_{\tau=0}}{s!\gamma!}\partial_w^{m-k+s+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\right\|$$

$$\ll\sum_{\substack{|\gamma|+s+r+q=p\\ r\ge 1,p,\alpha,\beta,\ell,}}\frac{p!t^{2p+\ell+|\alpha+\beta|}|\zeta|^{p+|\beta|}}{(p+\ell+|\alpha|)!(p+|\beta|)!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}$$

$$\times\frac{(C_2^{1+r+s+\ell'+|\alpha'+\beta'+\gamma|}r!\ell'!\alpha'!s!(\beta'+\gamma)!|\zeta|^{-s-r-|\beta'+\gamma|}}{s!\gamma!}$$

$$\times(m+1)\max_{0\le k\le m}\left\|\partial_w^{k+s+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\right\|$$

$$\ll\sum_{\substack{|\gamma|+s+r+q=p\\ r\ge 1,p,\alpha,\beta,\ell}}(*)(m+1)C_2^{1-r}(C_2t)^{s+2r+\ell'+|\gamma+\alpha'+\beta'|}$$

$$\times\frac{q!t^{2q+\ell''^*+|\alpha''^*+\beta''|}|\zeta|^{q+|\beta''|}}{(q+\ell''^*+|\alpha''^*|)!(q+|\beta''|)!}\max_{0\le k\le m}\left\|\partial_w^{k+\ell''^*}\partial_z^{\alpha''^*}\partial_\zeta^{\beta''}U_{-q}\right\|.$$

Here $\ell''^*=\ell''+s,\alpha''^*=\alpha''+\gamma$ and $(*)$ is given by

$$\frac{p!(q+s+\ell''+|\alpha''+\gamma|)!(q+|\beta''|)!r!\ell!\alpha!\beta!(\beta'+\gamma)!}{(p+\ell+|\alpha|)!(p+|\beta|)!q!\gamma!\ell''!\alpha''!\beta'!\beta''!}$$

$$=\frac{r!\ell'!\alpha'!\beta'!}{(r+\ell'+|\alpha'|)!|\beta'|!}\binom{\ell}{\ell'}\binom{\alpha}{\alpha'}\binom{p}{|\gamma|+s+r}\binom{\beta}{\beta'}\binom{\beta'+\gamma}{\beta'}$$

$$\times\binom{p+\ell+|\alpha|}{r+\ell'+|\alpha'|}^{-1}\binom{p+|\beta|}{|\gamma|+s+r+|\beta'|}^{-1}\binom{s+r+|\beta'+\gamma|}{|\beta'|}^{-1}\le 1,$$

where we used the inequality (4.10). Therefore we obtain the estimate (4.13).

Secondly we consider the non-regular type case. The proof for the regular type case is also available in this case if we check the following:

$$\max_{0\le k\le m}\left\|\partial_w^{k+s+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\right\|_{\mu+\ell+|\alpha+\beta|+p}$$

$$\le\max_{0\le k\le m}\left\|\partial_w^{k+s+\ell''}\partial_z^{\alpha''+\gamma}\partial_\zeta^{\beta''}U_{-q}\right\|_{\mu+k+s+\ell''+|\alpha''+\gamma+\beta''|+q-m+1}.$$

Indeed this inequality holds since $(\mu+\ell+|\alpha+\beta|+p)-(\mu+k+s+\ell''+|\alpha''+\gamma+\beta''|+q-m+1)=\ell'+|\alpha'+\beta'|+r-k+m-1\geq 0$ (use $r\geq 1$). This completes the proof. □

4.6. A construction of solutions

We recall our approximation process:

$$\begin{cases} LU_0=0, \\ LU_{k+1}=\{(L-\mathcal{L})-R\circ\}U_k \quad (k=0,1,2,\dots). \end{cases} \tag{4.15}$$

We assume that the coefficients $A_k^0(w,z,\zeta)$ $(k=0,1,\dots,m; A_0^0\equiv w)$ of L and $\mathcal{L}$ are holomorphic functions defined in D_ν of homogeneous degree 0 with respect to ζ for some $\nu>0$. Further we also assume that the microdifferential operator $R=\sum_{k=0}^m R_k(w,z,D_w,D_z)D_w^{m-k}$ with $\mathrm{ord}(R_k)\leq -1$ is defined in D_ν.

THEOREM 4.8. *Let $U_0\equiv U_{00}$ be any holomorphic solution of $LU_0=0$ in $D_\nu(\supset\{|w|\leq 1\}\times\{(\overset{\circ}{x};i\overset{\circ}{\eta})\})$ with homogeneous degree 0 concerning ζ. We can choose a series U_k $(k=1,2,\dots,)$ of solutions of (4.15) such that each $U_k=\sum_{j=-\infty}^0 U_{jk}$ is a formal symbol defined in a neighborhood of $D_{\nu/2}$ satisfying*

$$\partial_w^\ell U_k|_{w=0}=0\ (\ell=0,1,\dots,m-2,\ k\geq 1).$$

Then $U\equiv\sum_{k=0}^\infty U_k$ converges in $N_m(\cdot;t)$-norm uniformly on $\{(z;\zeta)\in V_{\nu/2}\}$, and it gives a solution of

$$\mathcal{L}U+R\circ U=0.$$

Further ord$(U_k)\leq -k$ $(\forall k\geq 0)$. In particular, $\sigma_0(U)=U_0$.

PROOF. As seen at (2.23), under the condition $A_1^0(0,\overset{\circ}{x},i\overset{\circ}{\eta})\neq 0,-1,-2,\cdots$ we can construct U_k successively at least in a neighborhood of $w=0$. Then each U_k extends analytically to $D_{\nu/2}$ because it satisfies a linear ordinary differential equation. Then by Propositons 4.4, 4.6, 4.7 we obtain the following estimates:

$$\begin{aligned}
&N_m(U_{k+1};t)\\
&\ll \Phi(t)\Big\{N_0(R\circ U_k;t)+N_0((\mathcal{L}-L)U_k;t)+\sum_{j=0}^{m-2}K(\partial_w^j U_{k+1}|_{w=0};t)\Big\}\\
&\ll \Phi(t)\{(\psi_1(t)+\psi_2(t))N_m(U_k;t)\}\\
&\ll\cdots\ll\{\Phi(t)(\psi_1(t)+\psi_2(t))\}^{k+1}N_m(U_0;t).
\end{aligned}$$

Therefore $\sum_{k=0}^{\infty}N_m(U_k;t)$ has a convergent majorant series

$$[1-\Phi(t)\{\psi_1(t)+\psi_2(t)\}]^{-1}N_m(U_0;t).$$

This completes the proof. □

THEOREM 4.9. *Let $U_0\equiv U_{00}$ be any holomorphic solution of $LU_0=0$ in $\{w\in\Omega\}\cap D_\nu(\supset\{w\in\Omega\}\times\{(\mathring{x};i\mathring{\eta})\})$ with homogeneous degree 0 concerning ζ. We can choose a series U_k $(k=1,2,\dots,)$ of solutions of (4.15) such that each $U_k=\sum_{j=-\infty}^{0}U_{jk}$ is a formal symbol defined in a neighborhood of $\{w\in\Omega\}\cap D_{\nu/2}$ satisfying*

$$\partial_w^\ell U_k|_{w=1}=0\ (\ell=0,1,\dots,m-1,\ k\geq 1).$$

Then $U\equiv\sum_{k=0}^{\infty}U_k$ converges in $N_m^\mu(\cdot;t)$-norm uniformly on $\{(z;\zeta)\in V_{\nu/2}\}$ for any sufficiently large μ, and it gives a solution of

$$\mathcal{L}U+R\circ U=0.$$

Further ord$(U_k)\leq -k$ $(\forall k\geq 0)$. In particular, $\sigma_0(U)=U_0$.

PROOF. We can construct U_k successively in a neighborhood of $w=1$, then each U_k extends analytically to $\{w\in\Omega\}\cap D_{\nu/2}$. By using Propositions 4.5, 4.6, 4.7 we obtain the following estimates in the same way as above:

$$N_m^\mu(U_{k+1};t)\ll\{\Phi(t)(\psi_1(t)+\psi_2(t))\}^{k+1}N_m^\mu(U_0;t),$$

$$\sum_{k=0}^{\infty}N_m^\mu(U_k;t)\ll[1-\Phi(t)\{\psi_1(t)+\psi_2(t)\}]^{-1}N_m^\mu(U_0;t).$$

This completes the proof. □

5. Boundary Values and the Main Theorems

To state the main theorems, we need some theorem concerning boundary values for operators obtained in Section 4.

