J. Math. Sci. Univ. Tokyo
9 (2002), 347–404.

Fourier-Jacobi Expansion of Eisenstein Series on Unitary Groups of Degree Three

By Atsushi Murase and Takashi Sugano

Abstract. After reformulating Shintani's theory of Fourier-Jacobi expansion of automorphic forms on $U(2,1)$ in the adelic setting, we show that the primitive components of holomorphic Eisenstein series are expressed in terms of the periods of primitive theta functions and critical values of Hecke L-functions.

§0. Introduction

0.1 The object of this paper is two-fold. We first reformulate, in the adelic setting, Shintani's theory of Fourier-Jacobi expansion for holomorphic automorphic forms on a unitary group $G = U(2,1)$ defined over a totally real number field F. The main ingredient is the primitive theta functions first introduced by Shintani ([Shin]) and studied later by Glaubermann–Rogawski ([GlRo]) and ourselves ([MS]). For another approach to the theory of Fourier-Jacobi expansion, we refer to [PS], [GeRo] and [Is].

The second object is to calculate explicitly the primitive components of Fourier-Jacobi expansion of holomorphic Eisenstein series on G. It is to be noted that, in his thesis ([Hi]), Hickey obtained a similar result in the case of $F = \mathbf{Q}$ by a method somewhat different from ours.

0.2 We now summarize our results in the simplest case where the base field F is $\mathbf{Q}$. Let K be an imaginary quadratic field of discriminant D, $\mathcal{O}_K$ the integer ring of K and σ the nontrivial automorphism of $K/\mathbf{Q}$. Denote by G the unitary group of a Hermitian matrix

$$S = \begin{bmatrix} 0 & 0 & 1/\kappa \\ 0 & 1 & 0 \\ -1/\kappa & 0 & 0 \end{bmatrix},$$

1991 *Mathematics Subject Classification.* Primary 11F55; Secondary 11F67, 11F70.
Key words: Automorphic forms, Fourier-Jacobi expansion, Theta functions, Eisenstein series, Unitary groups.

where $\kappa = \sqrt{D}$. Note that $\mathrm{sgn}(S) = (2,1)$. Let N and R be subgroups of G given by

$$N_{\mathbf{Q}} = \left\{ (w,x) := \begin{bmatrix} 1 & \kappa w^\sigma & x + \dfrac{\kappa}{2} w w^\sigma \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \mid w \in K, x \in \mathbf{Q} \right\}$$

and

$$R_{\mathbf{Q}} = \left\{ nt := n \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid n \in N_{\mathbf{Q}}, t \in K^1 \right\},$$

where $K^1 = \{t \in K^\times \mid tt^\sigma = 1\}$. For $a \in K^\times$, we put

$$\mathbf{d}(a) = \begin{bmatrix} a^\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} \in G_{\mathbf{Q}}.$$

Define an action of G_∞ on $\mathcal{D} = \{{}^t(z,w) \in \mathbf{C}^2 \mid (z - \overline{z})/\kappa - w\overline{w} > 0\}$ and an automorphic factor $j\colon G_\infty \times \mathcal{D} \to \mathbf{C}^\times$ in a usual way (see §1.5). Let $\mathcal{K}_\infty = \{g \in G_\infty \mid g_\infty\langle Z_0\rangle = Z_0\}$ be a maximal compact subgroup of G_∞, where $Z_0 = {}^t(\kappa/2, 0) \in \mathcal{D}$. We put $\mathcal{K}_f = \{g \in G_{\mathbf{A},f} \mid g \cdot L_f = L_f\}$, where $L = \mathcal{O}_K^3, L_f = L \otimes_{\mathbf{Z}} \mathbf{Z}_f$ ($\mathbf{Z}_f = \prod_{p<\infty} \mathbf{Z}_p$) and $G_{\mathbf{A},f}$ is the finite part of $G_{\mathbf{A}}$. Then $\mathcal{K}_f$ is a maximal open compact subgroup of $G_{\mathbf{A},f}$.

For a positive even integer l, let $\mathcal{A}_l(\mathcal{K}_f)$ be the space of smooth functions f on $G_{\mathbf{Q}}\backslash G_{\mathbf{A}}$ satisfying

(i) $f(gk_fk_\infty) = j(k_\infty, Z_0)^{-l} f(g) \quad (g \in G_{\mathbf{A}}, k_f \in \mathcal{K}_f, k_\infty \in \mathcal{K}_\infty)$

(ii) For any $g_f \in G_{\mathbf{A},f}$, $j(g_\infty, Z_0)^l f(g_\infty g_f)$ is holomorphic in $g_\infty\langle Z_0\rangle$.

We call $\mathcal{A}_l(\mathcal{K}_f)$ the space of holomorphic automorphic forms of weight l on $\mathcal{K}_f$. Let $f \in \mathcal{A}_l(\mathcal{K}_f)$. For $m \in \mathbf{Q}$ and a nonzero fractional ideal $\mathfrak{a}$ of K, we put

$$f_{\mathfrak{a}}^m(r) = \int_{\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}} \psi_m(-x)\, f((0,x)r\mathbf{d}(\alpha_f))\, dx \quad (r \in R_{\mathbf{A}}).$$

Here dx is the Haar measure on $\mathbf{Q}_{\mathbf{A}}$ normalized by $\mathrm{vol}(\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}) = 1$, ψ_m is the additive character of $\mathbf{Q}\backslash\mathbf{Q}_{\mathbf{A}}$ with $\psi_m(x_\infty) = \mathbf{e}_m[x_\infty] :=$

$\exp(2\pi\sqrt{-1}mx_\infty)$ for $x_\infty \in \mathbf{R}$, and we choose an element α_f of $K^\times_{\mathbf{A},f}$ such that $\mathfrak{a}_f = \alpha_f \mathcal{O}_{K,f}$ ($\mathfrak{a}_f = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{Z}_f, \mathcal{O}_{K,f} = \mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_f$). Then f is determined by $\{f^m_{\mathfrak{a}}\}_{m,\mathfrak{a}}$. Note that $f^m_{\mathfrak{a}} = 0$ unless m is nonnegative and $mN(\mathfrak{a})$ is integral.

Throughout this section, we let m be a positive rational number. Let $\mathbf{T}^m_{hol}$ be the space of smooth functions Θ on $R_{\mathbf{Q}}\backslash R_{\mathbf{A}}$ satisfying

(i) $\Theta((0,x)r) = \psi_m(x)\,\Theta(r)\ \ (x \in \mathbf{Q_A}, r \in R_{\mathbf{A}})$

(ii) $\Theta(rt_\infty) = \Theta(r)\ \ (r \in R_{\mathbf{A}}, t_\infty \in K^1_\infty)$

(iii) For any $r_f \in R_{\mathbf{A},f}$, $w_\infty \mapsto \mathbf{e}_m\left[-\dfrac{\kappa}{2}\, w_\infty\overline{w_\infty}\right]\Theta((w_\infty,0)r_f)$ is holomorphic on $\mathbf{C}$.

We call $\mathbf{T}^m_{hol}$ the space of *holomorphic theta functions* of index m.

We next define a metaplectic representation of $K^1_{\mathbf{A}}$ on $\mathbf{T}^m_{hol}$. Let $\mathcal{X}$ be the set of Hecke characters χ of K with $\chi|_{\mathbf{Q}^\times_{\mathbf{A}}} = \omega$ and $\mathcal{X}_0 = \{\chi \in \mathcal{X} \mid \chi(z_\infty) = |z_\infty|/z_\infty\ (z_\infty \in K^\times_\infty = \mathbf{C}^\times)\}$, where ω denotes the quadratic Hecke character of $\mathbf{Q}$ corresponding to $K/\mathbf{Q}$. Let $\chi \in \mathcal{X}_0$. For a prime v of $\mathbf{Q}$, we put $\chi_v = \chi|_{K_v^\times}$, where $K_v = K \otimes_{\mathbf{Q}} \mathbf{Q}_v$. Let $t_v \in K^1_v$ and $\Theta \in \mathbf{T}^m_{hol}$. If $t_v = 1$, we put $\mathcal{M}'_{\chi_v}(t_v)\,\Theta = \Theta$. If $t_v \neq 1$, we put

$$\begin{aligned}
&\mathcal{M}'_{\chi_v}(t_v)\,\Theta(r)\\
&\quad = \lambda_{K_v}(\psi_m)^{-1}\,\chi_v\left(\frac{1-t_v}{\kappa}\right)\left|N_{K/\mathbf{Q}}(1-t_v)\right|_v^{1/2}\\
&\qquad \times \int_{K_v}\psi_m\left(\frac{1}{2}\langle w_v, t_v w_v\rangle\right)\Theta(r\,((1-t_v)w_v,0))\,dw_v \quad (r \in R_{\mathbf{A}})\,.
\end{aligned}$$

Here $\lambda_{K_v}(\psi_m)$ is the Weil constant (see §2.9), $\langle w_v, w'_v\rangle = Tr_{K_v/\mathbf{Q}_v}(\kappa w_v^\sigma w'_v)$ $(w_v, w'_v \in K_v)$ and dw_v is the self-dual Haar measure of K_v with respect to the pairing $(w_v, w'_v) \mapsto \psi_m(\langle w_v, w'_v\rangle)$. For $t = (t_v)_v \in K^1_{\mathbf{A}}$, we put $\mathcal{M}'_\chi(t) = \bigotimes_v \mathcal{M}'_{\chi_v}(t_v)$. Denote by ρ' the right translation of $N_{\mathbf{A},f}$ on $\mathbf{T}^m_{hol}$. Then $\mathcal{M}'_\chi$ defines a smooth representation of $K^1_{\mathbf{A}}$ on $\mathbf{T}^m_{hol}$ satisfying $\mathcal{M}'_\chi(t)\circ\rho'(n_f)\circ\mathcal{M}'_\chi(t^{-1}) = \rho'(tn_ft^{-1})$ for $t \in K^1_{\mathbf{A}}, n_f \in N_{\mathbf{A},f}$ (see §2.9–11).

