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Non-uniqueness in an Overdetermined Cauchy Problem for the Wave Equation

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Abstract. In this paper, we prove non-uniqueness in an overdetermined Cauchy problem for the wave equation in quasi-analytic ultradistribution category. In our Cauchy problem we give initial values at one space point. It is an inverse problem to reconstruct the wave from observation at one space point.

§0. Introduction

In this paper, we study uniqueness in an overdetermined Cauchy problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = u_\alpha(t) \ \text{ for any } \alpha, \end{cases} \tag{0.1}$$

where Δ is the Laplacian on $\mathbb{R}^n$, $n \geq 2$.

This is an inverse problem to reconstruct the wave from observation at one space point. This problem was first introduced by L.Ehrenpreis [E], who called it "the Watergate Problem" for fun. He proved uniqueness in this problem in distribution category, employing expansion by harmonic functions. As for uniqueness, F.John [J] also proved it globally with respect to general real analytic time-like curves. For distribution solutions, another uniqueness result was proved by M.Nacinovich [N] in a different way.

In this paper, we study this uniqueness in more generalized classes of functions. In 1993, S.Tanabe and T.Takiguchi [TT] proved that

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^\alpha u|_{x=x_0} = 0 \ \text{ for any } \alpha \end{cases} \tag{0.2}$$

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would imply that $u = 0$ in a neighborhood of $x = x_0$ if u is a non-quasi-analytic ('NQA' for short) ultradistribution. In the same paper, they introduced a counterexample by A.Kaneko which yields that uniqueness in this Cauchy problem does not hold for hyperfunctions.

For uniqueness in the Cauchy problem (0.1), the case where u is a quasi-analytic ('QA' for short) ultradistribution is left open, which we study in this paper.

§1. Ultradistributions

In this section, we review the definition of ultradistributions. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $M_p, p = 0, 1, \cdots$, be a sequence of positive numbers.

DEFINITION 1.1. $f \in \mathcal{E}(\Omega) = C^\infty(\Omega)$ is called an *ultradifferentiable function* of class $\{M_p\}$ (resp. (M_p)) if for any compact subset $K \subset \Omega$ there exist constants h and C (resp. for any K and for any $h > 0$ there exists some C) such that

$$\sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|} \quad \text{for all } \alpha \tag{1.1}$$

holds. Denote the set of the ultradifferentiable functions of class $\{M_p\}$ (resp. (M_p)) on Ω by $\mathcal{E}^{\{M_p\}}(\Omega)$ (resp. $\mathcal{E}^{(M_p)}(\Omega)$) and denote by $\mathcal{D}^*(\Omega)$ the set of all functions in $\mathcal{E}^*(\Omega)$ with support compact in Ω, where $* = \{M_p\}$ or (M_p).

For a compact subset $K \subset \Omega$ let

$$\mathcal{D}^*_K = \{\varphi \in \mathcal{D}^*(\mathbb{R}^n) \ ; \ \mathrm{supp} f \subset K\}, \tag{1.2}$$

and we define

$$\mathcal{D}_K^{\{M_p\},h} = \{\varphi \in \mathcal{D}_K^{\{M_p\}} \ ; \ \exists C \text{ such that } \sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|}\}. \tag{1.3}$$

These spaces are endowed with natural structure of locally convex spaces.

For NQA classes, we impose the following conditions on M_p.
$(M.0)$ (normalization)

$$M_0 = M_1 = 1.$$

$(M.1)$ (logarithmic convexity)

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad p = 1, 2, \cdots .$$

$(M.2)$ (stability under ultradifferential operators)

$$\exists G, \quad \exists H \text{ such that } M_p \leq GH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \cdots .$$

$(M.3)$ (strong non-quasi-analyticity)

$$\exists G \text{ such that } \sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Gp \frac{M_p}{M_{p+1}}, \quad p = 1, 2, \cdots .$$

$(M.2)$ and $(M.3)$ are often replaced by the following weaker conditions respectively;
$(M.2)'$ (stability under differential operators)

$$\exists G, \quad \exists H \text{ such that } M_{p+1} \leq GH^p M_p, \quad p = 0, 1, \cdots .$$

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty.$$

We note that if $\sigma > 1$ then the Gevrey sequence

$$M_p = (p!)^\sigma \tag{1.4}$$

satisfies all the above conditions. For more details about NQA ultradifferentiable functions and NQA ultradistributions confer [Ko1] and [Ko2].

Assume that M_p satisfy $(M.0)$, $(M.1)$ and $(M.3)'$. We denote by $\mathcal{D}^{*\prime}(\Omega)$ the strong dual space of $\mathcal{D}^*(\Omega)$. $\mathcal{D}^{*\prime}(\Omega)$ is called the space of *NQA ultradistributions* of class $*$ defined on Ω, where $* = \{M_p\}$ or (M_p). $\mathcal{E}^{*\prime}(\Omega)$ is the space of compactly supported ultradistributions of class $*$ defined on Ω.