The non-regular type solution constructed in Theorem 4.9 is written as

$$U = \sum_{j=-\infty}^{0} U_j(w, z, D_z).$$

If $A_1^0(0, z, \zeta) \not\equiv$ any integer, we can take a singular solution at $w = 0$ of $LU_0 = 0$ as the principal symbol U_0 of U; that is, a solution of the form

$$U_0(w, z, \zeta) = w^{m-1-A_1^0(0,z,\zeta)}\Big(1 + \sum_{\ell=1}^{\infty} c_\ell(z, \zeta) w^\ell\Big),$$

where $A_1^0(0, z, \zeta), c_\ell(z, \zeta)$ $(\ell \geq 1)$ are holomorphic functions of homogeneous degree 0 with respect to ζ. Then the lower order terms also have stronger singularities at $w = 0$ in general. Since $N_m^\mu(U; t)$ has a convergent majorant series, we have the estimates

$$|U_{-p}| \leq C^{p+1} p! |w|^{-\mu-p} |\zeta|^{-p} \ \ (w \in \Omega,\ p \geq 0),$$

for some fixed $\mu, C > 0$. We show that the operators of this type have boundary values on any $\mathbb{R}$-conic and real analytic hypersurface H passing through $w = 0$. To simplify the situation, by using Lemma 2.5 we reduce H to $H = \{\operatorname{Im} w = 0\}$ under some quantized contact transformation preserving $\mathcal{CO}_Z$.

The main idea of the proof is in decomposing the kernel function of U into 2 kernel functions with double phases. We prepare an elementary inequality about integrations of holomorphic functions of w:

LEMMA 5.1. *Let $f(w)$ be a holomorphic function defined in a neighborhood of $G = \{w \in \mathbb{C}; r_0 < \operatorname{Im} w \leq r_1, |\operatorname{Re} w| < r_1\}$ satisfying an estimate*

$$|f(w)| \leq C |\operatorname{Im} w|^{-\mu} \ \ (\forall w \in G)$$

for some constants $C, \mu, r_1 > 0$ and r_0 $(0 \leq r_0 < r_1 \leq 1)$. Choose an positive integer p as $\mu < p \leq \mu + 1$. Then the $(p+1)$-times integration

$$g_p(w) \equiv \int_{ir_1}^{w} \frac{(w - w')^p}{p!} f(w') dw'$$

is holomorphic in a neighborhood of G, continuous up to $\operatorname{Im} w = r_0$, and satisfies

$$|g_p(w)| \le C(2^p+1)/p! \ (\forall w \in G).$$

PROOF.

$$\begin{aligned}
|g_p(w)| &\le \Big| \int_{ir_1}^{u+ir_1} \frac{(w-w')^p}{p!} f(w')dw' \Big| + \Big| \int_{u+ir_1}^{u+iv} \frac{(w-w')^p}{p!} f(w')dw' \Big| \\
&\le r_1 \frac{(2r_1)^p}{p!} C r_1^{-\mu} + \int_v^{r_1} \frac{|v'-v|^p}{p!} C|v'|^{-\mu} dv' \\
&\le \frac{(2r_1)^p}{p!} C r_1^{1-\mu} + \int_v^{r_1} \frac{C}{p!} |v'|^{p-\mu} dv' \le C \frac{(2^p+1)}{p!}. \ \square
\end{aligned}$$

THEOREM 5.2. *Let $U = \sum_{j=-\infty}^{0} U_j(w, z, \zeta)$ be a classical formal symbol of a pseudo-differential operator with order ≤ 0 defined in an $\mathbb{R}$-conic open set*

$$W_r \equiv \Big\{ (w, z; *, \zeta) \in T^*X; \operatorname{Im} w > 0, |w| < r, |z| < \kappa, \quad |\zeta_j| < \rho|\zeta_n| \ (1 \le \forall j \le n-1), |\operatorname{Re} \zeta_n| < \delta \operatorname{Im} \zeta_n \Big\} \tag{5.1}$$

for some $r, \kappa, \rho, \delta > 0 \, (\delta < 1)$. We suppose that $U_j \in \mathcal{O}(W_r)$ $(\forall j \le 0)$ and that there exists some constants $C, \mu > 0$ satisfying the following inequalities:

$$|U_{-p}(w, z, \zeta)| \le C^{p+1} p! |\operatorname{Im} w|^{-p-\mu} |\zeta|^{-p} \ \text{ on } W_r \ (\forall p \ge 0). \tag{5.2}$$

Then, for a sufficiently large number

$$\lambda > \max\{1, 2560C/r\} \tag{5.3}$$

we have 2 holomorphic functions $E^{(k)}(w, z, z - z^*, s)$ $(k = 1, 2)$ defined in

$$\mathcal{W}^{(1)} \equiv \Big\{ (w, z, z-z^*, s) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}; |w| < r/40, \quad \max\{0, -80 \operatorname{Im} w/\lambda r\} < \operatorname{Im} s \le \lambda^{-1}, |\operatorname{Re} s| < \lambda^{-1}, \quad |z| < \kappa, \ |z_j - z_j^*| > \rho^{-1} |z_n - z_n^*| \ (j = 1, \dots, n-1) \Big\}, \tag{5.4}$$

and

$$\mathcal{W}^{(2)} \equiv \Big\{(w, z, z - z^*, s) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}; |\operatorname{Re} s| < \lambda^{-1}, \\ |w - ir/80| < r/320, \quad -(320\lambda)^{-1} < \operatorname{Im} s \leq \lambda^{-1}, \\ |z| < \kappa, \ |z_j - z_j^*| > \rho^{-1}|z_n - z_n^*| \ (j = 1, \ldots, n-1)\Big\}. \tag{5.5}$$

respectively satisfying the following:

$$\sum_{k=1}^{2} E^{(k)}(w, z, z - z^*, s) = \sum_{|\alpha'| \geq 0, p \geq 0} \frac{\alpha'!}{|\alpha'|!} \Big(\prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j + 1}} \Big) \\ \times \int_{i/\lambda}^{s} \frac{(s - s^*)^{p+\nu+3}}{(p + \nu + 3)!} ds^* \int_{\lambda}^{\infty} U_{-p,\alpha'}(w, z, it) \cdot (it)^{p-2} e^{its^*} \frac{dt}{2\pi} \tag{5.6}$$

on $\mathcal{W}^{(1)} \cap \mathcal{W}^{(2)} \equiv \mathcal{W}^{(3)}$*:*

$$\mathcal{W}^{(3)} = \Big\{(w, z, z - z^*, s) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}; \\ 0 < \operatorname{Im} s \leq \lambda^{-1}, \ |\operatorname{Re} s| < \lambda^{-1}, \ |w - ir/80| < r/320, \\ |z| < \kappa, \ |z_j - z_j^*| > \rho^{-1}|z_n - z_n^*| \ (j = 1, \ldots, n-1)\Big\}. \tag{5.7}$$

Here we expand each U_{-p} *in* W_r *as follows:*

$$U_{-p}(w, z, \zeta) = \sum_{\alpha' \geq 0} U_{-p,\alpha'}(w, z, \zeta_n)(\zeta'/\zeta_n)^{\alpha'},$$

with $\zeta'/\zeta_n = (\zeta_1/\zeta_n, \ldots, \zeta_{n-1}/\zeta_n)$. *Further* ν *is some positive integer.*

PROOF. Each $U_{-p,\alpha'}(w, z, \zeta_n)$ is holomorphic in $W_r^{(1)} \equiv$

$\{(w, z; \zeta_n) \in \mathbb{C}^{n+1} \times \mathbb{C}; \operatorname{Im} w > 0, |w| < r, |z| < \kappa, |\operatorname{Re} \zeta_n| < \delta \operatorname{Im} \zeta_n\}$

and satisfies

$$|U_{-p,\alpha'}(w, z, \zeta_n)| \leq p! C^{p+1} \rho^{-|\alpha'|} |\operatorname{Im} w|^{-p-\mu} |\zeta_n|^{-p} \tag{5.8}$$

on $W_r^{(1)}$. By the preceding lemma, we get holomorphic functions $V_{p,\alpha'}(w, z, \zeta_n) \in \mathcal{O}(W_{r/2}^{(1)})$ satisfying

$$\begin{cases} \partial_w^{p+\nu+1} V_{p,\alpha'}(w, z, \zeta_n) = U_{-p,\alpha'}(w, z, \zeta_n), \\ |V_{p,\alpha'}(w, z, \zeta_n)| \leq C^{p+1} 2^{p+\nu+1} \rho^{-|\alpha'|} |\zeta_n|^{-p} \end{cases} \tag{5.9}$$

on $W^{(1)}_{r/2}$ for all p, α'. Here ν is the integer satisfying $\mu < \nu \leq \mu + 1$. Take a conformal mapping $\tilde{w} = \varphi(w)$:

$$\varphi : \{w \in \mathbb{C}; \operatorname{Im} w > 0, |w| < r/2\} \xrightarrow{\sim} \{\tilde{w} \in \mathbb{C}; |\tilde{w}| < 1\}$$

such that $\varphi(0) = 1$; for example,

$$\begin{aligned} \varphi(w) &= \Big\{ i \Big(\frac{r + 2w}{r - 2w} \Big)^2 + 1 \Big\} \Big/ \Big\{ \Big(\frac{r + 2w}{r - 2w} \Big)^2 + i \Big\} \\ &= \Phi\Big(- 2i(\sqrt{2} - 1)w/r \Big) \Phi\Big(2i(\sqrt{2} + 1)w/r \Big), \end{aligned} \tag{5.10}$$

where $\Phi(t) = (1 + t)(1 - t)^{-1}$. Then by expanding $V_{p,\alpha'}(\varphi^{-1}(\tilde{w}), z, \zeta_n)$ into a power series of $\tilde{w}$, we have expansions

$$V_{p,\alpha'}(w, z, \zeta_n) = \sum_{\ell=0}^{\infty} V_{p,\alpha',\ell}(z, \zeta_n) \varphi(w)^{\ell} \quad (\forall (w, z, \zeta_n) \in W^{(1)}_{r/2}). \tag{5.11}$$