For $\chi \in \mathcal{X}_0$, set

$$\mathbf{T}^m_{hol,\chi} = \{\Theta \in \mathbf{T}^m_{hol} \mid \mathcal{M}'_\chi(t)\,\Theta(r) = \Theta(rt)\ \ (r \in R_{\mathbf{A}}, t \in K^1_{\mathbf{A}})\}\,.$$

Then the following facts hold:

(0.1) $\mathbf{T}^m_{hol,\chi}$ is an irreducible $N_{\mathbf{A},f}$-module.

(0.2) $\mathbf{T}^m_{hol} = \bigoplus_{\chi \in \mathcal{X}_0} \mathbf{T}^m_{hol,\chi}$.

Note that we define $\mathbf{T}^m_{hol,\chi}$ in a different way in §2 and that both definitions are equivalent (see §2.25 and §2.24). For a nonzero fractional ideal $\mathfrak{a}$ of K and $\chi \in \mathcal{X}_0$, we put

$$\begin{aligned} \mathbf{T}^m_{hol}(\mathfrak{a}) &= \{\Theta \in \mathbf{T}^m_{hol} \mid \Theta(rr_0) = \Theta(r) \\ &\qquad \text{for any } r \in R_{\mathbf{A}} \text{ and } r_0 \in R(\mathfrak{a})_f\} \\ \mathbf{T}^m_{hol}(\mathfrak{a}, \chi) &= \mathbf{T}^m_{hol}(\mathfrak{a}) \cap \mathbf{T}^m_{hol,\chi}\,, \end{aligned}$$

where $R(\mathfrak{a})_f$ is an open compact subgroup of $R_{\mathbf{A},f}$ given by

$$\begin{aligned} R(\mathfrak{a})_f &= \{nt \mid n \in N(\mathfrak{a})_f, t \in \mathcal{O}^1_{K,f} = K^1_{\mathbf{A},f} \cap \mathcal{O}^\times_{K,f}\} \\ N(\mathfrak{a})_f &= \{(w,x) \in N_{\mathbf{A},f} \mid w \in \mathfrak{a}_f, x + \frac{\kappa}{2} ww^\sigma \in N(\mathfrak{a}_f)\mathcal{O}_{K,f}\}\,. \end{aligned}$$

Note that $f^m_{\mathfrak{a}} \in \mathbf{T}^m_{hol}(\mathfrak{a})$ for $f \in \mathcal{A}_l(\mathcal{K}_f)$.

We define the primitive part $\mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi)$ of $\mathbf{T}^m_{hol}(\mathfrak{a}, \chi)$ to be the space of $\Theta \in \mathbf{T}^m_{hol}(\mathfrak{a}, \chi)$ such that

$$\int_{N(\mathfrak{b})_f} \rho'(n)\Theta \, d_{\mathfrak{b}} n = 0$$

holds for any fractional ideal $\mathfrak{b}$ of K with $\mathfrak{b} \supset \mathfrak{a}$ and $\mathfrak{b} \neq \mathfrak{a}$, where $d_{\mathfrak{b}} n$ is the Haar measure on $N(\mathfrak{b})_f$ normalized by $\mathrm{vol}(N(\mathfrak{b})_f) = 1$. It is known that

$$\dim_{\mathbf{C}} \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi) \leq 1$$

([Shin], [GlRo]; see also Theorem 3.4). Moreover we have

$$\dim_{\mathbf{C}} \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi) = 1 \iff \chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)\,,$$

where $\mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$ consists of $\chi \in \mathcal{X}_0$ satisfying certain conditions on the Artin conductor and the epsilon factor (for the precise statement, see §3.7 and §3.8). Then we have the following direct sum decomposition:

$$\mathbf{T}^m_{hol}(\mathfrak{a}) = \sum_{\mathfrak{b}} \sum_{\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{b},m)} \mathbf{T}^m_{hol,prim}(\mathfrak{b}, \chi)\,,$$

where $\mathfrak{b}$ runs over the fractional ideals of K with $\mathfrak{b} \supset \mathfrak{a}$ and $mN(\mathfrak{b})$ integral.

Let $f \in \mathcal{A}_l(\mathcal{K}_f), \chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$ and $\Theta \in \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi), \Theta \neq 0$. We call the inner product

$$(f^m_{\mathfrak{a}}, \Theta) = \int_{R_{\mathbf{Q}} \backslash R_{\mathbf{A}}} f^m_{\mathfrak{a}}(r) \overline{\Theta(r)}\, dr$$

the *primitive* $(\mathfrak{a}, \chi)$*-component* of f with respect to Θ (dr is normalized by $\mathrm{vol}(R_{\mathbf{Q}} \backslash R_{\mathbf{A}}) = 1$).

We now state the main results of the paper. Let $l > 4$ be an even integer and Ω a Hecke character of K satisfying $\Omega|_{\mathcal{O}^\times_{K,f}} = 1$ and $\Omega(z_\infty) = (z_\infty/|z_\infty|)^l$ for $z_\infty \in K^\times_\infty$. Let $E_\Omega \in \mathcal{A}_l(\mathcal{K}_f)$ be the holomorphic Eisenstein series of weight l attached to Ω (see §4.2). Let $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$. Denote by $A(\chi)$ the set of finite primes p with $p|D, \mathrm{ord}_p(mN(\mathfrak{a})) = 0, a(\chi_p) = 2\,\mathrm{ord}_p(D) - 1$, where $a(\chi_p)$ is the p-exponent of the Artin conductor of χ_p (see §3.5). For $p \in A(\chi)$, let Π_p be a prime element of K_p. We define the period of $\Theta \in \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi)$ by

$$I(\Theta) = \int_{K^1 \backslash K^1_{\mathbf{A}}} \Theta(t)\, d^\times t\,,$$

where $d^\times t$ is normalized by $\mathrm{vol}(K^1 \backslash K^1_{\mathbf{A}}) = 1$.

Main Theorem. *Let $\Theta \in \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi) - \{0\}$. Then the primitive $(\mathfrak{a}, \chi)$-component of E_Ω with respect to Θ is given by*

$$((E_\Omega)^m_{\mathfrak{a}}, \Theta) = c(\Omega) \prod_{p \in A(\chi)} (1 + \Omega(\Pi_p)\, p^{1-l/2}) \cdot \frac{L(\chi\Omega; (l-1)/2)}{L(\omega; l-1) L(\Omega; l/2)} \cdot \overline{I(\Theta)}\,,$$

where

$$c(\Omega) = \frac{(2\pi\sqrt{-1})^l}{(l-1)!}\, m^{l-2} |D|^{-1} N(\mathfrak{a})^{l/2-2}\, \mathbf{e}_m \left[\frac{\kappa}{2}\right] \Omega^{-1}(\alpha^\sigma_f)\,.$$

The following criterion for the nonvanishing of $I(\Theta)$ is due to T. Yang ([Yan]).

THEOREM. *Let* $\chi \in \mathcal{X}_{0,prim}^{+}(\mathfrak{a}, m)$ *and* $\Theta \in \mathbf{T}_{hol,prim}^{m}(\mathfrak{a}, \chi) - \{0\}$. *Then we have* $I(\Theta) \neq 0$ *if and only if* $L\Big(\chi; \dfrac{1}{2}\Big) \neq 0$.

This implies that the primitive $(\mathfrak{a}, \chi)$-component of E_{Ω} vanishes if and only if $L\Big(\chi; \dfrac{1}{2}\Big) \neq 0$. By virtue of Shimura's results on critical values of Hecke L-functions ([Shim1], [Shim3]), we obtain

THEOREM. *Let* $\chi \in \mathcal{X}_{0,prim}^{+}(\mathfrak{a}, m)$ *and suppose that* $L\Big(\chi; \dfrac{1}{2}\Big) \neq 0$. *Let* $\Theta_{\mathfrak{a},\chi}$ *be the element of* $\mathbf{T}_{hol,prim}^{m}(\mathfrak{a}, \chi)$ *with* $I(\Theta_{\mathfrak{a},\chi}) = \pi^{-1}\, \mathbf{e}_m \left[-\dfrac{\kappa}{2}\right] \overline{L\Big(\chi; \dfrac{1}{2}\Big)}$. *Then* $((E_{\Omega})_{\mathfrak{a}}^{m}, \Theta_{\mathfrak{a},\chi})$ *is algebraic.*

0.3 The paper is organized as follows. The first three sections are devoted to an adelic reformulation with some refinement of the theory of Fourier-Jacobi expansion for automorphic forms on G mainly due to Shintani. In §1, we recall basic facts about holomorphic automorphic forms on G, their Fourier-Jacobi expansions and theta functions appearing as coefficients of the expansion. The second section is of expository nature. We first recall several basic facts about the lattice model after [MVW] and [MS]. We next study a relation between the lattice model and the space of theta functions. The main object of this section is to prove the facts (0.1) and (0.2). Though the content of this section might be known, we give its detailed account since it is not found in the literature. In §3, we recall the definition of primitive theta functions, which plays an essential role in Shintani's theory of Fourier-Jacobi expansion, and summarize their basic properties (the uniqueness and existence) after [Shin], [GlRo] and [MS]. In §4, we state the main result of the paper (Theorem 4.4): The primitive components of a holomorphic Eisenstein series on G are expressed in terms of critical values of Hecke L-functions and periods of primitive theta functions. In §5, we first give a criterion for non-vanishing of a primitive component. This follows from a relation between periods of primitive theta functions and central critical values of certain Hecke L-functions, which is essentially due to T. Yang ([Yan]). We next show that the primitive components of the Eisenstein series, under a suitable normalization of primitive theta functions, are algebraic. This fact seems to be closely related to the theory of the arithmeticity of automorphic forms on unitary groups due to Shimura ([Shim2]) and Yamauchi ([Yam]). The remaining part of the paper is devoted to the proof of Theorem 4.4. In §6, we reduce the proof of the theorem to the calculation of certain local integrals. After preparing several notation in §7, we carry out the calculation of the local integrals in §§8–9 and complete the proof of Theorem 4.4 in §10.

