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Splitting of Singular Fibers in Certain Holomorphic Fibrations

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Abstract. We will consider splitting of certain singular fibers. The singular fibers dealt with in this paper are the singular fibers in certain holomorphic fibration of genus g whose total space is diffeomorphic to $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}^2}$. We study how these singular fibers split into Lefschetz type singular fibers. We also study the monodromies about the new Lefschetz type singular fibers obtained by splitting.

1. Introduction

The purpose of this paper is to consider splitting of certain singular fibers. In case of genus one, splitting of degenerate elliptic curves was studied in [Mo]. In case of genus two, splitting of degenerate genus two curves was studied in [Ho]. And in case of higher genera, a proof applicable to hyperelliptic singular fibers was recently given by [AA].

Although it is important to study splitting of singular fibers, it is also important to study the monodromies about the new singular fibers obtained by splitting. In [Ma5], Y. Matsumoto took up certain degenerate genus two curves and he studied splitting of these degenerate genus two curves and the monodromies about the new singular fibers obtained by splitting, by using a computer.

Now in this paper, we take up singular fibers in certain holomorphic fibration of genus g and we study how these singular fibers split into Lefschetz type singular fibers and the monodromies about the new Lefschetz type singular fibers obtained by splitting.

The holomorphic fibration of genus g over a 2-sphere dealt with in this paper is constructed as follows (cf. [Ma2]).

Let Σ_g be the closed oriented surface of genus g. Let $\omega_g : \Sigma_g \to \Sigma_g$ be the hyperelliptic involution shown in Figure 1 and $\tau : S^2 \to S^2$ the 180°

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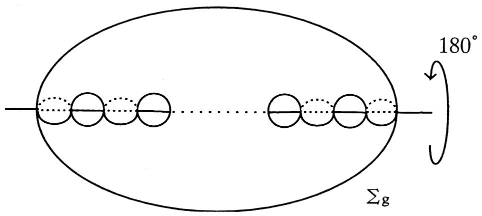

FIGURE 1.

rotation of a 2-sphere about the axis through the poles. The quotient space $\Sigma_g \times S^2/\omega_g \times \tau$ has $2(2g+2)$ singular points. Blowing up these singularities, we obtain the compact complex surface M_g which is diffeomorphic to $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}^2}$. And we also obtain the holomorphic fibration of genus g $f_g : M_g \to S^2/\tau \cong S^2$, where f_g is the map induced by the projection to the second factor $\Sigma_g \times S^2 \to S^2$. This holomorphic fibration $f_g : M_g \to S^2$ contains two singular fibers over the north and the south poles of S^2. They are topologically equivalent and the monodromy about each singular fiber is the hyperelliptic involution ω_g. Therefore, we denote either singular fiber by F_{ω_g}.

We explain the structure of F_{ω_g}. The quotient space Σ_g/ω_g is homeomorphic to S^2 with $2g+2$ cusps corresponding to the fixed points of ω_g. By blowing up $\Sigma_g \times S^2/\omega_g \times \tau$ to M_g, $2g+2$ spheres with self-intersection number -2 are inserted at these cusps on Σ_g/ω_g. Thus F_{ω_g} is constructed and the shape of F_{ω_g} is shown in Figure 2, where segments stand for 2-spheres and the numbers attached to them are the multiplicities of f_g about the irreducible components. Our purpose is to consider splitting of this singular fiber F_{ω_g}. We study how F_{ω_g} splits into Lefschetz type singular fibers.

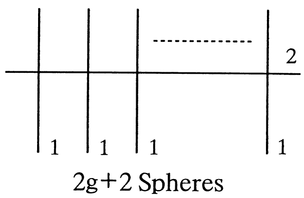

FIGURE 2.

And we also study the monodromies about the new Lefschetz type singular fibers obtained by splitting. In case of genus two, the result on splitting of the singular fiber F_{ω_2} is stated in [Ma2] (See Fact 1 in [Ma2]).

The author is deeply grateful to Professor Yukio Matsumoto for his valuable advice.

2. Definitions

Let $Diff^+(\Sigma_g)$ be the group of all orientation preserving self-diffeomorphisms of Σ_g with the C^∞-topology and $Diff_0^+(\Sigma_g)$ the subgroup of $Diff^+(\Sigma_g)$ consisting of all self-diffeomorphisms isotopic to the identity. We denote $Diff^+(\Sigma_g)/Diff_0^+(\Sigma_g)$ by $\mathcal{M}_g$. $\mathcal{M}_g$ is called the mapping class group of genus g.

We review the definition of a Lefschetz fibration (cf. [Ma2]).

Definition 2.1. Let M and B be compact oriented (not necessarily closed) smooth manifolds of dimensions 4 and 2, respectively. A smooth map $f : M \to B$ is called a *Lefschetz fibration of genus* g if the following conditions are satisfied:

(i) $\partial M = f^{-1}(\partial B)$;

(ii) there is a finite set of points $b_1, b_2, \cdots, b_n$ (called the critical values of f) in $IntB$ $(= B - \partial B)$ such that $f|f^{-1}(B - \{b_1, \cdots, b_n\}) : f^{-1}(B - \{b_1, \cdots, b_n\}) \to B - \{b_1, \cdots, b_n\}$ is a smooth fiber bundle with the fiber diffeomorphic to Σ_g;

(iii) for each i ($1 \leq i \leq n$), there exists a single point $p_i \in f^{-1}(b_i)$ such that

(a) $(df)_p : T_p(M) \to T_{f(p)}(B)$ is onto for any $p \in f^{-1}(b_i) - \{p_i\}$,

(b) about p_i (resp. b_i), there exist local complex coordinates z_1, z_2 with $z_1(p_i) = z_2(p_i) = 0$ (resp. local complex coordinate ξ with $\xi(b_i) = 0$), so that f is locally written as $\xi = f(z_1, z_2) = z_1 z_2$;

(iv) no fiber contains a (-1)-sphere, which is a smoothly embedded 2-sphere with self-intersection number -1.

We call a fiber $f^{-1}(b)$ a *singular fiber* if $b \in \{b_1, \cdots, b_n\}$ or else a *general fiber*. Also we call M the *total space*, B the *base space*, and f the *projection*.

In Condition (iii) (b), we tacitly require the orientation of M (resp. B) to coincide with the canonical orientation determined by the local complex coordinates z_1, z_2 (resp. local complex coordinate ξ) about the critical point p_i (resp. b_i).

Let $f : M \to B$ be a Lefschetz fibration of genus g over a connected base space B. Fix a base point $b_0 \in B \setminus (\partial B \cup \{b_1, \cdots, b_n\})$ and identify the general fiber $f^{-1}(b_0)$ with Σ_g by an orientation preserving diffeomorphism $\Phi : \Sigma_g \to f^{-1}(b_0)$. Then we obtain a monodromy representation and denote this representation by

$$\rho : \pi_1(B \setminus \{b_1, \cdots, b_n\}, b_0) \to \mathcal{M}_g.$$

The monodromy representation ρ becomes a homomorphism if $\mathcal{M}_g$ is assumed, by convention, to act on Σ_g from the right. Now, we will adopt this convention.

We review the elementary transformation (cf. [Ma2]). Suppose that the base space B is the 2-disk D or the 2-sphere S^2. Then the monodromy representation $\rho : \pi_1(B \setminus \{b_1, \cdots, b_n\}, b_0) \to \mathcal{M}_g$ can be given by an n-tuple

$$(g_1, g_2, \cdots, g_n),$$

where $g_i = \rho(\gamma_i), i = 1, 2, \cdots n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$ are the loops drawn on B as shown in Figure 3. Let $\epsilon_i : (B \setminus \{b_1, \cdots, b_n\}, b_0) \to (B \setminus \{b_1, \cdots, b_n\}, b_0)$ and $\epsilon_i^{-1} : (B \setminus \{b_1, \cdots, b_n\}, b_0) \to (B \setminus \{b_1, \cdots, b_n\}, b_0)$ be the homeomorphisms shown in Figure 4. ϵ_i is called the *i-th elementary homeomorphism* and ϵ_i^{-1} is called its inverse. When we change the loops by ϵ_i or ϵ_i^{-1}, the

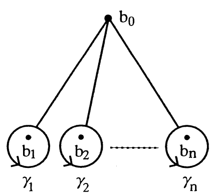

Figure 3.

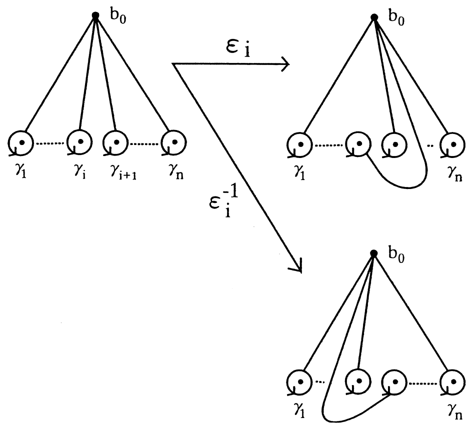

FIGURE 4.

n-tuple $(g_1, g_2, \cdots, g_n)$ is changed into

$$(g_1, \cdots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \cdots, g_n)$$

or

$$(g_1, \cdots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \cdots, g_n),$$

respectively. We call these transformations the *i-th elementary transformation* or its inverse. The elementary transformations do not change the total monodromy

$$g_1 g_2 \cdots g_n.$$

We notice that the n-tuple

$$(g_1, \cdots, g_i, g_{i+1}, \cdots, g_n)$$

is changed into

$$(g_1, \cdots, g_{i+1}, g_i, \cdots, g_n)$$

by the i-th elementary transformation or its inverse if g_i and g_{i+1} are commutative in $\mathcal{M}_g$: $g_i g_{i+1} = g_{i+1} g_i \in \mathcal{M}_g$.

Therefore we will say that *g_i and g_{i+1} in the n-tuple $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_n)$ are commutative by the i-th elementary transformation or its inverse* when g_i and g_{i+1} are commutative in $\mathcal{M}_g$.

We give several Lefschetz singular fibers of type I. A Lefschetz singular fiber of type I is obtained by pinching a non-separating simple closed curve on Σ_g to a point (See Figure 5). And its monodromy is the negative Dehn twist about this curve (See Figure 6).

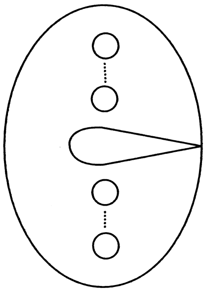

Figure 5.

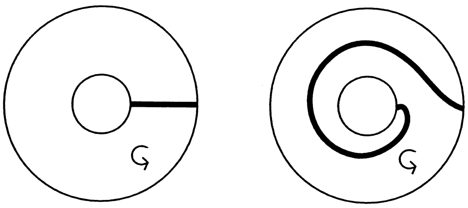

Figure 6.

For each i ($i = 1, 2, \cdots, 2g+1$), we take the Lefschetz singular fiber of type I obtained by pinching the curve C_i in Figure 7 and denote the monodromy about this singular fiber by ζ_i. For each j ($j = 1, 2, \cdots, g$), we take the Lefschetz singular fiber of type I obtained by pinching the curve C'_j in Figure 8 and denote the monodromy about this singular fiber by β_j. Then we notice that two monodromies β_j and β_k ($j, k = 1, 2, \cdots, g$) are commutative in $\mathcal{M}_g$.

Similarly, for each j ($j = 1, 2, \cdots, g$), we take the Lefschetz singular fibers of type I obtained by pinching the curves C''_j and C'''_j in Figure 9 and denote the monodromies about these singular fibers by η_j and δ_j, respectively. We also notice that two monodromies η_j and η_k (or δ_j and δ_k)

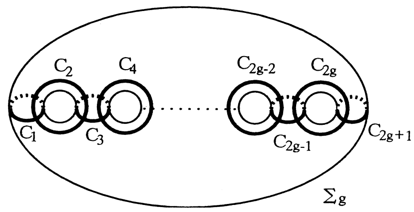

FIGURE 7.

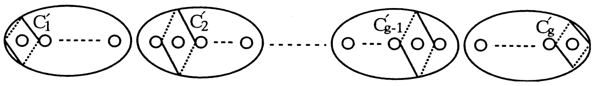

FIGURE 8.

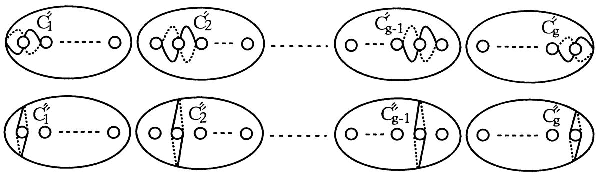

FIGURE 9.

$(j, k = 1, 2, \cdots, g)$ are commutative in $\mathcal{M}_g$.

We will extend the notion of Lefschetz fibrations to a more general class of fiber spaces as follows (cf. [Ma2]).

DEFINITION 2.2. Let M and B be compact oriented (not necessarily closed) smooth manifolds of dimension 4 and 2, respectively. A smooth map $f : M \to B$ is called a *locally analytic fibration of genus g* if it satisfies Conditions (i) and (ii) of Definition 2.1 and the following Condition (iii′) instead of (iii):

(iii′) for each i $(1 \le i \le n)$ and at each point $p \in f^{-1}(b_i)$, the germ (f, p) is conjugate via (not necessarily orientation preserving) diffeomorphisms to the germ at 0 of a holomorphic function $\mathbb{C}^2 \to \mathbb{C}$. Moreover, there exists at least one critical point of f on the fiber $f^{-1}(b_i)$.

In Condition (iii′) of above definition, if the local diffeomorphisms giving the conjugations of the germs are always orientation preserving, we call such a fibration a *locally holomorphic fibration.*

We give the definition of *splitting* (cf. [Ma2]).

Definition 2.3. Let $f^1 : M^1 \to D$ be a locally analytic fibration of genus g over a 2-disk D with a singular fiber F^1. Suppose there is another locally analytic fibration $f^2 : M^2 \to D$ of genus g over the same 2-disk, with the set of singular fibers $\{F^2_1, F^2_2, \cdots, F^2_r\}$. Moreover, suppose there exists an orientation preserving diffeomorphism $H : M^1 \to M^2$ which commutes with f^1 and f^2 on the boundary: $(f^2|\partial M^2)\circ(H|\partial M^1) = f^1|\partial M^1$. Then we say that the singular fiber F^1 *splits* into the set of singular fibers $F^2_1, F^2_2, \cdots, F^2_r$.

It is easy to see that if a singular fiber F_i of a locally analytic fibration $f : M \to B$ splits into $F_{i1}, F_{i2}, \cdots, F_{ir}$, then we can change the projection f within an arbitrarily small neighborhood of F_i (without changing M and B) so that the new projection f' has singular fibers $F_{i1}, F_{i2}, \cdots, F_{ir}$ in place of F_i.

The splitting defined above is called *splitting in a weak sense.* The splitting in a stronger sense is discussed in [Ma5], where singular fibers split through certain perturbation of the projection map.

3. Main Theorem

Let N be a noncompact complex surface and D an open disk centered at the origin in $\mathbb{C}$. To consider splitting of the singular fiber F_{ω_g}, we concretely construct in the next section a proper surjective holomorphic map $\varphi : N \to D$ which satisfies the following conditions:

1. $\varphi|_{\varphi^{-1}(D\setminus\{0\})} : \varphi^{-1}(D \setminus \{0\}) \to D \setminus \{0\}$ is a smooth fiber bundle with fiber Σ_g;
2. $\varphi^{-1}(0) = F_{\omega_g}$.

Our main theorem is the following:

Theorem 3.1. *By perturbation of the map φ and deformation of the complex structure of N with some parameter, F_{ω_g} splits, in the strong sense,* *into $2g$ Lefschetz singular fibers of type I and a singular fiber F' which is not a Lefschetz singular fiber. Furthermore, the singular fiber F' splits, in the weak sense, into $2g+2$ Lefschetz singular fibers of type I.*

Therefore F_{ω_g} splits into $2(2g+1)$ Lefschetz singular fibers of type I. Suppose that $b_1, b_2, \cdots, b_{2(2g+1)}$ are the new critical values in D obtained by the splitting. When we fix a small positive real number b_0 in D as a base point and take loops shown in Figure 3, the monodromy representation $\rho : \pi_1(D \setminus \{b_1, \cdots, b_{2(2g+1)}\}, b_0) \to \mathcal{M}_g$ is given by the following $2(2g+1)$-tuple of Dehn twists which are divided into 5 groups:

$$\begin{aligned}(3.1)\qquad &(\text{ the group of } \beta_i s \text{ , the group of } \eta_i s \text{ , the group of } \zeta_{2i+1} s \text{ ,}\\ &\quad \text{the group of } \zeta_{2i} s \text{ , the group of } \delta_i s \text{).}\end{aligned}$$

If the genus g is even, then the $2(2g+1)$-tuple (3.1) is

$$\begin{aligned}(3.2)\quad (&\beta_1, \beta_3, \beta_5, \cdots, \beta_{g-3}, \beta_{g-1}, \quad \eta_{g-1}, \eta_{g-3}, \cdots, \eta_5, \eta_3, \eta_1,\\ &\zeta_1, \zeta_1, \zeta_3, \zeta_3, \zeta_5, \zeta_5, \cdots, \zeta_{2g-1}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1},\\ &\zeta_{2\times 2}, \zeta_{2\times 4}, \zeta_{2\times 6}, \cdots, \zeta_{2(g-2)}, \zeta_{2g}, \quad \delta_g, \delta_{g-2}, \cdots, \delta_6, \delta_4, \delta_2);\end{aligned}$$

If the genus g is odd, then the $2(2g+1)$-tuple (3.1) is

$$\begin{aligned}(3.3)\quad (&\beta_2, \beta_4, \beta_6, \cdots, \beta_{g-3}, \beta_{g-1}, \quad \eta_{g-1}, \eta_{g-3}, \cdots, \eta_6, \eta_4, \eta_2,\\ &\zeta_1, \zeta_1, \zeta_3, \zeta_3, \zeta_5, \zeta_5, \cdots, \zeta_{2g-1}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1},\\ &\zeta_{2\times 1}, \zeta_{2\times 3}, \zeta_{2\times 5}, \cdots, \zeta_{2(g-2)}, \zeta_{2g}, \quad \delta_g, \delta_{g-2}, \cdots, \delta_5, \delta_3, \delta_1).\end{aligned}$$

Furthermore, by applying some elementary transformations and their inverses to the $2(2g+1)$-tuples (3.2) and (3.3), we can arrange these $2(2g+1)$-tuples to

$$(\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_3, \zeta_2, \zeta_1).$$

REMARK 3.2. Two monodromies in each group of (3.1) are commutative in $\mathcal{M}_g$. Therefore two adjacent monodromies in each group are commutative by a suitable elementary transformation or its inverse.