Here $V_{p,\alpha',\ell}(z, \zeta_n)$'s are holomorphic functions in

$$W^{(2)} \equiv \{\zeta_n \in \mathbb{C}; |z| < \kappa, |\operatorname{Re} \zeta_n| < \delta \operatorname{Im} \zeta_n\}$$

satisfying

$$|V_{p,\alpha',\ell}(z, \zeta_n)| \leq 2^{\nu} (2C)^{p+1} \rho^{-|\alpha'|} |\zeta_n|^{-p}$$

on $W^{(2)}$ for all p, α', ℓ. Therefore we have

$$U_{-p} = \sum_{\alpha', \ell} \partial_w^{p+\nu+1} \{\varphi(w)^{\ell}\} \cdot (\zeta'/\zeta_n)^{\alpha'} V_{p,\alpha',\ell}(z, \zeta_n). \tag{5.12}$$

Now for a large positive constant $\lambda > 1$ we introduce 2 kernel functions for $V_{p,\alpha',\ell}(z, \zeta_n) \zeta_n^{p-2}$:

$$A^{(1)}_{p,\alpha',\ell}(z, s) \equiv \int_{\lambda(\ell+1)}^{\infty} V_{p,\alpha',\ell}(z, it) \cdot (it)^{p-2} e^{its} \frac{dt}{2\pi}, \tag{5.13}$$

$$A^{(2)}_{p,\alpha',\ell}(z, s) \equiv \int_{\lambda}^{\lambda(\ell+1)} V_{p,\alpha',\ell}(z, it) \cdot (it)^{p-2} e^{its} \frac{dt}{2\pi}, \tag{5.14}$$

which are holomorphic functions defined in

$$W^{(3)} \equiv \{s \in \mathbb{C}; |z| < \kappa, \operatorname{Im} s > -\delta |\operatorname{Re} s| \}$$

with the estimates:

$$|A^{(1)}_{p,\alpha',\ell}(z,s)| \le 2^{\nu}(2C)^{p+1}\rho^{-|\alpha'|}e^{-\lambda(\ell+1)\operatorname{Im}s}/\pi \tag{5.15}$$

on $W^{(3)}$ for all p, α', ℓ. Further $A^{(2)}_{p,\alpha',\ell}(z,s)$ are entire functions satisfying

$$|A^{(2)}_{p,\alpha',\ell}(z,s)| \le 2^{\nu}(2C)^{p+1}\rho^{-|\alpha'|}e^{\lambda(\ell+1)(-\operatorname{Im}s)_+}/\pi \tag{5.16}$$

on $\{s \in \mathbb{C}\}$ for all p, α', ℓ. Here $(t)_+ = t$ $(\forall t \ge 0)$, $= 0$ $(\forall t < 0)$. Therefore for each $k = 1, 2$ and each p, α' the series

$$A^{(k)}_{p,\alpha'}(w,z,s) \equiv \sum_{\ell=0}^{\infty} A^{(k)}_{p,\alpha',\ell}(z,s)\varphi(w)^{\ell} \tag{5.17}$$

converges locally uniformly in

$$\begin{cases} \{|\varphi(w)| < e^{\lambda \operatorname{Im}s}\} & (\forall s \in W^{(3)}) \quad \text{for } k = 1, \\ \{|\varphi(w)|e^{\lambda(-\operatorname{Im}s)_+} < 1\} & (\forall s \in \mathbb{C}) \quad \text{for } k = 2. \end{cases} \tag{5.18}$$

We note that $\varphi(w)$ at (5.10) is holomorphic in $|w| < (\sqrt{2}-1)r/2$ and

$$\begin{aligned} \log|\Phi(t)| &= \frac{1}{2}\log\Big(\Phi\Big(\frac{2\operatorname{Re}t}{1+|t|^2}\Big)\Big) = \sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\Big(\frac{2\operatorname{Re}t}{1+|t|^2}\Big)^{2\ell+1} \\ &\le 4(\operatorname{Re}t)_+ - (-\operatorname{Re}t)_+ \end{aligned} \tag{5.19}$$

for $\forall t$ $(|t| \le 1/4)$. Consequently if $|w| < (\sqrt{2}-1)r/8$, we have

$$\log|\varphi(w)| \le (6\sqrt{2}+10)(-\operatorname{Im}w)_+/r - (10-6\sqrt{2})(\operatorname{Im}w)_+/r.$$

Hence $A^{(1)}_{p,\alpha'}(w,z,s)$'s are holomorphic in

$$\begin{aligned} &\{(w,z,s) \in \mathbb{C} \times W^{(3)}; |w| < (\sqrt{2}-1)r/8, |z| < \kappa, \\ &\qquad (10+6\sqrt{2})(-\operatorname{Im}w)_+ < \lambda r \operatorname{Im}s/2\} \\ \supset &\{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w| < r/20, |z| < \kappa, \\ &\qquad (-\operatorname{Im}w)_+ < \frac{\lambda r}{40}\operatorname{Im}s\} \equiv W^{(4)}_{\lambda} \end{aligned} \tag{5.20}$$

because $|\varphi(w)|e^{-\lambda \operatorname{Im} s} \leq e^{-\lambda \operatorname{Im} s/2} < 1$ on $W_\lambda^{(4)}$. At the same time we have the following estimates:

$$|A_{p,\alpha'}^{(1)}(w,z,s)| \leq \frac{2^\nu (2C)^{p+1}}{\pi\rho^{|\alpha'|}(1-e^{-\lambda \operatorname{Im} s/2})} \leq \frac{e2^{\nu+1}(2C)^{p+1}}{\pi\lambda\rho^{|\alpha'|}\operatorname{Im} s}$$

on $W_\lambda^{(4)} \cap \{0 < \operatorname{Im} s \leq 1/\lambda\}$. Consequently we obtain

$$|\partial_w^{p+\nu+1} A_{p,\alpha'}^{(1)}(w,z,s)| \leq (p+\nu+1)! \frac{re2^{\nu+1}(2C)^{p+1}}{80\pi\rho^{|\alpha'|}} \Big(\frac{80}{\lambda r \operatorname{Im} s}\Big)^{p+\nu+2}$$

on

$$W_\lambda^{(5)} \equiv \{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w| < r/40, |z| < \kappa, \max\{0, -80 \operatorname{Im} w/(\lambda r)\} < \operatorname{Im} s \leq 1/\lambda, |\operatorname{Re} s| < \lambda^{-1}\}. \tag{5.21}$$

Further $A_{p,\alpha'}^{(2)}(w,z,s)$'s are holomorphic in

$$\begin{aligned}&\{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w| < (\sqrt{2}-1)r/8, |z| < \kappa, \\ &\qquad \lambda(-\operatorname{Im} s)_+ - (10-6\sqrt{2}) \operatorname{Im} w/(2r) < 0\} \\ &\supset \{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w| < r/20, |z| < \kappa, \\ &\qquad \operatorname{Im} w > 0, -\operatorname{Im} w/(2\lambda r) < \operatorname{Im} s\} \equiv W_\lambda^{(6)}\end{aligned} \tag{5.22}$$

and satisfy the following estimates

$$|A_{p,\alpha'}^{(2)}(w,z,s)| \leq \frac{2^\nu (2C)^{p+1}}{\pi\rho^{|\alpha'|}(1-e^{-\operatorname{Im} w/(2r)})} \leq \frac{re2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}\operatorname{Im} w}$$

on $W_\lambda^{(6)}$ because $|\varphi(w)|e^{\lambda(-\operatorname{Im} s)_+} \leq e^{-\operatorname{Im} w/(2r)} < 1$ on $W_\lambda^{(6)}$. Consequently, setting

$$W_\lambda^{(6)} \supset W_\lambda^{(7)} \equiv \{(w,z,s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |z| < \kappa, |w - ir/80| < r/320, \ \operatorname{Im} s > -(320\lambda)^{-1}\}, \tag{5.23}$$

we get the following estimates:

$$|\partial_w^{p+\nu+1} A_{p,\alpha'}^{(2)}(w,z,s)| \leq (p+\nu+1)! \frac{re2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}(r/320)^{p+\nu+2}} \tag{5.24}$$

on $W_\lambda^{(7)}$. Fixing the initial point $s = i/\lambda$, we apply again the preceding lemma to $\partial_w^{p+\nu+1} A_{p,\alpha'}^{(k)}(w,z,s)$ for $k = 1,2$. That is, we have holomorphic functions $E_{p,\alpha'}^{(1)}(w,z,s) \in \mathcal{O}(W_\lambda^{(5)})$ satisfying

$$(5.25) \qquad \begin{cases} \partial_s^{p+\nu+4} E_{p,\alpha'}^{(1)}(w,z,s) = \partial_w^{p+\nu+1} A_{p,\alpha'}^{(1)}(w,z,s), \\ |E_{p,\alpha'}^{(1)}(w,z,s)| \le \dfrac{2^{p+\nu+4} r e 2^{\nu+1}(2C)^{p+1}}{80\pi\rho^{|\alpha'|}} \Big(\dfrac{80}{\lambda r}\Big)^{p+\nu+2} \end{cases}$$