Notation. Let E be an algebraic number field with the integer ring $\mathcal{O}_E$. For a prime v of E, we denote by E_v the completion of E at v. Let $|\cdot|_v$ be the valuation of E_v given by $d(ax_v) = |a|_v dx_v \quad (a \in E_v^\times)$, where dx_v is a Haar measure on E_v. Let $\mathfrak{p}$ be a finite prime of E and $\mathcal{O}_{E,\mathfrak{p}}$ the integer ring of $E_\mathfrak{p}$. We put $q_\mathfrak{p} = \#(\mathcal{O}_{E,\mathfrak{p}}/\pi_\mathfrak{p}\mathcal{O}_{E,\mathfrak{p}})$, where $\pi_\mathfrak{p}$ is a prime element of $E_\mathfrak{p}$. Let $\mathrm{ord}_\mathfrak{p}\colon E_\mathfrak{p}^\times \to \mathbf{Z}$ be the additive valuation of $E_\mathfrak{p}$ normalized by $\mathrm{ord}_\mathfrak{p}(\pi_\mathfrak{p}) = 1$. For a nontrivial character ψ of $E_\mathfrak{p}$, we denote by $n(\psi)$ the largest integer n such that ψ is trivial on $\pi_\mathfrak{p}^{-n}\mathcal{O}_{E,\mathfrak{p}}$. By an ideal of E (resp. $E_\mathfrak{p}$), we always mean a nonzero fractional ideal of E (resp. $E_\mathfrak{p}$). Denote by $E_\mathbf{A}$ the adele ring of E and by $E_{\mathbf{A},f}$ (resp. E_∞) the finite (resp. infinite) part of $E_\mathbf{A}$. We put $\mathcal{O}_{E,f} = \prod_{\mathfrak{p}<\infty} \mathcal{O}_{E,\mathfrak{p}} \subset E_{\mathbf{A},f}$. Let $w \in E_\mathbf{A}$. For a finite prime $\mathfrak{p}$ of E, let $w_\mathfrak{p} \in E_\mathfrak{p}$ be the $\mathfrak{p}$-component of w. We also put $w_\infty = (w_v)_{v|\infty} \in E_\infty$. Denote by $\mathcal{S}(E_\mathbf{A})$ the space of Schwartz–Bruhat functions on $E_\mathbf{A}$. By a Hecke character of E, we mean a unitary character of $E_\mathbf{A}^\times$ trivial on $E^\times$. For a Hecke character ξ of E, $L(\xi; s)$ stands for the Hecke L-function attached to ξ.

Let G be an algebraic group defined over E. For a prime v of E, we write G_v for the group of E_v-rational points of G. Let $G_\mathbf{A}$ be the adelization of G, and $G_{\mathbf{A},f}$ (resp. G_∞) the finite (resp. infinite) part of $G_\mathbf{A}$.

For a locally compact abelian group H, we denote by $H^\wedge$ the group of unitary characters of H, continuous homomorphisms from H to $\mathbf{C}^1 = \{z \in \mathbf{C} \mid z\overline{z} = 1\}$. For $z \in \mathbf{C}$, we put $\mathbf{e}[z] = \exp(2\pi\sqrt{-1}z)$. For a set X, ch_X stands for the characteristic function of X. For $x \in \mathbf{R}$, $[x]$ denotes the largest integer n with $n \leq x$. We denote by $\mathrm{diag}(a_1, \ldots, a_n)$ the diagonal matrix with the i-th diagonal component a_i. We denote by $\mathbf{Z}_{\geq 0}$ the set of nonnegative integers. For a condition P, we set $\delta(P) = 1$ if P is satisfied and $\delta(P) = 0$ otherwise.

§1. Automorphic Forms on $U(2,1)$ and Fourier-Jacobi Expansion

1.1 Let F be a totally real number field of degree n. We put $dx_\infty = |d_F|^{-1/2}\, d'x_\infty$, where d_F is the discriminant of F and $d'x_\infty$ is the usual Lebesgue measure on $F_\infty = F \otimes_{\mathbf{Q}} \mathbf{R} \simeq \mathbf{R}^n$. For a finite prime $\mathfrak{p}$ of F, let $dx_\mathfrak{p}$ be the Haar measure on $F_\mathfrak{p}$ normalized by $\int_{\mathcal{O}_{F,\mathfrak{p}}} dx_\mathfrak{p} = 1$. Put $dx = dx_\infty \prod_{\mathfrak{p}<\infty} dx_\mathfrak{p}$. Then we have $\mathrm{vol}(F_\mathbf{A}/F) = 1$. The idele norm $|a|_\mathbf{A}$ of $a \in F_\mathbf{A}^\times$ is defined by $d(ax) = |a|_\mathbf{A} dx$. We let $\mathcal{D}_F$ be the different of F:

$$\mathcal{D}_F^{-1} = \{x \in F \mid Tr_{F/\mathbf{Q}}(xy) \in \mathbf{Z} \text{ for any } y \in \mathcal{O}_F\}\,.$$

1.2 Let K be a totally imaginary quadratic extension of F. Denote by σ the nontrivial automorphism of K/F. Let $\mathcal{D}_{K/F}$ be the different of K/F:

$$\mathcal{D}_{K/F}^{-1} = \{z \in K \mid Tr_{K/F}(zw) \in \mathcal{O}_F \text{ for any } w \in \mathcal{O}_K\}\,.$$

For a finite prime $\mathfrak{p}$ of F and an ideal $\mathfrak{a}$ of K, we put $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$. Let $\mathfrak{a}_f = \mathfrak{a} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,f} = \prod_{\mathfrak{p}<\infty} \mathfrak{a}_\mathfrak{p}$.

1.3 Throughout the paper, we fix an element κ of $K^\times$ satisfying $\kappa^\sigma = -\kappa$ and a Hermitian matrix

$$S = \begin{bmatrix} & & 1/\kappa \\ & 1 & \\ -1/\kappa & & \end{bmatrix}.$$

Let $G = U(S)$ be the unitary group of S defined over F:

$$G_F = \{g \in GL_3(K) \mid {}^t g^\sigma S g = S\}\,.$$

1.4 Let $\{\tau_i\}_{1\le i\le n}$ be the CM-type of K such that $\mathrm{Im}(\tau_i(\kappa)) > 0$ $(1 \le i \le n = [F:\mathbf{Q}])$. We embed K into $K_\infty := K \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{C}^n$ by $z \mapsto (\tau_1(z), \ldots, \tau_n(z))$. Denote by ∞_i the infinite prime of F corresponding to the embedding $\tau_i|_F : F \to \mathbf{R}$ and identify F_{∞_i} (resp. $K_{\infty_i} = K \otimes_F F_{\infty_i}$)

with $\mathbf{R}$ (resp. $\mathbf{C}$) via the embedding τ_i. In what follows, we often write $z^{(i)}$ for $\tau_i(z)$ $(z \in K)$ to simplify the notation.

1.5 For $i = 1, \ldots, n$, let

$$\begin{aligned} S^{(i)} &= \begin{bmatrix} & & 1/\kappa^{(i)} \\ & 1 & \\ -1/\kappa^{(i)} & & \end{bmatrix} \\ G_\infty^{(i)} &= \{g \in GL_3(\mathbf{C}) \mid {}^t\overline{g} S^{(i)} g = S^{(i)}\}. \end{aligned}$$

Note that the signature of $S^{(i)}$ is $(2,1)$. The Lie group $G_\infty^{(i)}$ acts on a Hermitian symmetric domain $\mathcal{Z}^{(i)} = \{Z = {}^t(z,w) \in \mathbf{C}^2 \mid \dfrac{1}{\kappa^{(i)}}(z - \overline{z}) - w\overline{w} > 0\}$ as follows. For $Z = {}^t(z,w) \in \mathcal{Z}^{(i)}$, we put $Z\tilde{\ } = {}^t(z,w,1) \in \mathbf{C}^3$. For $g \in G_\infty^{(i)}$ and $Z \in \mathcal{Z}^{(i)}$, define $g\langle Z\rangle \in \mathcal{Z}^{(i)}$ and $j^{(i)}(g,Z) \in \mathbf{C}^\times$ by $g \cdot Z\tilde{\ } = j^{(i)}(g,Z) \cdot (g\langle Z\rangle)\tilde{\ }$. Then $(g,Z) \mapsto g\langle Z\rangle$ (resp. $(g,Z) \mapsto j^{(i)}(g,Z)$) defines an action of $G_\infty^{(i)}$ on $\mathcal{Z}^{(i)}$ (resp. a holomorphic automorphic factor). This action is transitive. Let $\mathcal{K}_\infty^{(i)}$ be the stabilizer subgroup of $Z_0^{(i)} = {}^t(\kappa^{(i)}/2, 0) \in \mathcal{Z}^{(i)}$ in $G_\infty^{(i)}$. Then $\mathcal{K}_\infty^{(i)} \simeq U(2) \times U(1)$, where $U(m) = \{g \in GL_m(\mathbf{C}) \mid {}^t\overline{g}g = 1_m\}$.