REMARK 3.3. Compare the above theorem with Example 3.14 in [AA].

Recall that the holomorphic fibration $f_g : M_g \to S^2$ has two singular fibers, or two F_{ω_g}s, and that M_g is diffeomorphic to $\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}^2}$. By making each of these singular fibers split, we obtain the following corollary.

Corollary 3.4. *By splitting of two F_{ω_g}s, we can obtain the Lefschetz fibration of genus g*

$$\mathbb{CP}^2\#(4g+5)\overline{\mathbb{CP}^2} \to S^2$$

with $4(2g+1)$ singular fibers of type I. And the monodromy representation is given by the $4(2g+1)$-tuple

$$(\zeta_1, \zeta_2, \cdots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_2, \zeta_1, \zeta_1, \zeta_2, \cdots, \\ \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_2, \zeta_1).$$

Therefore the total monodromy becomes

$$(\zeta_1\zeta_2\zeta_3\cdots\zeta_{2g}\zeta_{2g+1}^2\zeta_{2g}\cdots\zeta_3\zeta_2\zeta_1)^2 = 1 \tag{3.4}$$

and (3.4) *corresponds to the well known defining relation of the mapping class group $\mathcal{M}_g$.*

4. The Construction of The Map $\varphi : N \to D$

We will distinguish several complex planes $\mathbb{C}$ by denoting them as $\mathbb{C}_v, \mathbb{C}_t$, etc., where v or t represents the variable in the plane.

We begin with the construction of a holomorphic function

$$h : \{v \in \mathbb{C} | v \neq \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \cdots, \frac{1}{\alpha_{g-1}}, \frac{1}{\alpha_g}\} \to \mathbb{C}_{\tilde{v}}.$$

By taking a small positive real number α_g which satisfies $\alpha_g \ll 1$, we perturb $\tilde{v} = v^{g+1}$ to

$$\tilde{v} = v^g \frac{v - \alpha_g}{1 - \alpha_g v}.$$

Next, by taking a smaller positive real number α_{g-1} which satisfies $\alpha_{g-1} \ll \alpha_g$, we perturb $\tilde{v} = v^g \frac{v-\alpha_g}{1-\alpha_g v}$ to

$$\tilde{v} = v^{g-1} \frac{v - \alpha_{g-1}}{1 - \alpha_{g-1} v} \frac{v - \alpha_g}{1 - \alpha_g v}.$$

Similarly, by taking a smaller positive real number α_{g-2} which satisfies $\alpha_{g-2} \ll \alpha_{g-1}$, we perturb $\tilde{v} = v^{g-1}\frac{v-\alpha_{g-1}}{1-\alpha_{g-1}v}\frac{v-\alpha_g}{1-\alpha_g v}$ to

$$\tilde{v} = v^{g-2}\frac{v-\alpha_{g-2}}{1-\alpha_{g-2}v}\frac{v-\alpha_{g-1}}{1-\alpha_{g-1}v}\frac{v-\alpha_g}{1-\alpha_g v},$$

and so on.

By such perturbations, we obtain the final function

$$\tilde{v} = h(v) = v\frac{(v-\alpha_1)(v-\alpha_2)(v-\alpha_3)\cdots(v-\alpha_{g-1})(v-\alpha_g)}{(1-\alpha_1 v)(1-\alpha_2 v)(1-\alpha_3 v)\cdots(1-\alpha_{g-1}v)(1-\alpha_g v)} \tag{4.1}$$

which satisfies the following conditions:

(i) $0 < \alpha_1 \ll \alpha_2 \ll \alpha_3 \ll \cdots \ll \alpha_{g-1} \ll \alpha_g \ll 1$;

(ii) $h(\bar{v}) = \overline{h(v)}$ for the conjugate $\bar{v}$ of v;

(iii) $h(\frac{1}{v}) = \frac{1}{h(v)}$;

(iv) $h(v) \fallingdotseq v^{g+1}$ for v which satisfies $\alpha_g \ll |v| \ll \frac{1}{\alpha_g}$.

We denote the critical points of h by

$$\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_g, \frac{1}{\gamma_g}, \cdots, \frac{1}{\gamma_3}, \frac{1}{\gamma_2}, \frac{1}{\gamma_1} \tag{4.2}$$

as shown in Figure 10.

Since $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_{g-1}, \alpha_g$ satisfy the above condition (i), we may suppose that the critical values $h(\gamma_1), h(\gamma_2), h(\gamma_3), \cdots, h(\gamma_{g-1}), h(\gamma_g)$ satisfy the following:

$$\begin{aligned} &-1 < h(\gamma_g) < h(\gamma_{g-2}) < \cdots < h(\gamma_6) < h(\gamma_4) < h(\gamma_2) < 0 \\ &\qquad 0 < h(\gamma_1) < h(\gamma_3) < h(\gamma_5) < \cdots < h(\gamma_{g-3}) < h(\gamma_{g-1}) < 1 \end{aligned} \tag{4.3}$$

if g is even;

$$\begin{aligned} &-1 < h(\gamma_g) < h(\gamma_{g-2}) < \cdots < h(\gamma_5) < h(\gamma_3) < h(\gamma_1) < 0 \\ &\qquad 0 < h(\gamma_2) < h(\gamma_4) < h(\gamma_6) < \cdots < h(\gamma_{g-3}) < h(\gamma_{g-1}) < 1 \end{aligned} \tag{4.4}$$

if g is odd.

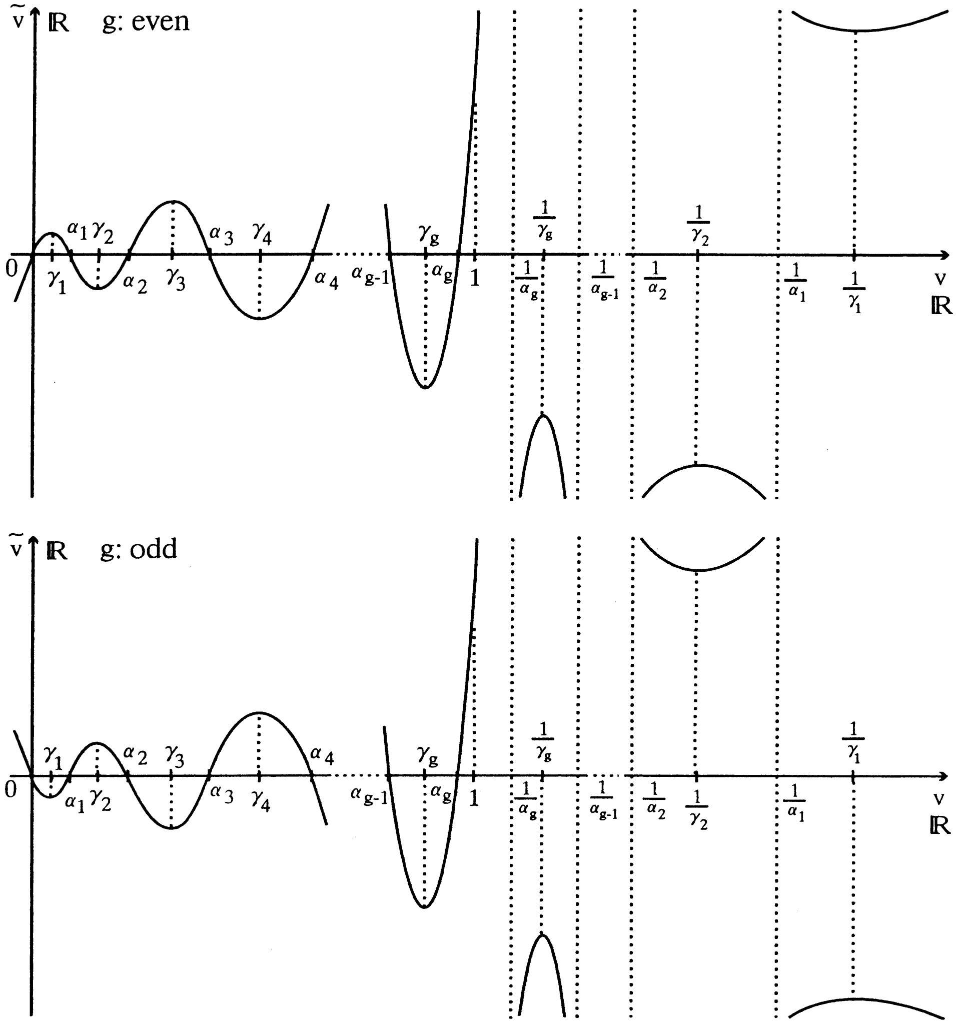

FIGURE 10. The graph of $h : \mathbb{R} \to \mathbb{R}$.

By (4.3),(4.4) and the above condition (iii), we also obtain that

$$
\begin{aligned}
(4.5)\quad & h(\frac{1}{\gamma_2}) < h(\frac{1}{\gamma_4}) < h(\frac{1}{\gamma_6}) < \cdots < h(\frac{1}{\gamma_{g-2}}) < h(\frac{1}{\gamma_g}) < -1, \\
& 1 < h(\frac{1}{\gamma_{g-1}}) < h(\frac{1}{\gamma_{g-3}}) < \cdots < h(\frac{1}{\gamma_5}) < h(\frac{1}{\gamma_3}) < h(\frac{1}{\gamma_1})
\end{aligned}
$$

if g is even and

$$(4.6)\quad h(\frac{1}{\gamma_1}) < h(\frac{1}{\gamma_3}) < h(\frac{1}{\gamma_5}) < \cdots < h(\frac{1}{\gamma_{g-2}}) < h(\frac{1}{\gamma_g}) < -1,$$
$$1 < h(\frac{1}{\gamma_{g-1}}) < h(\frac{1}{\gamma_{g-3}}) < \cdots < h(\frac{1}{\gamma_6}) < h(\frac{1}{\gamma_4}) < h(\frac{1}{\gamma_2})$$

if g is odd.

Take two subsets of $\mathbb{C}^2$

$$N'_{uv} = \{(u,v) \in \mathbb{C}^2 | v \neq \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \cdots, \frac{1}{\alpha_{g-1}}, \frac{1}{\alpha_g}\},$$

$$N'_{xy} = \{(x,y) \in \mathbb{C}^2 | y \neq \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \cdots, \frac{1}{\alpha_{g-1}}, \frac{1}{\alpha_g}\}$$

and attach N'_{uv} to N'_{xy} by the biholomorphic mapping

$$\Phi : \{(u,v) \in N'_{uv} | v \neq 0, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{g-1}, \alpha_g\} \to$$
$$\{(x,y) \in N'_{xy} | y \neq 0, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_{g-1}, \alpha_g\}$$

given by

$$(x,y) = \Phi(u,v) = (uh(v), \frac{1}{v}).$$

We denote $N'_{uv} \cup_\Phi N'_{xy}$ by N'.

By this attaching, $\{(x,y)|x=0\} \cup \{(u,v)|u=0\}$ in N' becomes the 1-dimensional complex projective space, therefore we denote this by $\mathbb{CP}^1_{vy}$.

We define a well defined function $\varphi : N' \to \mathbb{C}_t$ as follows:

$$(4.7)\qquad \begin{cases} \varphi(u,v) = u^2 h(v) & \text{on } N'_{uv} \\ \varphi(x,y) = x^2 h(y) & \text{on } N'_{xy}. \end{cases}$$

Then the divisor $\varphi = 0$ in N' is the 2-sphere, or $\mathbb{CP}^1_{vy}$, stuck with $2(g+1)$ complex planes as shown in Figure 11. Also, the shape of a fiber $\varphi = t \neq 0$ is shown in Figure 11.

Take the subset of N'

$$N'' = \{(u,v) \in N'_{uv} | |u| < R, |v| < 1+\delta\} \cup \{(x,y) \in N'_{xy} | |x| < R, |y| < 1+\delta\}$$

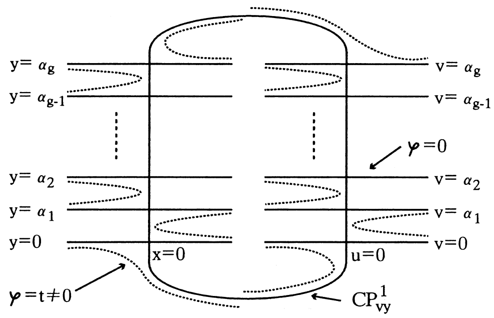

FIGURE 11. The shape of the divisor $\varphi = 0$ and a general fiber $\varphi = t \neq 0$.

and restrict the domain N' of φ to N'', where R is a sufficiently large real number and δ is a small real number. Then the divisor $\varphi = 0$ in N'' is the 2-sphere, or $\mathbb{CP}^1_{vy}$, stuck with $2(g+1)$ disks.

By attaching $2(g+1)$ 2-handles to N'', we extend $\varphi : N'' \to \mathbb{C}_t$ to $N'' \cup$ 2- handles $\to \mathbb{C}_t$ so that the divisor $\varphi = 0$ in $N'' \cup$ 2- handles is the 2-sphere stuck with $2(g+1)$ spheres, as follows.

We denote points in $\mathbb{CP}^1_{vy} \cap\, 2(g+1)$ disks $(\subset \{\varphi = 0\})$ by

$$
\begin{aligned}
&P_0 = (0,0) = (u,v), \quad P_i = (0,\alpha_i) = (u,v) \quad (i = 1,2,\cdots,g) \\
&Q_0 = (0,0) = (x,y), \quad Q_i = (0,\alpha_i) = (x,y) \quad (i = 1,2,\cdots,g).
\end{aligned} \tag{4.8}
$$

When we put $k = h(v)$ (or $k = h(y)$), we obtain a new local coordinate (u,k) (or (x,k)) at P_j (or Q_j) instead of (u,v) (or (x,y)). On this new coordinate, φ is given by

$$\varphi(u,k) = u^2 k$$

or

$$\varphi(x,k) = x^2 k.$$

Take a 2-handle $\Delta^2 = \{(\sigma,\tau) \mid |\sigma| < \delta', |\tau| < \delta''\}$ and attach this 2-handle to the above local coordinate by

$$\sigma = u^{-1}, \tau = u^2 k \tag{4.9}$$

or

$$\sigma = x^{-1}, \tau = x^2 k. \tag{4.10}$$

By defining a map $\Delta^2 \to \mathbb{C}_t$ as

$$(\sigma, \tau) \to \tau, \tag{4.11}$$

we can extend $\varphi : N'' \to \mathbb{C}_t$ to $\varphi : N'' \cup \Delta^2 \to \mathbb{C}_t$.

Thus we can extend $\varphi : N'' \to \mathbb{C}_t$ to $\varphi : N'' \cup\, 2(g+1)$ 2-handles $\to \mathbb{C}_t$. Moreover the divisor $\varphi = 0$ in $N'' \cup\, 2(g+1)$ 2-handles becomes the 2-sphere with multiplicity 2 stuck with $2(g+1)$ 2-spheres with multiplicity 1, that is F_{ω_g}.

We define a projection

$$\Pi : N' \to \mathbb{CP}^1_{vy}$$

as

$$\begin{cases} \Pi(u, v) = v & \text{on } N'_{uv} \\ \Pi(x, y) = y & \text{on } N'_{xy}. \end{cases} \tag{4.12}$$

Take a small 2-disk D in $\mathbb{C}_t$ centered at the origin and fix a fiber $\varphi = t$ which satisfies $0 \neq t \in D$. Then the projection $\Pi : N'' \cap \{\varphi = t\} \to \mathbb{CP}^1_{vy}$ is naturally extended to $\Pi : \{\varphi = t\} \to \mathbb{CP}^1_{vy}$.

If $\Pi^{-1}(v)$ (or $\Pi^{-1}(y)$) for v (or y) $\in \mathbb{CP}^1_{vy}$ is a subset on N'', then

$$\Pi^{-1}(v) = \{(u', v') \in N'_{uv} | u'^2 = \frac{t}{h(v)}, v' = v\} \tag{4.13}$$

(or

$$\Pi^{-1}(y) = \{(x', y') \in N'_{xy} | x'^2 = \frac{t}{h(y)}, y' = y\}), \tag{4.14}$$

from (4.7).