on $W_\lambda^{(5)}$ for all p, α', and $E_{p,\alpha'}^{(2)}(w,z,s) \in \mathcal{O}(W_\lambda^{(7)} \cap \{|s - i/\lambda| < 2/\lambda\})$ satisfying

$$(5.26) \qquad \begin{cases} \partial_s^{p+\nu+4} E_{p,\alpha'}^{(2)}(w,z,s) = \partial_w^{p+\nu+1} A_{p,\alpha'}^{(2)}(w,z,s), \\ |E_{p,\alpha'}^{(2)}(w,z,s)| \le \dfrac{r e 2^{\nu+1}(2C)^{p+1}}{\pi\rho^{|\alpha'|}(r/320)^{p+\nu+2}} (2/\lambda)^{p+\nu+4} \end{cases}$$

on $W_\lambda^{(7)} \cap \{|s - i/\lambda| < 2/\lambda\})$ for all p, α'. Choose λ (> 1) as

$$(5.27) \qquad 1280C/(\lambda r) \le 1/2.$$

Then we can introduce 2 kernel functions $E^{(k)}(w,z,z-z^*,s)$ for $k = 1,2$ as follows:

$$(5.28) \qquad E^{(k)} \equiv \sum_{|\alpha'|\ge 0, p\ge 0} \frac{\alpha'!}{|\alpha'|!} E_{p,\alpha'}^{(k)}(w,z,s) \prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j+1}}.$$

Here, $E^{(k)}$ are holomorphic in $\mathcal{W}^{(k)}$ at (5.4), (5.5), respectively for $k = 1,2$. On the other hand, from (5.13), (5.14) and (5.11) we obtain the following:

$$(5.29) \qquad \begin{aligned} \sum_{k=1}^{2} \partial_w^{p+\nu+1} A_{p,\alpha'}^{(k)}(w,z,s) &= \partial_w^{p+\nu+1}\Big(\sum_{\ell=0}^{\infty}\sum_{k=1}^{2} A_{p,\alpha',\ell}^{(k)}(z,s)\varphi(w)^\ell\Big) \\ &= \partial_w^{p+\nu+1}\Big(\int_\lambda^\infty V_{p,\alpha'}(w,z,it)\cdot(it)^{p-2}e^{its}\frac{dt}{2\pi}\Big) \\ &= \int_\lambda^\infty U_{-p,\alpha'}(w,z,it)\cdot(it)^{p-2}e^{its}\frac{dt}{2\pi} \end{aligned}$$

for any $(w, z, s) \in W_\lambda^{(5)} \cap W_\lambda^{(7)} \cap \{|s - i\lambda^{-1}| < 2\lambda^{-1}\}$

$$\begin{aligned}=&\{(w, z, s) \in \mathbb{C}^{n+1} \times \mathbb{C}; |w - ir/80| < r/320,\\ &|z| < \kappa,\ 0 < \operatorname{Im} s \leq \lambda^{-1},\ |\operatorname{Re} s| < \lambda^{-1}\}.\end{aligned}$$

Hence we have our conclusion (5.6). □

By using the expressions of the kernel functions obtained in the preceding theorem, we prove that $U(w, x, D_x)f(x)$ has a boundary value at $w = 0$ from $\operatorname{Im} w > 0$ for any microfunction $f(x)$. To do so, we introduce actions of $E^{(*)}(w, z, z - z^*, s)$ on holomorphic functions $F(z)$ similar to the Bony-Schapira actions of microdifferential operators on holomorphic functions [4] (also see [7] concerning the action of $\mathcal{E}_X^{\mathbb{R}}$).

Definition 5.3. We inherit the notation from the preceding theorem. Let $F(z)$ be a holomorphic function defined in

$$\begin{aligned}(5.30)\quad \Omega \equiv&\{z \in \mathbb{C}^n; |z'| < r', |\operatorname{Re} z_n| < (3\lambda)^{-1} + r', r' > \operatorname{Im} z_n > k|\operatorname{Im} z'|\}\\ &\cup \cup_{\sigma=\pm 1}\{z \in \mathbb{C}^n; |z'| < r', |z_n - \sigma/(3\lambda)| < r'\}\end{aligned}$$

for positive small constants r' ($r' < 1/(3\lambda)$), and k ($< \rho/(2n)$). Let $E(w, z, z - z^*, s)$ be a holomorphic kernel function defined in $\mathcal{W}^{(1)}$ at (5.4). For a sufficiently small $\varepsilon > 0$, we define a holomorphic function $(E * F)_{\lambda,\varepsilon}(w, z)$ depending on λ, ε by

$$(5.31)\qquad \int_{i\varepsilon}^{z_n} dz_n^* \int_\gamma dz^{*\prime} \int_{-1/(3\lambda)}^{1/(3\lambda)} E(w, z, z - z^*, z_n^* - z_n^{**})F(z^{*\prime}, z_n^{**})dz_n^{**}.$$

Here the path for z_n^* is the line segment

$$z_n^*(t) = z_n + t(i\varepsilon - z_n)\ \ (0 \leq t \leq 1)$$

combining z_n with $i\varepsilon$, $\gamma = \{z_j^* = z_j + R(z_n, z_n^*(t))e^{i\theta_j}\ (0 \leq \theta_j \leq 2\pi); j = 1, \ldots, n-1\}$ with some $R(z_n, z_n^*(t)) > \rho^{-1}|z_n - z_n^*(t)| = t\rho^{-1}|z_n - i\varepsilon|$. Further the path for z_n^{**} is the line graph passing through

$$-(3\lambda)^{-1},\ -(3\lambda)^{-1} + ih\operatorname{Im} z_n^*(t),\ (3\lambda)^{-1} + ih\operatorname{Im} z_n^*(t),\ (3\lambda)^{-1}$$

for a constant h $(1/2 < h < 1)$. That is,

$$z_n^{**}(\theta;t) = \begin{cases} -1/(3\lambda) + 3ih\theta \operatorname{Im} z_n^*(t) & (0 \le \theta \le 1/3), \\ (2\theta - 1)/\lambda + ih \operatorname{Im} z_n^*(t) & (1/3 \le \theta \le 2/3), \\ 1/(3\lambda) + 3ih(1-\theta) \operatorname{Im} z_n^*(t) & (2/3 \le \theta \le 1) \end{cases}$$

Indeed this integral is well-defined if $\operatorname{Im} w > 0, |w| < r/40, |z| < \kappa$, $|\operatorname{Re} z_n| < (3\lambda)^{-1}, 0 < \operatorname{Im} z_n \le \varepsilon < r'$ and the following sets are contained in Ω :

$$\Big\{(z_1 + t\omega_1, \dots, z_{n-1} + t\omega_{n-1}, q + ih(\operatorname{Im} z_n + t(\varepsilon - \operatorname{Im} z_n))); 0 \le t \le 1,$$
$$|q| \le (3\lambda)^{-1}\ (q \in \mathbb{R}), |\omega_1| \le \rho^{-1}|z_n - i\varepsilon|, \dots, |\omega_{n-1}| \le \rho^{-1}|z_n - i\varepsilon|\Big\},$$

$$\Big\{(z_1 + t\omega_1, \dots, z_{n-1} + t\omega_{n-1}, \pm(3\lambda)^{-1} + iq(\operatorname{Im} z_n + t(\varepsilon - \operatorname{Im} z_n)));$$
$$0 \le t \le 1, 0 \le q \le h, |\omega_1| \le \rho^{-1}|z_n - i\varepsilon|, \dots, |\omega_{n-1}| \le \rho^{-1}|z_n - i\varepsilon|\Big\}.$$

The former set is contained in Ω if $\varepsilon < r'$ and

$$|\operatorname{Re} z_n| < (2h-1)(\varepsilon - \operatorname{Im} z_n),\ \operatorname{Im} z_n > (k/h)|\operatorname{Im} z'|,$$
$$|z'| + (n/\rho)|z_n - i\varepsilon| < r'.$$

The latter set is contained in Ω if $|\operatorname{Im} z_n| \le \varepsilon < r'$ and

$$|z'| + (n/\rho)|z_n - i\varepsilon| < r'.$$

Hence we obtain the following lemma:

LEMMA 5.4. *Let* ε (> 0) *be smaller than* $\min\{\kappa/2, (1 + 2n/\rho)^{-1} r'\}$. *Then* $(E * F)_{\lambda,\varepsilon}(w, z)$ *is holomorphic in*

$$\Big\{\operatorname{Im} w > 0, |w| < r/40,\ |z'| < \varepsilon, |z_n| < \varepsilon,$$
$$\operatorname{Im} z_n > k|\operatorname{Im} z'|, |\operatorname{Re} z_n| < \varepsilon - \operatorname{Im} z_n\Big\} \tag{5.32}$$