In this paper, we study QA ultradistributions. Let N_p, $p = 0, 1, \cdots$, be a sequence of positive numbers. We impose the following conditions ((QA) and (NA)) instead of $(M.3)$ or $(M.3)'$;
(QA) (quasi-analyticity)

$$N_p \geq p!, \ p = 0, 1, \cdots , \quad \sum_{p=1}^{\infty} \frac{N_{p-1}}{N_p} = \infty.$$

Let N_p be a sequence of positive numbers satisfying (QA). If

$$\liminf_{p\to\infty} \sqrt[p]{\frac{p!}{N_p}} > 0 \tag{1.5}$$

then $\mathcal{E}^{\{N_p\}}$ is the class of analytic functions. We impose the condition that N_p would not define the analytic class, namely,
(NA) (non-analyticity)

$$\lim_{p\to\infty} \sqrt[p]{\frac{p!}{N_p}} = 0.$$

If the sequence N_p satisfies $(M.1)$ and (QA), the sets $\mathcal{D}^{(N_p)}$ and $\mathcal{D}^{\{N_p\}}$ are $\{0\}$ (cf. [C]), however, we define the sheaves $\mathcal{D}^{*\prime}$ of QA ultradistributions of class $*$, where $* = \{N_p\}$ or (N_p).

For a sequence M_p of positive numbers, we define its *associated functions*. For $t > 0$, let

$$\widetilde{M}(t) := \sup_k \frac{t^k}{M_k}, \tag{1.6}$$

$$M(t) := \sup_k \log \frac{t^k}{M_k}. \tag{1.7}$$

$$M^*(t) := \sup_k \frac{t^k k!}{M_k}. \tag{1.8}$$

H.Komatsu [Ko1] proved that for M_p satisfying $(M.1)$, $(M.2)$ and $(M.3)$, it is necessary and sufficient for $f \in \mathcal{D}^{(M_p)\prime}$ (resp. $f \in \mathcal{D}^{\{M_p\}\prime}$) that

$$f(x) = F_1(x + i\Gamma_1 0) + \cdots + F_m(x + i\Gamma_m 0), \tag{1.9}$$

where $i := \sqrt{-1}$, Γ_j, $j = 1, \cdots, m$ are open cones in $\mathbb{R}^n$, $F_j \in \mathcal{O}(\{z \in \mathbb{C}^n \ ; \ z \in \mathbb{R}^n + i\Gamma_j, \ |\mathrm{Im} z| < \exists\varepsilon\})$, $j = 1, \cdots, m$, for which, for any compact set $K \subset \mathbb{R}^n$ there exist constants L and C (resp. for any $L > 0$ there exists C) such that

$$\sup_{x\in K} |F_j(x + iy)| \leq C\widetilde{M}(L/|y|). \tag{1.10}$$

We apply Komatsu's idea to define QA ultradistributions.

Definition 1.2. Let N_p satisfy $(M.1)$, $(M.2)$, (QA) and (NA). We define that $f \in \mathcal{D}^{*\prime}$ if and only if f is expressed as a boundary value of holomorphic functions (1.9) with the estimates (1.10).

Note that we have two ways to define $\mathcal{E}^{*\prime}$ for QA case, both of which coincide with each other.

For a function defined on $\mathbb{R}^n$, its Fourier-Laplace transform is

$$\widehat{f}(\zeta) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\zeta} dx, \qquad \zeta \in \mathbb{C}^n. \tag{1.11}$$

The Paley-Wiener theorem for NQA ultradistributions are proved by H.Komatsu (Theorem 1.1 in [Ko2]). We extend this theorem for QA ultradistributions which are not hyperfunctions. Note that the Paley-Wiener theorem for hyperfunctions are known (Theorem 8.1.1 in [Ka]).

Proposition 1.3 (the Paley-Wiener theorem for ultradistributions). *Assume that a sequence M_p of positive numbers satisfies $(M.0)$, $(M.1)$, $(M.2)'$ and (NA). For a compact convex set $K \subset \mathbb{R}^n$, the following conditions are equivalent.*

i) $\widehat{f}$ *is the Fourier-Laplace transform of* $f \in \mathcal{E}_K^{(M_p)\prime}$ (*resp.* $f \in \mathcal{E}_K^{\{M_p\}\prime}$), *where* $\mathcal{E}_K^{*}{}'$ *is the set of ultradistributions of the class* $*$ *whose supports are contained in* K.

ii) *There exist* $L > 0$ *and* $C > 0$ (*resp. for any* $L > 0$, *there exists* $C > 0$) *such that*

$$|\widehat{f}(\xi)| \le C\widetilde{M}(L|\xi|), \qquad \xi \in \mathbb{R}^n \tag{1.12}$$

and for any $\varepsilon > 0$ *there exists* C_ε *such that*

$$|\widehat{f}(\zeta)| \le C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|), \qquad \zeta \in \mathbb{C}^n \tag{1.13}$$

where

$$H_K(\zeta) := \sup_{x\in K}(x \cdot \mathrm{Im}\ \zeta) \tag{1.14}$$

is the support function of K.

iii) *There exist* $L > 0$ *and* $C > 0$ (*resp. for any* $L > 0$, *there exists* $C > 0$) *such that*

$$|\widehat{f}(\zeta)| \le C\widetilde{M}(L|\zeta|)e^{H_K(\zeta)}, \qquad \zeta \in \mathbb{C}^n. \tag{1.15}$$

PROOF. Equivalence of i) and ii) and details are discussed in [T].

