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Extension of Tight Contact Structures from Tori to Solid Tori

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Abstract. We study tight contact structures on the solid torus $S^1 \times D^2$. We construct contact structures on the solid torus depending only on the given characteristic foliations on its boundary torus $S^1 \times \partial D^2$. Then, it is shown that tight contact structures on the solid torus, which admit transverse meridian curves on its boundary torus with a self-linking number -1 are determined up to an isotopy by their characteristic foliations on its boundary torus.

0. Introduction

The classification of contact structures on manifolds is a basic problem for a long time since J. Martinet showed that any closed orientable 3-manifold admits a contact structure (see [Mar]). Ya. Eliashberg introduced two notions, *tight* and *overtwisted* structures, for contact structures on 3-manifolds. It is shown in [El1] that the classification up to isotopy of overtwisted oriented contact structures on a compact 3-manifold coincides with the homotopy classification of oriented tangent plane fields. Then only the classification of tight contact structures is left to be open. There are some results for the classifications of tight contact structures on $S^1 \times \mathbb{R}^2, B^3, \mathbb{R}^3, T^3, L(p,q), T^2 \times I$ and so on (see [El2], [El3], [El4], [Et], [Gi2], [H1], [H2], [K]), but this problem is still open for some manifolds, as far as the author knows.

In this paper, we study tight contact structures on the solid torus $S^1 \times D^2$. S. Makar-Limanov partially classifies up to contact diffeomorphism tight contact structures on $S^1 \times D^2$ with "normal" boundaries, under some conditions in [Mak]. The normal boundary means that the characteristic foliation on the boundary torus has no singular point and no Reeb

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component (see [Mak] for the definition). An isotopy classification of tight contact structures in a wider class is considered in this paper. We treat tight contact structures on $S^1 \times D^2$ for which the characteristic foliations on $S^1 \times \partial D^2$ may have singular points satisfying some conditions (see the statement of Theorem A). Recently K. Honda and E. Giroux classify tight contact structures on the solid torus from another point of view (see [H1], [H2], [Gi3]). Their methods give the 3-dimensional contact topology a remarkable progress. Their methods are based on the theory of convex surfaces developed by E. Giroux in [Gi1]. They observed dividing curves on those surfaces, where contact planes are vertical. Those represent the situations of contact structure near the surfaces well, and are flexible. In this paper, we observe carefully characteristic foliations themselves. They are very sensitive and rigid, in a sense. Our approach would give another point of view to understanding tight contact structures. The result of this paper can not be induced from their statement immediately.

Our and the other results are affected by the following classification theorem for tight contact structures on the 3-ball B^3 in [El3].

THEOREM (Eliashberg). *Tight contact structures on the 3-ball B^3 which induce the same characteristic foliation on the boundary sphere ∂B^3 are isotopic relatively to the boundary each other.*

The argument of S. Makar-Limanov in [Mak] is also based on this. He showed a version of the above theorem for balls with transverse corners, and reduced the problem for tight contact structures on the solid torus to it. We consider a version of the above theorem for tight contact structures on solid tori. The above Theorem is proved by the method of extending tight contact structures from the sphere to the ball in [El3]. Similarly, we consider extending tight contact structures from the torus to the solid torus, and classifying tight contact structures on $S^1 \times D^2$. However, some difficulties arise when we treat contact structures on the solid torus. Meridian curves on the boundary torus which are transverse to the contact structure may have several values of self-linking number as transverse knots. This implies that there exist tight contact structures on $S^1 \times D^2$ with the same characteristic foliation on its boundary $S^1 \times \partial D^2$, but which are not isotopic to each other. In this paper, we first consider a simple case when there exists a transverse meridian curve on the boundary torus, with self-linking number -1. Then it is proved that such tight contact structures on the solid torus depends only on their characteristic foliations on the boundary torus.

First of all, we review several basic notions in order to state the results.

A *contact structure* on a 3-manifold is a completely non-integrable tangent plane field. The complete non-integrability implies that this tangent plane field is defined, at least locally, by a 1-form α which satisfies the inequality $\alpha \wedge d\alpha \neq 0$ everywhere. The sign of the form $\alpha \wedge d\alpha$ depends only on the contact structure. Then it defines an orientation of the base manifold. When the base manifold is oriented, we can distinguish whether the contact structure is positive or negative. In this paper, we treat only positively oriented contact structures.

Let F be an embedded surface in a contact 3-manifold (M, ξ). The contact structure ξ traces on F a 1-dimensional singular foliation. Such a foliation is called the *characteristic foliation* on F with respect to ξ, and F_ξ denotes it. When $\xi = \{\alpha = 0\}$ and F are oriented, F_ξ is oriented by the vector field X on F which satisfies $X \lrcorner \mathrm{vol}_F = i^*\alpha$, where vol_F is a volume form of F, and $i\colon F \hookrightarrow M$ is the inclusion mapping. Generically, the characteristic foliation F_ξ has a finite number of singular points where F is tangent to ξ. A singular point is called *positive* or *negative* depending on whether the orientations of F and ξ coincide at the point or not. Generically, the index of the vector field which defines the characteristic foliation locally at a singular point is ± 1. We call a singular point *elliptic* if its index is $+1$, and *hyperbolic* if it is -1. Because of the non-integrability of contact structures, the characteristic foliation F_ξ has topologically the focus type singularity at elliptic points, and F_ξ has the saddle type singularity at hyperbolic points. These singularities are called simple singularities.

A contact structure ξ is called *tight* if for any embedded disc $\mathcal{D} \subset (M, \xi)$ the characteristic foliation $\mathcal{D}_\xi$ has no limit cycle. A contact structure which is not tight is called *overtwisted* (see [El3]). In this paper, a tight contact structure ξ defined near $S^1 \times \partial D^2 \subset S^1 \times D^2$ means that the characteristic foliation $\mathcal{D}_\xi$ defined near ∂D has no limit cycle for any embedded disc $\mathcal{D} \subset S^1 \times D^2$.

A closed immersed oriented curve $\gamma\colon S^1 \to (M, \xi)$ is called positively (resp., negatively) *transverse* to ξ if $\gamma^*\alpha(\partial/\partial\theta)$ is positive (resp., negative), where $\partial/\partial\theta$ gives the positive orientation of S^1. For transverse knots a transverse isotopy invariant is defined (see [B], [El3], [El5]). Let Γ be a transverse knot in a contact manifold (M, ξ) which is homologous to zero. Fix a relative homology class $\beta \in H_2(M, \Gamma)$. Let F be a surface bounded by Γ which represents β : $[F] = \beta \in H_2(M, \Gamma)$. Let ν be a non-vanishing vector field tangent to $\xi|_F$. Then ν is transverse to Γ, and we can perturb Γ slightly along ν to a curve Γ'. We define $l(\Gamma|\beta)$ to be the intersection number of Γ' and β. It is well defined and we call it the *self-linking number* of Γ with respect to β. We note that this invariant is independent of the orientation of knots.

The results of this paper are the following.