Further let $E'(w,z,z-z^*,s)$ be a holomorphic kernel function defined in $\mathcal{W}^{(2)}$ at (5.5). *Then $(E'*F)_{\lambda,\varepsilon}(w,z)$ is holomorphic in a neighborhood of*

$$\Big\{|w-ir/80|<r/320,\ z=0\Big\}.$$

Proof. We have only to prove the latter statement. In this case we modify the paths of integrations as follows:

$$z_n^*(t)=z_n+t(i\varepsilon-z_n)\quad(0\le t\le 1),$$

$$z_n^{**}(\theta;t)=\begin{cases}-1/(3\lambda)+3i\theta\psi(t) & (0\le\theta\le 1/3),\\ (2\theta-1)/\lambda+i\psi(t) & (1/3\le\theta\le 2/3),\\ 1/(3\lambda)+3i(1-\theta)\psi(t) & (2/3\le\theta\le 1),\end{cases}$$

where $\psi(t)=\max\{\operatorname{Im}z_n^*(t),\epsilon\}$ with some small $\epsilon>0$. If we choose $\epsilon<\min\{(640\lambda)^{-1},\varepsilon\}$, for any $z_n=iy_n$ ($y_n\in(-\epsilon,\varepsilon)$) and $t\in[0,1]$ we have

$$\psi(t)=\max\{y_n+t(\varepsilon-y_n),\epsilon\}>\frac{t}{2}(\varepsilon-y_n)\ge\frac{nk}{\rho}t(\varepsilon-y_n).$$

Therefore $(E'*F)_{\lambda,\varepsilon}(w,z)$ is holomorphic in a neighborhood of

$$\Big\{|w-ir/80|<r/320,\ z'=0,\operatorname{Re}z_n=0,-\epsilon<\operatorname{Im}z_n<\varepsilon\Big\}.$$

This completes the proof. □

The following is our secondary main result. K. Uchikoshi [15] used a similar method (Bronshtein's method) of considering boundary values of holomorphic pseudodifferential operators for constructing fundamental solutions of weakly hyperbolic microdifferential operators. However the situations are different from each other, and the proofs and results are completely independent.

The most part of the proof is devoted to the proof of the compatibility of actions of pseudo-differential operators.

Theorem 5.5. *Let $U=\sum_{j=-\infty}^{0}U_j(w,z,\zeta)$ be the classical formal symbol of the pseudo-differential operator treated in Theorem 5.2. Then for any microfunction $f(x)\in\mathcal{C}_N|_{(0;idx_n)}$, a section $U(w,x,D_x)f(x)\in\Gamma(\{w\in$*

$\mathbb{C}; \operatorname{Im} w > 0, |w| < r\} \times \{(0; idx_n)\}; \mathcal{CO}_Z)$ *has a boundary value at* $(0, 0; idx_n)$ *from* $\operatorname{Im} w > 0$ *in the sense of Definition 3.9.*

PROOF. Choose a large positive number λ as indicated in Theorem 5.2. Let $f(x)$ be any germ of $\mathcal{C}_N$ at $(0; idx_n)$. Then by the flabbiness of the sheaf of microfunctions, we can take a defining function $F(z) \in \mathcal{O}(\Omega)$ for $(2\pi i)^{1-n} \partial_{x_n}^{\nu+7} f(x)$ with a sufficiently small $r' > 0$, where Ω at (5.30) satisfies a tighter condition (because $0 < \delta < 1$):

$$0 < k < \min\{1, \rho\delta/(8n)\}. \tag{5.33}$$

Choose a small positive number ε as indicated in Lemma 5.4:

$$0 < \varepsilon < \min\{\kappa/2, (1 + 2n/\rho)^{-1} r'\}, \tag{5.34}$$

which will be replaced by a tighter condition. Then, by Lemma 5.4 we conclude that the boundary value $g(w, x) \equiv (E^{(1)} * F)_{\lambda,\varepsilon}(w, x', x_n + i0)$ is a section of

$$\Gamma(\{\operatorname{Im} w > 0, |w| < r/40, |x| < \varepsilon\}; \mathcal{BO}_Z).$$

On the other hand, by the latter result in Lemma 5.4 we have $[g(w, x)] = [((E^{(1)} + E^{(2)}) * F)_{\lambda,\varepsilon}(w, x', x_n + i0)]$ as a section of $\mathcal{CO}_Z$ over $\{|w - ir/80| < r/320\} \times \{(0; idx_n)\}$. Therefore considering Theorem 5.2 and the unique continuation property of sections of $\mathcal{CO}_Z$, we have only to show that $U(w, x, D_x) f(x) = [(E^{(0)} * F)_{\lambda,\varepsilon}(w, x', x_n + i0)]$ on a neighborhood of $\{|w - ir/80| < r/320\} \times \{(0; idx_n)\}$. Here

$$E^{(0)}(w, z, z - z^*, s) \equiv \sum_{|\alpha'| \geq 0, p \geq 0} \frac{\alpha'!}{|\alpha'|!} \Big(\prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j+1}} \Big)$$
$$\times \int_{i/\lambda}^{s} \frac{(s - s^*)^{p+\nu+3}}{(p + \nu + 3)!} ds^* \int_{\lambda}^{\infty} U_{-p,\alpha'}(w, z, it) \cdot (it)^{p-2} e^{its^*} \frac{dt}{2\pi}.$$

Set

$$K_{p,\alpha'}(w, z, s) \equiv \int_{\lambda}^{\infty} U_{-p,\alpha'}(w, z, it) \cdot (it)^{p-2} e^{its} \frac{dt}{2\pi}. \tag{5.35}$$

Noting the estimates (5.8), we obtain the following: $K_{p,\alpha'}(w, z, s)$'s are holomorphic in

$$W' \equiv \{|w - ir/80| < r/320, |z| < \kappa, \operatorname{Im} s > -\delta |\operatorname{Re} s|\}, \tag{5.36}$$

and continuous up to $s = 0$ with estimates

$$|K_{p,\alpha'}(w, z, s)| \le p! C^{p+1} \rho^{-|\alpha'|} (160/r)^{p+\mu} / \pi \ \text{ on } W'.$$

Hence for $\ell = 0, 1, 2, \dots$

$$K^{(\ell)}_{p,\alpha'}(w, z, s) \equiv \int_0^s \frac{(s - s')^\ell}{\ell!} K_{p,\alpha'}(w, z, s') ds' \tag{5.37}$$

are holomorphic functions in W' with estimates

$$|K^{(\ell)}_{p,\alpha'}(w, z, s)| \le \frac{p! |s|^{\ell+1}}{\pi(\ell+1)! \rho^{|\alpha'|}} C^{p+1} (160/r)^{p+\mu} \ \text{ on } W'. \tag{5.38}$$

Therefore we have

$$\begin{aligned} E^{(0)}(w, z, z - z^*, s) = &\sum_{|\alpha'| \ge 0, p \ge 0} \frac{\alpha'!}{|\alpha'|!} \Big(\prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j + 1}} \Big) \\ &\times \Big(K^{(p+\nu+3)}_{p,\alpha'}(w, z, s) - \int_0^{i/\lambda} \frac{(s - s^*)^{p+\nu+3}}{(p+\nu+3)!} K_{p,\alpha'}(w, z, s^*) ds^* \Big). \end{aligned} \tag{5.39}$$

Since $\lambda^{-1} < r/(2560C)$, the first term of (5.39) is holomorphic in

$$\begin{aligned} \mathcal{W}^{(4)} \equiv \Big\{ &(w, z, z - z^*, s) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}; |s| < 16\lambda^{-1}, \\ &|w - ir/80| < r/320, \ \ \operatorname{Im} s > -\delta |\operatorname{Re} s|, \ |z| < \kappa, \\ &|z_j - z_j^*| > \rho^{-1} |z_n - z_n^*| \ (j = 1, \dots, n-1) \Big\}. \end{aligned} \tag{5.40}$$

Further the second term of (5.39) is holomorphic in

$$\begin{aligned} \mathcal{W}^{(5)} \equiv \Big\{ &(w, z, z - z^*, s) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}; \\ &|w - ir/80| < r/320, \ \ |s| < 15\lambda^{-1}, \ |z| < \kappa, \\ &|z_j - z_j^*| > \rho^{-1} |z_n - z_n^*| \ (j = 1, \dots, n-1) \Big\}. \end{aligned} \tag{5.41}$$

Since $\mathcal{W}^{(5)}$ includes $\mathcal{W}^{(2)}$ at (5.5), the contribution of the second term to $[(E^{(0)} * F)_{\lambda,\varepsilon}(w, x', x_n + i0)]$ is 0. Thus, our problem reduces to the action of the following operator on $F(z)$:

(5.42)

$$G(w, z, z - z^*, s) = \sum_{|\alpha'| \ge 0, p \ge 0} \frac{\alpha'!}{|\alpha'|!} \Big(\prod_{j=1}^{n-1} \frac{(z_n - z_n^*)^{\alpha_j}}{(z_j^* - z_j)^{\alpha_j + 1}} \Big) K^{(p+\nu+3)}_{p,\alpha'}(w, z, s),$$

which is holomorphic in $\mathcal{W}^{(4)}$ and continuous up to $s = 0$. For any small positive numbers $c < (3\lambda)^{-1}$ and δ' $(\delta/2 \le \delta' < \delta)$, we put

$$(5.43) \qquad (G * F)_{c,\delta',\varepsilon}(w, z) = \int_{i\varepsilon}^{z_n} dz_n^* \int_\gamma dz^{*\prime} \int_{-c(1-i\delta')}^{c(1+i\delta')} G(w, z, z - z^*, z_n^* - z_n^{**}) F(z^{*\prime}, z_n^{**}) dz_n^{**}.$$