i) $\Rightarrow$ iii); Since $f \in \mathcal{E}^{*}_{K}{}'(\mathbb{R}^n) \subset \mathcal{E}^{*}{}'(\mathbb{R}^n)$, there exist constants h and C (resp. for any $h > 0$, there is a constant C) such that

$$|\langle \varphi, f \rangle| \le C \sup_{x \in K, \alpha} \frac{D^\alpha \varphi(x)}{h^{|\alpha|} M_{|\alpha|}}, \qquad \varphi \in \mathcal{E}^*(\mathbb{R}^n). \tag{1.16}$$

Let

$$\varphi(x) = \exp(-ix \cdot \zeta), \qquad \zeta = \xi + i\eta \in \mathbb{C}^n. \tag{1.17}$$

Then we have

$$|\widehat{f}(\zeta)| \le C \sup_{x \in K, \alpha} \frac{|\zeta|^{|\alpha|}}{h^{|\alpha|} M_{|\alpha|}} |e^{-ix\cdot\xi + x\cdot\eta}| \le C e^{H_K(\zeta)} \widetilde{M}\left(\frac{|\zeta|}{h}\right). \tag{1.18}$$

iii) $\Rightarrow$ ii); For any $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that

$$k! \le M_k \left(\frac{\varepsilon}{L}\right)^k \tag{1.19}$$

for $k > m$ by virtue of (NA). Therefore

$$M(L|\zeta|) = \sup_k \frac{L^k |\zeta|^k}{M_k} \le C \sup_k \frac{(\varepsilon|\zeta|)^k}{k!} \le C e^{\varepsilon|\zeta|}. \tag{1.20}$$

(1.20) and the Paley-Wiener theorem for hyperfunctions (Theorem 8.1.1 in [Ka]) give ii). □

REMARK 1.4. Note that our assumption on M_p is weaker than Theorem 2.2 in [Ko2]. Proposition 1.3 contains Theorem 2.2 in [Ko2], however, Proposition 1.3 does not mention hyperfunctions because of the assumption (NA). As we mentioned above, for hyperfunctions, the Paley-Wiener theorem is known.

§2. Summary of the Known Results

In this section, we review the known results on uniqueness for the Cauchy problem (0.1). As we mentioned in Introduction, this problem was first treated by L.Ehrenpreis (cf. [E]). He proved uniqueness in this problem in distribution category, however, it does not seem that we can find any printed matter about his method. S.Tanabe and T.Takiguchi proved that this uniqueness also holds in NQA ultradistribution category. We shortly review their proof.

LEMMA 2.1 (cf. Theorem 2.2 in [TT]). *Let M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$ and $(M.3)'$, f be an ultradistribution of class $*$ defined on $\mathbb{R}^n$, where $* = \{M_p\}$ or (M_p), and N be a neighborhood of $x = 0$. Assume*

$$(0; \pm dx_n) \notin WF_A(f), \tag{2.1}$$

where $x = (x', x_n) \in \mathbb{R}^n$, $x_n \in \mathbb{R}$ and $WF_A(f)$ is the analytic wave front set of f. Assume furthermore that the restrictions to $\{x_n = 0\} \cap N$ of f and all its derivatives in x_n vanish;

$$\partial_{x_n}^k f|_{\{x_n=0\}\cap N} = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Then $f = 0$ in some neighborhood of $x = 0$.

This lemma is an extension of J.Boman's local vanishing theorem for distributions (cf. [B1]). With this lemma, S.Tanabe-T.Takiguchi proved uniqueness in (0.1) in NQA ultradistribution category.

THEOREM 2.2 (Theorem 6.2 in [TT]). *Assume that u is a NQA ultradistribution satisfying (0.2). Then $u = 0$ in some neighborhood of $\{x = x_0\}$.*

The proof of this theorem is too short and easy to omit, which we introduce.

PROOF. Without loss of generality, we assume that $x_0 = 0$. Since all conormals to t-axis at 0 are non-characteristic, we have $\{(t, x; \tau, \xi) \mid \tau = 0,\ \xi \neq 0\} \cap WF_A(u) = \emptyset$. Therefore Lemma 2.1 gives the theorem. □

The proof of Theorem 2.2 yields a more generalized uniqueness theorem.

THEOREM 2.3. *Let $P(x,D)$ be a partial differential operator with QA coefficients, $S\subset\mathbb{R}^n$ be a QA submanifold whose conormals are non-characteristic with respect to P. Assume that M_p satisfy $(M.0)$, $(M.1)$, $(M.2)'$, $(M.3)'$ and*
$(M.1)'$ (stability under compositions)

$$\exists H \text{ such that } \left(\frac{M_q}{q!}\right)^{1/(q-1)} \le H\left(\frac{M_p}{p!}\right)^{1/(p-1)}, \quad 2\le q\le p.$$

If for $x_0\in S$ and $f\in\mathcal{D}^{*\prime}(\mathbb{R}^n)$, where $*=\{M_p\}$ or (M_p), $P(x,D)f=0$ and that any derivatives along normals to x_0 restricted to x_0 vanish then $f=0$ in a neighborhood of x_0.

The idea to prove this theorem is almost the same as Theorem 2.2, however, in Theorem 2.3, the wave front set in (2.1) should be replaced by the QA one. The counterpart of Theorem 2.2 for QA wave front set in (2.1) is proved by J.Boman [B2].