If $\Pi^{-1}(v)$ (or $\Pi^{-1}(y)$) is a subset on a 2-handle Δ^2, then

$$\Pi^{-1}(v) = \{(\sigma, \tau) \in \Delta^2 | \sigma^2 = \frac{h(v)}{t}, \tau = t\} \tag{4.15}$$

(or

$$\Pi^{-1}(y) = \{(\sigma, \tau) \in \Delta^2 | \sigma^2 = \frac{h(y)}{t}, \tau = t\}), \tag{4.16}$$

from (4.9), (4.10) and (4.11). By (4.13),(4.14),(4.15) and (4.16), we know that the projection $\Pi : \{\varphi = t\} \to \mathbb{CP}^1_{vy}$ is the double branched-covering map with the following 2(g+1) branch points:

$$v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g, y = 0, \alpha_1, \alpha_2, \cdots, \alpha_g, \quad \text{on} \quad \mathbb{CP}^1_{vy}.$$

And the inverse image under Π of any branch point is on a 2-handle and satisfies

$$\Pi^{-1}(\text{ branch point }) = \{(\sigma, \tau) = (0, t)\}.$$

Thus, the fiber $\{\varphi = t\}$ is diffeomorphic to Σ_g.

And when we denote $\varphi^{-1}(D)$ by N, we can obtain the map $\varphi : N \to D$ which satisfies the following conditions:

1. $\varphi|_{\varphi^{-1}(D \setminus \{0\})} : \varphi^{-1}(D \setminus \{0\}) \to D \setminus \{0\}$ is a smooth fiber bundle with fiber Σ_g;
2. $\varphi^{-1}(0) = F_{\omega_g}$.

Thus we finish the construction of the map $\varphi : N \to D$.

5. Perturbation of The Map φ

We perturb $\varphi : N' \to \mathbb{C}_t$ to $\varphi_\epsilon : N' \to \mathbb{C}_t$ as follows:

$$(5.1) \qquad \begin{cases} \varphi_\epsilon(u, v) = u(u - \epsilon)h(v) & \text{on } N'_{uv} \\ \varphi_\epsilon(x, y) = x(xh(y) - \epsilon) & \text{on } N'_{xy}, \end{cases}$$

where ϵ is the parameter of perturbation.

On N'_{uv}, the divisor $\varphi_\epsilon = 0$ has the following components:

$$u = 0, u = \epsilon, v = 0, v = \alpha_1, v = \alpha_2, \cdots, v = \alpha_g.$$

And on N'_{xy}, the divisor $\varphi_\epsilon = 0$ has two components:

$$x = 0 \text{ and } xh(y) - \epsilon = 0.$$

The shape of the divisor $\varphi_\epsilon = 0$ is shown in Figure 12.

As before, by attaching $2(g+1)$ 2-handles to N'', we extend $\varphi_\epsilon : N'' \to \mathbb{C}_t$ to $N'' \cup$ 2-handles $\to \mathbb{C}_t$ so that divisors $\varphi_\epsilon = t$ in $N'' \cup$ 2-handles are

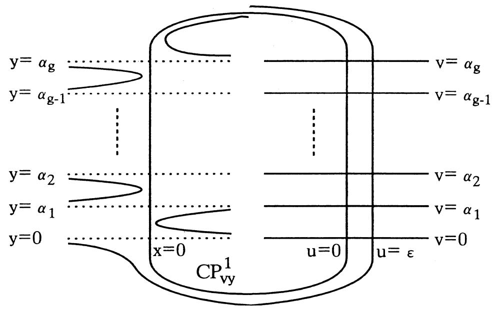

FIGURE 12. The divisor $\varphi_\epsilon = 0$ in N''.

compact. However, we need to deform the attaching maps of the 2-handles from the original attaching maps, according to the parameter ϵ.

On local coordinates (u, k) and (x, k) at P_j and Q_j defined in the previous section, φ_ϵ is given by

$$\varphi_\epsilon(u, k) = u(u - \epsilon)k$$

and

$$\varphi_\epsilon(x, k) = x(xk - \epsilon),$$

respectively. Take a 2-handle $\Delta^2 = \{(\sigma, \tau)||\sigma| < \delta', |\tau| < \delta''\}$ and attach this 2-handle to the local coordinate (u, k) or (x, k) by

$$\sigma = u^{-1}, \tau = u(u - \epsilon)k \tag{5.2}$$

or

$$\sigma = x^{-1}, \tau = x(xk - \epsilon). \tag{5.3}$$

By defining a map $\Delta^2 \to \mathbb{C}_t$ as

$$(\sigma, \tau) \to \tau, \tag{5.4}$$

we extend $\varphi_\epsilon : N'' \to \mathbb{C}_t$ to $\varphi_\epsilon : N'' \cup \Delta^2 \to \mathbb{C}_t$. And we can extend $\varphi_\epsilon : N'' \to \mathbb{C}_t$ to $\varphi_\epsilon : N'' \cup 2(g + 1)$ 2-handles $\to \mathbb{C}_t$.

Attaching maps (5.2) and (5.3) depend on ϵ. Therefore we denote the manifold obtained by attaching these $2(g + 1)$ 2-handles by

$$N'' \cup_\epsilon 2(g + 1) \text{ 2-handles.}$$

If $\epsilon = 0$, then the attaching maps (5.2) and (5.3) are the original attaching maps (4.9) and (4.10) and $N'' \cup_0 2(g+1)$ 2-handles is the original manifold obtained in the previous section. Now, when $\varphi_0 = \varphi$ is perturbed to φ_ϵ by ϵ, the complex structure of $N'' \cup_0 2(g+1)$ 2-handles is also deformed to $N'' \cup_\epsilon 2(g+1)$ 2-handles.

The shape of divisor $\varphi_\epsilon = 0$ in $N'' \cup_\epsilon 2(g+1)$ 2-handles for $\epsilon \neq 0$ is shown in Figure 13. As shown in Figure 13, the divisor $\varphi_\epsilon = 0$ is the singular fiber obtained by pinching each of the circles in Figure 13 to a point.

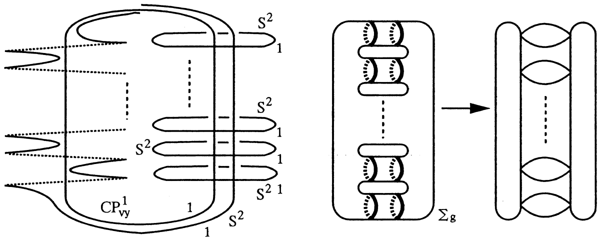

FIGURE 13. The shape of the divisor $\varphi_\epsilon = 0$.

6. New Singular Fibers of φ_ϵ

To seek other singular fibers of φ_ϵ, we will study the critical points of φ_ϵ. Recall the critical points of h are $\gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_g, \frac{1}{\gamma_g}, \cdots, \frac{1}{\gamma_3}, \frac{1}{\gamma_2}, \frac{1}{\gamma_1}$. we compute the critical points of φ_ϵ on N'.

On N'_{uv},

$$\frac{\partial \varphi_\epsilon}{\partial u} = (2u - \epsilon)h(v)$$

and

$$\frac{\partial \varphi_\epsilon}{\partial v} = u(u-\epsilon)\frac{\partial h}{\partial v}(v).$$

By solving $\frac{\partial \varphi_\epsilon}{\partial u} = 0$ and $\frac{\partial \varphi_\epsilon}{\partial v} = 0$, we obtain

(6.1) $$(u,v) = (0,0), (0,\alpha_1), (0,\alpha_2), \cdots, (0,\alpha_g), \\ (\epsilon,0), (\epsilon,\alpha_1), (\epsilon,\alpha_2), \cdots, (\epsilon,\alpha_g)$$

as the critical points of φ_ϵ on the divisor $\varphi_\epsilon = 0$. Moreover, we obtain

(6.2) $$(u,v) = (\frac{\epsilon}{2}, \gamma_1), (\frac{\epsilon}{2}, \gamma_2), \cdots, (\frac{\epsilon}{2}, \gamma_g)$$

as the critical points which are not on the divisor $\varphi_\epsilon = 0$.

On N'_{xy},

$$\frac{\partial \varphi_\epsilon}{\partial x} = (2xh(y) - \epsilon)$$

and

$$\frac{\partial \varphi_\epsilon}{\partial y} = x^2 \frac{\partial h}{\partial y}(y).$$

By solving $\frac{\partial \varphi_\epsilon}{\partial y} = 0$ and $\frac{\partial \varphi_\epsilon}{\partial x} = 0$, we obtain

$$(x, y) = (\frac{\epsilon}{2}\frac{1}{h(\gamma_1)}, \gamma_1), (\frac{\epsilon}{2}\frac{1}{h(\gamma_2)}, \gamma_2), \cdots, (\frac{\epsilon}{2}\frac{1}{h(\gamma_g)}, \gamma_g) \tag{6.3}$$

and these are not on the divisor $\varphi_\epsilon = 0$.

We compute the critical values of these critical points. Obviously, the critical points in (6.1) have the same critical value 0.

The critical values of critical points in (6.2) are the following:

$$\varphi_\epsilon(\frac{\epsilon}{2}, \gamma_i) = -\frac{\epsilon^2}{4}h(\gamma_i) \quad (i = 1, 2, \cdots, g). \tag{6.4}$$

And the critical values of critical points in (6.3) are the following:

$$\varphi_\epsilon(\frac{\epsilon}{2}\frac{1}{h(\gamma_i)}, \gamma_i) = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)} \quad (i = 1, 2, \cdots, g). \tag{6.5}$$

We guess that $2g$ fibers

$$\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i) \quad (i = 1, 2, \cdots, g), \varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)} \quad (i = 1, 2, \cdots, g) \tag{6.6}$$

are Lefschetz type singular fibers, which will be verified in Section 9.

Suppose that ϵ is a small positive real number. Then, by (4.3),(4.4), the critical values in (6.4) satisfy the following:

$$\begin{aligned} -\frac{\epsilon^2}{4} < -\frac{\epsilon^2}{4}h(\gamma_{g-1}) < -\frac{\epsilon^2}{4}h(\gamma_{g-3}) < \cdots < -\frac{\epsilon^2}{4}h(\gamma_5) < \\ -\frac{\epsilon^2}{4}h(\gamma_3) < -\frac{\epsilon^2}{4}h(\gamma_1) < 0 < -\frac{\epsilon^2}{4}h(\gamma_2) < -\frac{\epsilon^2}{4}h(\gamma_4) < \\ -\frac{\epsilon^2}{4}h(\gamma_6) < \cdots < -\frac{\epsilon^2}{4}h(\gamma_{g-2}) < -\frac{\epsilon^2}{4}h(\gamma_g) < \frac{\epsilon^2}{4} \end{aligned} \tag{6.7}$$

if g is even;

$$(6.8)\quad -\frac{\epsilon^2}{4} < -\frac{\epsilon^2}{4}h(\gamma_{g-1}) < -\frac{\epsilon^2}{4}h(\gamma_{g-3}) < \cdots < -\frac{\epsilon^2}{4}h(\gamma_6) < -\frac{\epsilon^2}{4}h(\gamma_4) < -\frac{\epsilon^2}{4}h(\gamma_2) < 0 < -\frac{\epsilon^2}{4}h(\gamma_1) < -\frac{\epsilon^2}{4}h(\gamma_3) < -\frac{\epsilon^2}{4}h(\gamma_5) < \cdots < -\frac{\epsilon^2}{4}h(\gamma_{g-2}) < -\frac{\epsilon^2}{4}h(\gamma_g) < \frac{\epsilon^2}{4}$$

if g is odd. And, by (4.5),(4.6) and the condition (iii) in Section 4, the critical values in (6.5) satisfy the following:

$$(6.9)\quad -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_1)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_3)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_5)} < \cdots < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-3})} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-1})} < -\frac{\epsilon^2}{4},$$
$$\frac{\epsilon^2}{4} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_g)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-2})} < \cdots < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_6)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_4)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_2)}$$

if g is even;

$$(6.10)\quad -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_2)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_4)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_6)} < \cdots < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-3})} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-1})} < -\frac{\epsilon^2}{4},$$
$$\frac{\epsilon^2}{4} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_g)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_{g-2})} < \cdots < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_5)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_3)} < -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_1)}$$

if g is odd.

7. Branch Points of The Double Branched-Covering Map Π

Let ϵ be a small positive real number and D a 2-disk which contains all critical values of φ_ϵ. Recall the projection $\Pi : N' \to \mathbb{CP}^1_{vy}$ defined by (4.12). For any t which satisfies $0 \neq t \in D$, we can naturally extend $\Pi : N'' \cap \{\varphi_\epsilon = t\} \to \mathbb{CP}^1_{vy}$ to $\Pi : \{\varphi_\epsilon = t\} \to \mathbb{CP}^1_{vy}$ as follows.

On the 2-handle $\Delta^2 = \{(\sigma,\tau) | |\sigma| < \delta', |\tau| < \delta''\}$ attached to the local coordinate (u,k) at P_j, the fiber $\varphi_\epsilon = t$ satisfies

$$\{\varphi_\epsilon = t\} \cap \Delta^2 = \{(\sigma,\tau) | \tau = t\}. \tag{7.1}$$

By (5.2) and $k = h(v)$, we obtain

$$h(v) = k = \frac{\sigma^2\tau}{1-\epsilon\sigma}. \tag{7.2}$$

From (7.2), we can define the projection Π on $\{\varphi_\epsilon = t\} \cap \Delta^2 = \{(\sigma,\tau) | \tau = t\}$ by

$$\Pi(\sigma, \tau = t) = h^{-1}\Big(\frac{\sigma^2 t}{1-\epsilon\sigma}\Big). \tag{7.3}$$

Similarly, on the 2-handle $\Delta^2 = \{(\sigma,\tau) | |\sigma| < \delta', |\tau| < \delta''\}$ attached to the local coordinate (x,k) at Q_j, the fiber $\varphi_\epsilon = t$ satisfies (7.1). By (5.3) and $k = h(y)$, we obtain

$$h(y) = k = \sigma(\sigma\tau + \epsilon). \tag{7.4}$$

From (7.4), we can define the projection Π on $\{\varphi_\epsilon = t\} \cap \Delta^2 = \{(\sigma,\tau) | \tau = t\}$ by

$$\Pi(\sigma, \tau = t) = h^{-1}(\sigma(\sigma t + \epsilon)). \tag{7.5}$$

And this projection $\Pi : \{\varphi_\epsilon = t\} \to \mathbb{CP}^1_{vy}$ becomes a double branched-covering map, as follows.

First, we study $\Pi^{-1}(v)$ for any $v \in \mathbb{CP}^1_{vy}$.

On N'_{uv}, the fiber $\varphi_\epsilon = t$ is given by

$$\varphi_\epsilon(u,v) = u(u-\epsilon)h(v) = t. \tag{7.6}$$

From (7.6), we obtain the quadratic equation on the unknown u on N'_{uv}

$$h(v)u^2 - h(v)\epsilon u - t = 0. \tag{7.7}$$

By (7.3), we obtain the equation

$$v = \Pi(\sigma, t) = h^{-1}\Big(\frac{\sigma^2 t}{1-\epsilon\sigma}\Big) \tag{7.8}$$

on the 2-handle attached to the local coordinate at P_j. From (7.8), we obtain

$$h(v) = \frac{\sigma^2 t}{1-\epsilon\sigma}.$$

By rewriting this equation, we obtain the quadratic equation on the unknown σ on the 2-handle attached to the local coordinate at P_j

$$-t\sigma^2 - h(v)\epsilon\sigma + h(v) = 0. \tag{7.9}$$

We will consider two cases below.

Case 1: $v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$.

By $h(v) = 0$, the quadratic equation (7.9) becomes

$$-t\sigma^2 = 0.$$

By $t \neq 0$, we obtain the double root

$$\sigma = 0.$$

Therefore v becomes a branch point on $\mathbb{CP}^1_{vy}$. $\Pi^{-1}(v)$ is on a 2-handle and satisfies

$$\Pi^{-1}(v) = \{(\sigma, \tau) = (0, t)\}.$$

Case 2: $v \neq 0, \alpha_1, \alpha_2, \cdots, \alpha_g$.

By the relation $\sigma = u^{-1}$ in (5.2), the quadratic equation (7.9) is equivalent to the quadratic equation (7.7). So we mainly treat the quadratic equation (7.7), instead of the quadratic equation (7.9).

The discriminant of (7.7) is

$$\tilde{\Delta}_t(v) = h(v)^2\epsilon^2 + 4th(v).$$

By $h(v) \neq 0$, we put

$$\Delta_t(v) = \frac{\tilde{\Delta}_t(v)}{h(v)}.$$

Then the discriminant $\Delta_t(v)$ satisfies

$$\Delta_t(v) = h(v)\epsilon^2 + 4t = 0 \tag{7.10}$$

if and only if $\Pi^{-1}(v)$ is a singleton, namely, a branch point of Π.

Next, we study $\Pi^{-1}(y)$ for any $y \in \mathbb{CP}^1_{vy}$.

On N'_{xy}, the fiber $\varphi_\epsilon = t$ is given by

$$\varphi_\epsilon(x, y) = x(xh(y) - \epsilon) = t. \tag{7.11}$$

From (7.11), we obtain the quadratic equation on the unknown x on N'_{xy}

$$h(y)x^2 - \epsilon x - t = 0. \tag{7.12}$$

By (7.5), we obtain the equation

$$y = \Pi(\sigma, t) = h^{-1}(\sigma(\sigma t + \epsilon)) \tag{7.13}$$

on the 2-handle attached to the local coordinate at Q_j. From (7.13), we obtain

$$h(y) = \sigma(\sigma t + \epsilon).$$

By rewriting this equation, we obtain the quadratic equation on the unknown σ on the 2-handle attached to the local coordinate at Q_j

$$-\sigma^2 t - \sigma\epsilon + h(y) = 0. \tag{7.14}$$

We will consider two cases below.