We define an action of $G_\infty = G_\infty^{(1)} \times \cdots \times G_\infty^{(n)}$ on $\mathcal{Z} = \mathcal{Z}^{(1)} \times \cdots \times \mathcal{Z}^{(n)}$ in a natural manner. Put $j(g,Z) = \prod_{1 \le i \le n} j^{(i)}(g^{(i)}, Z^{(i)}) \in \mathbf{C}^\times$ for $g = (g^{(i)}) \in G_\infty$ and $Z = (Z^{(i)}) \in \mathcal{Z}$. The stabilizer subgroup of $Z_0 = (Z_0^{(i)})$ in G_∞ is $\mathcal{K}_\infty = \mathcal{K}_\infty^{(1)} \times \cdots \times \mathcal{K}_\infty^{(n)}$.

1.6 Let $L = \{{}^t(z_1, z_2, z_3) \in K^3 \mid z_1 \in \kappa\mathcal{D}_{K/F}^{-1},\ z_2, z_3 \in \mathcal{O}_K\}$. Then L is a maximal $\mathcal{D}_{K/F}^{-1}$-integral lattice of K^3 with respect to the Hermitian form S. For each finite prime $\mathfrak{p}$ of F, we put $\mathcal{K}_\mathfrak{p} = \{g \in G_\mathfrak{p} \mid gL_\mathfrak{p} = L_\mathfrak{p}\}$, where $L_\mathfrak{p} = L \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$. Then $\mathcal{K}_\mathfrak{p}$ is a maximal open compact subgroup of $G_\mathfrak{p}$. We put $\mathcal{K}_f = \prod_{\mathfrak{p} < \infty} \mathcal{K}_\mathfrak{p} \subset G_{\mathbf{A},f}$.

1.7 Let l be an even positive integer. We denote by $\mathcal{A}_l(\mathcal{K}_f)$ the space of functions f on $G_\mathbf{A}$ satisfying the following conditions:

(i) $f(\gamma g k_f k_\infty) = j(k_\infty, Z_0)^{-l} f(g)$ $(\gamma \in G_F, g \in G_\mathbf{A}, k_f \in \mathcal{K}_f, k_\infty \in \mathcal{K}_\infty)$.

(ii) For any $g_f \in G_{\mathbf{A},f}$, $j(g_\infty, Z_0)^l f(g_\infty g_f)$ is holomorphic in $g_\infty\langle Z_0\rangle$.

We call $\mathcal{A}_l(\mathcal{K}_f)$ the space of *holomorphic automorphic forms* of weight l on $\mathcal{K}_f$.

1.8 Let N be an algebraic subgroup of G such that

$$N_F = \left\{ (w,x) := \begin{bmatrix} 1 & \kappa w^\sigma & x + \dfrac{\kappa}{2} w w^\sigma \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \mid w \in K, x \in F \right\}.$$

The multiplication law of N is given by

$$(w,x)(w',x') = (w+w', x+x' + \frac{1}{2}\langle w, w'\rangle)\,, \tag{1.1}$$

where

$$\langle w, w'\rangle = Tr_{K/F}(\kappa w^\sigma w')\,. \tag{1.2}$$

Thus N is the Heisenberg group associated with $(K, \langle,\rangle)$.

Let R be an algebraic subgroup of G such that

$$R_F = \left\{ (w,x)\cdot \mathrm{diag}(1,t,1) \mid w \in K, x \in F, t \in K^1 \right\},$$

where $K^1 = \{t \in K^\times \mid tt^\sigma = 1\}$. We henceforth identify R with a semidirect product of N and K^1, and write $(w,x)t$ for $(w,x)\,\mathrm{diag}(1,t,1)$ if there is no fear of confusion. Under this notation, we have $t(w,x)t^{-1} = (tw, x)$ $(t \in K^1, (w,x) \in N)$. The center of R is $\{(0,x)\}$. For $\alpha \in K^\times$, put $\mathbf{d}(\alpha) = \mathrm{diag}(\alpha^\sigma, 1, \alpha^{-1}) \in G_F$. Then we have the Iwasawa decomposition

$$G_{\mathbf{A}} = R_{\mathbf{A}}\,\{\mathbf{d}(\alpha) \mid \alpha \in K_{\mathbf{A}}^\times\}\,\mathcal{K}_f\,\mathcal{K}_\infty\,. \tag{1.3}$$

1.9 Denote by $\psi_{\mathbf{Q}}$ the additive character of $\mathbf{Q}\backslash\mathbf{Q_A}$ with $\psi_{\mathbf{Q}}(x_\infty) = \mathbf{e}[x_\infty]$ $(x_\infty \in \mathbf{R})$. For $m \in F$, we define an additive character ψ_m of $F\backslash F_{\mathbf{A}}$ by $\psi_m(x) = \psi_{\mathbf{Q}}(Tr_{F/\mathbf{Q}}(mx))$ $(x \in F_{\mathbf{A}})$. Let $f \in \mathcal{A}_l(\mathcal{K}_f)$. Then f admits the *Fourier-Jacobi expansion*

$$f(g) = \sum_{m \in F} f^m(g) \quad (g \in G_{\mathbf{A}})\,, \tag{1.4}$$

where

$$f^m(g) = \int_{F\backslash F_{\mathbf{A}}} f((0,x)g)\,\psi_m(-x)\,dx\,. \tag{1.5}$$

We easily see that f^m is left R_F-invariant.

1.10 Let $\mathfrak{a}$ be an ideal of K. We set

$$f^m_{\mathfrak{a}}(r) = f^m(r\,\mathbf{d}(\alpha_f)) \qquad (r \in R_{\mathbf{A}}), \tag{1.6}$$

where α_f is an element of $K^\times_{\mathbf{A},f}$ such that $\mathfrak{a}_f = \alpha_f\,\mathcal{O}_{K,f}$. Note that the right-hand side of (1.6) is independent of the choice of α_f. For $m \in F$ and $w = (w_1, \ldots, w_n) \in K_\infty = \mathbf{C}^n$, we put

$$\mathbf{e}_m[w] = \exp\left(2\pi\sqrt{-1}(m^{(1)}w_1 + \cdots + m^{(n)}w_n)\right)\,.$$

The following facts are easily verified.

1.11 Lemma. *Let $f \in \mathcal{A}_l(\mathcal{K}_f)$ and $\mathfrak{a}$ an ideal of K.*

(i) *We have $f^m_{\mathfrak{a}} = 0$ unless m is totally non-negative.*

(ii) *For a fixed $t \in K^1_{\mathbf{A}}$, $f^0_{\mathfrak{a}}((w,0)t)$ is a constant function in $w \in K_{\mathbf{A}}$.*

(iii) *For $r_f \in R_{\mathbf{A},f}$, $\mathbf{e}_m\left[-\frac{\kappa}{2}\,w_\infty w_\infty^\sigma\right] f^m_{\mathfrak{a}}((w_\infty,0)r_f)$ is holomorphic in $w_\infty \in K_\infty$.*

1.12 Lemma. *Let $f \in \mathcal{A}_l(\mathcal{K}_f), t_\infty \in K^1_\infty$ and $\alpha = \alpha_\infty\alpha_f \in K^\times_{\mathbf{A}}$. Then we have*

$$f^m(rt_\infty\mathbf{d}(\alpha)) = \left(\prod_{i=1}^n \alpha_\infty^{(i)}\right)^l \mathbf{e}_m\left[\frac{\kappa}{2}\,(\alpha_\infty\alpha_\infty^\sigma - 1)\right]\cdot f^m_{\mathfrak{a}}(r) \qquad (r \in R_{\mathbf{A}})\,,$$

where $\mathfrak{a}$ is the ideal of K corresponding to α_f ($\mathfrak{a} = K \cap (\alpha_f\mathcal{O}_{K,f} \times K_\infty)$).

Remark. In view of (1.3) and Lemma 1.12, $f \in \mathcal{A}_l(\mathcal{K}_f)$ is determined by $\{f^m_{\mathfrak{a}}\}_{m,\mathfrak{a}}$.

1.13 Let m be a totally positive element of F. Let $\mathbf{T}^m$ be the space of smooth functions $\Theta\colon R_F\backslash R_{\mathbf{A}} \to \mathbf{C}$ which satisfy the following properties:

(1.7) $\Theta((0,x)r) = \psi_m(x)\,\Theta(r) \quad (x \in F_{\mathbf{A}}, r \in R_{\mathbf{A}})$.

(1.8) For any $r \in R_{\mathbf{A}}$, $t_\infty \mapsto \Theta(rt_\infty)$ is K^1_∞-finite.

We call $\mathbf{T}^m$ the space of *smooth theta functions* of index m. Let $\mathbf{T}^m_{hol}$ be the subspace of $\mathbf{T}^m$ consisting of $\Theta \in \mathbf{T}^m$ satisfying the following properties:

(1.9) For any $r \in R_{\mathbf{A}}$, we have $\Theta(rt_\infty) = \Theta(r)$ $(t_\infty \in K^1_\infty)$.