A.Kaneko proved that there exists a hyperfunction $u(t,x)\not\equiv 0$ in a neighborhood of $\{x=x_0\}$ satisfying (0.2), which implies that uniqueness in (0.1) does not hold for hyperfunctions (cf. [TT]). We shortly review his counterexample. Consider the Cauchy problem,

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}-\Delta u=0,\\ u|_{x_1=0}=\varphi(t,x'), \quad \left.\dfrac{\partial u}{\partial x_1}\right|_{x_1=0}=0,\end{cases} \tag{2.3}$$

where $x=(x_1,x')\in\mathbb{R}^n,\ x_1\in\mathbb{R}$. φ is a famous counterexample by M.Sato (cf. Note 3.3 in [Ka]) satisfying

$$\varphi \text{ contains } x' \text{ as holomorphic parameters at } x'=0, \tag{2.4}$$

$$\partial_{x'}^{\alpha}\varphi|_{x'=0}=0 \quad \text{for all } \alpha, \tag{2.5}$$

$$0\in \operatorname{supp} f, \tag{2.6}$$

where $(t,x')\in\mathbb{R}^n$. Since the equation (0.1) is partially hyperbolic modulo $x'=0$ and initial data contain x' as holomorphic parameters, we have a local hyperfunction solution u (cf. [LK]). Simple calculation yields

$$\partial_{x,t}^{\alpha}u|_{x=0}=0, \quad \text{for any } \alpha. \tag{2.7}$$

We modify A.Kaneko's idea and prove that uniqueness in (0.1) does not hold for QA ultradistributions neither.

§3. Ultradistribution Solution for Partially Hyperbolic Partial Differential Equations

In this section, we study solvability of partially hyperbolic partial differential equations in ultradistribution category. This solvability is one of the main tools to prove non-uniqueness in the Cauchy problem (0.1) in QA ultradistribution category.

We denote $x = (x_1, x') = (x_1, x'', x''') \in \mathbb{R}^n$, where $x'' = (x_2, \cdots, x_{k+1})$, $x''' = (x_{k+2}, \cdots, x_n)$. Let $P(D)$ be an m-th order linear partial differential operator with constant coefficients and $p_m(D)$ be its principal part. We assume that $\{x_1 = 0\}$ is non-characteristic with respect to P. We consider the complexification $z = x + iy$ of $x \in \mathbb{R}^n$ and apply similar notations for x'' and x'''. We put

$$\Omega_A := \{x'' \in \mathbb{R}^k \ ; \ |x''| < A\}, \tag{3.1}$$

$$U_A := \{z''' \in \mathbb{C}^{n-k-1} \ ; \ |z'''| < A\}, \tag{3.2}$$

$$T_A := \{x_1 \in \mathbb{R} \ ; \ |x_1| < A\}. \tag{3.3}$$

Let M_p, $p = 0, 1, \cdots$, be a sequence of positive numbers satisfying $(M.0)$, $(M.1)$ and $(M.2)'$. We denote by $\mathcal{D}^{*\prime}\mathcal{O}(\Omega_A \times U_A)$ the space of ultradistributions of the class $*$ defined on $\Omega_A \times U_A \subset \mathbb{R}^k \times \mathbb{C}^{n-k-1}$ containing $z''' \in U_A$ as holomorphic parameters. For the definition of hyperfunctions and holomorphic parameters, confer [Ka]. In the same way, we define $\mathcal{D}^{*\prime}\mathcal{O}(T_A \times \Omega_A \times U_A)$. We apply the same notations for $\mathcal{E}^{*\prime}\mathcal{O}$.

Our main purpose in this section is to prove the following theorem.

Theorem 3.1. *Let P be a partial differential operator defined above. Assume that the sequence M_p satisfies $(M.0)$, $(M.1)$, $(M.2)$ and (NA). Then the following conditions are equivalent.*

i) *For any $A > 0$, there exist such $0 < a$, $0 < B < A$ that the initial value problem*

$$\begin{cases} P(D)u(x) = 0, \\ \partial_{x_1}^j u|_{x_1=0} = u_j(x'', z'''), \quad j = 0, 1, \cdots, m-1, \end{cases} \tag{3.4}$$

where $u_j \in \mathcal{E}^{*\prime}\mathcal{O}(\Omega_A \times U_A)$, allows an ultradistribution solution $u(x_1, x'', z''')$ of class $* = (M_p)$ (resp. $\{M_p\}$) defined on $T_a \times \Omega_B \times U_B$, whose support is compact in x'', which contains $z''' \in U_B$ as holomorphic parameters.

ii) *There exist constants* $\beta,\ \gamma,\ C, l$ (*resp. there exist* $\beta,\ \gamma$ *and for any* l *there exists* C) *such that*

$$|\mathrm{Im}\zeta_1| \leq \beta|\mathrm{Im}\zeta''| + \gamma|\zeta'''| + M(l|\zeta''|) + C, \tag{3.5}$$

for $P(\zeta) = 0$.