Case 1′: $y = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$.

By $h(y) = 0$, the quadratic equation (7.14) becomes

$$-\sigma^2 t - \sigma\epsilon = 0.$$

By solving this equation, we obtain

$$\sigma = 0, -\frac{\epsilon}{t}.$$

Therefore y is not a branch point on $\mathbb{CP}^1_{vy}$. And $\Pi^{-1}(y)$ satisfies

$$\Pi^{-1}(y) = \{(\sigma, \tau) = (0, t)\} \cup \{(\sigma, \tau) = (-\frac{\epsilon}{t}, t)\}.$$

By the relation $\sigma = x^{-1}$ in (5.3) and (7.5), the point $(\sigma, \tau) = (-\frac{\epsilon}{t}, t)$ corresponds to the point in N'_{xy}

$$(\sigma^{-1}, \Pi(\sigma, \tau)) = (-\frac{t}{\epsilon}, y).$$

Remark that y is a branch point on $\mathbb{CP}^1_{vy}$ if $\epsilon = 0$; (See Section 4).

Case 2′: $y \neq 0, \alpha_1, \alpha_2, \cdots, \alpha_g$.

By the relation $\sigma = x^{-1}$ in (5.3), the quadratic equation (7.14) is equivalent to the quadratic equation (7.12). So we mainly treat the quadratic equation (7.12), instead of the quadratic equation (7.14).

The discriminant of (7.12) is

$$\Delta_t(y) = \epsilon^2 + 4th(y).$$

Then the discriminant $\Delta_t(y)$ satisfies

$$\Delta_t(y) = \epsilon^2 + 4th(y) = 0 \tag{7.15}$$

if and only if $\Pi^{-1}(y)$ is a singleton, namely, a branch point of Π.

The equation (7.15) is equivalent to the equation (7.10) by the relation $v = \frac{1}{y}$ on $\mathbb{CP}^1_{vy}$. Therefore, we can consider equations (7.10) and (7.15) as the same equation $\Delta_t = 0$ on $\mathbb{CP}^1_{vy}$. And, by solving the equation $\Delta_t = 0$, we can obtain branch points on $\mathbb{CP}^1_{vy}$.

Thus we obtain the following lemma:

Lemma 7.1. *For any t which satisfies $0 \neq t \in D$,*

$$\Pi : \{\varphi_\epsilon = t\} \to \mathbb{CP}^1_{vy}$$

is a double branched-covering map. We obtain

$$v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$$

as $g+1$ branch points on $\mathbb{CP}^1_{vy}$. The inverse image $\Pi^{-1}(v)$ for each v ($v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$) is on a 2-handle and satisfies

$$\Pi^{-1}(v) = \{(\sigma, \tau) | \sigma = 0, \tau = t\}.$$

The other branch points on $\mathbb{CP}^1_{vy}$ are obtained by solving the equation $\Delta_t = 0$.

When t moves in D, the branch points obtained by the equation $\Delta_t = 0$ also move in $\mathbb{CP}^1_{vy}$. But the branch points $v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$ do not move. Therefore we will call the branch points obtained by the equation $\Delta_t = 0$ *the moving branch points* and the branch points $v = 0, \alpha_1, \alpha_2, \cdots, \alpha_g$ *the fixed branch points*.

Remark 7.2. In the early stage of the study of splitting of the singular fiber F_{ω_g}, the author solved the equation $\Delta_t = 0$ by using *Mathematica* and studied how the moving branch points move on $\mathbb{CP}^1_{vy}$ when t moves from the smallest critical value of φ_ϵ to the largest critical value of φ_ϵ on the real axis in D. But in the final proof given below, the author has succeeded in avoiding computer calculation.

8. Moving Branch Points on $\mathbb{CP}^1_{vy}$

Let ϵ be a small positive real number and D the 2-disk given in the previous section. Then the critical values of φ_ϵ satisfy (6.7),(6.8),(6.9) and (6.10). In this section, we study how the moving branch points move on $\mathbb{CP}^1_{vy}$ when t moves from the smallest critical value of φ_ϵ to the largest critical value of φ_ϵ on the real axis in D.

Recall that the holomorphic function $h : \{v \in \mathbb{C} | v \neq \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \cdots, \frac{1}{\alpha_{g-1}}, \frac{1}{\alpha_g}\} \to \mathbb{C}_{\tilde{v}}$ given by (4.1) satisfies the following conditions:

(i) $0 < \alpha_1 \ll \alpha_2 \ll \alpha_3 \ll \cdots \ll \alpha_{g-1} \ll \alpha_g \ll 1$;

(ii) $h(\bar{v}) = \overline{h(v)}$ for the conjugate $\bar{v}$ of v;

(iii) $h(\frac{1}{v}) = \frac{1}{h(v)}$;

(iv) $h(v) \risingdotseq v^{g+1}$ for v which satisfies $\alpha_g \ll |v| \ll \frac{1}{\alpha_g}$.

And recall that the critical values of the critical points (4.2) satisfy (4.3), (4.4), (4.5), (4.6).

Since the function h satisfies the above condition (iii), we can extend this function to the map

$$h : \mathbb{CP}^1_{vy} \to \mathbb{CP}^1_{\tilde{v}\tilde{y}}$$

defined by

$$\tilde{v} = h(v) \tag{8.1}$$

and

$$\tilde{y} = h(y). \tag{8.2}$$

Then the map $h : \mathbb{CP}^1_{vy} \to \mathbb{CP}^1_{\tilde{v}\tilde{y}}$ becomes a $(g+1)$-fold branched-covering map with $2g$ branch points. The branch points of h on $\mathbb{CP}^1_{vy}$ are the critical points of h

$$v = \gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_g, \frac{1}{\gamma_g}, \cdots, \frac{1}{\gamma_3}, \frac{1}{\gamma_2}, \frac{1}{\gamma_1}$$

and the branching order of h at each branch point is 2.

We study the $(g+1)$-fold branched-covering map h. For each i ($i = 1, 2, 3, \cdots, g$), we take the loop l_i based at $\tilde{v} = 0$ which goes around the branch point $\tilde{v} = h(\gamma_i)$ as shown in Figure 14. When $\tilde{v}$ goes around from the base point 0 to itself on the loop l_i in $\mathbb{C}_{\tilde{v}}$ ($\subset \mathbb{CP}^1_{\tilde{v}\tilde{y}}$), $h^{-1}(\tilde{v})$ moves on $\mathbb{C}_v$ ($\subset \mathbb{CP}^1_{vy}$) as shown in Figure 15, where we put $\alpha_0 = 0$. As shown in Figure

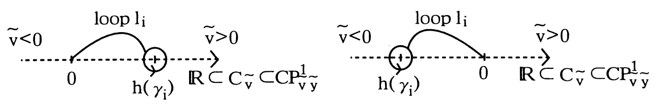

Figure 14.

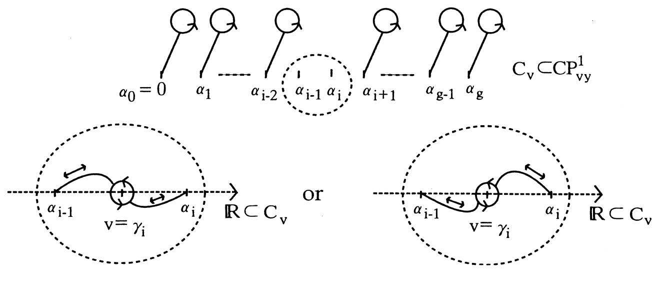

Figure 15. The shape of $h^{-1}(l_i)$.

15, α_{i-1} and α_i are interchanged to α_i and α_{i-1} by the loop l_i. Therefore we denote the permutation obtained by the loop l_i as

$$(8.3)\quad \begin{pmatrix} 0, & \alpha_1, & \alpha_2, & \cdots, & \alpha_{i-2}, & \alpha_{i-1}, & \alpha_i, & \alpha_{i+1}, & \cdots, & \alpha_g \\ 0, & \alpha_1, & \alpha_2, & \cdots, & \alpha_{i-2}, & \alpha_i, & \alpha_{i-1}, & \alpha_{i+1}, & \cdots, & \alpha_g \end{pmatrix}_{l_i}.$$

Case (1): g is even.

We take two loops l' and l'' based at $\tilde{v} = 0$ as shown in Figure 16 (i). Then we have the following relations:

$$l' = l_g \cdot l_{g-2} \cdot l_{g-4} \cdots l_6 \cdot l_4 \cdot l_2 \quad \in \pi_1(\mathbb{C}_{\tilde{v}} \setminus \{\text{ branch points of } h\}, \tilde{v} = 0)$$

and

$$l'' = l_1 \cdot l_3 \cdot l_5 \cdots l_{g-5} \cdot l_{g-3} \cdot l_{g-1} \quad \in \pi_1(\mathbb{C}_{\tilde{v}} \setminus \{\text{ branch points of } h\}, \tilde{v} = 0).$$

Therefore the permutations obtained by the loops l' and l'' are the following:

$$(8.4)\quad \begin{pmatrix} 0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6, & \cdots, & \alpha_{g-2}, & \alpha_{g-1}, & \alpha_g \\ 0, & \alpha_2, & \alpha_1, & \alpha_4, & \alpha_3, & \alpha_6, & \alpha_5, & \cdots, & \alpha_{g-3}, & \alpha_g, & \alpha_{g-1} \end{pmatrix}_{l'}$$

and

$$(8.5)\quad \begin{pmatrix} 0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6, & \cdots, & \alpha_{g-2}, & \alpha_{g-1}, & \alpha_g \\ \alpha_1, & 0, & \alpha_3, & \alpha_2, & \alpha_5, & \alpha_4, & \alpha_7, & \cdots, & \alpha_{g-1}, & \alpha_{g-2}, & \alpha_g \end{pmatrix}_{l''}.$$

And when $\tilde{v}$ goes around from the base point 0 to itself on the loop l' or the loop l'' in $\mathbb{C}_{\tilde{v}}$, $h^{-1}(\tilde{v})$ moves on $\mathbb{C}_v$ as shown in Figure 16 (ii).

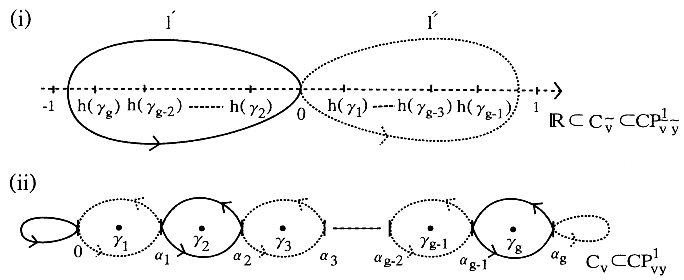

FIGURE 16. The shapes of $l', l'', h^{-1}(l')$ and $h^{-1}(l'')$.

When we take the loop $l' \cdot l''$, the permutation obtained by the loop $l' \cdot l''$ is the following:

$$(8.6)\quad \begin{pmatrix} 0, & \alpha_1, & \alpha_2, & \alpha_3, & \alpha_4, & \alpha_5, & \alpha_6, & \cdots, & \alpha_{g-3}, & \alpha_{g-2}, & \alpha_{g-1}, & \alpha_g \\ \alpha_1, & \alpha_3, & 0, & \alpha_5, & \alpha_2, & \alpha_7, & \alpha_4, & \cdots, & \alpha_{g-1}, & \alpha_{g-4}, & \alpha_g, & \alpha_{g-2} \end{pmatrix}_{l'\cdot l''}.$$

We study $h^{-1}(\tilde{v}=1)$. By $h(v=1)=1$, we have $1 \in h^{-1}(\tilde{v}=1)$. The function h has the condition (iv). Therefore by solving the equation $h(v) \fallingdotseq v^{g+1}=1$, we obtain

$$v \fallingdotseq e^{\frac{2\pi n i}{g+1}}, \quad (n=1,2,3,\cdots,g)$$

and $v=1$ as the elements of $h^{-1}(\tilde{v}=1)$.

We take the loop based at $\tilde{v}=1$

$$l=\{\tilde{v}=e^{i\theta} | 0 \le \theta \le 2\pi\}.$$

By $h(v) \fallingdotseq v^{g+1}$, when $\tilde{v}$ goes around from the base point 1 to itself on the loop l, $h^{-1}(\tilde{v})$ moves on $\mathbb{C}_v$ as shown in Figure 17. Therefore the permutation obtained by the loop l becomes as follows:

$$(8.7)\qquad \begin{pmatrix} 1, & e^{\frac{2\pi i}{g+1}}, & e^{\frac{4\pi i}{g+1}}, & \cdots, & e^{\frac{2(g-1)\pi i}{g+1}}, & e^{\frac{2g\pi i}{g+1}} \\ e^{\frac{2\pi i}{g+1}}, & e^{\frac{4\pi i}{g+1}}, & e^{\frac{6\pi i}{g+1}}, & \cdots, & e^{\frac{2g\pi i}{g+1}}, & 1 \end{pmatrix}_{l}.$$

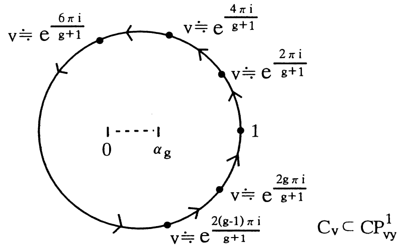

Figure 17.

We take the disk

$$D_{\tilde{v}}=\{\tilde{v} \in \mathbb{C}_{\tilde{v}} | \; |\tilde{v}| \le 1\}.$$

We draw the arc L which joins $v=\alpha_g$ to $v=1$ on $\mathbb{C}_v$ as shown in Figure 18 (i).

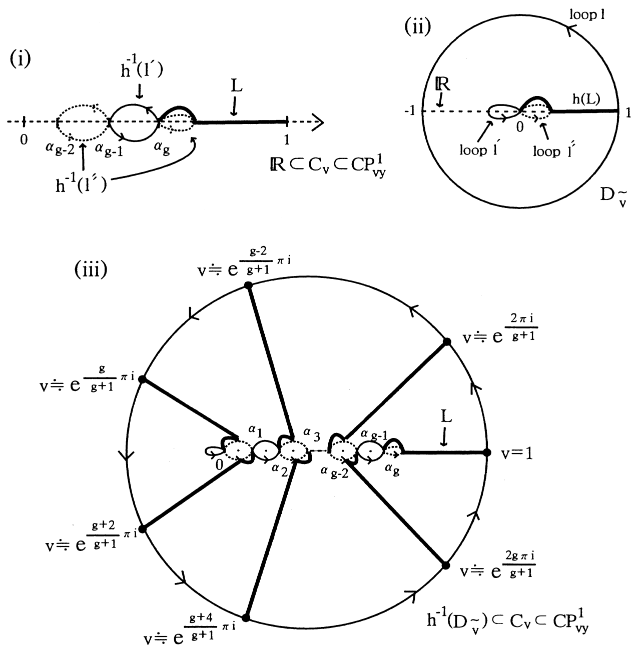

FIGURE 18.

From the graph of $h : \mathbb{R}(\subset \mathbb{C}_v) \to \mathbb{R}(\subset \mathbb{C}_{\tilde{v}})$ shown in Figure 10, we understand that $h(L)$ becomes as shown in Figure 18 (ii). Since the loop $l' \cdot l''$ has the permutation (8.6) and the loop l has the permutation (8.7), the inverse image of $D_{\tilde{v}}$ with the loops l, l', l'' and $h(L)$ under the map $h : \mathbb{C}_v\ (\subset \mathbb{CP}^1_{vy}) \to \mathbb{C}_{\tilde{v}}\ (\subset \mathbb{CP}^1_{\tilde{v}\tilde{y}})$ becomes as shown in Figure 18 (iii).

By the relation $v = \frac{1}{y}$ on $\mathbb{CP}^1_{vy}$, we can identify the branch points of h

$$v = \frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{1}{\gamma_3}, \cdots, \frac{1}{\gamma_g}$$

with

$$y = \gamma_1, \gamma_2, \gamma_3, \cdots, \gamma_g.$$

Also, by the relation $\tilde{v} = \frac{1}{\tilde{y}}$ on $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$, we can identify

$$\tilde{v} = h(\frac{1}{\gamma_1}), h(\frac{1}{\gamma_2}), h(\frac{1}{\gamma_3}), \cdots, h(\frac{1}{\gamma_g})$$

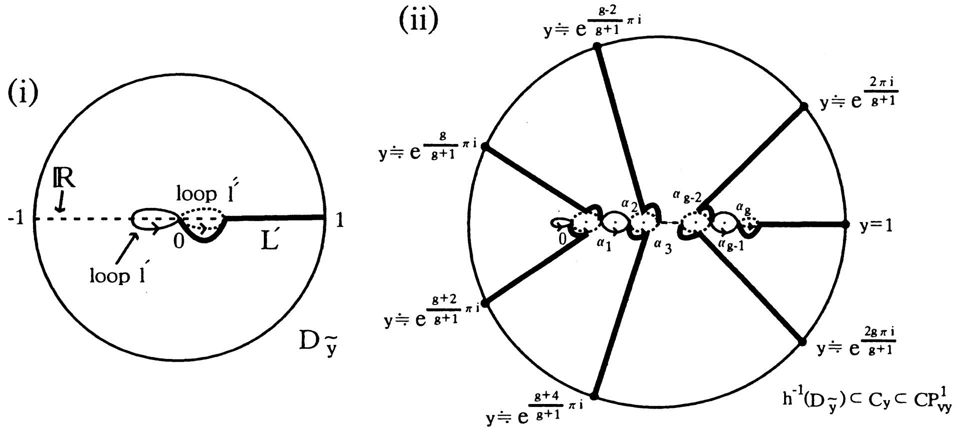

Figure 19.

with

$$\tilde{y} = h(\gamma_1), h(\gamma_2), h(\gamma_3), \cdots, h(\gamma_g).$$

On $\mathbb{C}_{\tilde{y}}$ ($\subset \mathbb{CP}^1_{\tilde{v}\tilde{y}}$), we take the same loops l' and l'' shown in Figure 16 (i) and the disk

$$D_{\tilde{y}} = \{\tilde{y} \in \mathbb{C}_{\tilde{y}} |\ |\tilde{y}| \leq 1\}$$

with the loops l', l'' and the arc L' shown in Figure 19 (i). Then the inverse image of $D_{\tilde{y}}$ with the loops l', l'' and the arc L' under the map $h : \mathbb{C}_y$ ($\subset \mathbb{CP}^1_{vy}$) $\to \mathbb{C}_{\tilde{y}}$ ($\subset \mathbb{CP}^1_{\tilde{v}\tilde{y}}$) becomes as shown in Figure 19 (ii).