THEOREM A. *Let ξ be a germ of a tight contact structure on the solid torus $S^1 \times D^2$ along the boundary torus $S^1 \times \partial D^2$. We suppose that singular points of the characteristic foliation $(S^1 \times \partial D)_\xi$ are simple and that closed orbits are a finite number of limit cycles which are isotopic to each other by orientation preserving isotopies. Then ξ can be extended, to a contact structure $\tilde{\xi}$ on $S^1 \times D^2$, for which there exists a transverse meridian curve C in $S^1 \times \partial D^2$ with self-linking number $l(C) = -1$. This construction depends only on the characteristic foliation on $S^1 \times \partial D^2$.*

Moreover, the following theorem implies that contact structures on the solid torus $S^1 \times D^2$ with a transverse meridian $C \subset S^1 \times \partial D^2$ of self-linking number $l(C) = -1$ depend only on their characteristic foliation on $S^1 \times \partial D^2$. It is also proved by S. Makar-Limanov in [Mak], in the case when the boundaries are normal with a different method.

THEOREM B. *Let ξ_0, ξ_1 be tight contact structures on the solid torus $S^1 \times D^2$ with the same characteristic foliation $\mathcal{F} = (S^1 \times \partial D^2)_{\xi_0} = (S^1 \times \partial D^2)_{\xi_1}$ on the boundary torus. Suppose that a positively transverse meridian curve C exists in $S^1 \times \partial D^2$, which self-linking number is $l(C) = -1$ for both of ξ_0 and ξ_1. Then ξ_0 and ξ_1 are isotopic relatively to $S^1 \times \partial D^2$.*

In the following section we study the taming functions for characteristic foliations on a 2-torus. The taming function on a 2-sphere were studied by Ya. Eliashberg in [El3]. The S^1-valued taming function for a characteristic foliation on a 2-torus is defined in this section. The taming function inherits some properties of the characteristic foliation (see Section. 1.1 for the definition).

Section 2 is devoted to the extension of tight contact structures from $S^1 \times \partial D^2$ to $S^1 \times D^2$. A tight contact structure on $S^1 \times D^2$ with a transverse meridian of self-linking number -1 is constructed depending only on the characteristic foliation on $S^1 \times \partial D^2$. We obtain a germ of strictly pseudoconvex embeddings of $S^1 \times D^2$ along $S^1 \times \partial D^2$ into $\mathbb{C}^2$ with respect to the standard complex structure J_0. The contact structure induced by complex tangencies of the above germ induce the same characteristic foliation on $S^1 \times \partial D^2$ as the given one. We show Theorem A in this section.

Theorem B is proved in Section 3. The condition on self-linking numbers of transverse meridians is essential for the proof. We show that there exists a family of strictly pseudoconvex embeddings for a family of characteristic foliations satisfying certain conditions. Thus there exists a family of contact structures on $S^1 \times D^2$ for a family of characteristic foliations on $S^1 \times \partial D^2$. Then we show that all contact structures having the same characteristic foliation and a positively transverse meridian with self-linking number -1 is path connected.

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1. Taming Function

1.1. Definition of the taming function

Let $S^1 \times D^2$ be a solid torus embedded into a contact 3-manifold (M, ξ), and $T := S^1 \times \partial D^2$ its boundary. Let X be a vector field generating the characteristic foliation T_ξ. We assume that X has a finite number of singular points which are simple or of "birth-death" type.

Definition. We say that an S^1-valued function $f\colon T \to S^1$ *tames* the characteristic foliation T_ξ if the following properties are satisfied.

(1) The generating field X is gradient like. In other words, the inequality $df(X) \geq \varepsilon \cdot \|df\|^2$ holds for a positive constant ε and a Riemannian metric on T. The singularities of X coincide with the critical points of f in particular.

(2) Positive (resp., negative) elliptic points of T_ξ are local minima (resp., maxima) of f.

(3) When we pass through a critical value of f corresponding to a hyperbolic point in the positive direction of S^1, the number of connected component of f increases (resp., decreases) if the point is negative (resp., positive).

We call a taming function $f\colon T \to S^1$ of T_ξ the *ℓ-taming* function if it has the following properties,

- there is a longitude curve $l \subset T$ where f is monotone,
- there is a regular value $a \in S^1$ of f which level set $f^{-1}(a)$ is a meridian curve $C \subset T$.

In the following of this paper, we consider only ℓ-taming functions, and suppose $a = [0] \in S^1$.

1.2. Existence of taming functions

The following proposition guarantees the existence of a taming function for certain characteristic foliations.

PROPOSITION 1.1. *Let $S^1 \times D^2$ be a solid torus embedded into a tight contact 3-manifold (M, ξ) and $T := S^1 \times \partial D^2$ its boundary. Suppose that singular points of T_ξ are elliptic or hyperbolic. In addition, we suppose that all closed leaves of T_ξ have the same orientation if they exist. Then, there exists an ℓ-taming function $f\colon T \to S^1$ for T_ξ.*

To prove this proposition, we need the following lemmas. First, we introduce the Elimination Lemma which is a result due to E. Giroux in an improved version by D. Fuchs (see [Gi1], [El3]).

LEMMA 1.2 (Elimination Lemma). *Let F be an embedded surface in a contact 3-manifold (M, ξ), and $p, q \in F$ be elliptic and hyperbolic points of F_ξ with the same sign. If there is a trajectory γ of F_ξ joining p and q, then there exists a C^0-small isotopy $h_t\colon F \to (M, \xi)$, $t \in [0, 1]$, which has the following properties.*

(1) *h_t is the identity on γ and outside a neighborhood U of γ.*

(2) $h_0 = \mathrm{id}$.

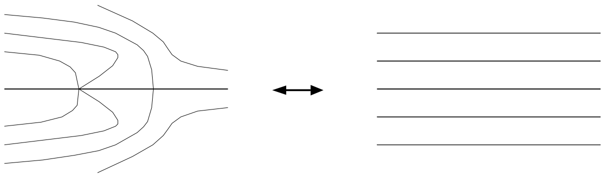

Figure 1.

(3) *The characteristic foliation* $\tilde{F}_\xi$ *on* $\tilde{F} := h_1(F)$ *has no singular point in* $\tilde{F} \cap U$.

(see Fig.1)

The converse of this lemma is easily shown. This means that it is easy to create singular points (see [El3]).

LEMMA 1.3. *By a* C^0*-small isotopy of the embedded surface near a non-singular point, a pair of elliptic and hyperbolic singular points with the same sign can be created.*

S. Makar-Limanov studies in [Mak] characteristic foliations on a torus T embedded in a tight contact 3-manifold (M, ξ).

LEMMA 1.4. T *can be deformed by a* C^0*-small isotopy* $h_t \colon M \to M$ *so that* $h_1(T)$ *is normal, that is,* $h_1(T)_\xi$ *is non-singular and without Reeb components.*

We note that singular points are canceled by the Elimination Lemma 1.2.

The following lemma is essential for the proof of Proposition 1.1. This shows the nature of the characteristic foliations on a 2-torus for tight contact structures. This lemma treats not only generic case but also non-generic case for the later use.

LEMMA 1.5. *Let* $T = S^1 \times \partial D^2$ *be the* 2*-torus which is the boundary of a solid torus embedded into a tight contact* 3*-manifold* (M, ξ)*. We suppose that singular points of* T_ξ *are elliptic or hyperbolic, and there is at most one separatrix connection between hyperbolic points. In addition, we suppose that all closed leaves of* T_ξ *have the same orientation if they exist. Then,* *there exists a meridian curve C in T which is transverse to ξ. Moreover by a C^0-small perturbation of T fixing a neighborhood of C, all singular points of T_ξ can be canceled.*

We note that there are transverse meridians which can not be fixed when singularities are eliminated. It is important for the following arguments that we can eliminate all singular points fixing a transverse meridian curve.