PROOF. The proof of i) $\Rightarrow$ ii);

By virtue of Holmgren's uniqueness theorem, the solution to (3.4) is compactly supported in x'' when x_1 is bounded. Sato's fundamental theorem yields that the solution u contains x_1 as a real analytic parameter. Hence $u|_{x_1=a'}$, $(|a'| < a)$ is defined as an element of $\mathcal{E}^{*\prime}$ in x''. We have a mapping

$$\begin{cases} \left(\mathcal{E}^{*\prime}[K]\widehat{\otimes}\mathcal{O}(U_A)\right)^m & \rightarrow \quad \left(\mathcal{E}^{*\prime}[L]\widehat{\otimes}\mathcal{O}(U_B)\right)^m \\ \qquad \{u_j\}_{j=0}^{m-1} & \mapsto \quad \{\partial_{x_1}^j u|_{x_1=a'}\}_{j=0}^{m-1}, \end{cases} \tag{3.6}$$

where $|a'| < a,\ K, L \subset \mathbb{R}^n$ are compact and $K \subset\subset L$. $\mathcal{E}^{*\prime}[K]\widehat{\otimes}\mathcal{O}(U_A)$ is the space of QA ultradistributions, with support in K in x'', containing $z''' \in U_A$ as holomorphic parameters. By virtue of Proposition 1.3, $\mathcal{E}^{*\prime}[K]\widehat{\otimes}\mathcal{O}(U_A)$ is a Fréchet space with the semi-norms

$$\|v\|_{K,A'} := \sup_{\zeta''\in\mathbb{C}^k,\ |z'''|<A'} \left|\widehat{v}(\zeta'', z''')\frac{1}{\widetilde{M}(l|\zeta''|)}e^{-H_K(\zeta'')}\right|, \tag{3.7}$$

for $A' < A$ (for $\{M_p\}$ class, $l > 0$ is also a parameter of the semi-norms).

(3.6) is continuous by the closed graph theorem. In fact, our Cauchy problem (3.4) admits $\mathcal{O}(\widetilde{L})'\widehat{\otimes}\mathcal{O}(U_B)$-valued solution, where $\widetilde{L} \subset \mathbb{C}^k$ is a polydisc. This solution depends continuously on the initial data. Therefore we obtain a continuous inclusion

$$\mathcal{E}^{*\prime}[L]\widehat{\otimes}\mathcal{O}(U_B) \subset \mathcal{O}(\widetilde{L})'\widehat{\otimes}\mathcal{O}(U_B) \tag{3.8}$$

for $L \subset \widetilde{L}$. Hence (3.6) has a closed graph.

For $\forall B' < B$, there exist $A' < A,\ C$ such that

$$\sum_{j=1}^{m-1} \|\partial_{x_1}^j u|_{x_1=a'}\|_{L,B'} \leq C \sum_{j=0}^{m-1} \|u_j\|_{K,A'}. \tag{3.9}$$

Let

$$(u_0, \cdots, u_{m-1}) = (0, \cdots, 0, \delta(x'')e^{iz'''\cdot\zeta'''}), \tag{3.10}$$

and suppose that $v(x_1, x'', \zeta''')$ be the solution of a Cauchy problem

$$\begin{cases} P(D_1, D'', \zeta''')v(x_1, x'', \zeta''') = 0, \\ \partial_{x_1}^j v|_{x_1=0} = 0, \qquad j = 0, \cdots, m-2 \\ \partial_{x_1}^{m-1} v|_{x_1=0} = \delta(x''). \end{cases} \tag{3.11}$$

As for solvability of (3.11), consider the Fourier-Laplace transform $\widehat{v}(x_1, \zeta'', \zeta''')$ of v with respect to x'' and solve the Cauchy problem

$$\begin{cases} P(D_1, \zeta'', \zeta''')\widehat{v}(x_1, \zeta'', \zeta''') = 0, \\ \partial_{x_1}^j \widehat{v}|_{x_1=0} = \widehat{v}_j(\zeta'') \qquad j = 0, \cdots, m-1, \end{cases} \tag{3.12}$$

for fixed $\zeta'' \in \mathbb{C}^k$, $\zeta''' \in \mathbb{C}^{n-k-1}$. Uniqueness of (3.12) implies that the solution to (3.4) with initial data (3.10) has the form

$$u(x_1, x'', z''') = v(x_1, x'', \zeta''')e^{iz'''\cdot\zeta'''}. \tag{3.13}$$

We have

$$\begin{aligned} (3.14)\quad & P(D_1, D'', \zeta''')\left(Y(a' - x_1)Y(x_1)v(x_1, x'', \zeta''')\right) \\ &= (-i)^m p_m(N)\delta(x_1)\delta(x'') + \\ &\quad + i\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1} P_j(D'', \zeta''')(D_1^{m-k-1-j}\delta(a' - x_1)D_1^k v|_{x_1=a'}) \\ &= (-i)^m p_m(N)\delta(x_1)\delta(x'') + f_{a'}(x_1, x'', \zeta'''), \end{aligned}$$

where $N = (1, 0, \cdots, 0)$, $P_j(D')$ is the coefficients of D_1^{m-j} in $P(D)$ and $f_{a'}$ is defined in (3.14).

For (ζ_1, ζ'') satisfying $P(\zeta_1, \zeta'', \zeta''') = 0$, take the Fourier-Laplace transform of (3.14) with respect to x_1. Then we have

$$(-i)^m p_m(N) + \widehat{f_{a'}}(\zeta_1, \zeta'', \zeta''') = 0. \tag{3.15}$$

Since there holds

$$\widehat{f_{a'}}(\zeta_1,\zeta'',\zeta''') = ie^{-ia'\zeta_1}\sum_{k=0}^{m-1}\sum_{j=0}^{m-k-1} P_j(\zeta')\zeta_1^{m-k-1-j}\widehat{D_1^k v}(a',\zeta'',\zeta'''), \tag{3.16}$$

we have

$$\left|e^{ia'\zeta_1}\widehat{f_{a'}}(\zeta_1,\zeta'',\zeta''')\right| \le C(1+|\zeta_1|+|\zeta'|)^{m-1}\Big|\sum_{j=0}^{m-1}\widehat{D_1^j v}(a',\zeta'',\zeta''')\Big|. \tag{3.17}$$