When we attach the disk $D_{\tilde{v}}$ with the loops l', l'' and the arc $h(L)$ to the disk $D_{\tilde{y}}$ with the loops l', l'' and the arc L' by the relation $\tilde{v} = \frac{1}{\tilde{y}}$, we obtain $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ with loops and an arc as shown in Figure 20 (i). And when we attach the inverse image of $D_{\tilde{v}}$ with the loops l', l'' and the arc $h(L)$ to the inverse image of $D_{\tilde{y}}$ with the loops l', l'' and the arc L' by the relation $v = \frac{1}{y}$, we obtain $\mathbb{CP}^1_{vy}$ with arcs as shown in Figure 20 (ii). $\mathbb{CP}^1_{vy}$ with arcs shown in Figure 20 (ii) is the inverse image of $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ with loops and the arc shown in Figure 20 (i) under the covering map $h : \mathbb{CP}^1_{vy} \to \mathbb{CP}^1_{\tilde{v}\tilde{y}}$.

Next we will study the moving branch points.

From the equation $\Delta_t(v) = 0$ in (7.10), we obtain the equation

$$\tilde{v} = h(v) = -\frac{4t}{\epsilon^2}.$$

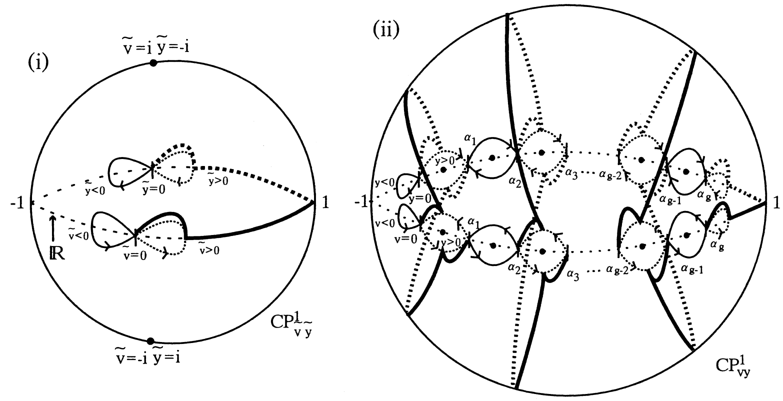

FIGURE 20.

For any t which satisfies $0 \neq t \in D$, the moving branch points for the fiber $\varphi_\epsilon = t$ are obtained by

$$h^{-1}\left(\tilde{v} = -\frac{4t}{\epsilon^2}\right).$$

And by the map $\mu : D \to \mathbb{C}_{\tilde{v}}$ $(\subset \mathbb{CP}^1_{\tilde{v}\tilde{y}})$ defined by

$$\tilde{v} = \mu(t) = -\frac{4t}{\epsilon^2},$$

we identify a point $t \in D$ with a point $\tilde{v} \in \mu(D)$ $(\subset \mathbb{C}_{\tilde{v}})$.

By this identification, we can identify the critical values of φ_ϵ

$$t = -\frac{\epsilon^2}{4} h(\gamma_i) \quad (i = 1, 2, \cdots, g)$$

in (6.4) with the branch points of h on $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$

$$\tilde{v} = h(\gamma_i) \quad \left(= \mu\left(-\frac{\epsilon^2}{4} h(\gamma_i)\right)\right)$$

and the the critical values of φ_ϵ

$$t = -\frac{\epsilon^2}{4} \frac{1}{h(\gamma_i)} \quad (i = 1, 2, \cdots, g)$$

in (6.5) with the branch points of h on $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$

$$\tilde{y} = h(\gamma_i) \left(= \frac{1}{\mu\left(-\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}\right)} \right).$$

And for any $\tilde{v} \in \mu(D \setminus \{0\})$, the moving branch points for the fiber $\varphi_\epsilon = \mu^{-1}(\tilde{v})$ are obtained by

$$h^{-1}(\tilde{v}).$$

To know how the moving branch points move on $\mathbb{CP}^1_{vy}$ when t moves from the smallest critical value of φ_ϵ to the largest critical value of φ_ϵ on the real axis in D, we may study how $h^{-1}(\tilde{v})$ moves on $\mathbb{CP}^1_{vy}$ when $\tilde{v}$ moves from $\tilde{v} = \infty$ to itself on the real axis in $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ as shown in Figure 21.

On the real axis in the disk $D_{\tilde{v}}$ with the loops l', l'' and the arc $h(L)$, we move $\tilde{v}$ from 1 to the intersection point $\tilde{v}_0$ of the loop l'' and $h(L)$ other

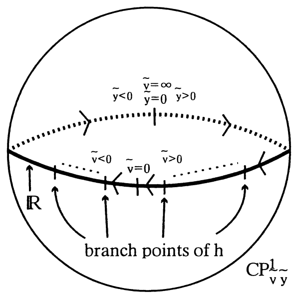

Figure 21.

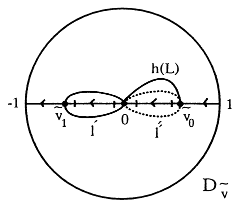

Figure 22.

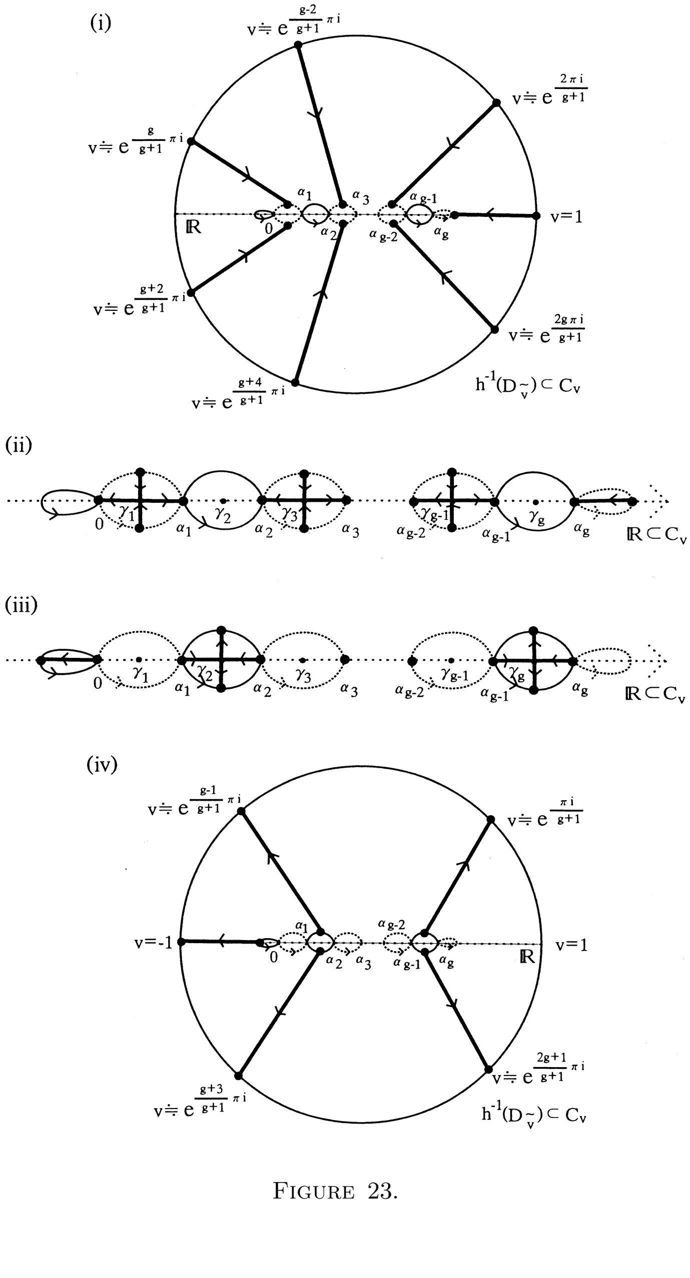

FIGURE 23.

than 0 as shown in Figure 22. Then $h^{-1}(\tilde{v})$ moves as shown in Figure 23 (i), on $h^{-1}(h(L))$.

Next, on the real axis in $D_{\tilde{v}}$, we move $\tilde{v}$ from $\tilde{v}_0$ to 0 as shown in Figure 22. Recall that the branching order of h at each branch point is 2. Therefore, from the graph of $h : \mathbb{R}(\subset \mathbb{C}_v) \to \mathbb{R}(\subset \mathbb{C}_{\tilde{v}})$ shown in Figure 10, we understand that $h^{-1}(\tilde{v})$ moves on $\mathbb{C}_v$ as shown in Figure 23 (ii).

On the real axis in $D_{\tilde{v}}$, we move $\tilde{v}$ from 0 to the intersection point $\tilde{v}_1$ of the loop l' and the real axis other than 0 as shown in Figure 22. From the graph of $h : \mathbb{R}(\subset \mathbb{C}_v) \to \mathbb{R}(\subset \mathbb{C}_{\tilde{v}})$ shown in Figure 10, we understand that $h^{-1}(\tilde{v})$ moves as shown in Figure 23 (iii).

On the real axis in $D_{\tilde{v}}$, we move $\tilde{v}$ from $\tilde{v}_1$ to -1 as shown in Figure 22. Recall $h(v) \risingdotseq v^{g+1}$ for v which satisfies $\alpha_g \ll |v| \ll \frac{1}{\alpha_g}$. Therefore we obtain

$$v \risingdotseq e^{\frac{(2n+1)\pi i}{g+1}}, \quad (n = 0, 1, 2, \cdots, \frac{g}{2} - 1, \frac{g}{2} + 1, \cdots, g)$$

and $v = -1$ as elements of $h^{-1}(-1)$. So $h^{-1}(\tilde{v})$ moves as shown in Figure 23 (iv).

Thus when $\tilde{v}$ moves from 1 to -1 on the real axis in $D_{\tilde{v}}$, $h^{-1}(\tilde{v})$ moves

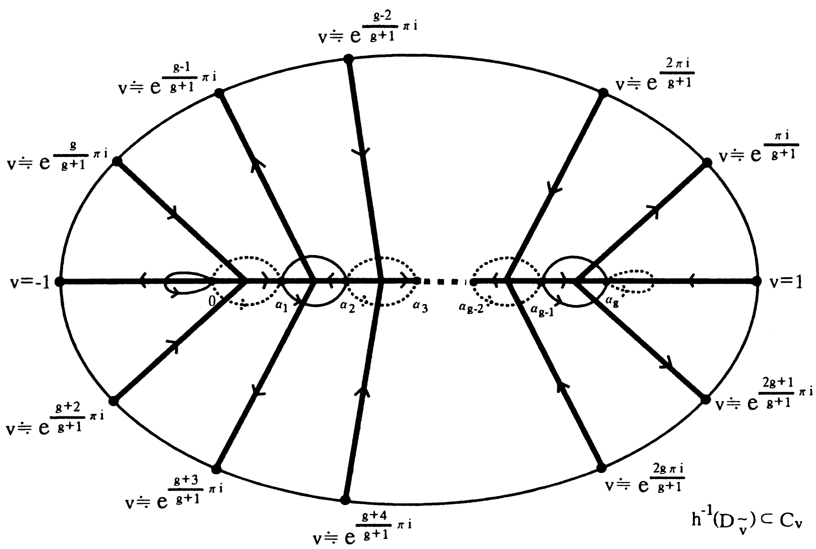

Figure 24.

as shown in Figure 24.

$h^{-1}(\mathbb{R} \cap D_{\tilde{v}})$ $(\subset \mathbb{C}_v)$ becomes the arcs shown in Figure 24. Similarly, $h^{-1}(\mathbb{R} \cap D_{\tilde{y}})$ $(\subset \mathbb{C}_y)$ becomes the same arcs shown in Figure 24. When we attach $h^{-1}(\mathbb{R} \cap D_{\tilde{v}})$ to $h^{-1}(\mathbb{R} \cap D_{\tilde{y}})$ by the relation $v = \frac{1}{y}$, we obtain the arcs shown in Figure 25 on $\mathbb{CP}^1_{vy}$. And the arcs shown in Figure 25 are the inverse image of the real axis on $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ under the covering map h. When $\tilde{v}$ moves from $\tilde{v} = \infty$ to itself on the real axis in $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ as shown in Figure 21, $h^{-1}(\tilde{v})$ moves as shown in Figure 25.

Case (2): g is odd.

As before, we can study the $(g+1)$-fold branched-covering map h and how the moving branch points move on $\mathbb{CP}^1_{vy}$. Here, we state the result only. When $\tilde{v}$ moves from $\tilde{v} = \infty$ to itself on the real axis in $\mathbb{CP}^1_{\tilde{v}\tilde{y}}$ as shown

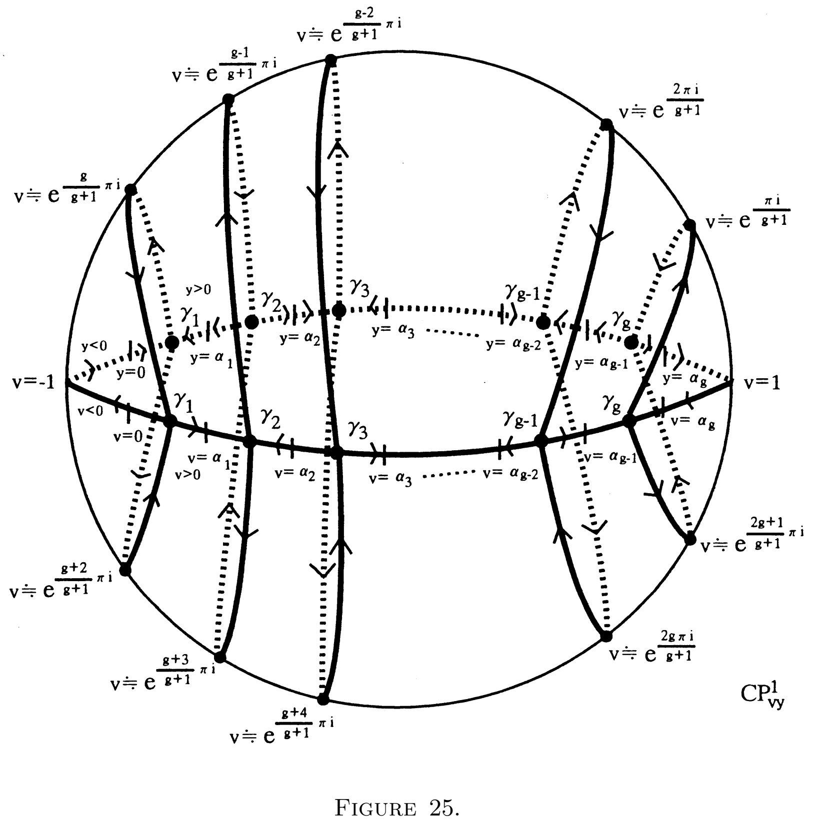

FIGURE 25.

in Figure 21, $h^{-1}(\tilde{v})$ moves as shown in Figure 26.

9. Vanishing Cycles and Monodromies

Let ϵ be a small positive real number and D the 2-disk given in Section 7. Then the critical values of φ_ϵ satisfy (6.7),(6.8),(6.9) and (6.10). In this section, by using Figure 25 and Figure 26, we study vanishing cycles and monodromies about singular fibers of φ_ϵ.

Let $b_0 \in D$ be a sufficiently small positive real number. We fix the fiber $\varphi_\epsilon = b_0$ as the base fiber.

Case (1): g is even.