PROOF. First, we consider the case when T is generic, that is to say, T_ξ has no separatrix connecting two hyperbolic points. By Lemma 1.4, we obtain a normal foliation $\mathcal{F}$ on perturbed T. We note that $\mathcal{F}$ never admits meridian closed orbit because of the tightness of ξ. Then we can take a meridian curve C on T which is transverse to $\mathcal{F}$. Recall that $\mathcal{F}$ is obtained from T_ξ by the Elimination Lemma 1.2. The support of the corresponding C^0-small perturbation is contained in an arbitrary small neighborhood of trajectories of T_ξ joining elliptic and hyperbolic points with the same sign. Then, T_ξ can be obtained from $\mathcal{F}$ by Lemma 1.3. If the above curve C intersects the support, we can replace C not to intersect it. In fact, as the foliation is trivial sufficiently near the support, the replacement can be done preserving transversality to the foliation (see Fig.2). After all, we obtain a meridian curve C transverse to T_ξ, which does not intersect the supports of the perturbation of eliminations.

Next, we consider the non-generic case. There are separatrices connecting hyperbolic points. We have supposed that there is exactly one separatrix connecting two hyperbolic points. There are the following four cases.

(1) The separatrix is unstable for the positive hyperbolic point and stable for the negative one.

(2) The separatrix is stable for the positive hyperbolic point and unstable for the negative one.

(3) The separatrix connects two positive hyperbolic points.

(4) The separatrix connects two negative hyperbolic points.

Other separatrices of these connected hyperbolic points are connected with elliptic points or go to limit cycles. If either of these hyperbolic points are connected with an elliptic point with the same sign, this connection between

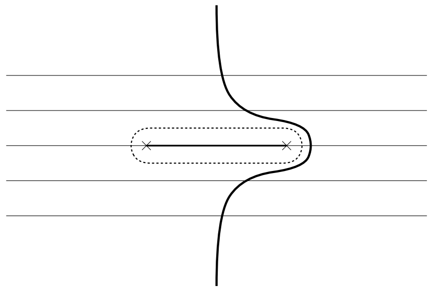

Figure 2.

hyperbolic points are canceled by the Elimination Lemma 1.2. Then, we can apply the same argument as the generic case above. Therefore we have only to consider the cases that the stable separatrices of positive hyperbolic points come from limit cycles (repellers), and that the unstable separatrices of negative hyperbolic points go to limit cycles (attractors). We are going to show that these cases would not occur.

Let us consider in each case. First, we consider Case (1). According to the tightness of ξ, closed leaves of the characteristic foliation T_ξ are non-nullhomotopic and belong to the same torus knot class. Then we consider an annulus which boundaries are an attractor and a repeller. Stable separatrices of the positive hyperbolic point must come from the same repeller. By applying Lemma 1.3 near this repeller, these separatrices come from the same positive elliptic point. An closed orbit is obtained by the Elimination Lemma 1.2. This contradicts the tightness. Thus this case never occurs. (see Fig.3) We can prove that the other cases never occur in similar ways. We can find the situation where both of the stable (unstable) separatrices of a hyperbolic point come from (reach to) the same repeller (attractor), by breaking a separatrix connection between hyperbolic points if necessary. $\square$

It was shown by Ya. Eliashberg in [El3] that there exists an $\mathbb{R}$-valued taming function $\overline{f}\colon S^2 \to \mathbb{R}$ for a 2-sphere S^2 embedded into a tight contact 3-manifold (M,ξ) if all singular points of S^2_ξ are elliptic or hyperbolic. We can prove Proposition 1.1 by reducing the problem to this case.

Proof of Proposition 1.1. Let $T := S^1 \times \partial D^2$ be the boundary of an embedded solid torus in a tight contact 3-manifold (M,ξ). There exists

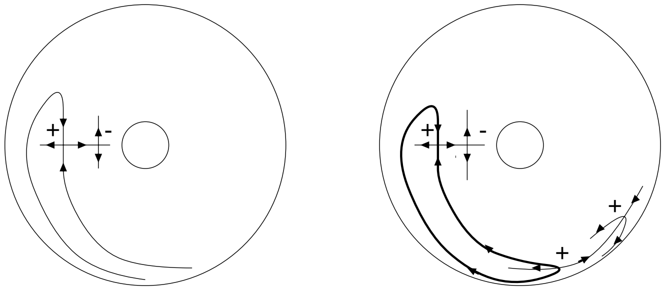

Figure 3. Case(1).

a meridian curve $C \subset T$ transverse to T_ξ, according to Lemma 1.5. Then we can take a sufficiently small tubular neighborhood $N \subset T$ of γ with respect to the characteristic foliation T_ξ, that is, N is diffeomorphic to $S^1 \times [0,1]$ and leaves of N_ξ correspond to fibres $\{x\} \times [0,1]$. Let $S_\pm$ be the components of ∂N where trajectories of N_ξ look inwards or outwards respectively. We set $A := T \setminus N$. It is an embedded open annulus. We consider a singular foliation $\mathcal{F}$ on a 2-sphere S^2 obtained as follows. Let $p_\pm \in S^2$ be the north and south poles. We can identify $S^2 \setminus \{p_+, p_-\}$ with the annulus $A = T \setminus N$. The foliation $\mathcal{F}$ is identified with A_ξ on $S^2 \setminus \{p_+, p_-\}$. The points $p_\pm$ are considered to be elliptic points of $\mathcal{F}$. We note that this singular foliation $\mathcal{F}$ on S^2 determines a germs on tight contact structure. In fact, we can eliminate all singular point of A_ξ fixing both ends, according to Lemma 1.5. Then we can apply the argument of Eliashberg in [El3] to the foliation $\mathcal{F}$ on S^2. We obtain a taming function $\tilde{f}\colon S^2 \to \mathbb{R}$ for $\mathcal{F}$, which satisfies $\tilde{f}(p_+) = 2\pi - \varepsilon$, $\tilde{f}(p_-) = \varepsilon$ for some small $\varepsilon > 0$. From this function, we obtain a taming function $\overline{f}\colon \overline{A} \to \mathbb{R}$ for A_ξ, which satisfies $\overline{f}(S_+) = 2\pi - \varepsilon$, $\overline{f}(S_-) = \varepsilon$. Last of all, by extending $\overline{f}$ to N, we obtain an S^1-valued taming function for T_ξ, $f\colon T \to S^1$, which satisfies that $f(C) = [0] = [2\pi] \in S^1$.

We note that the meridian curve C is the unique component of a level set $f^{-1}([0])$, from the construction. Further, there exists a segment $\bar{l} \subset \overline{T \setminus N}$ connecting two boundary components of $\partial(\overline{T \setminus N})$, where f is monotone, according to the construction. Since the characteristic foliation N_ξ is trivial and f is monotone on each leaf, the segment $\bar{l}$ is extended to a closed curve $l \subset T$ so that l is a longitude of T and that f is monotone on l. Thus we show that f is the required ℓ-taming function. $\square$

1.3. Existence of families of taming functions

The main result of this section is the following.