(1.10) For any $r_f \in R_{\mathbf{A},f}$, $w_\infty \mapsto \mathbf{e}_m\left[-\frac{\kappa}{2}\, w_\infty w_\infty^\sigma\right]\Theta((w_\infty, 0)\, r_f)$ is holomorphic on K_∞.

We call $\mathbf{T}^m_{hol}$ the space of *holomorphic theta functions* of index m. In view of Lemma 1.11 and Lemma 1.12, we have $f^m_{\mathfrak{a}} \in \mathbf{T}^m_{hol}$ for $f \in \mathcal{A}_l(\mathcal{K}_f)$.

Let $\widetilde{\mathbf{T}}^m$ be the completion of $\mathbf{T}^m$ with respect to the inner product

$$(\Theta, \Theta') = \int_{R_F\backslash R_{\mathbf{A}}} \Theta(r)\,\overline{\Theta'(r)}\,dr \qquad (\Theta, \Theta' \in \mathbf{T}^m)\,,$$

where the Haar measure dr on $R_{\mathbf{A}}$ is normalized by $\mathrm{vol}(R_F\backslash R_{\mathbf{A}}) = 1$. Then $\widetilde{\mathbf{T}}^m$ is the Hilbert space of square integrable functions Θ on $R_F\backslash R_{\mathbf{A}}$ satisfying (1.7). We call $\widetilde{\mathbf{T}}^m$ the space of L^2-*theta functions* of index m. Let ρ' be the right translation of $R_{\mathbf{A}}$ on $\widetilde{\mathbf{T}}^m$. Note that $\mathbf{T}^m$ and $\mathbf{T}^m_{hol}$ are $R^*_{\mathbf{A}}$-stable under ρ', where

(1.11)
$$R^*_{\mathbf{A}} = N_{\mathbf{A},f}\cdot K^1_{\mathbf{A}} = R_{\mathbf{A},f}K^1_\infty \subset R_{\mathbf{A}}\,.$$

1.14 Let $\mathfrak{p}$ be a finite prime of F and $\mathfrak{a}_{\mathfrak{p}}$ an ideal of $K_{\mathfrak{p}}$. We put

(1.12)
$$N(\mathfrak{a}_{\mathfrak{p}}) = \Big\{(w,x) \in N_{\mathfrak{p}} \mid w \in \mathfrak{a}_{\mathfrak{p}},\ x + \frac{\kappa}{2}\,ww^\sigma \in \left(\kappa\mathcal{D}^{-1}_{K/F}\right)_{\mathfrak{p}} N_{K/F}(\mathfrak{a}_{\mathfrak{p}})\Big\}$$

and

(1.13)
$$R(\mathfrak{a}_{\mathfrak{p}}) = \{nt \mid n \in N(\mathfrak{a}_{\mathfrak{p}}), t \in \mathcal{O}^1_{K,\mathfrak{p}}\}\,,$$

where $\mathcal{O}^1_{K,\mathfrak{p}} = K^1_{\mathfrak{p}} \cap \mathcal{O}^\times_{K,\mathfrak{p}}$.

For an ideal $\mathfrak{a}$ of K, we set $N(\mathfrak{a})_f = \prod_{\mathfrak{p}<\infty} N(\mathfrak{a}_\mathfrak{p})$ and $R(\mathfrak{a})_f = \prod_{\mathfrak{p}<\infty} R(\mathfrak{a}_\mathfrak{p})$. Let

$$\mathbf{T}^m_{hol}(\mathfrak{a}) = \{\Theta \in \mathbf{T}^m_{hol} \mid \rho'(r)\,\Theta = \Theta \ \text{ for any } r \in R(\mathfrak{a})_f\}. \tag{1.14}$$

It is easy to see that $f^m_\mathfrak{a} \in \mathbf{T}^m_{hol}(\mathfrak{a})$ for $f \in \mathcal{A}_l(\mathcal{K}_f)$.

1.15 For each finite prime $\mathfrak{p}$ of F, we choose and fix an element $\nu_\mathfrak{p}$ of $(\mathcal{D}^{-1}_{K/F})_\mathfrak{p}$ satisfying the following two conditions:

(i) $\nu_\mathfrak{p} + \nu^\sigma_\mathfrak{p} = 1$.

(ii) If $\mathfrak{p}$ ramifies in K/F, we have $\nu_\mathfrak{p} \in \Pi_\mathfrak{p}(\mathcal{D}^{-1}_{K/F})_\mathfrak{p}$, where $\Pi_\mathfrak{p}$ is a prime element of $K_\mathfrak{p}$.

Set $\nu_f = (\nu_\mathfrak{p})_{\mathfrak{p}<\infty} \in (\mathcal{D}^{-1}_{K/F})_f$. For $w \in K_{\mathbf{A},f}$, we put

$$x_w = \frac{\kappa}{2}\,(\nu_f - \nu^\sigma_f)\,ww^\sigma \ \in F_{\mathbf{A},f}\,. \tag{1.15}$$

Then

$$N(\mathfrak{a})_f = \left\{(w, x + x_w) \mid w \in \mathfrak{a}_f, x \in F_{\mathbf{A},f} \cap \left(\kappa\mathcal{D}^{-1}_{K/F}\, N_{K/F}(\mathfrak{a})\right)_f\right\}. \tag{1.16}$$

Let $\mathfrak{f}(\mathfrak{a}, m)$ be an ideal of F given by

$$\mathfrak{f}(\mathfrak{a}, m) = F \cap \left(m\mathcal{D}_F(\kappa\mathcal{D}^{-1}_{K/F})N_{K/F}(\mathfrak{a})\right). \tag{1.17}$$

In view of (1.16), we have the following:

1.16 Lemma. *We have* $\mathbf{T}^m_{hol}(\mathfrak{a}) = \{0\}$ *unless* $\mathfrak{f}(\mathfrak{a}, m)$ *is integral.*

§2. Lattice Model and Intertwining Operators

2.1 In this section, we first summarize several facts about the lattice model. We refer to [MVW] and [MS] for the local theory of the lattice model. We next study a relation between the lattice model and the space of (smooth, holomorphic or L^2-) theta functions.

2.2 From now on, we fix a totally positive element m of F. Let

$$\mathcal{L} = \frac{1}{m\kappa^2}\,\mathcal{D}_F^{-1} + \frac{\kappa}{2}\,\mathcal{O}_F \tag{2.1}$$

be an $\mathcal{O}_F$-lattice of K. Then $\mathcal{L}$ is self-dual with respect to the pairing $(w, w') \mapsto \psi_m(\langle w, w'\rangle)$, where $\langle , \rangle$ is given by (1.2). Put $\mathcal{L}_{\mathfrak{p}} = \mathcal{L} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,\mathfrak{p}}$ for each finite prime $\mathfrak{p}$ of F and put $\mathcal{L}_f = \mathcal{L} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,f} = \prod_{\mathfrak{p}<\infty} \mathcal{L}_{\mathfrak{p}}$.

For each prime v of F, let dw_v be the Haar measure on $K_v = K \otimes_F F_v$ self-dual with respect to the pairing $(w_v, w_v') \mapsto \psi_{m,v}(\langle w_v, w_v'\rangle)$, where $\psi_{m,v} = \psi_m|_{F_v}$. Let $dw = \prod_v dw_v$ be the product measure on $K_{\mathbf{A}}$. We note that $\mathrm{vol}(K\backslash K_{\mathbf{A}}) = 1$.

REMARK. The Haar measure dz_∞ on $K_\infty = \mathbf{C}^n$ is given by

$$dz_\infty = \prod_{i=1}^{n} \frac{2m^{(i)}\kappa^{(i)}}{\sqrt{-1}}\, dx^{(i)} dy^{(i)} \qquad (z_\infty = (z_\infty^{(i)})_{1\le i\le n},\, z_\infty^{(i)} = x^{(i)} + \sqrt{-1}y^{(i)}),$$

where $dx^{(i)}$ and $dy^{(i)}$ are the usual Lebesgue measures on $\mathbf{R}$. We also note that $\displaystyle\int_{\mathcal{L}_{\mathfrak{p}}} dz_{\mathfrak{p}} = 1$.

For $k = (k_1, \ldots, k_n) \in (\mathbf{Z}_{\ge 0})^n$ and $z = (z^{(1)}, \ldots, z^{(n)}) \in K_\infty$, we write $z^k = \prod_{i=1}^n (z^{(i)})^{k_i}$. The following elementary facts are used in later discussion.