By Figure 25, the moving branch points $h^{-1}(\mu(b_0))$ and the fixed branch points become as shown in Figure 27. Also, since the projection $\Pi : \{\varphi_\epsilon =$

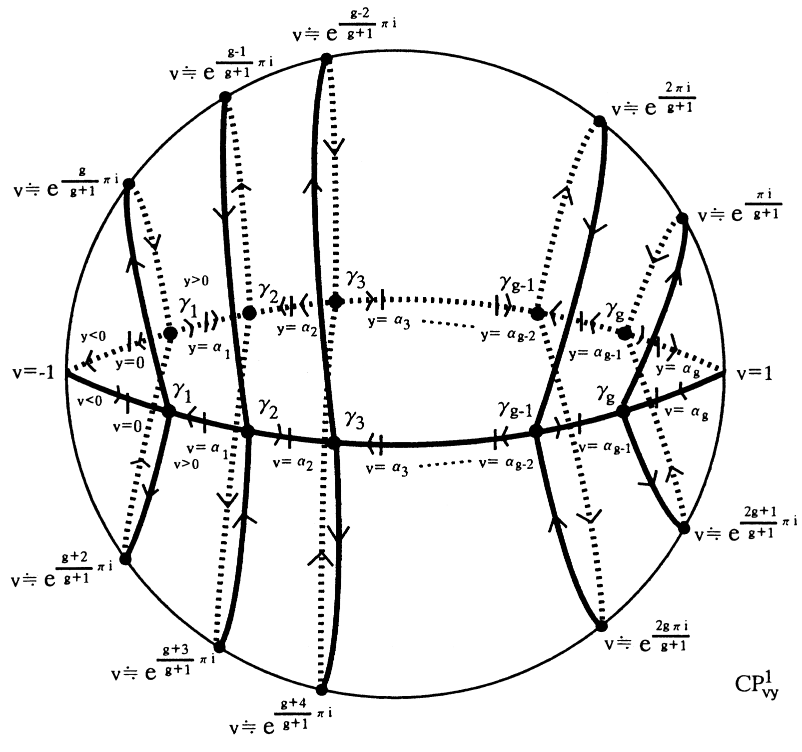

Figure 26.

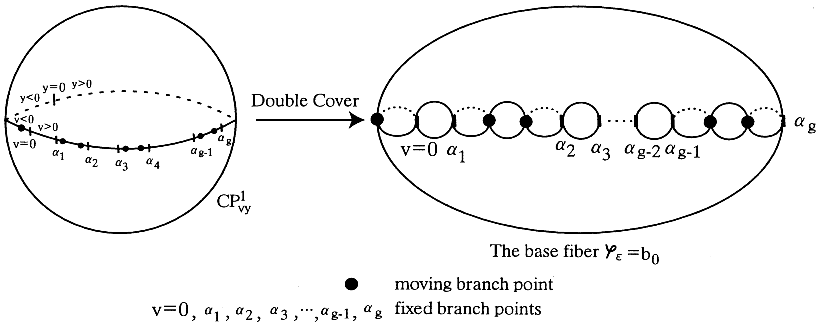

FIGURE 27.

$b_0\} \to \mathbb{CP}^1_{vy}$ is the double branched-covering map, the fiber $\varphi_\epsilon = b_0$ becomes as shown in Figure 27.

Case 1: $t = -\frac{\epsilon^2}{4}h(\gamma_2)$.

We study the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_2)$ on the base fiber $\varphi_\epsilon = b_0$. We move t from b_0 to $-\frac{\epsilon^2}{4}h(\gamma_2)$ on the real axis in D as shown in Figure 28 (i). Then from Figure 25, $h^{-1}(\mu(t))$ moves as shown in Figure 28 (ii).

We denote the two moving branch points on the fiber $\varphi_\epsilon = b_0$ by A, B as shown in Figure 28 (ii). Remark that the branch point on $\mathbb{CP}^1_{vy}$ mapped to

$$\tilde{v} = h(\gamma_2) = \mu\left(t = -\frac{\epsilon^2}{4}h(\gamma_2)\right)$$

is $v = \gamma_2$. When t is moving from b_0 to $-\frac{\epsilon^2}{4}h(\gamma_2)$, the two moving branch points which started from A and B are approaching to the branch point $v = \gamma_2$. And when t arrives at $-\frac{\epsilon^2}{4}h(\gamma_2)$, the two moving branch points collide at $v = \gamma_2$.

We take the locus AB shown in Figure 28 (ii). Then the inverse image of the locus AB under the projection Π becomes the circle shown in Figure 28 (iii). Now when t moves from b_0 to $-\frac{\epsilon^2}{4}h(\gamma_2)$, this circle shrinks to a point. And when t arrives at $-\frac{\epsilon^2}{4}h(\gamma_2)$, the shrunk circle becomes just one point. Therefore the circle shown in Figure 28 (iii) is the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_2)$ on the base fiber $\varphi_\epsilon = b_0$. And this circle is the curve C_4 in Figure 7.

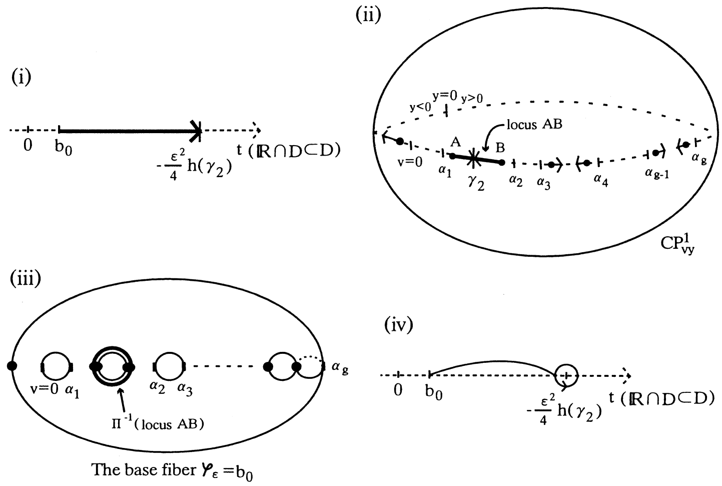

Figure 28.

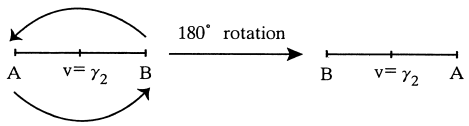

Figure 29.

Next, we take the loop shown in Figure 28 (iv).

When t goes around from b_0 to itself on this loop, the locus AB rotates as shown in Figure 29. When we lift the "180° rotation" twist about the locus AB to the base fiber $\varphi_\epsilon = b_0$ by the projection Π, we obtain the negative Dehn twist about the circle $\Pi^{-1}(\text{locus } AB) = C_4$.

Therefore the monodromy about this loop is ζ_4. Thus we understand that the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_2)$ is a Lefschetz singular fiber of type I.

Case 2: $t = -\frac{\epsilon^2}{4}h(\gamma_i)$ $(i = 4, 6, 8, \cdots, g-2, g)$.

We study the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ on the base fiber $\varphi_\epsilon = b_0$. Remark that the branch point on $\mathbb{CP}^1_{vy}$ mapped to

$$\tilde{v} = h(\gamma_i) = \mu\left(t = -\frac{\epsilon^2}{4}h(\gamma_i)\right)$$

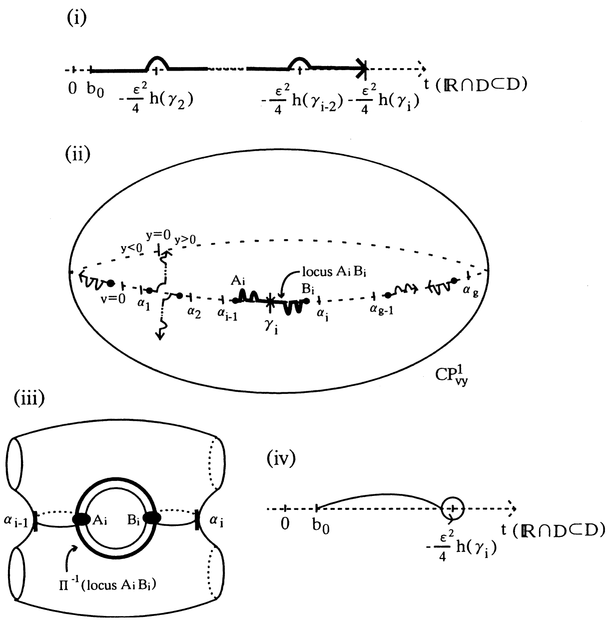

FIGURE 30.

is $v = \gamma_i$.

We take the arc, which joins b_0 to $-\frac{\epsilon^2}{4}h(\gamma_i)$, shown in Figure 30 (i). We move t from b_0 to $-\frac{\epsilon^2}{4}h(\gamma_i)$ on this arc. Then from Figure 25, the two moving branch points which started from points A_i, B_i shown in Figure 30 (ii) approach to the branch point $v = \gamma_i$ and collide at it as shown in Figure 30 (ii). Therefore when we take the locus A_iB_i, Π^{-1}(locus A_iB_i) becomes the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$. The shape of Π^{-1}(locus A_iB_i) is shown in Figure 30 (iii). This curve is the curve C_{2i} in Figure 7.

When we take the loop shown in Figure 30 (iv), the monodromy about this loop becomes ζ_{2i}.

Thus we understand that the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ is a Lefschetz singular fiber of type I.

Case 3: $t = -\frac{\epsilon^2}{4}h(\gamma_i)$ $(i = 1, 3, 5, \cdots, g-3, g-1)$.

We study the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ on the base

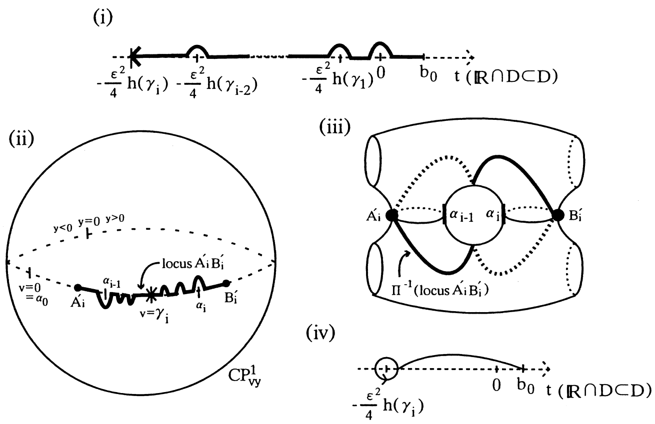

FIGURE 31.

fiber $\varphi_\epsilon = b_0$. Remark that the branch point on $\mathbb{CP}^1_{vy}$ mapped to

$$\tilde{v} = h(\gamma_i) = \mu\left(t = -\frac{\epsilon^2}{4}h(\gamma_i)\right)$$

is $v = \gamma_i$.

We take the arc, which joins b_0 to $-\frac{\epsilon^2}{4}h(\gamma_i)$, shown in Figure 31 (i). We move t from b_0 to $-\frac{\epsilon^2}{4}h(\gamma_i)$ on this arc. Then from Figure 25, the two moving branch points which started from points A'_i, B'_i shown in Figure 31 (ii) approach to the branch point $v = \gamma_i$ and collide at it as shown in Figure 31 (ii). Therefore when we take the locus $A'_iB'_i$, Π^{-1}(locus $A'_iB'_i$) becomes the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$. The shape of Π^{-1}(locus $A'_iB'_i$) is shown in Figure 31 (iii). This curve is the curve C''_i in Figure 9.

When we take the loop shown in Figure 31 (iv), the monodromy about this loop becomes η_i.

Thus we understand that the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ is a Lefschetz singular fiber of type I.

Case 4: $t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ $(i = 1, 3, 5, \cdots, g-3, g-1)$.

We study the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ on the base

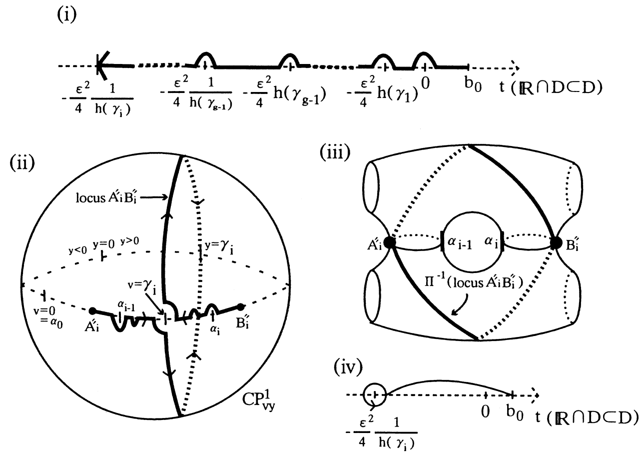

FIGURE 32.

fiber $\varphi_\epsilon = b_0$. Remark that the branch point on $\mathbb{CP}^1_{vy}$ mapped to

$$\tilde{y} = h(\gamma_i) = \frac{1}{\mu\left(t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}\right)}$$

is $y = \gamma_i$.

We take the arc, which joins b_0 to $-\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$, shown in Figure 32 (i). We move t from b_0 to $-\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ on this arc. Then from Figure 25, the two moving branch points which started from points A''_i, B''_i shown in Figure 32 (ii) approach to the branch point $y = \gamma_i$ and collide at it as shown in Figure 32 (ii). Therefore when we take the locus $A''_i B''_i$, $\Pi^{-1}(\text{locus } A''_i B''_i)$ becomes the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$. The shape of $\Pi^{-1}(\text{locus } A''_i B''_i)$ is shown in Figure 32 (iii). This curve is the curve C'_i in Figure 8.

When we take the loop shown in Figure 32 (iv), the monodromy about this loop becomes β_i.

Thus we understand that the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ is a Lefschetz singular fiber of type I.

Case 5: $t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ $(i = 2, 4, 6, \cdots, g-2, g)$.

We study the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ on the base

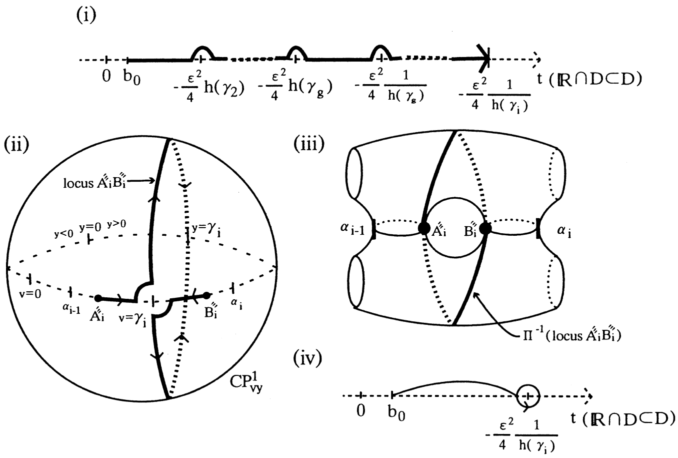

Figure 33.

fiber $\varphi_\epsilon = b_0$. Remark that the branch point on $\mathbb{CP}^1_{vy}$ mapped to

$$\tilde{y} = h(\gamma_i) = \frac{1}{\mu\left(t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}\right)}$$

is $y = \gamma_i$.

We take the arc, which joins b_0 to $-\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$, shown in Figure 33 (i). We move t from b_0 to $-\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ on this arc. Then from Figure 25, the two moving branch points which started from points A_i''', B_i''' shown in Figure 33 (ii) approach to the branch point $y = \gamma_i$ and collide at it as shown in Figure 33 (ii). Therefore when we take the locus $A_i'''B_i'''$, $\Pi^{-1}(\text{locus } A_i'''B_i''')$ becomes the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$. The shape of $\Pi^{-1}(\text{locus } A_i'''B_i''')$ is shown in Figure 33 (iii). This curve is the curve C_i''' in Figure 9.

When we take the loop shown in Figure 33 (iv), the monodromy about this loop becomes δ_i.

Thus we understand that the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ is a Lefschetz singular fiber of type I.

Case (2): g is odd.

By using Figure 26, we can study vanishing cycles and monodromies about singular fibers of φ_ϵ. We state the result only.

Case 1: $t = -\frac{\epsilon^2}{4}h(\gamma_i)$ $(i = 1, 3, 5, \cdots, g-2, g)$.

We take the loop shown in Figure 30 (iv). Then the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ is the curve C_{2i} in Figure 7. And the monodromy about this loop is ζ_{2i}.

Case 2: $t = -\frac{\epsilon^2}{4}h(\gamma_i)$ $(i = 2, 4, 6, \cdots, g-3, g-1)$.

We take the loop shown in Figure 31 (iv). Then the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}h(\gamma_i)$ is the curve C''_i in Figure 9. And the monodromy about this loop is η_i.

Case 3: $t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ $(i = 2, 4, 6, \cdots, g-3, g-1)$.

We take the loop shown in Figure 32 (iv). Then the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ is the curve C'_i in Figure 8. And the monodromy about this loop is β_i.

Case 4: $t = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ $(i = 1, 3, 5, \cdots, g-2, g)$.

We take the loop shown in Figure 33 (iv). Then the vanishing cycle about the fiber $\varphi_\epsilon = -\frac{\epsilon^2}{4}\frac{1}{h(\gamma_i)}$ is the curve C'''_i in Figure 9. And the monodromy about this loop is δ_i.

Finally, we study the vanishing cycle and monodromy about the fiber $\varphi_\epsilon = 0$. As shown in Figure 13, the fiber $\varphi_\epsilon = 0$ is the singular fiber obtained by pinching each of the circles in Figure 13 to a point. Therefore, when we take the loop shown in Figure 34, the vanishing cycle about the fiber $\varphi_\epsilon = 0$ on the base fiber $\varphi_\epsilon = b_0$ is given by the curves in Figure 7

$$C_1, C_1, C_3, C_3, C_5, C_5, \cdots, C_{2g+1}, C_{2g+1}.$$

And the monodromy about the fiber $\varphi_\epsilon = 0$ becomes

$$\zeta_1\zeta_1\zeta_3\zeta_3\zeta_5\zeta_5\cdots\zeta_{2g+1}\zeta_{2g+1}.$$

Thus we obtain the following lemma:

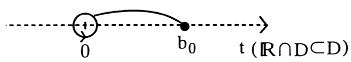

FIGURE 34.