Proposition 1.6. *Let $T := S^1 \times \partial D^2$ be the boundary of a solid torus embedded into a 3-manifold M. Let $\{\xi_t\}_{t\in[0,1]}$ be a smooth family of tight contact structures defined near $S^1 \times \partial D^2$, which satisfy that all closed orbits of the characteristic foliation T_{ξ_t} have the same orientation if they exist. Suppose that there exist a finite number of values $0 < t_1 < t_2 < \ldots < t_n < 1$, and a finite number of disjoint closed intervals $I_1, I_2, \ldots, I_k \subset [0,1]$, which satisfy that T_{ξ_t} is generic for $t \in [0,1] \setminus (\{t_1, t_2, \ldots t_n\} \cup I_1 \cup I_2 \cup \ldots I_k)$. In addition, we suppose that the characteristic foliation T_{ξ_t} admits either a "birth-death" type singular point or a separatrix connection between hyperbolic points for $t = t_1, t_2, \ldots, t_n$, and that T_{ξ_t} is normal for $t \in \operatorname{int} I_j, j = 1, 2, \ldots, k$ and each interval contains a point at which T_{ξ_t} is not generic.*

Then, for any two S^1-valued functions $f_0, f_1 \colon T \to S^1$ which tame T_{ξ_0}, T_{ξ_1} respectively, there exists a family of S^1-valued taming functions $\{f_t\}_{t\in[0,1]}$ including the above f_0 and f_1, each of which tames T_{ξ_t} for all $t \in [0,1]$.

A similar result for $S^2 = \partial D^3$ is proved by Ya. Eliashberg in [El3]. The proof of the proposition above is parallel to the arguments in [El3]. The following lemma is a key for the proof of this proposition in the case for a torus.

Lemma 1.7. *Let ξ be a tight contact structure defined near $S^1 \times \partial D^2$. Suppose that all singular points of $T_\xi = (S^1 \times \partial D)_\xi$ are non-degenerate. Let f_0, f_1 be two taming functions for the same T_ξ. Then, there exists a smooth family of taming functions $\{f_t\}_{t\in[0,1]}$ for T_ξ containing f_0, f_1.*

Proof. We will prove that any taming function for a characteristic foliation can be deformed, as taming functions, to the same one. Let f be a taming function for T_ξ. Fix a regular value $[0] \in S^1$ of f. We may suppose that the level set $f^{-1}([0])$ is a meridian curve on T by homotopy among taming functions. Suppose that $\operatorname{Im} f \subset (0, 2\pi)$ in the complement of this meridian. We note that all taming functions for T_ξ have the same critical points. Therefore, taming functions for T_ξ with a fixed regular value $[0] \in S^1$, for which the level set is a meridian curve, are homotopic among taming functions if they have the same ordering of corresponding critical value. Let $c_1 < c_2$ are consecutive critical values. If c_2 corresponds to a positive elliptic point, we can change the order of critical values of critical points corresponding to c_1 and c_2 by a homotopy among taming functions, whichever type a critical point corresponding to c_1 may have. In fact, we can make a critical value corresponding to a positive elliptic point arbitrary small by a homotopy among taming functions, because taming functions have positive elliptic points as their minimal points. In the case when c_1 corresponds to a negative elliptic point, we can change the order similarly. Then we can arrange the function f so that it has critical values corresponding to positive and negative elliptic points near $[0]$ and $[2\pi]$ respectively, and those corresponding to hyperbolic points between them. We can exchange the order of critical values corresponding to hyperbolic points unless they are connected by separatrices. Eventually, we obtain the same ordering of critical values.

The homotopy class in taming functions is independent of the choice of a fixed regular value, or a transverse meridian curve. In fact, although the change of the regular value alter the order of critical points, we can adjust it in the same manner mentioned above. □

The existence of a taming function for a characteristic foliation, with respect to a tight contact structure, on a sphere are proved by Ya. Eliashberg in [El3]. The following elementary lemmas, which he use, are valid in the case of a torus.

LEMMA 1.8. *Let ξ be a contact structure defined near $T = S^1 \times \partial D^2 \subset M$, and $f\colon T \to S^1$ a taming function for T_ξ. Suppose that the contact structure $\tilde{\xi}$ is sufficiently C^0-close to ξ and coincides with ξ near singular points of T_ξ. Then, f is also a taming function for $T_{\tilde{\xi}}$.*

LEMMA 1.9. *Let $\{\xi_t\}_{t\in[-1,1]}$ be a family of contact structures near T, which satisfies that T_{ξ_t} is generic for $t \neq 0$, and that T_{ξ_0} has a "birth-death" type singular point $p \in T$ and T_{ξ_t} are isomorphic outside an arbitrary small neighborhood of p in T for all $t \in [-1,0]$. If the characteristic foliation $T_{\xi_{-1}}$ admits a taming function f, then there exists a smooth family of functions $\{f_t\colon T \to S^1\}_{t\in[-1,1]}$ which satisfies that f_t tames T_{ξ_t} for each $t \in [-1,1]$ and $f_{-1} = f$.*

Now, we are ready to prove the main result of this section. Proposition 1.6 can be proved in a similar manner as in [El3]. In order to make this paper reasonably self-contained, we give an outline of the proof.

Proof of Proposition 1.6. We construct the family of taming functions near $t = t_1, \dots, t_n$. First, suppose that $T_{\xi_{t_j}}$ has a separatrix connection between hyperbolic points for $t = t_j$. There is a taming function f_{t_j} for $T_{\xi_{t_j}}$ according to Proposition 1.1. Then, on account of Lemma 1.8, there is a family of taming functions $\{f_t\}_t$ for T_{ξ_t} defined for t close to t_j, which includes f_{t_j}. Next, suppose that $T_{\xi_{t_k}}$ has a "birth-death" type singular points for $t = t_k$. There exists a family of taming functions $\{f_t\}_t$ for T_{ξ_t} defined for t close to t_k, including f_{t_k} according to Lemma 1.9.

It is clear that a non-singular taming function exists for a normal characteristic foliation. Moreover, there exists a family of non-singular taming functions for a family of normal characteristic foliations. Then, there exists a family of taming functions $\{f_t\}_t$ for T_{ξ_t} on $t \in I_j$.

We may assume that there exists a transverse meridian $C \subset T$ for a given open interval (t_i, t_{i+1}), that is, C is transverse to ξ_s for any $s \in (t_i, t_{i+1})$. Let $A \subset T$ be an annulus obtained as a complement of a tubular neighborhood of C, in the same way as in the proof of Proposition 1.1. If a taming function f_s for $s \in (t_i, t_{i+1})$ is given, there is a family of taming functions $\{\overline{f}_t\}_{t\in[t_i,t_{i+1}]}$ on A including $f_s|_A$ because A_{ξ_t} are equivalent topologically for all $t \in (t_i, t_{i+1})$. We can easily extend it to a family of taming functions on T. Moreover, by Lemma 1.7, there is a family of taming functions $\{f_t\}_{t\in[t_i,t_{i+1}]}$ containing f_{s_1}, f_{s_2} for given two taming functions f_{s_1} and f_{s_2} for $T_{\xi_{s_1}}$ and $T_{\xi_{s_2}}$, $s_1, s_2 \in (t_i, t_{i+1})$. Thus we can extend the family of taming functions for all $t \in [0, 1]$. $\square$

2. Extension of Contact Structures from the Torus to the Solid Torus

In this section, we prove Theorem A. We consider constructing contact structures on a solid torus $S^1 \times D^2$ from generic characteristic foliations on its boundary torus which satisfy the condition of Theorem A. In this case, there exist the taming functions for the given characteristic foliation from Proposition 1.1. Moreover, we construct a smooth family of contact structures for a smooth family of pairs of characteristic foliations and their taming functions. To be precise, we construct smooth family of strictly pseudoconvex embedding of $S^1 \times D^2$ into $\mathbb{C}^2$. As a corollary, it is proved that the constructed contact structures depend only on their characteristic foliations because Lemma 1.7 guarantees the existence of a family of taming functions for a characteristic foliation. This implies Theorem A. At the end of this section, we check the self-linking number of a transverse meridian.