2.3 LEMMA.

(i) *For $k, l \in (\mathbf{Z}_{\ge 0})^n$ and $z \in K_\infty$, we have*

$$\int_{K_\infty} \mathbf{e}_m[\kappa w w^\sigma + \kappa w^\sigma z]\,(w^\sigma)^k\,(z+w)^l\,dw = \delta_{k,l} \prod_{i=1}^{n} k_i!\, \left(\frac{2\pi m^{(i)}\kappa^{(i)}}{\sqrt{-1}}\right)^{-k_i}.$$

(ii) *For a polynomial $P(z)$ in $z \in K_\infty$, we have*

$$\int_{K_\infty} \mathbf{e}_m[\kappa w w^\sigma - \kappa z^\sigma w]\,\overline{P(w)}\,dw = \overline{P(z)}\,.$$

PROOF. It suffices to consider the case $n = 1$. Denote by $f(z)$ the integral of (i). Expanding $\mathbf{e}_m[\kappa \overline{w} z]$ and $(z+w)^l$ into power series in $\overline{w}$ and w respectively, we obtain

$$f(z) = \sum_{j=0}^{\infty} \sum_{r=0}^{l} \frac{l!}{(l-r)!\,r!\,j!} (2\pi\sqrt{-1}m\kappa)^j \, z^{j+l-r} \int_{\mathbf{C}} \mathbf{e}_m[\kappa w \overline{w}] \, \overline{w}^{k+j} \, w^r \, dw \, .$$

Since the last integral is equal to $\delta_{k+j,r}\,(k+j)! \left(\frac{2\pi m\kappa}{\sqrt{-1}}\right)^{-k-j}$, we have $f(z) = 0$ if $k > l$. Suppose that $k \le l$. Then we have

$$\begin{aligned} f(z) &= z^{l-k} \left(\frac{2\pi m\kappa}{\sqrt{-1}}\right)^{-k} \sum_{j=0}^{l-k} \frac{l!}{j!\,(k+j-l)!} (-1)^j \\ &= \delta_{l,k}\, k! \left(\frac{2\pi m\kappa}{\sqrt{-1}}\right)^{-k} , \end{aligned}$$

which proves (i). The assertion (ii) is similarly proved. □

2.4 Let $\mathfrak{p}$ be a finite prime of F. Define a smooth irreducible representation $(\rho_{\mathfrak{p}}, V_{\mathfrak{p}}^m)$ of $N_{\mathfrak{p}}$ as follows:

$$V_{\mathfrak{p}}^m = \{\Phi \in \mathcal{S}(K_{\mathfrak{p}}) \mid \Phi(z+l) = \psi_{m,\mathfrak{p}}\left(\frac{1}{2}\langle z, l\rangle + \frac{1}{4}\langle l, l^\sigma\rangle\right) \Phi(z) \quad (z \in K_{\mathfrak{p}}, l \in \mathcal{L}_{\mathfrak{p}})\}$$

$$\rho_{\mathfrak{p}}(n)\,\Phi(z) = \psi_{m,\mathfrak{p}}\left(\frac{1}{2}\langle z, w\rangle + x\right) \Phi(z+w) \quad (n = (w,x) \in N_{\mathfrak{p}}, \Phi \in V_{\mathfrak{p}}^m) .$$

The Stone–von Neumann theorem asserts that any smooth irreducible representation of $N_{\mathfrak{p}}$ with central character $(0,x) \mapsto \psi_{m,\mathfrak{p}}(x)$ is equivalent to $\rho_{\mathfrak{p}}$. Let $\Phi_{0,\mathfrak{p}}$ be an element of $V_{\mathfrak{p}}^m$ given by

$$\Phi_{0,\mathfrak{p}}(z) = \begin{cases} \psi_{m,\mathfrak{p}}\left(\frac{1}{4}\langle z, z^\sigma\rangle\right) & \cdots \quad z \in \mathcal{L}_{\mathfrak{p}} \\ 0 & \cdots \quad z \in K_{\mathfrak{p}} - \mathcal{L}_{\mathfrak{p}} \, . \end{cases}$$

2.5 Let V_f^m be the space of $\Phi_f \in \mathcal{S}(K_{\mathbf{A},f})$ satisfying

$$\Phi_f(z_f + l_f) = \psi_{m,f}\left(\frac{1}{2}\langle z_f, l_f\rangle + \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right) \Phi(z_f) \tag{2.2}$$

for $z_f \in K_{\mathbf{A},f}, l_f \in \mathcal{L}_f$. Then V_f^m is identified with the restricted tensor product of $V_{\mathfrak{p}}^m$ over $\mathfrak{p} < \infty$ with respect to $\{\Phi_{0,\mathfrak{p}}\}_{\mathfrak{p}<\infty}$. Let V_∞^m be the space of functions Φ_∞ on $K_\infty = \mathbf{C}^n$ such that $\mathbf{e}_m\left[-\frac{\kappa}{2} z_\infty z_\infty^\sigma\right]\Phi_\infty(z_\infty)$ is a polynomial in $\overline{z_\infty^{(1)}}, \ldots, \overline{z_\infty^{(n)}}$. The space V_∞^m is called the Fock space (cf. [Ig]). We consider $V^m = V_f^m \otimes V_\infty^m$ as a subspace of $\mathcal{S}(K_{\mathbf{A}})$.

Let $\widetilde{V}^m$ be the completion of V^m with respect to the inner product given by

$$(\Phi, \Phi') = \int_{K_{\mathbf{A}}} \Phi(z)\overline{\Phi'(z)}\, dz \qquad (\Phi, \Phi' \in V^m)\,.$$

Then $\widetilde{V}^m = \widetilde{V}_f^m \otimes \widetilde{V}_\infty^m$, where

$$\begin{aligned} \widetilde{V}_f^m &= \{\Phi_f \in L^2(K_{\mathbf{A},f}) \mid \Phi_f \text{ satisfies (2.2)}\} \\ \widetilde{V}_\infty^m &= \{\Phi_\infty \in L^2(K_\infty) \mid \mathbf{e}_m\left[-\frac{\kappa}{2}\, z_\infty z_\infty^\sigma\right] \Phi_\infty(z_\infty) \\ &\qquad \text{is anti-holomorphic in } z_\infty\}\,. \end{aligned}$$

For $n = (w, x) \in N_{\mathbf{A}}, z \in K_{\mathbf{A}}$ and $\Phi \in \widetilde{V}^m$, we put

$$\rho(n)\,\Phi(z) = \psi_m\left(\frac{1}{2}\langle z, w\rangle + x\right)\Phi(z+w)\,.$$

Then ρ defines an irreducible unitary representation of $N_{\mathbf{A}}$ on $\widetilde{V}^m$ satisfying $\rho(0,x) = \psi_m(x)\,\mathrm{Id}_{\widetilde{V}^m}$ for $x \in F_{\mathbf{A}}$. Note that V^m is $N_{\mathbf{A},f}$-stable under ρ. We call V^m (resp. $\widetilde{V}^m$) the smooth (resp. L^2-) lattice model.

2.6 Let $\Phi_0 = \Phi_{0,f} \otimes \Phi_{0,\infty} \in V^m$, where

$$\begin{aligned} \Phi_{0,f}(z_f) &= \prod_{\mathfrak{p}<\infty} \Phi_{0,\mathfrak{p}}(z_{\mathfrak{p}}) \\ &= \begin{cases} \psi_{m,f}\left(\frac{1}{4}\langle z_f, z_f^\sigma\rangle\right) & \cdots \quad z_f \in \mathcal{L}_f \\ 0 & \cdots \quad z_f \in K_{\mathbf{A},f} - \mathcal{L}_f \end{cases} \\ \Phi_{0,\infty}(z_\infty) &= \mathbf{e}_m\left[\frac{\kappa}{2}\, z_\infty z_\infty^\sigma\right] \qquad (z_\infty \in K_\infty)\,. \end{aligned}$$

We note that $\Phi_{0,f}$ satisfies

$$\rho\left(l_f, \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right)\Phi_{0,f} = \Phi_{0,f} \qquad (l_f \in \mathcal{L}_f)\,, \tag{2.3}$$

and is characterized by this condition up to constant multiples.

2.7 PROPOSITION.

(i) *For* $\Phi \in \widetilde{V}^m$, *we have*

$$\Phi(z) = (\rho(z,0)\,\Phi, \Phi_0) \qquad (z \in K_{\mathbf{A}}).$$

(ii) *For* $\Phi, \Phi' \in \widetilde{V}^m$, *we have*

$$\int_{K_{\mathbf{A}}} \Phi(w)\,\overline{\rho(w,0)\,\Phi'(z)}\,dw = (\Phi, \Phi') \cdot \Phi_0(z) \qquad (z \in K_{\mathbf{A}}).$$

PROOF. To prove the proposition, we may (and do) suppose that $\Phi = \Phi_f \otimes \Phi_\infty \in V^m$ and $\Phi' = \Phi'_f \otimes \Phi'_\infty \in V^m$, since V^m is dense in $\widetilde{V}^m$. It is easily verified that

$$\Phi_f(z_f) = (\rho(z_f,0)\Phi_f, \Phi_{0,f}) \quad (z_f \in K_{\mathbf{A},f}).$$

Let $\Phi_\infty(z) = \Phi_{0,\infty}(z)\overline{P(z)}$, where $P(z)$ is a polynomial in z. By Lemma 2.3 (ii), we have

$$\begin{aligned}
(\rho(z,0)\Phi_\infty, \Phi_{0,\infty}) &= \int_{K_\infty} \mathbf{e}_m\left[\frac{1}{2}\langle w,z\rangle\right]\,\Phi_\infty(w+z)\,\overline{\Phi_{0,\infty}(w)}\,dw \\
&= \Phi_{0,\infty}(z)\int_{K_\infty} \mathbf{e}_m[\kappa w w^\sigma - \kappa w z^\sigma]\,\overline{P(w)}\,dw \\
&= \Phi_{0,\infty}(z)\,\overline{P(z)} = \Phi_\infty(z),
\end{aligned}$$

which proves (i). To prove (ii), put

$$\Phi''_f(z_f) = \int_{K_{\mathbf{A},f}} \Phi_f(w_f)\overline{\rho(w_f,0)\,\Phi'_f(z_f)}\,dw_f \quad (z_f \in K_{\mathbf{A},f}).$$