In view of (3.13), we obtain

$$\begin{aligned}
&\sup_{\zeta_1\in\mathbb{C},\ \zeta''\in\mathbb{C}^k,\ |\zeta'''|\le B'} \frac{1}{(1+|\zeta_1|+|\zeta'|)^{m-1}}\times \\
&\quad\times\left|e^{ia'\zeta_1}\widehat{f_{a'}}(\zeta_1,\zeta'',\zeta''')\frac{e^{iz'''\cdot\zeta'''}}{\widetilde{M}(l|\zeta''|)e^{H_L(\zeta'')}}\right| \\
&\le C\sum_{j=0}^{m-1}\left\|\partial_{x_1}^j u|_{x_1=a'}\right\|_{L,B'} \\
&\le C'\sum_{j=0}^{m-1}\left\|u_j\right\|_{K,A'} = C'\left\|\delta(x'')e^{iz'''\cdot\zeta'''}\right\|_{K,A'} \\
&= C'\sup_{\zeta''\in\mathbb{C}^k,\ |z'''|<A'}\left|e^{iz'''\cdot\zeta'''}\right| \le C'e^{A'|\zeta'''|}.
\end{aligned} \tag{3.18}$$

Assume that $L\subset\{|x''|\le r\}$. Then

$$\frac{1}{(1+|\zeta_1|+|\zeta'|)^{m-1}}\left|\widehat{f_{a'}}(\zeta_1,\zeta'',\zeta''')\right|e^{-r|\mathrm{Im}\zeta''|-B'|\zeta'''|+a'|\mathrm{Im}\zeta_1|}\frac{1}{\widetilde{M}(l|\zeta''|)} \le C'e^{A'|\zeta'''|}. \tag{3.19}$$

Let $\zeta=(\zeta_1,\zeta'',\zeta''')$ satisfy $P(\zeta)=0$. It holds that there exists such $C>0$ that

$$|\zeta_1|\le C(1+|\zeta''|+|\zeta'''|), \tag{3.20}$$

by the non-characteristic property. Therefore, we obtain

$$e^{-r|\mathrm{Im}\zeta''|-(A'+B')|\zeta'''|+a'|\mathrm{Im}\zeta_1|} \le C\widetilde{M}(l|\zeta''|)(1+|\zeta''|+|\zeta'''|)^{m-1}, \tag{3.21}$$

hence

$$|\mathrm{Im}\zeta_1| \le \beta|\mathrm{Im}\zeta''| + \gamma|\zeta'''| + M(l|\zeta''|) + C. \tag{3.22}$$

The proof of ii) $\Rightarrow$ i);

Fix a compact set $K \subset \Omega_A$ and for a fixed $\zeta''' \in \mathbb{C}^{n-k-1}$ let

$$u_j(x'', z''') = v_j(x'')e^{iz'''\cdot\zeta'''}, \qquad v_j \in \mathcal{E}^{*\prime}[K]. \tag{3.23}$$

Consider a Cauchy problem

$$\begin{cases} P(D_1, D'', \zeta''')v(x_1, x'', \zeta''') = 0, \\ \partial_{x_1}^j v|_{x_1=0} = v_j(x''), \quad j = 0, 1, \cdots, m-1. \end{cases} \tag{3.24}$$

The support of the solution to this problem is compact in x'' if x_1 is bounded. Consider the Fourier-Laplace transform in x'' and we obtain a Cauchy problem (3.12) for an ordinary differential equation. For $l = 0, \cdots, m-1$ there exists a solution to a Cauchy problem

$$\begin{cases} P(D_1, \zeta'', \zeta''')f_l = 0, \\ \partial_{x_1}^j f_l|_{x_1=0} = \delta_{jl}, \qquad j = 0, \cdots, m-1. \end{cases} \tag{3.25}$$

By the non-characteristic property we have (3.20) for $P(\zeta_1, \zeta'', \zeta''') = 0$. By Lemma 12.7.7 in [H], the Cauchy problem (3.25) has a unique solution and

$$\begin{aligned} &|f_l(x_1, \zeta'', \zeta''')| \\ &\quad \le 2^m \big(C(1 + |\zeta''| + |\zeta'''|) + 1\big)^{m-l} \\ &\qquad \times \exp\{(\beta|\mathrm{Im}\zeta''| + \gamma|\zeta'''| + M(l|\zeta'''|) + C)|x_1|\}. \end{aligned} \tag{3.26}$$

The solution to (3.12) is given by

$$\widehat{v}(x_1, \zeta'', \zeta''') = \sum_{j=0}^{m-1} \widehat{v_j}(\zeta'')f_j(x_1, \zeta'', \zeta'''). \tag{3.27}$$

By Proposition 1.3,

$$|\widehat{v_j}(\zeta'')| \le C\widetilde{M}(l|\zeta''|)e^{H_K(\zeta'')}. \tag{3.28}$$