Here the path $z_n^* = z_n + t(i\varepsilon - z_n)$ $(0 \le t \le 1)$ for z_n^* and γ are chosen in the same way with Definition 5.3. Further the path $z_n^{**} = z_n^{**}(\theta; t)$ for z_n^{**} is the line graph passing through

$$-c(1 - i\delta'),\ z_n^*,\ c(1 + i\delta')$$

That is, for $-1 \le \theta \le 1, 0 \le t \le 1$ we have

$$(5.44) \qquad \begin{aligned} z_n^{**}(\theta; t) &= (1 - |\theta|)(z_n + t(i\varepsilon - z_n)) + c\theta + ic\delta'|\theta| \\ &= c\theta + (1 - |\theta|)(1 - t) \operatorname{Re} z_n \\ &\quad + i(c\delta'|\theta| + (1 - |\theta|)(\varepsilon t + (1 - t) \operatorname{Im} z_n)). \end{aligned}$$

This integral is well-defined if $|w - ir/80| < r/320, |z| < \kappa$, $|z_n| + \varepsilon + 2c < 16\lambda^{-1}, \delta|\operatorname{Re} z_n| - \operatorname{Im} z_n < (\delta - \delta')c$ and the following set is contained in Ω :

$$\Big\{(z_1 + t\omega_1, \dots, z_{n-1} + t\omega_{n-1}, z_n^{**}(\theta; t));\ -1 \le \theta \le 1,$$
$$0 \le t \le 1, |\omega_1| \le \rho^{-1}|z_n - i\varepsilon|, \dots, |\omega_{n-1}| \le \rho^{-1}|z_n - i\varepsilon|\Big\},$$

Indeed, this set is contained in Ω if $|z'| + (n/\rho)|z_n - i\varepsilon| < r'$, $|\operatorname{Re} z_n| + c < (3\lambda)^{-1}$ and if for any $\theta \in [-1, 1], t \in [0, 1]$ we have

$$k|\operatorname{Im} z'| + \frac{nkt}{\rho}|z_n - i\varepsilon| < c\delta'|\theta| + (1 - |\theta|)(\varepsilon t + (1 - t) \operatorname{Im} z_n) < r'$$

$$\iff \begin{cases} k|\operatorname{Im} z'| + nkt|z_n - i\varepsilon|/\rho < \operatorname{Im} z_n + t(\varepsilon - \operatorname{Im} z_n)\ (\forall t \in [0, 1]), \\ k|\operatorname{Im} z'| + nk|z_n - i\varepsilon|/\rho < c\delta', \\ c\delta'|\theta| + (1 - |\theta|)(\varepsilon t + (1 - t) \operatorname{Im} z_n) < r'\ (\forall t, \forall|\theta| \in [0, 1]). \end{cases}$$

Hence, these conditions are all satisfied under (5.33),(5.34) and $\delta/2 \le \delta' < \delta$ if

$$\begin{aligned} &\varepsilon + c < \min\{(3\lambda)^{-1}, r'\}, \varepsilon < (\rho/n + 2)^{-1}c, \\ &|w - ir/80| < r/320, |z'| < \varepsilon, |z_n| < \varepsilon, \\ &k|\operatorname{Im} z'| < \operatorname{Im} z_n, |\operatorname{Re} z_n| < \varepsilon - \operatorname{Im} z_n. \end{aligned}$$

Replace the condition (5.34) by

$$(5.45)\qquad c < \frac{\min\{(3\lambda)^{-1}, r'\}}{2+3n/\rho},\ \varepsilon < \min\Big\{\frac{\kappa}{2}, \frac{r'}{1+2n/\rho}, \frac{c}{\rho/n+2}\Big\}.$$

Therefore $(G*F)_{c,\delta',\epsilon}(w,z)$ is holomorphic in

$$\{|w-ir/80| < r/320, |z'| < \varepsilon, |z_n| < \varepsilon,\\ k|\operatorname{Im} z'| < \operatorname{Im} z_n, |\operatorname{Re} z_n| < \varepsilon - \operatorname{Im} z_n\}.$$

We claim here that

$$[(G*F)_{\lambda,\varepsilon}(w,x',x_n+i0)] = [(G*F)_{c,\delta',\varepsilon}(w,x',x_n+i0)]$$

on a neighborhood of $\{|w-ir/80| < r/320\} \times \{(0; idx_n)\}$. To prove this, it is sufficient to show that the functions

$$\int_{i\varepsilon}^{z_n} dz_n^* \int_\gamma dz^{*\prime} \int_{c\sigma+ic\delta'}^{\sigma(3\lambda)^{-1}} G(w,z,z-z^*,z_n^*-z_n^{**})F(z^{*\prime},z_n^{**})dz_n^{**}$$

extend holomorphically to $\{|w-ir/80| < r/320\} \times \{z=0\}$ for all $\sigma = \pm 1$. Take the line graph Γ_σ passing through

$$c\sigma + ic\delta',\ \sigma(3\lambda)^{-1} + ic\delta',\ \sigma(3\lambda)^{-1}$$

as the paths of integration. Indeed, these integrals are well-defined for $(w,z) \in \{|w-ir/80| < r/320\} \times \{z=0\}$ because Ω includes

$$\Big\{(t\omega_1,\dots,t\omega_{n-1},z_n^{**});\ z_n^{**} \in \Gamma_\sigma, 0 \le t \le 1,\\ |\omega_1| \le \rho^{-1}\varepsilon,\dots,|\omega_{n-1}| \le \rho^{-1}\varepsilon\Big\}$$

under the conditions (5.45). This proves our claim above.

As a last step of the proof, we eliminate the variable z_n^* from the integral expression (5.43). When we restrict the variables (w,z) to a neighborhood of $\{|w-ir/80| < r/320\} \times \{z'=0, z_n = 3i\varepsilon/4\}$, we can deform the path $z_n^{**}(\theta;t)$ at (5.44) for z_n^{**} to the line graph passing through

$$-c+ic\delta',\ i\varepsilon/2,\ c+ic\delta'.$$

Indeed, this deformation is possible if $\varepsilon + 2c < 16\lambda^{-1}$ and the following set is contained in Ω :

$$\Big\{(\omega_1, \dots, \omega_{n-1}, c\theta + i(c\delta'|\theta| + (1-|\theta|)\varepsilon/2)); \\ -1 \le \theta \le 1, |\omega_1| \le \rho^{-1}\varepsilon/4, \dots, |\omega_{n-1}| \le \rho^{-1}\varepsilon/4\Big\}.$$

Hence, all of these conditions are satisfied under (5.45) because $n\varepsilon/(4\rho) < r'$ and

$$nk\varepsilon/(4\rho) < c\delta'|\theta| + (1-|\theta|)\varepsilon/2 < r' \ (\forall|\theta| \in [0,1]) \\ \iff \begin{cases} nk\varepsilon/(4\rho) < \min\{c\delta', \varepsilon/2\}, \\ \max\{c\delta', \varepsilon/2\} < r'. \end{cases}$$

Since this new path for z_n^{**} does not depend on z_n^*, we can exchange the order of integration in the integral $(G*F)_{c,\delta',\varepsilon}(w,z)$: That is, for any (w,z) as above, we have

$$(5.46) \qquad (G*F)_{c,\delta',\varepsilon}(w,z) \\ = \int_{-c(1-i\delta')}^{c(1+i\delta')} d\tau \int_\gamma dz^{*\prime} F(z^{*\prime},\tau) \int_{i\varepsilon-\tau}^{z_n-\tau} G(w,z,z'-z^{*\prime}, z_n-\tau-s, s)ds \\ = \int_{-c(1-i\delta')}^{c(1+i\delta')} d\tau \int_\gamma F(z^{*\prime},\tau)(H(w,z'-z^{*\prime}, z_n-\tau, z_n-\tau) \\ - H(w,z'-z^{*\prime}, z_n-\tau, i\varepsilon-\tau))dz^{*\prime}$$

Here

$$(5.47) \qquad H(w,z'-z^{*\prime}, s_1, s_2) \equiv \int_0^{s_2} G(w,z,z'-z^{*\prime}, s_1-s, s)ds$$

is holomorphic in

$$\mathcal{W}^{(6)} \equiv \Big\{(w,z,z'-z^{*\prime}, s_1, s_2) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^{n-1} \times \mathbb{C}^2; |s_2| < 16\lambda^{-1}, \\ |w - ir/80| < r/320, \ \ \operatorname{Im} s_2 > -\delta|\operatorname{Re} s_2|, \ |z| < \kappa, \\ |z_j - z_j^*| > \rho^{-1}\max\{|s_1 - s_2|, |s_1|\} \ (j = 1, \dots, n-1)\Big\}.$$

Therefore the second term of (5.46) extends holomorphically to

$$\{|w - ir/80| < r/320\} \times \{(z', z_n); z' = 0, z_n = it \ (0 \le t \le 3\varepsilon/4)\}$$

if the following sets are contained in Ω for all $t \in [0, 3\varepsilon/4]$:

$$\begin{aligned}&\Big\{(\omega_1, \dots, \omega_{n-1}, c\theta + i(c\delta'|\theta| + (1-|\theta|)\varepsilon/2));\ -1 \le \theta \le 1,\\ &\max_{1\le j\le n-1} |\omega_j| \le \frac{\max\{|t-\varepsilon|, c|\theta| + |t - (c\delta'|\theta| + (1-|\theta|)\varepsilon/2|\}}{\rho}\Big\}\\ &\subset \Big\{(\omega_1, \dots, \omega_{n-1}, c\theta + i(c\delta'|\theta| + (1-|\theta|)\varepsilon/2));\\ &\qquad -1 \le \theta \le 1,\ \max_{1\le j\le n-1} |\omega_j| \le \frac{\varepsilon + 2c|\theta|}{\rho}\Big\}\end{aligned}$$

Hence these sets are contained in Ω if $n(\varepsilon + 2c)/\rho < r'$ and

$$\begin{aligned}&nk(\varepsilon + 2c|\theta|)/\rho < c\delta'|\theta| + (1-|\theta|)\varepsilon/2 < r'\ (\forall|\theta| \in [0,1])\\ &\iff \begin{cases} nk\varepsilon/\rho < \varepsilon/2,\ nk(\varepsilon+2c)/\rho < c\delta',\\ \max\{c\delta', \varepsilon/2\} < r'.\end{cases}\end{aligned}$$

Indeed all these conditions are fullfilled under the conditions (5.45); we essentially required the tight condition (5.33) for $nk(\varepsilon + 2c)/\rho < c\delta'$.