Let $z_f \in K_{\mathbf{A},f}$ and $l_f \in \mathcal{L}_f$. Since $\Phi_f \in V_f^m$, $\Phi''_f(z_f + l_f)$ is equal to

$$\begin{aligned}
&\int_{K_{\mathbf{A},f}} \Phi_f(w_f - l_f)\,\psi_m\left(-\frac{1}{2}\langle z_f + l_f, w_f - l_f\rangle\right)\overline{\Phi'_f(z_f + w_f)}\,dw_f \\
&= \psi_m\left(\frac{1}{2}\langle z_f, l_f\rangle + \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right) \\
&\quad \times \int_{K_{\mathbf{A},f}} \Phi_f(w_f)\,\psi_m\left(-\frac{1}{2}\langle z_f, w_f\rangle\right)\overline{\Phi'_f(z_f + w_f)}\,dw_f \\
&= \psi_m\left(\frac{1}{2}\langle z_f, l_f\rangle + \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right)\,\Phi''_f(z_f).
\end{aligned}$$

On the other hand, since $\rho(w_f, 0)\Phi'_f \in V_f^m$ for $w_f \in K_{\mathbf{A},f}$, we have

$$\Phi''_f(z_f + l_f) = \psi_m\left(-\frac{1}{2}\langle z_f, l_f\rangle - \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right)\Phi''_f(z_f)\,.$$

These two facts imply that Φ''_f is a constant multiple of $\Phi_{0,f}$. Since $\Phi''_f(0) = (\Phi_f, \Phi'_f)$, we obtain

$$\Phi''_f = (\Phi_f, \Phi'_f)\,\Phi_{0,f}\,.$$

By using a standard argument and Lemma 2.3, we obtain

$$\int_{K_\infty} \Phi_\infty(w)\,\overline{\rho(w,0)\Phi'_\infty(z)}\,dw = (\Phi_\infty, \Phi'_\infty)\,\Phi_{0,\infty}(z) \qquad (z \in K_\infty)\,,$$

which completes the proof of (ii). □

2.8 Denote by $\omega = \omega_{K/F} \in (F_\mathbf{A}^\times/F^\times)^\wedge$ the quadratic Hecke character of F corresponding to K/F by class field theory. Let

$$\mathcal{X} = \{\chi \in (K_\mathbf{A}^\times/K^\times)^\wedge \mid \chi|_{F_\mathbf{A}^\times} = \omega\} \tag{2.4}$$

$$\mathcal{X}_v = \{\chi_v \in (K_v^\times)^\wedge \mid \chi_v|_{F_v^\times} = \omega_v\}\,, \tag{2.5}$$

where ω_v is the restriction of ω to $F_v^\times$ for each prime v of F.

2.9 We now recall a splitting of metaplectic representation of $K_\mathbf{A}^1$ on V^m (or on $\widetilde{V}^m$) given in [MS]. For each prime v of F, let $\lambda_{K_v}(\psi_{m,v}) \in \mathbf{C}^\times$ be the Weil constant attached to $(K_v/F_v, \psi_{m,v})$. By definition, we have

$$\begin{aligned}&\int_{K_v} \varphi(z_v)\,\psi_{m,v}(z_v z_v^\sigma)\,dz_v\\ &= \lambda_{K_v}(\psi_{m,v})\,\left|N_{K/F}(\kappa)\right|_v^{1/2}\int_{K_v}\widehat{\varphi}(z_v)\,\psi_{m,v}(\kappa^2\,z_v z_v^\sigma)\,dz_v\end{aligned}$$

for $\varphi \in \mathcal{S}(K_v)$, where $\widehat{\varphi}$ is the Fourier transform of φ with respect to the pairing $(z, w) \mapsto \psi_{m,v}(\langle z, w\rangle)$. The following facts are well-known:

(i) $\lambda_{K_v}(\psi_{m,v})^2 = \omega_v(-1)$ for every v.

(ii) If $v = \mathfrak{p}$ is finite and $K_\mathfrak{p}/F_\mathfrak{p}$ is not ramified, we have $\lambda_{K_\mathfrak{p}}(\psi_{m,\mathfrak{p}}) = \omega_\mathfrak{p}\left(\pi_\mathfrak{p}^{n(\psi_{m,\mathfrak{p}})}\right)$.

(iii) If v is infinite, $\lambda_{K_v}(\psi_{m,v}) = \sqrt{-1}$.

(iv) $\prod_v \lambda_{K_v}(\psi_{m,v}) = 1$.

For each prime v of F, let V_v^m (resp. $\widetilde{V}_v^m$) be the v-component of V^m (resp. $\widetilde{V}^m$). For $\chi_v \in \mathcal{X}_v$ and $t_v \in K_v^1$, we define an endomorphism $\mathcal{M}_{\chi_v}(t_v)$ of V_v^m by

$$\mathcal{M}_{\chi_v}(t_v)\,\Phi_v = \begin{cases} \Phi_v & \cdots \quad t_v = 1 \\ \lambda_{K_v}(\psi_{m,v})^{-1}\,\chi_v\left(\dfrac{1-t_v}{\kappa}\right)\,\left|N_{K_v/F_v}(1-t_v)\right|_v^{1/2} & \\ \times \displaystyle\int_{K_v} \psi_{m,v}\left(\frac{1}{2}\langle w_v, t_v w_v\rangle\right)\,\rho_v((1-t_v)w_v, 0)\,\Phi_v\,dw_v & \cdots \quad t_v \neq 1 \end{cases}$$

for $\Phi_v \in V_v^m$. The following fact is proved in [MS, §4].

2.10 Proposition. *The mapping $t_v \mapsto \mathcal{M}_{\chi_v}(t_v)$ defines a smooth representation of K_v^1 on V_v^m and extends to a unitary representation of K_v^1 on $\widetilde{V}_v^m$.*

2.11 For $\chi \in \mathcal{X}$, define a unitary representation $\mathcal{M}_\chi$ of $K_{\mathbf{A}}^1$ on $\widetilde{V}^m$ by

$$\mathcal{M}_\chi(t)\,\Phi = \prod_v \mathcal{M}_{\chi_v}(t_v)\,\Phi_v \qquad (t = (t_v) \in K_{\mathbf{A}}^1, \Phi = \prod_v \Phi_v \in \widetilde{V}^m)\,.$$

We use the same notation $\mathcal{M}_\chi$ to denote a representation of $R_{\mathbf{A}}$ on $\widetilde{V}^m$ given by

$$\mathcal{M}_\chi(nt)\,\Phi = \rho(n)\,\mathcal{M}_\chi(t)\,\Phi \qquad (n \in N_{\mathbf{A}}, t \in K_{\mathbf{A}}^1, \Phi \in \widetilde{V}^m)\,.$$

Let $\xi \in (K_{\mathbf{A}}^\times/K^\times)^\wedge$ be a Hecke character of K trivial on $F_{\mathbf{A}}^\times$. For $t \in K_{\mathbf{A}}^1$, we put $\xi^1(t) = \xi(z)$ where we choose $z \in K_{\mathbf{A}}^\times$ so that $t = z^\sigma/z$. Then ξ^1 does not depend on the choice of z and defines a character of $K_{\mathbf{A}}^1/K^1$. Recall that $R_{\mathbf{A}}^* = N_{\mathbf{A},f}K_{\mathbf{A}}^1$. The following fact is easily verified.

2.12 Lemma.

(i) *Let* $\chi \in \mathcal{X}$. *The space* V^m *is* $R^*_{\mathbf{A}}$*-stable under* $\mathcal{M}_\chi$ *and the representation of* $R^*_{\mathbf{A}}$ *on* V^m *is smooth.*

(ii) *For* $\chi, \chi' \in \mathcal{X}$, *we have* $\mathcal{M}_{\chi'}(t) = (\chi/\chi')^1(t)\,\mathcal{M}_\chi(t)$ *for* $t \in K^1_{\mathbf{A}}$.

2.13 For $k = (k_1, \ldots, k_n) \in (\mathbf{Z}_{\geq 0})^n$ and $z_\infty \in K_\infty$, we put

$$\Phi_{k,\infty}(z_\infty) = c_k \cdot \mathbf{e}_m \left[\frac{\kappa}{2}\, z_\infty z_\infty^\sigma \right] (z_\infty^\sigma)^k \,,$$

where

$$c_k = \prod_{i=1}^n \frac{1}{\sqrt{k_i!}} \left(\frac{2\pi m^{(i)} \kappa^{(i)}}{\sqrt{-1}} \right)^{k_i/2} .$$

Then $\{\Phi_{k,\infty} \mid k \in (\mathbf{Z}_{\geq 0})^n\}$ forms an orthonormal basis of $\widetilde{V}^m_\infty$.

Let $\chi \in \mathcal{X}$ and put $\chi_\infty = \chi|_{K_\infty^\times}$. There exist integers $a_1(\chi), \ldots, a_n(\chi)$ such that

$$\chi_\infty(z_\infty) = \prod_{i=1}^n \left(z_\infty^{(i)} / |z_\infty^{(i)}| \right)^{a_i(\chi)} \qquad (z_\infty = (z_\infty^{(i)}) \in K_\infty^\times)\,. \tag{2.6}$$

Since $\chi_{\infty_i}(-1) = \omega_{\infty_i}(-1) = -1$, we have $a_i(\chi) \equiv 1 \pmod 2$ $(1 \leq i \leq n)$.