LEMMA 9.1. *By the perturbation of φ to φ_ϵ for a small positive real number ϵ, F_{ω_g} splits into $2g$ Lefschetz singular fibers of type I and the singular fiber $\varphi_\epsilon = 0$. When we fix a sufficiently small positive real number*

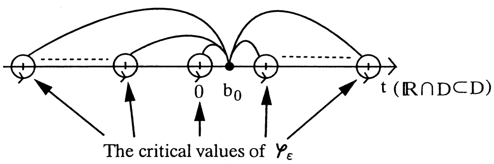

Figure 35.

b_0 as a base point and take loops as shown in Figure 35, the monodromy representation is given by the following $(2g+1)$-tuple:

$$(9.1)\quad \begin{aligned}(&\beta_1, \beta_3, \beta_5, \cdots, \beta_{g-3}, \beta_{g-1}, \quad \eta_{g-1}, \eta_{g-3}, \cdots, \eta_5, \eta_3, \eta_1,\\ &\zeta_1\zeta_1\zeta_3\zeta_3\zeta_5\zeta_5\cdots\zeta_{2g-1}\zeta_{2g-1}\zeta_{2g+1}\zeta_{2g+1},\\ &\zeta_{2\times 2}, \zeta_{2\times 4}, \zeta_{2\times 6}, \cdots, \zeta_{2(g-2)}, \zeta_{2g}, \quad \delta_g, \delta_{g-2}, \cdots, \delta_6, \delta_4, \delta_2)\end{aligned}$$

if the genus g is even;

$$(9.2)\quad \begin{aligned}(&\beta_2, \beta_4, \beta_6, \cdots, \beta_{g-3}, \beta_{g-1}, \quad \eta_{g-1}, \eta_{g-3}, \cdots, \eta_6, \eta_4, \eta_2,\\ &\zeta_1\zeta_1\zeta_3\zeta_3\zeta_5\zeta_5\cdots\zeta_{2g-1}\zeta_{2g-1}\zeta_{2g+1}\zeta_{2g+1},\\ &\zeta_{2\times 1}, \zeta_{2\times 3}, \zeta_{2\times 5}, \cdots, \zeta_{2(g-2)}, \zeta_{2g}, \quad \delta_g, \delta_{g-2}, \cdots, \delta_5, \delta_3, \delta_1)\end{aligned}$$

if the genus g is odd.

10. Splitting of the Fiber $\varphi_\epsilon = 0$

We study splitting of the fiber $\varphi_\epsilon = 0$ in this section. We fix one of the $2(g+1)$ transverse self-intersection points in the fiber $\varphi_\epsilon = 0$ and denote this fixed self-intersection point by P as shown in Figure 36. When we put $l = u(u-\epsilon)$ and $k = h(v)$, we obtain a new local coordinate with a boundary at P

$$\{(l,k) \mid |l| \le \delta''',\ |k| \le \delta'''\}$$

instead of $\{(u,v) \mid (u,v) \in N'_{uv}\}$. On this new local coordinate (l,k), φ_ϵ is given by

$$t = \varphi_\epsilon(l,k) = lk.$$

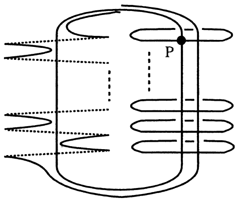

FIGURE 36. The fiber $\varphi_\epsilon = 0$ and the fixed self-intersection point P.

Suppose that λ is a sufficiently small positive real number which satisfies $\frac{\lambda}{\delta'''} < \delta'''$ and take the 2-disk D' in D defined by

$$D' = \{t \in D| \ |t| \leq \lambda\}.$$

Then we obtain the following lemma based on Theorem 4.1 in [Ma4].

LEMMA 10.1. *We can deform the structure of the fibration $\varphi_\epsilon : \varphi_\epsilon^{-1}(D') \to D'$, without altering it in a neighborhood of $\partial(\varphi_\epsilon^{-1}(D'))$, so that the resulting fibration $\tilde{\varphi}' : \tilde{\varphi}'^{-1}(D') \to D'$ has one Lefschetz singular fiber of type I and one singular fiber with $2g+1$ transverse self-intersection points.*

PROOF. We define a smooth 4-cell U with corners at P as follows:

$$U = \{(l, k) \mid |lk| \leq \lambda, |l| \leq \delta''', \ |k| \leq \delta'''\}.$$

Then we have $\varphi_\epsilon(U) = D'$.

We denote $Closure(\varphi_\epsilon^{-1}(D') \setminus U)$ by H. The intersection $U \cap H$ consists of two solid tori T^1, T^2 given as follows:

$$T^1 = \{(l, k) \mid |l| = \delta''', \ |k| \leq \frac{\lambda}{\delta'''}\},$$
$$T^2 = \{(l, k) \mid |l| \leq \frac{\lambda}{\delta'''}, \ |k| = \delta'''\}.$$

We denote the two solid tori T^1, T^2 by

$$T^1_U, T^2_U \quad (\subset U)$$

and

$$T_H^1, T_H^2 \quad (\subset H).$$

Let $\tilde{\psi} : T_U^1 \cup T_U^2 \to T_H^1 \cup T_H^2$ be the identity map given by

$$\tilde{\psi}(l,k) = (l,k)$$

and $\psi : D' \to D'$ the identity map. We define a projection $\tilde{\varphi} : U \cup_{\tilde{\psi}} H \to D'$ as

$$\begin{cases} \tilde{\varphi}(q) = \psi \circ \varphi_\epsilon(q) & q \in U \\ \tilde{\varphi}(q) = \varphi_\epsilon(q) & q \in H. \end{cases}$$

Obviously, we have

$$U \cup_{\tilde{\psi}} H = \varphi_\epsilon^{-1}(D'), \quad \tilde{\varphi} = \varphi_\epsilon.$$

The solid tori $T_U^1, T_U^2, T_H^1, T_H^2$ are foliated by circles as follows. The solid torus T_U^1 is foliated by the "sectional circles" $\{\varphi_\epsilon^{-1}(t) \cap T_U^1\}_{t \in D'}$, each of which is parametrized as

$$l = \delta''' e^{i\theta}, k = t\delta'''^{-1} e^{-i\theta},$$

where $0 \leq \theta \leq 2\pi$. And the solid torus T_U^2 is foliated by the "sectional circles" $\{\varphi_\epsilon^{-1}(t) \cap T_U^2\}_{t \in D'}$, each of which is parametrized as

$$l = t\delta'''^{-1} e^{-i\theta}, k = \delta''' e^{i\theta},$$

where $0 \leq \theta \leq 2\pi$. Similarly, the solid tori T_H^1, T_H^2 are foliated.

We fix $r \in Int(D')$ and call $\varphi_\epsilon^{-1}(0) \cap T_U^1, \varphi_\epsilon^{-1}(0) \cap T_U^2, \varphi_\epsilon^{-1}(r) \cap T_H^1$, $\varphi_\epsilon^{-1}(r) \cap T_H^2$ the *distinguished circles.* They are nothing but the sections of the fibers $\varphi_\epsilon = 0, \varphi_\epsilon = r$.

In T_H^1, the distinguished circle of T_H^1 and the image of distinguished circle of T_U^1 are situated as shown in Figure 37.

The diffeomorphism $\tilde{\psi}^1 := \tilde{\psi}|_{T_U^1} : T_U^1 \to T_H^1$ can be deformed via a leaf preserving isotopy

$$(\tilde{\psi}^1)_{t'} : T_U^1 \to T_H^1 \quad (0 \leq t' \leq 1)$$

so that the resulting diffeomorphism $(\tilde{\psi}^1)' := (\tilde{\psi}^1)_1$ maps $\varphi_\epsilon^{-1}(0) \cap T_U^1$ (the distinguished circle of T_U^1) to $\varphi_\epsilon^{-1}(r) \cap T_H^1$ (the distinguished circle of T_H^1). The isotopy $(\tilde{\psi}^1)_{t'}$ may be assumed not to alter $\tilde{\psi}^1$ near the boundary ∂T_U^1.

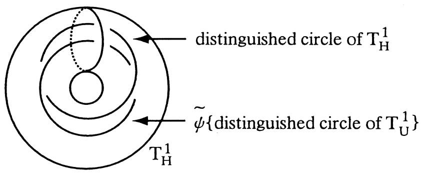

FIGURE 37.

And this leaf preserving isotopy $(\tilde{\psi}^1)_{t'}$ induces an isotopy

$$(\psi)_{t'} : D' \to D'$$

of $\psi : D' \to D'$. This isotopy $(\psi)_{t'}$, in turn, induces a leaf preserving isotopy

$$(\tilde{\psi}^2)_{t'} : T_U^2 \to T_H^2 \quad (0 \leq t' \leq 1)$$

of $\tilde{\psi}^2 := \tilde{\psi}|_{T_U^2} : T_U^2 \to T_H^2$. Then the resulting diffeomorphism $(\tilde{\psi}^2)' := (\tilde{\psi}^2)_1$ maps $\varphi_\epsilon^{-1}(0) \cap T_U^2$ (the distinguished circle of T_U^2) to $\varphi_\epsilon^{-1}(r) \cap T_H^2$ (the distinguished circle of T_H^2).

The isotopy

$$(\tilde{\psi})_{t'} = (\tilde{\psi}^1)_{t'} \cup (\tilde{\psi}^2)_{t'} : T_U^1 \cup T_U^2 \to T_H^1 \cup T_H^2$$

of $\tilde{\psi}$ gives a family of manifolds $U \cup_{(\tilde{\psi})_{t'}} H$ equipped with the projection $\tilde{\varphi}_{t'} : U \cup_{(\tilde{\psi})_{t'}} H \to D'$, which is defined by

$$\begin{cases} \tilde{\varphi}_{t'}(q) = (\psi)_{t'} \circ \varphi_\epsilon(q) & q \in U \\ \tilde{\varphi}_{t'}(q) = \varphi_\epsilon(q) & q \in H. \end{cases}$$

Each manifold $U \cup_{(\tilde{\psi})_{t'}} H$ in the family is diffeomorphic to $U \cup_{\tilde{\psi}} H = \varphi_\epsilon^{-1}(D')$ via a diffeomorphism which is the identity near the boundary. Also, near the boundary, $\tilde{\varphi}_{t'}$ always restricts to $\tilde{\varphi} = \varphi_\epsilon$. Thus the family $(U \cup_{(\tilde{\psi})_{t'}} H, \tilde{\varphi}_{t'})_{0 \leq t' \leq 1}$ is considered as giving a deformation of $\tilde{\varphi} = \varphi_\epsilon : U \cup_{\tilde{\psi}} H = \varphi_\epsilon^{-1}(D') \to D'$.

When we put $\tilde{\varphi}' = \tilde{\varphi}_1$ and $\tilde{\psi}' = (\tilde{\psi})_1$, the fibration $\tilde{\varphi}' : U \cup_{\tilde{\psi}'} H \to D'$ has the singular fiber $\tilde{\varphi}' = 0$ with $2g + 1$ transverse self-intersection points and the Lefschetz singular fiber of type I $\tilde{\varphi}' = r$.

This completes the proof of Lemma 10.1. □

By deforming the structure of the fibration $\varphi_\epsilon : \varphi_\epsilon^{-1}(D') \to D'$, without altering it in a neighborhood of $\partial(\varphi_\epsilon^{-1}(D'))$, the original fiber $\varphi_\epsilon = 0$ splits into one Lefschetz singular fiber and one singular fiber which is not Lefschetz singular fiber.

Also, by repeating similar argument for this singular fiber which is not Lefschetz singular fiber, this singular fiber splits into $2g+1$ Lefschetz singular fibers of type I.

Thus the original fiber $\varphi_\epsilon = 0$ splits into $2(g+1)$ Lefschetz singular fibers of type I and the monodromy about the fiber $\varphi_\epsilon = 0$

$$\zeta_1\zeta_1\zeta_3\zeta_3\zeta_5\zeta_5\cdots\zeta_{2g+1}\zeta_{2g+1}$$

splits into

$$\zeta_1, \zeta_1, \zeta_3, \zeta_3, \zeta_5, \zeta_5, \cdots, \zeta_{2g+1}, \zeta_{2g+1}.$$

Therefore F_{ω_g} splits into $2(2g+1)$ Lefschetz singular fibers of type I and the monodromy representation is given by (3.2) and (3.3) instead of (9.1) and (9.2).

11. Transformation of the $2(2g+1)$-tuples by Elementary Transformations

By splitting of F_{ω_g}, we have obtained the $2(2g+1)$-tuples (3.2) and (3.3). In this section, we arrange these to a simple $2(2g+1)$-tuple by elementary transformations.

We put

$$W(j,k) = \zeta_j\zeta_{j+1}\zeta_{j+2}\cdots\zeta_{2g}\zeta_{2g+1}\zeta_{2g+1}\zeta_{2g}\cdots\zeta_{k+2}\zeta_{k+1}\zeta_k \quad (\in \mathcal{M}_g),$$

where $j, k = 1, 2, \cdots, 2g+1$. Recall that $\mathcal{M}_g$ is assumed, by convention, to act on Σ_g from the right. Then we know that the curve $C_i'(\subset \Sigma_g)$ in Figure 8 satisfies

$$(C_i')\zeta_{2i-1}W(2i+1, 2i+2)^{-1} = C_{2i},$$

where C_{2i} is the curve in Figure 7. Therefore the monodromy β_i is given by

$$\beta_i = \zeta_{2i-1}W(2i+1, 2i+2)^{-1}\zeta_{2i}W(2i+1, 2i+2)\zeta_{2i-1}^{-1}. \tag{11.1}$$

On the curve C_i'' in Figure 9, we know that it satisfies

$$(C_i'')\zeta_{2i-1}\zeta_{2i+1} = C_{2i}.$$

Therefore the monodromy η_i is given by

$$\eta_i = \zeta_{2i-1}\zeta_{2i+1}\zeta_{2i}\zeta_{2i+1}^{-1}\zeta_{2i-1}^{-1}. \tag{11.2}$$

On the curve C_i''' in Figure 9, we know that it satisfies

$$(C_i''')W(2i+1, 2i+1) = C_{2i}.$$

Therefore the monodromy δ_i is given by

$$\delta_i = W(2i+1, 2i+1)\zeta_{2i}W(2i+1, 2i+1)^{-1}. \tag{11.3}$$

Case (1): g is even.

Then we obtain the $2(2g+1)$-tuple (3.2). Since two monodromies in each group of (3.2) are commutative in $\mathcal{M}_g$ (cf. Remark 3.2), we can change the $2(2g+1)$-tuple (3.2) to

$$\begin{aligned}(11.4)\quad (&\beta_{g-1}, \beta_{g-3}, \beta_{g-5}, \cdots, \beta_5, \beta_3, \beta_1,\ \eta_{g-1}, \eta_{g-3}, \cdots, \eta_5, \eta_3, \eta_1,\\ &\zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-7}, \cdots, \zeta_9, \zeta_{11}, \zeta_9, \zeta_5, \zeta_7, \zeta_5, \zeta_1, \zeta_3, \zeta_1,\\ &\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}, \cdots, \zeta_{2g-9}, \zeta_{2g-5}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1},\\ &\zeta_4, \zeta_8, \zeta_{12}, \cdots, \zeta_{2(g-2)}, \zeta_{2g},\ \delta_2, \delta_4, \delta_6, \cdots, \delta_{g-2}, \delta_g)\end{aligned}$$

by elementary transformations. Since the two loops mapped to the monodromies β_{g-1} and δ_g in (11.4) adjoin in D, we take the following $2(2g+1)$-tuple, instead of the $2(2g+1)$-tuple (11.4):

$$\begin{aligned}(11.5)\quad (&\zeta_3, \zeta_7, \zeta_{11}, \zeta_{15}, \cdots, \zeta_{2g-9}, \zeta_{2g-5}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1},\\ &\zeta_4, \zeta_8, \zeta_{12}, \cdots, \zeta_{2(g-2)}, \zeta_{2g},\ \delta_2, \delta_4, \delta_6, \cdots, \delta_{g-2}, \delta_g,\\ &\beta_{g-1}, \beta_{g-3}, \beta_{g-5}, \cdots, \beta_5, \beta_3, \beta_1,\ \eta_{g-1}, \eta_{g-3}, \cdots, \eta_5, \eta_3, \eta_1,\\ &\zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-7}, \cdots, \zeta_9, \zeta_{11}, \zeta_9, \zeta_5, \zeta_7, \zeta_5, \zeta_1, \zeta_3, \zeta_1).\end{aligned}$$

For integers i and j which satisfy $1 \leq i, j \leq g$ and $i < j$, the monodromies ζ_{2i} and ζ_{2j+1} are commutative in $\mathcal{M}_g$, and for integers k and l which satisfy

$1 \leq k \leq g$, $1 \leq l \leq 2g+1$ and $2k+1 < l$, the monodromies δ_k and ζ_l are commutative. Therefore we can change the $2(2g+1)$-tuple (11.5) to

$$\begin{aligned}(11.6)\quad (&\zeta_3, \zeta_4, \delta_2, \zeta_7, \zeta_8, \delta_4, \zeta_{11}, \zeta_{12}, \delta_6, \cdots, \\ &\zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \delta_g, \\ &\beta_{g-1}, \beta_{g-3}, \beta_{g-5}, \cdots, \beta_5, \beta_3, \beta_1,\ \eta_{g-1}, \eta_{g-3}, \cdots, \eta_5, \eta_3, \eta_1, \\ &\zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-7}, \cdots, \zeta_9, \zeta_{11}, \zeta_9, \zeta_5, \zeta_7, \zeta_5, \zeta_1, \zeta_3, \zeta_1)\end{aligned}$$

by elementary transformations. For integers i and j which satisfy $1 \leq i, j \leq g$ and $i+1 < j$, the monodromies β_i and η_j are commutative. For integers k and l which satisfy $1 \leq k \leq g$, $1 \leq l \leq 2g+1$ and $2k+2 < l$, the monodromies β_k and ζ_l are commutative and the monodromies η_k and ζ_l are also commutative. Therefore we can change the $2(2g+1)$-tuple (11.6) to

$$\begin{aligned}(11.7)\quad (&\zeta_3, \zeta_4, \delta_2, \zeta_7, \zeta_8, \delta_4, \zeta_{11}, \zeta_{12}, \delta_6, \cdots, \\ &\zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \delta_g, \\ &\beta_{g-1}, \eta_{g-1}, \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-3}, \beta_{g-3}, \eta_{g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-7}, \cdots, \\ &\beta_5, \eta_5, \zeta_9, \zeta_{11}, \zeta_9, \beta_3, \eta_3, \zeta_5, \zeta_7, \zeta_5, \beta_1, \eta_1, \zeta_1, \zeta_3, \zeta_1)\end{aligned}$$

by elementary transformations.