First, we recall some fundamental notions. Let (W, J) be a real 4-dimensional almost complex manifold with an almost complex structure J, and $M \subset W$ a real hypersurface. Then M admits a tangent plane field ζ formed by its complex tangency: $\zeta := TM \cap J(TM)$. Suppose that M is oriented. This 3-manifold is said to be *strictly pseudoconvex* (J-convex) if ζ is a positive contact structure on M. Let $\phi\colon M \to (W, J)$ be an embedding which image $\phi(M) \subset (W, J)$ is a strictly pseudoconvex hypersurface. This embedding is also called the *strictly pseudoconvex* embedding. Let $\mathrm{ct}(\phi)$ denote the contact structure $(\phi^{-1})_*\{T(\operatorname{Im}\phi) \cap J(T(\operatorname{Im}\phi))\}$ on M, which is the pull back of the contact structure constructed by complex tangency of $\phi(M) \subset (W, J)$.

The main result of this section is formulated as follows. Let $(\mathbb{C}^2, J_0)$ be a complex 2-plane with the standard complex structure. Its coordinate is given as $(z_1, z_2) = \big(r_1 \cdot \exp(\sqrt{-1}\theta_1), r_2 \cdot \exp(\sqrt{-1}\theta_2)\big) \in \mathbb{C}^2$. We identify $S^1 \times \mathbb{R}^2$ with $K_c := \{r_2 = c\} \subset \mathbb{C}^2$ for some constant $c > 0$. We set $pr\colon \mathbb{C}^2 \setminus \{r_2 = 0\} \to S^1$ the projection to θ_2-elements; $(r_1, \theta_1, r_2, \theta_2) \mapsto \theta_2$, and $p\colon \mathbb{C}^2 \to \mathbb{R}$ the projection to r_2-elements; $(r_1, \theta_1, r_2, \theta_2) \mapsto r_2$.

PROPOSITION 2.1. (***a***) *Let $(f, \mathcal{F})$ be a pair of a singular foliation $\mathcal{F}$ on $T = S^1 \times \partial D^2$ and its taming function $f\colon T \to S^1$. Then there exists a strictly pseudoconvex embedding $\varphi\colon S^1 \times D^2 \to \mathbb{C}^2 \setminus \{r_2 = 0\}$ which satisfies*

- $\varphi(S^1 \times \partial D^2) \subset K_c$, *for sufficiently small constant $c > 0$,*
- $p \circ \varphi(S^1 \times \operatorname{int} D) > c$,
- $\operatorname{Im}\varphi$ *is contractible to $\{r_1 = 0, r_2 = 1\}$ in $\mathbb{C}^2 \setminus \{r_2 = 0\}$,*
- $(S^1 \times \partial D^2)_{\mathrm{ct}(\varphi)} = \mathcal{F}$,
- $pr \circ \varphi|_{S^1 \times \partial D^2} = f$.

(***b***) *Let $\{F_t = (f_t, \mathcal{F}_t)\}_{t \in [0,1]}$ be a smooth family of pairs of singular foliations $\mathcal{F}_t$ on $T = S^1 \times \partial D^2$ and their taming functions $f_t\colon T \to S^1$. Suppose that* *two strictly pseudoconvex embeddings* $\varphi_0, \varphi_1 \colon S^1 \times D^2 \to \mathbb{C}^2 \setminus \{r_2 = 0\}$ *satisfying the above condition for* F_0, F_1, *are given. Then, there exists a family of strictly pseudoconvex embeddings* $\{\varphi_t \colon S^1 \times D^2 \to \mathbb{C}^2 \setminus \{r_2 = 0\}\}_{t \in [0,1]}$ *containing the above* φ_0, φ_1, *which satisfies the following properties,*

- $(S^1 \times \partial D^2)_{\mathrm{ct}(\varphi_t)} = \mathcal{F}_t$,
- $pr \circ \varphi_t|_{S^1 \times \partial D^2} = f_t$,

for all $t \in [0,1]$.

To prove the Proposition above, we prepare the following lemmas.

LEMMA 2.2. *Let* $\{f_t \colon S^1 \times \partial D^2 \to S^1\}_{t \in [0,1]}$ *be a family of* S^1*-valued functions which satisfies that singular points of each function* f_t *are non-degenerate or of "birth-death" type, and that there are meridian curves* $C_t \subset S^1 \times \partial D^2$ *which are unique components of* $f^{-1}([0])$ *and longitude curves* $l_t \subset S^1 \times \partial D^2$ *where* f_t *are monotone. Suppose that there are two given embeddings* $\psi_0, \psi_1 \colon S^1 \times D^2 \to S^1 \times \mathbb{R}^2$ *which satisfy the following conditions,*

- $pr \circ \psi_i|_{S^1 \times \partial D^2} = f_i$,
- *The vector* $\partial/\partial\theta_2$ *is the normal vector looking inwards* (*resp., outwards*) *at positive* (*resp., negative*) *singular points of* f_i,

for $i = 0, 1$, *where* θ_2 *is the first coordinate of* $S^1 \times \mathbb{R}^2$. *Then there exists a family of embeddings* $\{\psi_t \colon S^1 \times D^2 \to S^1 \times \mathbb{R}^2\}_{t \in [0,1]}$ *containing the above* ψ_0, ψ_1, *and satisfying the following properties for all* $t \in [0,1]$.

- $pr \circ \psi_t|_{S^1 \times \partial D^2} = f_t$.
- *The vector* $\partial/\partial\theta_2$ *is the normal vector looking inwards* (*resp., outwards*) *at positive* (*resp., negative*) *singular points of* f_t.

PROOF. It is proved in a way similar to that used by Ya. Eliashberg in [El3]. Let Γ_{f_t} be the graph constructed by identifying a component of level set of f_t with a point. f_t induces a function $\overline{f_t} \colon \Gamma_{f_t} \to S^1$ on this graph. Then there exists a family of embeddings $\{\overline{\psi_t} \colon \Gamma_{f_t} \to S^1 \times \mathbb{R}^2\}_{t \in [0,1]}$ which satisfies $pr \circ \overline{\psi_t} = \overline{f_t}$. By taking a regular neighborhood of $\overline{\psi_t}(\Gamma_{f_t}) \subset S^1 \times \mathbb{R}^2$, we obtain a family of embeddings $\{\psi_t \colon S^1 \times D^2 \to S^1 \times \mathbb{R}^2\}_{t \in [0,1]}$ which satisfies the required condition. The space of embeddings $\psi\colon S^1\times D^2\to S^1\times\mathbb{R}^2$ satisfying the condition of the lemma for one function $f\colon S^1\times\partial D^2\to S^1$ is connected. In fact, we need not to care the knot classes because of the conditions of the functions f_t mentioned in this lemma. Thus we can take a family $\{\psi_t\}_{t\in[0,1]}$ which contains the given functions ψ_0,ψ_1. $\square$