Let $|x_1| < \beta'/\beta,\ \beta' < \beta$ then Proposition 3.6 in [Ko1] gives that there exist constants H, J such that

$$
\begin{aligned}
(3.29)\qquad &|\widehat{v}(x_1,\zeta'',\zeta''')| \\
&\le C'(1+|\zeta''|+|\zeta'''|)^m e^{\beta'|\mathrm{Im}\zeta''|} e^{\gamma|\zeta'''||x_1|} e^{2M(l|\zeta''|)} e^{H_K(\zeta'')} \\
&\le C'(1+|\zeta''|+|\zeta'''|)^m e^{\gamma|\zeta'''||x_1|} e^{M(Hl|\zeta''|)+\log J} e^{H_{K_{\beta'}}(\zeta'')},
\end{aligned}
$$

where

$$
(3.30)\qquad K_a := \{y\ ;\ |y-x| \le a,\ x \in K\}.
$$

Therefore

$$
(3.31)\qquad |\widehat{v}(x_1,\zeta'',\zeta''')| \le C(1+|\zeta'''|)^m e^{\gamma|\zeta'''||x_1|} \widetilde{M}(H'l|\zeta''|) e^{H_{K_{\beta'}}(\zeta'')}.
$$

Hence we obtain the solution $v \in \mathcal{D}^{*\prime}$ to (3.12) with support $K_{\beta'}$ in x''. We have a dense inclusion

$$
(3.32)\qquad \left\{\sum v_k(x'') e^{iz'''\cdot\zeta_k'''}\ ;\ v_k \in \mathcal{E}^{*\prime}[K]\right\} \subset \mathcal{E}^{*\prime}[K]\widehat{\otimes}\mathcal{O}(U_A).
$$

In fact, assume $A = 1$ for simplicity. For any $f(x'',z''') \in \mathcal{E}^{*\prime}[K]\widehat{\otimes}\mathcal{O}(U_1)$, we have

$$
\begin{aligned}
(3.33)\quad f(x'',z''') &= \frac{n!}{\pi^n}\int_{|w'''|<1} \frac{f(x'',w''')}{(1-z'''\overline{w'''})^{n+1}} |dw'''|^2 \\
&= \frac{1}{\pi^n}\int_{|w'''|<1} f(x'',w''') \left(\int_0^\infty t^n e^{-t(1-z'''\overline{w'''})} dt\right) |dw'''|^2
\end{aligned}
$$

Approximating all the integrands in the right hand side of (3.33) by Riemannian sums yields that the inclusion (3.32) is dense. Note that the solution depends continuously on the initial data. Hence Theorem 3.1 is proved. □

Remark 3.2. i) The counterparts of Theorem 3.1 for distributions and hyperfunctions are proved in [LK] in a stronger form. In Lee-Kaneko's theorems they do not assume that initial values and solutions are compactly supported in x''. For NQA ultradistributions, this extension is possible since $\mathcal{D}^{*\prime}\mathcal{O}$ is partially soft when $*$ defines NQA class. For QA case, we have to prove partial flabbiness of $\mathcal{D}^{*\prime}\mathcal{O}$ for this extension, which is left open.

ii) Note that our proof holds for the case where P is a convolution operator which acts on QA ultradistributions.

iii) What we claim in Theorem 3.1 is that we have a solution with holomorphic parameter in ultradistribution category, especially in QA ones. The author expresses his gratitude to Professors Kiyoomi Kataoka and Akira Kaneko for having meaningful discussions on the context of this section.

§4. Uniqueness of Functions with Microlocal Analyticity

In this section, we review the results on the following problem.

Problem 4.1. Let f be a function defined in $\mathbb{R}^n$ with

$$(0; \pm dx_n) \notin WF_A(f), \tag{4.1}$$

where $x = (x', x_n) \in \mathbb{R}^n,\ x_n \in \mathbb{R}$, N be a neighborhood of $x = 0$. Assume that the restrictions to $S = \{x_n = 0\} \cap N$ of f and all its derivatives vanish, that is,

$$\partial_{x_n}^k f|_S = 0 \quad \text{for all } k \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

Whether $f = 0$ in some neighborhood of $x = 0$ or not?

The answer to this problem is positive when f is a NQA ultradistribution (cf. Lemma 2.1 above and [TT]) and it is negative when f is a hyperfunction (cf. [Ka]). As we mentioned in Section 2, it is also known that Problem 4.1 is positively solved even if the wave front set of f in (4.1) is replaced by QA one, when f is a NQA ultradistribution (cf. [B2]).

In this section, we review J.Boman's answer to Problem 4.1, when f is a QA ultradistribution (cf. [B3]). He extended M.Sato's counterexample for hyperfunctions to QA ultradistributions and proved that the answer to Problem 4.1 is negative when f is a QA ultradistribution.

Proposition 4.2 (cf. Proposition 3 in [B3]). *There exists a QA ultradistribution $f(x', x'')$ on $\mathbb{R}^n$ satisfying the following conditions.*

$$f \text{ contains } x' \text{ as holomorphic parameters at } x' = 0, \tag{4.3}$$

$$\partial_{x'}^{\alpha}|_{x'=0} f = 0 \quad \text{for all } \alpha, \tag{4.4}$$

$$0 \in \operatorname{supp} f, \tag{4.5}$$

where $(x', x'') \in \mathbb{R}^n$.