For an n-tuple

$$(11.8)\qquad (\cdots, \eta_i, \zeta_{2i-1}, \zeta_{2i+1}, \zeta_{2i-1}, \cdots),$$

we change this to

$$(11.9)\qquad (\cdots, \zeta_{2i-1}, \zeta_{2i+1}, \zeta_{2i+1}^{-1}\zeta_{2i-1}^{-1}\eta_i\zeta_{2i-1}\zeta_{2i+1}, \zeta_{2i-1}, \cdots)$$

by elementary transformations. Then, by the relation (11.2), the $2(2g+1)$-tuple (11.9) becomes

$$(11.10)\qquad (\cdots, \zeta_{2i-1}, \zeta_{2i+1}, \zeta_{2i}, \zeta_{2i-1}, \cdots).$$

Therefore we can change the $2(2g+1)$-tuple (11.7) to

$$\begin{aligned}(11.11)\quad (&\zeta_3, \zeta_4, \delta_2, \zeta_7, \zeta_8, \delta_4, \zeta_{11}, \zeta_{12}, \delta_6, \cdots, \\ &\zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \delta_g, \\ &\beta_{g-1}, \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-2}, \zeta_{2g-3}, \beta_{g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-6}, \zeta_{2g-7}, \cdots, \\ &\beta_5, \zeta_9, \zeta_{11}, \zeta_{10}, \zeta_9, \beta_3, \zeta_5, \zeta_7, \zeta_6, \zeta_5, \beta_1, \zeta_1, \zeta_3, \zeta_2, \zeta_1)\end{aligned}$$

by elementary transformations.

We change the $2(2g+1)$-tuple (11.11)

$$(\cdots, \zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \delta_g, \\ \beta_{g-1}, \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-2}, \zeta_{2g-3}, \cdots)$$

to

$$(11.12) \quad (\cdots, \zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, W(2g+1, 2g)\delta_g W(2g+1, 2g)^{-1}, \\ \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \beta_{g-1}, \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-2}, \zeta_{2g-3}, \cdots)$$

by elementary transformations. By two relations (11.3) and

$$\zeta_{2g+1}\zeta_{2g}\zeta_{2g+1} = \zeta_{2g}\zeta_{2g+1}\zeta_{2g},$$

we obtain

$$W(2g+1, 2g)\delta_g W(2g+1, 2g)^{-1} = \zeta_{2g}.$$

Therefore the $2(2g+1)$-tuple (11.12) becomes

$$(11.13) \quad (\cdots, \zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \zeta_{2g-1}, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \beta_{g-1}, \\ \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-2}, \zeta_{2g-3}, \cdots).$$

We denote the sequence

$$\zeta_j, \zeta_{j+1}, \zeta_{j+2}, \cdots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_{k+2}, \zeta_{k+1}, \zeta_k$$

in an n-tuple $(\cdots, \zeta_j, \zeta_{j+1}, \zeta_{j+2}, \cdots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_{k+2}, \zeta_{k+1}, \zeta_k, \cdots)$ by

$$\tilde{W}(j, k),$$

where $j, k = 1, 2, \cdots, 2g+1$. Then the $2(2g+1)$-tuple (11.13) is given by

$$(11.14) \quad (\cdots, \zeta_{2g-5}, \zeta_{2g-4}, \delta_{g-2}, \tilde{W}(2g-1, 2g), \\ \beta_{g-1}, \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-2}, \zeta_{2g-3}, \cdots).$$

Here, we put $i = g-1$. Then the $2(2g+1)$-tuple (11.14) is given by

$$(11.15) \quad (\cdots, \zeta_{2i-3}, \zeta_{2i-2}, \delta_{i-1}, \tilde{W}(2i+1, 2i+2), \\ \beta_i, \zeta_{2i-1}, \zeta_{2i+1}, \zeta_{2i}, \zeta_{2i-1}, \cdots).$$

We change the $2(2g+1)$-tuple (11.15) to

$$(11.16)\quad (\cdots,\zeta_{2i-3},\zeta_{2i-2},\delta_{i-1},\tilde{W}(2i+1,2i+2),\\ \zeta_{2i-1},\zeta_{2i-1}^{-1}\beta_i\zeta_{2i-1},\zeta_{2i+1},\zeta_{2i},\zeta_{2i-1},\cdots)$$

by an elementary transformation. By the relation (11.1), the $2(2g+1)$-tuple (11.16) becomes

$$(11.17)\quad (\cdots,\zeta_{2i-3},\zeta_{2i-2},\delta_{i-1},\tilde{W}(2i+1,2i+2),\zeta_{2i-1},\\ W(2i+1,2i+2)^{-1}\zeta_{2i}W(2i+1,2i+2),\zeta_{2i+1},\zeta_{2i},\zeta_{2i-1},\cdots).$$

We change the $2(2g+1)$-tuple (11.17) to

$$(11.18)\quad \begin{aligned} &(\cdots,\zeta_{2i-3},\zeta_{2i-2},\delta_{i-1},\zeta_{2i-1},\\ &\ W(2i+1,2i+2)\cdot W(2i+1,2i+2)^{-1}\zeta_{2i}W(2i+1,2i+2)\\ &\ \cdot W(2i+1,2i+2)^{-1},\\ &\ \tilde{W}(2i+1,2i+2),\zeta_{2i+1},\zeta_{2i},\zeta_{2i-1},\cdots)\\ =\,&(\cdots,\zeta_{2i-3},\zeta_{2i-2},\delta_{i-1},\zeta_{2i-1},\zeta_{2i},\\ &\ \tilde{W}(2i+1,2i+2),\zeta_{2i+1},\zeta_{2i},\zeta_{2i-1},\cdots)\\ =\,&(\cdots,\zeta_{2i-3},\zeta_{2i-2},\delta_{i-1},\tilde{W}(2i-1,2i-1),\cdots) \end{aligned}$$

by elementary transformations. By the relation (11.3), the $2(2g+1)$-tuple (11.18) becomes

$$(11.19)\quad (\cdots,\zeta_{2i-3},\zeta_{2i-2},W(2i-1,2i-1)\zeta_{2i-2}W(2i-1,2i-1)^{-1},\\ \tilde{W}(2i-1,2i-1),\cdots).$$

We change the $2(2g+1)$-tuple (11.19) to

$$(11.20)\quad \begin{aligned} &(\cdots,\zeta_{2i-3},\zeta_{2i-2},\tilde{W}(2i-1,2i-1),\\ &\ W(2i-1,2i-1)^{-1}\cdot W(2i-1,2i-1)\zeta_{2i-2}W(2i-1,2i-1)^{-1}\\ &\ \cdot W(2i-1,2i-1),\cdots)\\ =\,&(\cdots,\zeta_{2i-3},\zeta_{2i-2},\tilde{W}(2i-1,2i-1),\zeta_{2i-2},\cdots)\\ =\,&(\cdots,\tilde{W}(2i-3,2i-2),\cdots) \end{aligned}$$

by elementary transformations.

By $i = g-1$, the $2(2g+1)$-tuple (11.20) becomes

$$(11.21)\quad (\zeta_3, \zeta_4, \delta_2, \zeta_7, \zeta_8, \delta_4, \zeta_{11}, \zeta_{12}, \delta_6, \cdots, \\ \zeta_{2g-9}, \zeta_{2g-8}, \delta_{g-4}, \tilde{W}(2g-5, 2g-4), \beta_{g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-6}, \zeta_{2g-7}, \\ \cdots, \beta_5, \zeta_9, \zeta_{11}, \zeta_{10}, \zeta_9, \beta_3, \zeta_5, \zeta_7, \zeta_6, \zeta_5, \beta_1, \zeta_1, \zeta_3, \zeta_2, \zeta_1).$$

Next we put $i = g-3$. Then the $2(2g+1)$-tuple (11.21) is denoted by (11.15). Therefore, by the elementary transformations which change the $2(2g+1)$-tuple (11.15) to the $2(2g+1)$-tuple (11.20), the $2(2g+1)$-tuple (11.21) is changed into

$$(11.22)\quad (\zeta_3, \zeta_4, \delta_2, \zeta_7, \zeta_8, \delta_4, \zeta_{11}, \zeta_{12}, \delta_6, \cdots, \\ \zeta_{2g-13}, \zeta_{2g-12}, \delta_{g-6}, \tilde{W}(2g-9, 2g-8), \beta_{g-5}, \zeta_{2g-11}, \zeta_{2g-9}, \zeta_{2g-10}, \zeta_{2g-11}, \\ \cdots, \beta_5, \zeta_9, \zeta_{11}, \zeta_{10}, \zeta_9, \beta_3, \zeta_5, \zeta_7, \zeta_6, \zeta_5, \beta_1, \zeta_1, \zeta_3, \zeta_2, \zeta_1).$$

Recursively, by repeating such elementary transformations for the $2(2g+1)$-tuple (11.22), we obtain the $2(2g+1)$-tuple

$$(11.23)\qquad (\tilde{W}(3,4), \beta_1, \zeta_1, \zeta_3, \zeta_2, \zeta_1).$$

We put $i = 1$. Then the $2(2g+1)$-tuple (11.23) is denoted by (11.15). Therefore, by the elementary transformations which change the $2(2g+1)$-tuple (11.15) to the $2(2g+1)$-tuple (11.18), the $2(2g+1)$-tuple (11.23) is changed into

$$(\tilde{W}(1,1)) = (\zeta_1, \zeta_2, \zeta_3, \cdots, \zeta_{2g}, \zeta_{2g+1}, \zeta_{2g+1}, \zeta_{2g}, \cdots, \zeta_3, \zeta_2, \zeta_1).$$

Case (2): g is odd.

Then we obtain the $2(2g+1)$-tuple (3.3). Since two monodromies in each group of (3.3) are commutative in $\mathcal{M}_g$, we can change the $2(2g+1)$-tuple (3.3) to

$$(11.24)\quad (\beta_{g-1}, \beta_{g-3}, \beta_{g-5}, \cdots, \beta_6, \beta_4, \beta_2,\ \eta_{g-1}, \eta_{g-3}, \cdots, \eta_6, \eta_4, \eta_2, \\ \zeta_{2g-3}, \zeta_{2g-1}, \zeta_{2g-3}, \zeta_{2g-7}, \zeta_{2g-5}, \zeta_{2g-7}, \cdots, \zeta_{11}, \zeta_{13}, \zeta_{11}, \zeta_7, \zeta_9, \zeta_7, \zeta_3, \zeta_5, \zeta_3, \zeta_1, \\ \zeta_1, \zeta_5, \zeta_9, \zeta_{13}, \cdots, \zeta_{2g-9}, \zeta_{2g-5}, \zeta_{2g-1}, \zeta_{2g+1}, \zeta_{2g+1}, \\ \zeta_2, \zeta_6, \zeta_{10}, \cdots, \zeta_{2(g-2)}, \zeta_{2g},\ \delta_1, \delta_3, \delta_5, \cdots, \delta_{g-2}, \delta_g).$$

Since the two loops mapped to the monodromies β_{g-1} and δ_g in (11.24) adjoin in D, we take the following $2(2g+1)$-tuple, instead of the $2(2g+1)$-tuple (11.24):

$$\begin{aligned}(11.25)\quad &(\zeta_1,\zeta_5,\zeta_9,\zeta_{13},\cdots,\zeta_{2g-9},\zeta_{2g-5},\zeta_{2g-1},\zeta_{2g+1},\zeta_{2g+1},\\ &\zeta_2,\zeta_6,\zeta_{10},\cdots,\zeta_{2(g-2)},\zeta_{2g},\ \delta_1,\delta_3,\delta_5,\cdots,\delta_{g-2},\delta_g,\\ &\beta_{g-1},\beta_{g-3},\beta_{g-5},\cdots,\beta_6,\beta_4,\beta_2,\ \eta_{g-1},\eta_{g-3},\cdots,\eta_6,\eta_4,\eta_2,\\ &\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-3},\zeta_{2g-7},\zeta_{2g-5},\zeta_{2g-7},\cdots,\\ &\zeta_{11},\zeta_{13},\zeta_{11},\zeta_7,\zeta_9,\zeta_7,\zeta_3,\zeta_5,\zeta_3,\zeta_1).\end{aligned}$$

As before, we can change the $2(2g+1)$-tuple (11.25) to

$$\begin{aligned}(11.26)\quad &(\zeta_1,\zeta_2,\delta_1,\zeta_5,\zeta_6,\delta_3,\zeta_9,\zeta_{10},\delta_5,\cdots,\\ &\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\zeta_{2g-1},\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\delta_g,\\ &\beta_{g-1},\eta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-3},\beta_{g-3},\eta_{g-3},\zeta_{2g-7},\zeta_{2g-5},\zeta_{2g-7},\cdots,\\ &\beta_6,\eta_6,\zeta_{11},\zeta_{13},\zeta_{11},\beta_4,\eta_4,\zeta_7,\zeta_9,\zeta_7,\beta_2,\eta_2,\zeta_3,\zeta_5,\zeta_3,\zeta_1).\end{aligned}$$

Recall the elementary transformation which changes (11.8) to (11.10). By using this elementary transformation several times, we can change the $2(2g+1)$-tuple (11.26) to

$$\begin{aligned}(11.27)\quad &(\zeta_1,\zeta_2,\delta_1,\zeta_5,\zeta_6,\delta_3,\zeta_9,\zeta_{10},\delta_5,\cdots,\\ &\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\zeta_{2g-1},\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\delta_g,\\ &\beta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-2},\zeta_{2g-3},\beta_{g-3},\zeta_{2g-7},\zeta_{2g-5},\zeta_{2g-6},\zeta_{2g-7},\cdots,\\ &\beta_6,\zeta_{11},\zeta_{13},\zeta_{12},\zeta_{11},\beta_4,\zeta_7,\zeta_9,\zeta_8,\zeta_7,\beta_2,\zeta_3,\zeta_5,\zeta_4,\zeta_3,\zeta_1).\end{aligned}$$

We change the $2(2g+1)$-tuple (11.27)

$$\begin{aligned}&(\cdots,\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\zeta_{2g-1},\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\\ &\qquad\delta_g,\beta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-2},\zeta_{2g-3},\cdots)\end{aligned}$$

to

$$
\begin{aligned}
&(\cdots,\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\zeta_{2g-1},W(2g+1,2g)\delta_g W(2g+1,2g)^{-1},\\
&\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\beta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-2},\zeta_{2g-3},\cdots)\\
&=(\cdots,\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\zeta_{2g-1},\zeta_{2g},\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\\
&\beta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-2},\zeta_{2g-3},\cdots)\\
&=(\cdots,\zeta_{2g-5},\zeta_{2g-4},\delta_{g-2},\tilde{W}(2g-1,2g),\\
&\beta_{g-1},\zeta_{2g-3},\zeta_{2g-1},\zeta_{2g-2},\zeta_{2g-3},\cdots).
\end{aligned}
\tag{11.28}
$$

We put $i=g-1$. Then, by the elementary transformations which change the $2(2g+1)$-tuple (11.15) to the $2(2g+1)$-tuple (11.20), the $2(2g+1)$-tuple (11.28) is changed into

$$
\begin{aligned}
&(\zeta_1,\zeta_2,\delta_1,\zeta_5,\zeta_6,\delta_3,\zeta_9,\zeta_{10},\delta_5,\cdots,\\
&\zeta_{2g-9},\zeta_{2g-8},\delta_{g-4},\tilde{W}(2g-5,2g-4),\beta_{g-3},\zeta_{2g-7},\zeta_{2g-5},\zeta_{2g-6},\zeta_{2g-7},\\
&\cdots,\beta_6,\zeta_{11},\zeta_{13},\zeta_{12},\zeta_{11},\beta_4,\zeta_7,\zeta_9,\zeta_8,\zeta_7,\beta_2,\zeta_3,\zeta_5,\zeta_4,\zeta_3,\zeta_1).
\end{aligned}
\tag{11.29}
$$

As before, by repeating such elementary transformations for the $2(2g+1)$-tuple (11.29), we obtain the final $2(2g+1)$-tuple

$$(\zeta_1,\zeta_2,\zeta_3,\cdots,\zeta_{2g},\zeta_{2g+1},\zeta_{2g+1},\zeta_{2g},\cdots,\zeta_3,\zeta_2,\zeta_1).$$

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