LEMMA 2.3. *Let $\{(\psi_t,\mathcal{F}_t)\}_{t\in[0,1]}$ be a family of pairs of embeddings $\psi_t\colon S^1\times D^2\to S^1\times\mathbb{R}^2=K_c\subset\mathbb{C}^2$ and characteristic foliations $\mathcal{F}_t$ on $T=S^1\times\partial D^2$ which are tamed by $pr\circ\psi_t|_{S^1\times\partial D^2}$ for each $t\in[0,1]$. Then there exists a family of 1-forms $\{\alpha_t\in ST^*\mathbb{C}^2\}_{t\in[0,1]}$ defined on $\psi_t(S^1\times\partial D^2)$, which defines hyperplane fields $\ker\alpha_t$ satisfying the followings,*

$$T_p\left(\psi_t(S^1\times\partial D^2)\right)\cap\left\{\ker\alpha_t\cap J_0(\ker\alpha_t)\right\}_p=T_p(\psi_{t*}\mathcal{F}_t)$$

at each non-singular point $p\in\psi_t(S^1\times\partial D^2)$ of $\psi_{t*}(\mathcal{F}_t)$, and

$$T_p\left(\psi_t(S^1\times\partial D^2)\right)=\left\{\ker\alpha_t\cap J_0(\ker\alpha_t)\right\}_p$$

at each singular point of $\mathcal{F}_t$.

PROOF. Let X_t be a vector field on $T=S^1\times\partial D^2$ defining the characteristic foliation $\mathcal{F}_t$. A hyperplane field $\ker\alpha_t$ is determined by the following relations,

$$\psi_{t*}(X_t)_p\in(\ker\alpha_t)_p\cap J_0(\ker\alpha_t)_p$$

for non-singular points $p\in\psi_t(S^1\times\partial D^2)$, and

$$(\ker\alpha_t)_p\cap J_0(\ker\alpha_t)_p=T_p\left(\psi_t(S^1\times\partial D^2)\right)$$

for singular points. Then we obtained a differential 1-form α_t up to multiplications of non-vanishing functions. $\square$

LEMMA 2.4. *Let $\{\psi_t,\varphi_t^\partial\}_{t\in[0,1]}$ be a family of pairs of embeddings $\psi_t\colon S^1\times D^2\to S^1\times\mathbb{R}^2=K_c\subset\mathbb{C}^2$ as in Lemma 2.2 and embeddings $\varphi_t^\partial\colon S^1\times\partial D^2\times[0,\varepsilon)\to\mathbb{C}^2$ for a small $\varepsilon>0$ which satisfy the following properties for each $t\in[0,1]$,*

- $\varphi_t^\partial|_{S^1\times\partial D^2\times\{0\}}=\psi_t|_{S^1\times\partial D^2}$,
- $p\circ\varphi_t^\partial|_{S^1\times\partial D^2\times(0,\varepsilon)}>c$,

- $\operatorname{Im}\varphi_t^{\partial} \subset (\mathbb{C}^2, J_0)$ *is strictly pseudoconvex with respect to the orientation which has* $\partial(\operatorname{Im}\varphi_t^{\partial})$ *as the standardly oriented tori in* $S^1 \times \mathbb{R}^2 = K_c$,

where $p\colon \mathbb{C}^2 \to \mathbb{R}$ *is the projection to the* r_2*-elements. Then there exists a family of pairs* $\{(\psi_t, \varphi_t)\}_{t\in[0,1]}$ *which consists of the above embeddings* ψ_t *and embeddings* $\varphi_t\colon S^1 \times D^2 \to \mathbb{C}^2$ *which satisfy the following properties for each* $t \in [0,1]$,

- $\varphi_t = \varphi_t^{\partial}$ *near* $S^1 \times \partial D^2$,
- $\operatorname{Im}\varphi_t \subset (\mathbb{C}^2, J_0)$ *are strictly pseudoconvex,*
- $\mathrm{ct}(\varphi_\mathrm{t})$ *is transverse to* $\gamma_0 := S^1 \times \{0\} \subset S^1 \times D^2 \subset S^1 \times \mathbb{R}^2 \subset \mathbb{C}^2$.

Proof. There exist a sufficiently large $R > 0$ and a sufficiently small $\delta > 0$, which satisfy the following properties for any $t \in [0,1]$,

- $\psi_t(S^1 \times D^2) \subset S^1 \times D(R) \subset S^1 \times \mathbb{R}^2 = K_c$, where $D(R) \subset \mathbb{R}^2$ is a disc centered at the origin with radius R,
- the hypersurface $\Sigma := \left\{r_2 - c = \delta\cdot\sqrt{R^2 - r_1^2}\right\} \subset \mathbb{C}^2$ intersects $\varphi_t^{\partial}(S^1 \times D^2 \times [0,\overline{\varepsilon}))$ transversely for a sufficiently small $\overline{\varepsilon} < \varepsilon$.

In fact, $\varphi_t^{\partial}(S^1 \times \partial D^2 \times [0,\overline{\varepsilon})) \subset \mathbb{C}^2$ are strictly pseudoconvex hypersurfaces with boundaries $\varphi_t^{\partial}(S^1 \times \partial D^2 \times \{0\}) = \psi_t(S^1 \times \partial D^2) \subset K_c \subset \mathbb{C}^2$, and $\Sigma \subset \mathbb{C}^2$ is a strictly pseudoconvex hypersurface which intersects $K_c \subset \mathbb{C}^2$ transversely at $S^1 \times \partial D(R) \subset S^1 \times \mathbb{R}^2 = K_c$. We note that the embedded tori $\varphi_t^{\partial}(S^1 \times \partial D^2 \times \{0\})$ and $S^1 \times \partial D(R) = \Sigma \cap K_c$ in $S^1 \times \mathbb{R}^2 = K_c$ are isotopic to each other, and have the same orientation induced standardly. The intersection $\Sigma \cap \operatorname{Im}\varphi_t^{\partial}$ bounds in Σ a solid torus which is strictly pseudoconvex. Let U_t denote it. Also, $\partial U_t := \Sigma \cup \operatorname{Im}\varphi_t^{\partial}$ and $\varphi_t^{\partial}(S^1 \times \partial D^2 \times \{0\})$ bound in $\operatorname{Im}\varphi$ a toric annulus, that is a domain bounded by two 2-tori and diffeomorphic to $T^2 \times I$, which is strictly pseudoconvex. Let V_t denote it. By a canonical smoothing procedure preserving pseudoconvexity, we obtain a strictly pseudoconvex solid torus W_t embedded into $\mathbb{C}^2$ because both U_t and V_t are strictly pseudoconvex. We note that W_t coincide with $\operatorname{Im}\varphi_t^{\partial}$ near $\partial W_t = \varphi_t^{\partial}(S^1 \times \partial D^2 \times \{0\})$. The embedding $\varphi_t\colon S^1 \times D^2 \to \mathbb{C}^2$ is defined so that φ_t has W_t as its image and coincides with φ_t^{∂} near $S^1 \times \partial D^2$ for each

$t \in [0,1]$. As R and δ are taken for all $t \in [0,1]$, $\{\varphi_t\}_{t\in[0,1]}$ is a smooth family. From the construction, W_t coincides with Σ near $\Gamma_0 := \{r_1 = 0\}$ and Σ is standard near Γ_0. Then the induced contact structures near $\varphi_t(S^1 \times \{0\})$ are standard. That is to say, they are transverse to $\varphi_t(S^1 \times \{0\})$. $\square$