M.Sato constructed a hyperfunction satisfying (4.3), (4.4) and (4.5) with $t = x' \in \mathbb{R},\ x = x'' \in \mathbb{R}$. He considered polynomials $p_k(z)$ which approximate $1/z$ in $\mathbb{C} \setminus (-\infty, 0]$ uniformly in the wider sense. Then

$$F(\tau, z) := \sum_{k=0}^{\infty} \frac{p_k(z)}{k!} \tau^k \in \mathcal{O}\left((\mathbb{C} \setminus (-\infty, 0]) \times \mathbb{C}\right) \tag{4.6}$$

is the defining function of the required counterexample

$$f(t, x) = F(t, x + i0) - F(t, x - i0). \tag{4.7}$$

(4.7) is the counterexample by M.Sato (cf. [Ka]).

The idea of J.Boman's extension is the following. Assume that N_p satisfies $(M.0)$, $(M.1)$, $(M.2)'$, (QA) and (NA). Let

$$E := \{z \in \mathbb{C}\ ;\ |z| < 1,\ \mathrm{Im} z \neq 0\}. \tag{4.8}$$

Take such polynomials $p_k(z)$ which approximate $1/z$ uniformly in the wider sense in E that

$$|F(\tau, z)| \leq C_r M^* \Big(\frac{r}{|\mathrm{Im} z|} \Big), \tag{4.9}$$

for $\forall r > 0,\ \exists C_r$, where F is defined in (4.6). (4.9) yields that (4.7) is a QA ultradistribution of class $\{N_p\}$ satisfying (4.3), (4.4) and (4.5). It is not difficult to construct a counterexample in (N_p) class applying the inclusion relation between $\{N_p\}$ and (N_p) classes.

§5. The Main Theorem

In this section, we prove that uniqueness in (0.1) does not hold in QA ultradistribution category, to prove which, Theorem 3.1 and Proposition 4.2 play important roles.

THEOREM 5.1. *Assume that the sequence* N_p *satisfies* $(M.0)$, $(M.1)$, $(M.2)$, (QA) *and* (NA). *There exists such a QA ultradistribution* $u(t, x)$ *of class* (N_p) *or* $\{N_p\}$ *satisfying* (0.2) *that* $u(t, x) \not\equiv 0$ *in any neighborhood of* $x = x_0$.

Proof. For simplicity, let us assume that $x_0 = 0$. Consider the Cauchy problem

$$(5.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ u|_{x_1=0} = \varphi(x', t), \quad \dfrac{\partial u}{\partial x_1}|_{x_1=0} = \psi(x', t) = 0, \end{cases}$$

where $x = (x_1, x') \in \mathbb{R}^n$ and φ is a QA ultradistribution satisfying (4.3), (4.4) and (4.5) with taking t for x''. It is not difficult to see that (3.5) is satisfied for $P(D) = \partial_t^2 - \Delta$. By the construction, φ is compactly supported in t. By virtue of Theorem 3.1, the Cauchy problem (5.1) has a local QA ultradistribution solution $u(t, x)$ near $x_1 = 0$. By (0.1),

$$(5.2)\qquad \partial_{x,t}^{\alpha} u = \sum_{\beta} c_{\beta} \partial_{x',t}^{\beta} \partial_{x_1} u + \sum_{\gamma} c_{\gamma} \partial_{x',t}^{\gamma} u,$$

where $c_\beta,\ c_\gamma = 1$ or -1. We have

$$(5.3)\qquad \partial_{x,t}^{\alpha} u|_{x_1=0} = \sum_{\gamma} c_{\gamma} \partial_{x',t}^{\gamma} \varphi = \sum_{\gamma',\gamma''} c_{\gamma',\gamma''} \partial_t^{\gamma'} \partial_{x'}^{\gamma''} \varphi.$$

Restricting both sides to $\{x' = 0\}$ gives us

$$(5.4)\qquad \partial_{x,t}^{\alpha} u|_{x=0} = \partial_t^{\gamma'} (\partial_{x'}^{\gamma''} \varphi|_{x'=0}) = 0,$$

because $\partial_{x'}^{\gamma''} \varphi|_{x'=0} = 0$. □

Theorem 5.1 completes the study of uniqueness of the Cauchy problem (0.1).

Remark 5.2. Even in NQA ultradistribution category, uniqueness does not hold if initial values are restricted to finite order. More strongly, we construct a counterexample in distribution category. Let $m \in \mathbb{N}$. We have a local distribution solution $u(t, x) \not\equiv 0$ to the Cauchy problem

$$(5.5)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ \partial_x^{\alpha} u|_{x=x_0} = 0 \ \text{ for } |\alpha| \le m. \end{cases}$$

In fact, for simplicity, we assume that $x_0 = 0$. Consider the Cauchy problem

$$(5.6)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0, \\ u|_{x_1=0} = (x_2 \cdots x_n)^{m+1} g(t), \quad \dfrac{\partial u}{\partial x_1}|_{x_1=0} = \psi(x', t) = 0, \end{cases}$$

where $g(t)$ is a distribution of one variable. By Theorem 2 or 3 in [LK], (5.6) has a distribution solution $u(t, x)$ near $x_1 = 0$. It is easy to show that $\partial_x^\alpha u|_{x=0} = 0$ for $|\alpha| \le m$.

In smoother classes where the counterpart of Theorem 3.1 holds, the counterpart of Remark 5.2 is proved. For example, C^∞, ultradifferential and analytic classes are those ones. Note also that the argument in this paper applies to a general linear partial differential equation with analytic coefficients and a real analytic submanifold whose conormals are non-characteristic.

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