Further we deform the path of integration for the first term of (5.46) to the line graph passing through

$$-c + ic\delta',\ z_n,\ c + ic\delta'.$$

Then we can extend the first term of (5.46) holomorphically to

$$\begin{aligned}&\{|w - ir/80| < r/320, |z'| < \varepsilon, |z_n| < \varepsilon,\\ &k|\operatorname{Im} z'| < \operatorname{Im} z_n, |\operatorname{Re} z_n| < \varepsilon - \operatorname{Im} z_n\}\end{aligned}$$

by the same estimates of domains with ones for $(G * F)_{c,\delta',\varepsilon}(w, z)$; indeed, the present estimates for domains exactly correspond to the case $t = 0$ at (5.43). Consequently the boundary value of

$$\int_{-c(1-i\delta')}^{c(1+i\delta')} d\tau \int_\gamma H(w, z' - z^{*\prime}, z_n - \tau, z_n - \tau) F(z^{*\prime}, \tau) dz^{*\prime}$$

coincides with $[(E^{(0)} * F)_{\lambda,\varepsilon}(w, x', x_n + i0)]$ as a section of $\mathcal{CO}_Z$ in a neighborhood of $\{|w - ir/80| < r/320\} \times \{(0; idx_n)\}$. On the other hand it is clear that

$$H(w,z,z-z^{*\prime},s,s)=\sum_{|\alpha'|\geq 0,p\geq 0}K_{p,\alpha'}^{(p+|\alpha'|+\nu+4)}(w,z,s)\prod_{j=1}^{n-1}\frac{\alpha_j!}{(z_j^*-z_j)^{\alpha_j+1}}$$

is the kernel function for $(2\pi i)^{n-1}U(w,z,D_z)D_{z_n}^{-\nu-7}$ (see [1]). Thus we have

$$[(E^{(0)}*F)_{\lambda,\varepsilon}(w,x',x_n+i0)]=$$
$$(2\pi i)^{n-1}U(w,x,D_x)D_{x_n}^{-\nu-7}((2\pi i)^{1-n}D_{x_n}^{\nu+7}f(x))=U(w,x,D_x)f(x)$$

on a neighborhood of $\{|w-ir/80|<r/320\}\times\{(0;idx_n)\}$. This completes the proof. □

REMARK 5.6. The growth order condition (5.2) for the lower order terms of $\sum_{j=-\infty}^{0}U_j(w,z,\zeta)$ is the best possible in the following sense: For any constant k $(1<k<2)$ there exists a classical formal symbol $U=\sum_{j=-\infty}^{0}U_j(w,z,\zeta)$ satisfying the following (1)–(3):

(1) $U_j\in\mathcal{O}(W_r)$ $(\forall j\leq 0)$.

(2) For some constants $C,\mu>0$ we have

$$|U_{-p}(w,z,\zeta)|\leq C^{p+1}p!|\operatorname{Im}w|^{-kp-\mu}|\zeta|^{-p}\ \text{ on } W_r\ (\forall p\geq 0).$$

(3) $U(w,x,D_x)\delta(x_n)$ does not have a boundary value at $(0,0;idx_n)$ in microfunctions of $(\operatorname{Re}w,x)$ from $\operatorname{Im}w>0$.

Indeed, we can give an explicit example as follows:

$$U_{-p}(w,z,\zeta)\equiv p!\{-i(w/i)^k\}^{-p-1}\zeta_n^{-p-1}, \tag{5.48}$$

where $p=0,1,2,\dots$ and $|\arg(w/i)|<\pi/2$. It is easy to see that the above conditions 1, 2 are satisfied and that

$$U(w,x,D_x)\delta(x_n)=\Big[\sum_{p=0}^{\infty}-\frac{1}{2\pi i}\{-i(w/i)^k\}^{-p-1}z_n^p\log z_n\Big]$$
$$=\Big[\frac{1}{2\pi i}\cdot\frac{\log z_n}{z_n+i(w/i)^k}\Big].$$

Here the equalities are valid for sections of $\mathcal{CO}_Z$ over $\{\operatorname{Im} w > \varepsilon\} \times \{(0; idx_n)\}$ with any small positive ε. Then by the following lemma we get the condition (3) above for U.

The following example is a variant of the example in [12] of second hyperfunctions:

LEMMA 5.7. *Let k be a constant satisfying $1 < k < 2$. Then the microfunction $f(w,x) = \dfrac{\log(x+i0)}{x+i(w/i)^k}$ extends to $\{(w,x;i\eta dx) \in \mathbb{C} \times T^*_{\mathbb{R}}\mathbb{C}; \operatorname{Im} w > 0, \eta > 0\}$ as a microfunction with holomorphic parameter w. However $f(w,x)$ never has a microfunction boundary value at $(0,0;idx)$ from $\operatorname{Im} w > 0$.*

PROOF. Consider a holomorphic function

$$F_1(w,z) = \frac{\log z - \log\{(w/i)^k/i\}}{z + i(w/i)^k}$$

defined in $\{(w,z) \in \mathbb{C}^2; \operatorname{Im} z > 0, \pi(1-k^{-1})/2 < \arg w < \pi\}$, where $-\pi < \arg\{(w/i)^k/i\} < \pi(k-1)/2 < \pi/2$. Further set

$$F_2(w,z) = F_1(w,z) + \frac{-2\pi i}{z + i(w/i)^k}.$$

Then $F_2(w,z) \in \mathcal{O}(\{(w,z) \in \mathbb{C}^2; \operatorname{Im} z > 0, 0 < \arg w < \pi(1+k^{-1})/2\})$. Hence the extension of $f(w,x)$ is given by

$$f(w,x) = \begin{cases} [F_1(w,x+i0)] & (\pi(1-k^{-1})/2 < \arg w < \pi), \\ [F_2(w,x+i0)] & (0 < \arg w < \pi(1+k^{-1})/2) \end{cases}$$

as microfunctions with holomorphic parameter w. We suppose here that $f(w,x)$ has a microfunction boundary value at $(0,0;idx)$ from $\operatorname{Im} w > 0$. Therefore, we have a holomorphic function $G(w,z)$ defined in $\{\operatorname{Im} w > 0, \operatorname{Im} z > 0, |w| + |z| < \varepsilon\}$ with some $\varepsilon > 0$ satisfying

$$f(w,x) = [G(w,x+i0)]$$

as sections of microfunctions with holomorphic parameter w in

$$\{(w,x;i\eta dx) \in \mathbb{C} \times T^*_{\mathbb{R}}\mathbb{C}; |w| + |x| < \varepsilon, \operatorname{Im} w > 0, \eta > 0\}.$$