Before the proof of Proposition 2.1, we recall the fundamental facts (see [A], [W], [EG] for precise). The radial vector field $X_0 := \{r_1\cdot(\partial/\partial r_1) + r_2\cdot(\partial/\partial r_2)\}/2$ satisfies $L_{X_0}\omega_0 = \omega_0$ for the standard symplectic structure $\omega_0 := r_1\cdot dr_1\wedge d\theta_1 + r_2\cdot dr_2\wedge d\theta_2$ on $\mathbb{C}^2$ compatible with the standard complex structure J_0. This vector field X_0 preserves symplectic structure ω_0 up to scalar multiplication. It is known that contact forms are induced on hypersurface $F \subset \mathbb{C}^2$ transverse to X_0 as $\alpha = (X_0 \lrcorner \omega_0)|_F$. It is easily checked that this induced contact structure is the same as what is induced by the complex tangency with respect to J_0. Namely, F is strictly pseudoconvex or pseudoconcave. The hypersurface $F \subset \mathbb{C}^2$ is strictly pseudoconvex if the given orientation on F coincides with that induced from those of $\mathbb{C}^2$ and X_0.

Now, we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. The existence of a contact structure for a pair of a characteristic foliation and its taming function (i.e., Proposition 2.1-(a)) is shown, for a family, in the proof of Proposition 2.1-(b). So, it is sufficient to prove (b).

The given family of functions, $\{f_t\colon S^1 \times \partial D^2 \to S^1\}_{t\in[0,1]}$, satisfies the condition of Lemma 2.2 because it is a family of taming functions. Let $\varphi_0, \varphi_1\colon S^1 \times D^2 \to \mathbb{C}^2 \setminus \{r_2 = 0\}$ be the given strictly pseudoconvex embeddings. By applying the flow of the radial symplectic vector field X_0, we may assume $\varphi_0(S^1 \times \partial D^2), \varphi_1(S^1 \times \partial D^2) \subset K_c$ for a constant $c > 0$. Let $\psi_0, \psi_1\colon S^1 \times D^2 \to S^1 \times \mathbb{R}^2$ be embeddings which images are domains in $S^1 \times \mathbb{R}^2 = K_c \subset \mathbb{C}^2$ bounded by $\varphi_i(S^1 \times \partial D^2)$, $i = 0, 1$, respectively, and which satisfy $\psi_i|_{S^1\times\partial D^2} = \varphi_i|_{S^1\times\partial D^2}$ for $i = 0, 1$. By the construction and the assumption, ψ_i satisfies the condition of Lemma 2.2. By using Lemma 2.2, we obtain a family of embeddings $\{\psi_t\colon S^1 \times D^2 \to S^1 \times \mathbb{R}^2\}_{t\in[0,1]}$ containing the given ψ_0 and ψ_1, with a property that $pr \circ \psi_t|_{S^1\times\partial D^2} = f_t$. Next, let $\alpha_i \in ST^*\mathbb{C}^2|_{\varphi_i(S^1\times\partial D^2)}$ be a differential 1-form which corresponds to $(\psi_i, \mathcal{F}_i)$ as in Lemma 2.3, for $i = 0, 1$. Each of them satisfies $(\ker\alpha_i)_p = T_p\varphi_i(S^1 \times D^2)$ for $p \in \varphi_i(S^1 \times \partial D^2) \subset$

$\mathbb{C}^2$. According to Lemma 2.3, we obtain a family of 1-forms $\{\alpha_t\}_{t\in[0,1]}$ containing the above α_0 and α_1. If we set $\mathrm{ct}(\alpha_t) := \ker\alpha_t \cap J_0(\ker\alpha_t)$, then $\psi_t(S^1 \times \partial D^2)_{\mathrm{ct}(\alpha_t)} = \psi_{t*}(\mathcal{F}_t)$ holds. We can take a family of embeddings $\{\varphi_t^\partial \colon S^1 \times \partial D^2 \times [0,\varepsilon) \to \mathbb{C}^2\}_{t\in[0,1]}$ for sufficiently small $\varepsilon > 0$, which satisfies the conditions of Lemma 2.4 and $(\ker\alpha_t)_p = T_p(\operatorname{Im}\varphi_t^\partial)$ for $p \in \varphi_t^\partial(S^1 \times \partial D^2 \times \{0\}) = \psi_t(S^1 \times \partial D^2)$ and each $t \in [0,1]$, as follows. We apply a similar argument to [El3], but we need a careful observation about the second condition of Lemma 2.4. We take a family $\{\varphi_t^\partial\}_{t\in[0,1]}$ of embeddings so that $\varphi_t^\partial|_{S^1\times\partial D^2\times\{0\}} = \psi_t|_{S^1\times\partial D^2}$, $T_p(\operatorname{Im}\varphi_t^\partial) = (\ker\alpha_t)_p$ at each point $p \in \varphi_t^\partial(S^1 \times \partial D^2 \times \{0\}) = \psi_t(S^1 \times \partial D^2)$. From the construction of α_t, we can take $\operatorname{Im}\varphi_t^\partial$ convex for sufficiently small $\varepsilon > 0$. We suppose that $\varphi_0^\partial, \varphi_1^\partial$ coincide with φ_0, φ_1 near $\psi_i(S^1 \times \partial D^2)$ for $i = 0, 1$. We have to note the second condition of Lemma 2.4, $p \circ \varphi_t^\partial|_{S^1\times\partial D^2\times(0,\varepsilon)} > c$. In the case considered now, $K_c = \{r_2 = c\}$ is strictly pseudoconvex for any $c > 0$. $T_p(\operatorname{Im}\varphi_t^\partial)$, $p \in \varphi_t^\partial(S^1 \times \partial D^2 \times \{0\})$ is described by the angle between X_0. By taking $c > 0$ sufficiently small, we can regard K_c and X_0 are almost parallel at $\varphi_t^\partial(S^1 \times \partial D^2 \times \{0\})$. Since $S^1 \times \partial D^2$ and $[0,1]$ are compact, we can take a sufficiently small $c > 0$ so that $p \circ \varphi_t^\partial(S^1 \times \partial D^2 \times (0,\varepsilon)) > c$ holds for any $t \in [0,1]$. By Lemma 2.4, we obtain a family of embeddings $\{\varphi_t \colon S^1 \times D^2 \to \mathbb{C}^2\}_{t\in[0,1]}$ containing the given φ_0, φ_1 which satisfies the following properties for all $t \in [0,1]$,

- $(S^1 \times \partial D^2)_{\mathrm{ct}(\varphi_t)} = (S^1 \times \partial D^2)_{(\psi_t^{-1})_*\mathrm{ct}(\alpha_t)} = \mathcal{F}_t$,
- $pr \circ \varphi_t|_{S^1\times\partial D^2} = pr \circ \psi_t|_{S^1\times\partial D^2} = f_t$.

Thus we complete the proof of Proposition 2.1. □

Last of all, we observe that the contact structures constructed as above depend only on the characteristic foliations up to isotopies. It implies Theorem A.

Corollary 2.5. *Let $\mathcal{F}$ be a characteristic foliation on $T = S^1 \times \partial D^2$ with taming functions f_0, f_1, and $\varphi_0, \varphi_1 \colon S^1 \times D^2 \to \mathbb{C}^2$ embeddings which are constructed from $(f_0, \mathcal{F}), (f_1, \mathcal{F})$ as above respectively. Then* $\mathrm{ct}(\varphi_0)$ *and* $\mathrm{ct}(\varphi_1)$ *are isotopic.*

Proof. According to Lemma 1.7, there exists a family $\{f_t\}_{t\in[0,1]}$ of taming functions for $\mathcal{F}$ containing given f_0 and f_1. By Proposition 2.1, there exists a family of embeddings $\{\varphi_t\}_{t\in[0,1]}$ containing φ_0 and φ_1. Then, $\mathrm{ct}(\varphi_t)$, $t \in [0,1]$, is a homotopy of contact structures between $\mathrm{ct}(\varphi_0)$ and $\mathrm{ct}(\varphi_1)$. Two contact structures $\mathrm{ct}(\varphi_0)$ and $\mathrm{ct}(\varphi_1)$ are isotopic by Gray's theorem (see [Gr], [El3]). □

Not appearing clearly, the base transverse meridian $C \subset S^1 \times \partial D^2$ is considered in the arguments above. It is a level set of taming functions. From the construction above, there exists an embedded disc $D \subset S^1 \times D^2$ spanning C, near which the obtained strictly pseudo convex embedding is standard. That is, the obtained contact structure is standard. Thus, the self linking number of C is $l(C) = -1$.

3. Proof of Theorem B

In this section we prove Theorem B. We will prove that all tight contact structures on $S^1 \times D^2$ which induce the same characteristic foliation on $S^1 \times \partial D^2$, and have a transverse meridian C on $S^1 \times \partial D^2$ with a self-linking number $l(C) = -1$, are isotopic.

First, we recall the S^1-Darboux theorem (see [Mar], [L]). The *standard contact structure* ξ_0 on $S^1 \times \mathbb{R}^2$ is defined to be a kernel of the *standard contact form* $\alpha_0 := d\theta + r^2 \cdot d\phi$, where $(\theta, r, \phi) \in S^1 \times \mathbb{R}^2$ is the cylindrical coordinate.

PROPOSITION 3.1 (the S^1-Darboux theorem). *Let ξ be a contact structure defined near a circle γ, which is transverse to γ. Then there exists a local diffeomorphism φ defined in a neighborhood of γ which satisfies $\varphi(\gamma) = S^1 \times \{0\}$ and $\varphi_*\xi = \xi_0$.*

Let $D(r) \subset \mathbb{R}^2$ be the disc centered at the origin with radius r. We note that with respect to the standard contact structure, the characteristic foliation $\left(S^1 \times D(r)\right)_{\xi_0}$ is non-singular, θ-invariant, and does not admit closed orbits which are homotopic to meridian curves.

Next, we review some results about transverse knots. Let (M, ζ) be a contact 3-manifold. The following is well known. (see [El5] for example)

LEMMA 3.2. *Any curve embedded in M can be made transverse to ζ by a C^0-small isotopy.*

We suppose that ζ is tight, in addition.

Lemma 3.3 (Eliashberg, [El5]). *Let Γ be a trivial transverse knot in (M, ζ), and fix a relative homology class $\beta \in H_2(M, \Gamma)$. The self-linking number of Γ with respect to β is $1 - 2k$ if and only if there is an embedded disc spanning Γ, which represent β and the characteristic foliation on which has only k positive elliptic points and $k - 1$ negative hyperbolic points as singular points.*

Now, we are ready to prove Theorem B.

Proof of Theorem B. Let ξ be a tight contact structure on $S^1 \times D^2$ and $C \subset S^1 \times \partial D^2$ a positively transverse meridian curve with a self-linking number $l(C) = -1$. According to Lemma 3.2, we may suppose that the core $\gamma_0 := S^1 \times \{0\}$ of the solid torus is positively transverse to ξ. There exists an embedded disc $\mathcal{D}$ in $S^1 \times D^2$ spanning C on which the characteristic foliation $\mathcal{D}_\xi$ has exactly one elliptic singular point $p \in \mathcal{D}$ by Lemma 3.3. As γ_0 is transverse to ξ, we may suppose that the singular point p is an intersection point of γ_0 and $\mathcal{D}$, and that γ_0 and $\mathcal{D}$ intersect only at this point. We note that all leaves of $\mathcal{D}_\xi$ starting from p intersect $\partial\mathcal{D} = C$ transversely. Then we can take a family $\{C_t\}_{t \in (0,1]}$ of concentric circles on $\mathcal{D} \setminus \{p\}$ transverse to $\mathcal{D}_\xi$, which satisfies

$$C_1 = C \subset S^1 \times \partial D^2, \quad C_t \to p \ (t \to 0).$$

In addition, we can take a family $\{U_t\}_{t \in (0,1]}$ of tubular neighborhoods of γ_0 in $S^1 \times D^2$, which satisfies,

- $U_1 = S^1 \times D^2, \quad U_t \to \gamma_0 \ (t \to 0)$,
- $\partial U_t \cap \mathcal{D} = C_t$ for all $t \in (0, 1]$.

We note that C_t is a transverse meridian of ∂U_t, and $l(C_t) = -1$ by Lemma 3.3. Let $\{\mathcal{F}_t\}_{t \in (0,1]}$ denote the family of characteristic foliations $\{(\partial U_t)_\xi\}_{t \in (0,1]}$.

By the S^1-Darboux theorem (Proposition 3.1), the contact structure ξ is isotopic to the standard structure near γ_0. Hence, the characteristic foliation $(\partial U_\varepsilon)_\xi$ for sufficiently small $\varepsilon \in (0, 1]$ is normal. Then it admits a taming function with C_ε as a level set. We can apply Proposition 1.1 and then Proposition 1.6. Then, there exists a family $\{f_t\colon \partial U_t \to S^1\}_{t\in[\varepsilon,1]}$ of taming functions for $\mathcal{F}_t$. For a family $\{(f_t, \mathcal{F}_t)\}_{t\in[\varepsilon,1]}$, there exists a family of embeddings $\{\varphi_t\colon U_t \to \mathbb{C}^2\}_{t\in[\varepsilon,1]}$ which satisfies the following properties by Proposition 2.1,

- $(\partial U_t)_{\mathrm{ct}(\varphi_t)} = \mathcal{F}_t$,
- $pr \circ \varphi_t|_{\partial U_t} = f_t$.

Let ζ_t denote contact structures on U_t defined by $\mathrm{ct}(\varphi_t)$. Define a family $\{\xi_t\}_{t\in[\varepsilon,1]}$ of contact structures on $S^1 \times D^2$ by

$$\xi_t := \begin{cases} \xi & \text{on} \quad (S^1 \times D^2) \setminus U_t, \\ \zeta_t & \text{on} \quad U_t. \end{cases}$$

It is a homotopy among contact structures between $\xi_\varepsilon = \xi$ and $\xi_1 = \zeta_1$ relatively to the boundary $S^1 \times \partial D^2$. According to Gray's theorem (see [Gr], [El3]), they are isotopic relatively to $S^1 \times \partial D^2$. Moreover, by Theorem A, ζ_1 depends only on $(S^1 \times \partial D^2)_\xi$. Therefore, all tight contact structures on $S^1 \times D^2$ with the characteristic foliation $\mathcal{F} = (S^1 \times \partial D^2)_\xi$ are isotopic to ζ_1.

This completes the proof of Theorem B. □

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