Since

$$\begin{aligned}0 &= \{x+i(w/i)^k\}\Big(f(w,x)-[G(w,x+i0)]\Big)\\ &= [\log(x+i0)-\{x+i(w/i)^k\}G(w,x+i0)],\end{aligned}$$

we conclude that

$$\begin{aligned}A(w,z) \equiv{}& \log z-\{z+i(w/i)^k\}G(w,z)\\ &\in \mathcal{O}(\{\operatorname{Im} w>0, \operatorname{Im} z>0, |w|+|z|<\varepsilon\})\end{aligned}$$

extends holomorphically to $\{|w|+|z|<\varepsilon', \operatorname{Im} z=0, \operatorname{Im} w>0\}$ with some smaller $\varepsilon'>0$. Therefore by Kashiwara's theorem on the local version of Bochner's tube theorem we can extend $A(w,z)$ to a holomorphic function $\tilde{A}(w,z)$ in

$$\Omega=\{|w|+|z|<\varepsilon'', |\operatorname{Im} z|<\varepsilon'' \operatorname{Im} w\}$$

for some smaller $\varepsilon''>0$. Set

$$P(r,\theta)=(re^{i\theta}, r^k e^{i(k\theta-\pi(k+1)/2)})\in\{(w,z)\in\mathbb{C}^2; z+i(w/i)^k=0\}$$

for $r>0, 0<\theta<\pi$. We note that $P(r,\theta)\in\Omega$ for any $\theta\in(0,\pi)$ with any sufficiently small $r>0$ because $k>1$. Therefore $H(w)\equiv\tilde{A}(w,-i(w/i)^k)$ is a holomorphic function in

$$W=\{w\in\mathbb{C}; 0<\arg w<\pi, 0<|w|<\varphi(\arg w)\}$$

with a positive valued continuous function $\varphi(\theta)$ on $(0,\pi)$. On the other hand, we have that

$$H(re^{i\theta})=A(re^{i\theta}, r^k e^{ik\{\theta-\pi(1+k^{-1})/2\}})=k\log r+ik\{\theta-\pi(1+k^{-1})/2\}$$

for $\pi(1+k^{-1})/2<\theta<\pi$, and that

$$H(re^{i\theta})=A(re^{i\theta}, r^k e^{ik\{\theta+\pi(3k^{-1}-1)/2\}})=k\log r+ik\{\theta+\pi(3k^{-1}-1)/2\}$$

for $0<\theta<\pi(1-k^{-1})/2$. That is, $H(w)-k\log w\in\mathcal{O}(W)$ coincides with 2 different constants $-\pi(k+1)i/2$ and $-\pi(k-3)i/2$ in the above domains, respectively. This contradicts with the connectedness of W. Thus $f(w,x)$ never have a microfunction boundary value at $(0,0;idx)$ from $\operatorname{Im} w>0$. $\square$

Here we return to our original subject for solving a microdifferential equation (1.2) in $\mathcal{CO}_Z$.

THEOREM 5.8. *We have a system* $\{U^{(\ell)}(w,z,D_z);\ell=1,...,m\}$ *of formal symbols of microdifferential operators defined around* $\overset{\circ}{p}$ *satisfying the conditions* $1\sim 5$ *in Introduction:*

(1) $U^{(1)}$ *is a multivalued section of* $\mathcal{E}_X$ *over* $\{(w,x;i\eta)\in T^*_ZX; 0<|w-\varphi(x,i\eta)|<r, |x-\overset{\circ}{x}|<r, |\eta-\overset{\circ}{\eta}|<r\}$ *for some small* $r>0$, *and* $U^{(\ell)}\in\mathcal{E}_X|_{\overset{\circ}{p}}$ *for* $\ell=2,...,m$.

(2) $U^{(\ell)}(w,z,D_z)$ $(\ell=1,...,m)$ *commute with* w.

(3) $P(w,z,D_w,D_z)U^{(\ell)}(w,z,D_z)=0$ $(\mathrm{mod}\,\mathcal{E}_X\cdot D_w)$ *for* $\ell=1,...,m$.

(4) *ord*$(U^{(\ell)})=0$ *for* $\ell=1,...,m$, *and holomorphic functions* $\{\sigma_0(U^{(\ell)})(w,x,i\eta);\ell=1,...,m\}$ *give a complete system of solutions in* $\{w-\varphi(x,i\eta)\neq 0\}$ *of the following linear ordinary differential equation :*

$$LU := \Big(\sum_{k=0}^{m}\sigma_0(A_k)(w,x,0,i\eta)\frac{\partial^{m-k}}{\partial w^{m-k}}\Big)U=0$$

(5) *For any microfunction* $f(x)\in\mathcal{C}_N|_{\rho(\overset{\circ}{p})}$, $U^{(1)}(w,x,D_x)f(x)$ *has a microfunction boundary value at* $w=\varphi(x,i\eta)$; *that is, a microfunction boundary value from any side of any* $\mathbb{R}$*-conic and real analytic hypersurface* H *of* T^*_ZX *passing through* $K=\{w-\varphi(x,i\eta)=0\}$.

PROOF. Let H be any $\mathbb{R}$-conic and real analytic hypersurface passing through $K=\{w-\varphi(x,i\eta)=0\}$ in T^*_ZX. Then, by Lemma 2.5, we can reduce the triple (K,H,P) to the case that $K=\{w=0\}, H=\{\mathrm{Im}\,w=0\}$ and P with $A_0(w,z,D_w,D_z)\equiv w$ under a quantized contact transformation $\mathcal{S}$, which preserves $\mathcal{CO}_Z$ and satisfies

$$\mathcal{S}^{-1}(D_w)\in\mathcal{E}_X\cdot D_{w^*}.$$

Further by taking the multiple of w by some non-zero constant as a new variable, we can suppose the coefficients $A_k(w,z,D_w,D_z)$ $(k=1,...,m)$ are defined in some neighborhood of $\{|w| \leq 1\} \times \{\overset{\circ}{p}\}$. Therefore under the assumption $\sigma_0(A_1)(\overset{\circ}{p}) \notin \mathbb{Z}$ we can construct a system $\{U_0^{(\ell)}; \ell = 1, ..., n\}$ of solutions of $LU = 0$ satisfying the following:

(1) $U_0^{(1)}(w, z, \zeta)$ has the following form in a neighborhood of $w = 0$:

$$U_0^{(1)}(w, z, \zeta) = w^{m-1-A_1^0(0,z,\zeta)}\Big(1 + \sum_{k=1}^{\infty} c_k(z, \zeta) w^k\Big),$$

where $A_1^0(0, z, \zeta) = \sigma_0(A_1)(0, z, 0, \zeta), c_k(z, \zeta)$ are holomorphic functions of homogeneous degree 0 with respect to ζ.

(2) $U_0^{(\ell)}(w, z, \zeta)$ $(2 \leq \ell \leq m)$ are holomorphic solutions in a neighborhood of $w = 0$ such that

$$\partial_w^k U_0^{(\ell)}(0, z, \zeta) = \delta_{k,\ell-2} \ \ (0 \leq \forall k \leq m-2).$$

It is clear that these solutions are of homogeneous degree 0 with respect to ζ, and that they uniquely extend to the solutions defined in a neighborhood of $\{|w| \leq 1\} \times \{\overset{\circ}{p}\}$ (for $\ell = 2, ..., m$), or a neighborhood of the universal covering of $\{0 < |w| \leq 1\} \times \{\overset{\circ}{p}\}$ (for $\ell = 1$). Therefore by Theorems 4.8, 4.9 we get the formal symbols $\{\sum_{j=-\infty}^{0} U_j^{(\ell)}; \ell = 1, ..., m\}$ of microdifferential operators satisfying the conditions (1)–(4). Further by Theorem 5.5 we also get the condition (5) for $U^{(1)}$.

On the other hand, as for the relationship with the original equation, we know the following: Let $\mathcal{S}$ be any quantized contact transformation satisfying $\mathcal{S}^{-1}(D_w) \in \mathcal{E}_X \cdot D_{w^*}$. If the original operator P is transformed into $P^* = \mathcal{S}^{-1}(P)$, for the solutions $U^{(\ell)}$ of $P^*U = 0 \bmod \mathcal{E}_X \cdot D_{w^*}$ $\mathcal{S}(U^{(\ell)})$ also become solutions of $PU = 0 \bmod \mathcal{E}_X \cdot D_w$. Considering Remark 2.6, we complete the proof. □

As a direct corollary, under the same notation with the preceding theorem we have the following:

THEOREM 5.9. *Let $q = (w', \overset{\circ}{x}; i\overset{\circ}{\eta})$ $(w' \neq \overset{\circ}{w})$ be a point of T_Z^*X sufficiently close to $\overset{\circ}{p}$, and $f(w, x) \in \mathcal{CO}_Z|_q$ be a solution of $Pf = 0$ in a neighborhood of q. Then there exists uniquely an m-vector*

$$(f_1(x), ..., f_m(x)) \in \mathcal{C}_N^m|_{(\overset{\circ}{x}; i\overset{\circ}{\eta})}$$

of microfunctions of x such that $f(w,x) = \sum_{\ell=1}^{m} U^{(\ell)}(w,x,D_x)f_\ell(x)$ in a neighborhood of q. In particular, $f(w,x)$ extends uniquely to a multi-valued $\mathcal{C}\mathcal{O}_Z$-soltuion of $Pf = 0$ around $\mathring{p}$ with microfunction boundary values at $\mathring{p}$.

Proof. We have only to show the following: The matrix

$$\left(\partial_w^{k-1} U^{(\ell)}(w',x,D_x)\right)_{k,\ell=1,\dots,m}$$

of microdifferential operators in $\mathcal{E}_Y$ is invertible at q as a morphism $\mathcal{C}_N^m \to \mathcal{C}_N^m$. Further this invertibility reduces to that of the matrix

$$\left(\partial_w^{k-1} \sigma_0(U^{(\ell)})(w',x,i\eta)\right)_{k,\ell=1,\dots,m}.$$

Indeed, since $\sigma_0(U^{(\ell)})(w,x,i\eta)$ ($\ell = 1,\dots,m$) are linearly independent solutions of $LU = 0$ from each other, we have the invertibility of this matrix. Thus the proof is complete. □

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Kiyoomi KATAOKA
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail: kiyoomi@ms.u-tokyo.ac.jp

Yoshiaki SATOH
Fujitsu Research Institute
1-16 Kaigan, Minato-ku
Tokyo 105-0022, Japan
E-mail: satouy@fri.fujitsu.com