2.14 LEMMA. *For* $t = (t_i)_{1 \leq i \leq n} \in K^1_\infty$, *we have*

$$\mathcal{M}_\chi(t)\,\Phi_{k,\infty} = \prod_{i=1}^n (t_i)^{k_i + (a_i(\chi)+1)/2} \cdot \Phi_{k,\infty} \,.$$

PROOF. We may assume that $t_i \neq 1$ for each i. For $z \in K_\infty$, we have

$$\begin{aligned}
&c_k^{-1}\,\mathcal{M}_\chi(t)\Phi_{k,\infty}(z) \\
&= \left(\frac{1}{\sqrt{-1}}\right)^n \chi_\infty\left(\frac{1-t}{\kappa}\right) \prod_{i=1}^n |1-t_i|^{-1} \\
&\quad \times \int_{K_\infty} \mathbf{e}_m \left[\frac{\kappa}{2} \left\{ \frac{1+t}{1-t}\, w w^\sigma + z^\sigma w - z w^\sigma + (w+z)(w+z)^\sigma \right\} \right] \\
&\qquad \times (w^\sigma + z^\sigma)^k \, dw \\
&= \left(\frac{1}{\sqrt{-1}}\right)^n \chi_\infty\left(\frac{1-t}{\kappa}\right) \prod_{i=1}^n |1-t_i|^{-1}\, \mathbf{e}_m \left[\frac{\kappa}{2}\, z z^\sigma \right] \cdot I \,,
\end{aligned}$$

where

$$I = \int_{K_\infty} \mathbf{e}_m \left[\frac{\kappa}{1-t} \, ww^\sigma + \kappa \, z^\sigma w \right] (w^\sigma + z^\sigma)^k \, dw \, .$$

A calculation similar to the proof of Lemma 2.3 shows that

$$I = \prod_{i=1}^{n} (1-t_i) t_i^{k_i} \cdot (z^\sigma)^k \, .$$

We thus obtain

$$\begin{aligned} \mathcal{M}_\chi(t)\Phi_{k,\infty}(z) &= \left(\frac{1}{\sqrt{-1}} \right)^n \chi_\infty \left(\frac{1-t}{\kappa} \right) \\ &\quad \times \prod_{i=1}^{n} |1-t_i|^{-1} (1-t_i) t_i^{k_i} \cdot \Phi_{k,\infty}(z) \\ &= \prod_{i=1}^{n} t_i^{k_i + (a_i(\chi)+1)/2} \, \Phi_{k,\infty}(z) \, , \end{aligned}$$

which proves the lemma. □

2.15 Corollary. *The space of K_∞^1-finite vectors in $\widetilde{V}_\infty^m$ coincides with V_∞^m.*

2.16 For $\chi \in \mathcal{X}$ and $\Phi \in V^m$, we put

$$\mathcal{T}_\chi^m \, \Phi(r) = \sum_{\xi \in K} (\mathcal{M}_\chi(r) \, \Phi)(\xi) \qquad (r \in R_\mathbf{A}) \, . \tag{2.7}$$

The theta series (2.7) is absolutely convergent, since $\mathcal{M}_\chi(r) \, \Phi \in \mathcal{S}(K_\mathbf{A})$. Set

$$C_0^m = \mathcal{T}_\chi^m \, \Phi_0(1) = \sum_{\xi \in K} \Phi_0(\xi) \, , \tag{2.8}$$

where Φ_0 is given in §2.6.

2.17 Lemma. *We have $C_0^m > 0$.*

Proof. We first observe that $C_0^m = \sum_{l \in \mathcal{L}} \varphi_0(l)$, where

$$\varphi_0(z) = \mathbf{e}_m \left[\frac{\kappa}{2} \, zz^\sigma - \frac{1}{4} \langle z, z^\sigma \rangle \right] \in \mathcal{S}(K_\infty) \quad (z \in K_\infty) \, .$$

Note that $\mathbf{e}_m\left[-\frac{1}{4}\langle l, l^\sigma\rangle\right] = \pm 1$ for $l \in \mathcal{L}$. A straightforward calculation shows that

$$\widehat{\varphi}_0(z) := \int_{K_\infty} \varphi_0(w)\, \mathbf{e}_m[\langle z, w\rangle]\, dw = 2^{n/2} \cdot \mathbf{e}_m\left[\kappa z z^\sigma - \frac{1}{2}\langle z, z^\sigma\rangle\right].$$

By Poisson summation formula, we obtain

$$C_0^m = \sum_{l\in\mathcal{L}} \widehat{\varphi}_0(l) = 2^{n/2} \sum_{l\in\mathcal{L}} \mathbf{e}_m[\kappa l l^\sigma] > 0\,. \square$$

2.18 PROPOSITION.

(i) *We have* $\mathcal{T}_\chi^m \Phi \in \mathbf{T}^m$ *for* $\Phi \in V^m$.

(ii) $\mathcal{T}_\chi^m$ *defines an* $R_{\mathbf{A}}^*$*-homomorphism of* V^m *(with the action* $\mathcal{M}_\chi$*) to* $\mathbf{T}^m$.

(iii) *For* $\chi, \chi' \in \mathcal{X}$ *and* $\Phi, \Phi' \in V^m$, *we have*

$$\mathcal{T}_{\chi'}^m \Phi((w,0)t) = (\chi/\chi')^1(t)\, \mathcal{T}_\chi^m \Phi((w,0)t) \quad (w \in K_{\mathbf{A}}, t \in K_{\mathbf{A}}^1)$$

and

$$(\mathcal{T}_\chi^m \Phi, \mathcal{T}_{\chi'}^m \Phi') = \delta_{\chi,\chi'} C_0^m \cdot (\Phi, \Phi')\,.$$

(iv) *We can extend* $\mathcal{T}_\chi^m$ *to an* $R_{\mathbf{A}}$*-homomorphism of* $\widetilde{V}^m$ *(with the action* $\mathcal{M}_\chi$*) to* $\widetilde{\mathbf{T}}^m$.

(v) *For* $\Theta \in \mathbf{T}^m$ *and* $\Phi \in V^m$, *we have*

$$\sum_{\chi\in\mathcal{X}} (\Theta, \mathcal{T}_\chi^m \Phi) = \int_{K_{\mathbf{A}}} \Theta((w,0))\, \overline{\Phi(w)}\, dw\,.$$

PROOF. Let $\Phi \in V^m$. It is straightforward to see that $\Theta = \mathcal{T}_\chi^m \Phi$ is smooth, left N_F-invariant and satisfies (1.7) and (1.8). To prove (i), it remains to show that $\Theta(tr) = \Theta(r)$ for $t \in K^1$ and $r \in R_{\mathbf{A}}$. Since the assertion is trivial for $t = 1$, we assume $t \neq 1$. We have

$$\Theta(tr) = \sum_{\xi\in K} \left(\mathcal{M}_\chi(t)\Phi'\right)(\xi)\,,$$

where $\Phi' = \mathcal{M}_\chi(r)\Phi \in \widetilde{V}^m$. By definition of $\mathcal{M}_\chi$, we have

$$(\mathcal{M}_\chi(t)\Phi')(\xi) = \int_{K_{\mathbf{A}}} \psi_m\left(\frac{1}{2}\langle(1-t)w, w-\xi\rangle\right)\Phi'((1-t)w+\xi)\,dw\,.$$

Changing the variable w into $(1-t)^{-1}(w-\xi)$, we have

$$\mathcal{M}_\chi(t)\Phi'(\xi) = \widehat{\Phi''}\left(\frac{\xi}{1-t}\right),$$

where $\widehat{\Phi''}$ is the Fourier transform of $\Phi''(w) = \psi_m\left(\frac{1}{2}Tr_{K/F}\left(\frac{\kappa}{1-t}ww^\sigma\right)\right)\Phi'(w)$ with respect to $\psi_m(\langle,\rangle)$. By Poisson summation formula, we obtain

$$\Theta(tr) = \sum_{\xi\in K}\widehat{\Phi''}\left(\frac{\xi}{1-t}\right) = \sum_{\xi\in K}\Phi''(\xi) = \sum_{\xi\in K}\Phi'(\xi) = \Theta(r)\,,$$

which proves (i). The second assertion of the proposition is easily verified. The first part of (iii) is immediate from Lemma 2.12 (ii). Let $\Phi, \Phi' \in V^m, \chi, \chi' \in \mathcal{X}$ and put $\Theta = \mathcal{T}_\chi^m\Phi$ and $\Theta' = \mathcal{T}_{\chi'}^m\Phi'$. Then we have

$$\begin{aligned}
&(\Theta, \Theta')\\
=\;& \int_{K\backslash K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t \sum_{\xi\in K}\psi_m\left(\frac{1}{2}\langle\xi, w\rangle\right)\\
&\times(\mathcal{M}_\chi(t)\Phi)(w+\xi)\,\overline{\Theta'((w,0)t)}\\
=\;& \int_{K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t\,(\mathcal{M}_\chi(t)\Phi)(w)\,\overline{\Theta'((w,0)t)}\\
=\;& \sum_{\xi\in K}\int_{K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t\,(\mathcal{M}_\chi(t)\Phi)(w)\,\overline{\rho(w,0)\,\mathcal{M}_{\chi'}(t)\Phi'(\xi)}\\
=\;& \sum_{\xi\in K}\Phi_0(\xi)\int_{K^1\backslash K^1_{\mathbf{A}}}(\mathcal{M}_\chi(t)\Phi, \mathcal{M}_{\chi'}(t)\Phi')\,d^\times t\\
&\qquad\qquad\qquad\qquad\text{(by Proposition 2.7 (ii))}\\
=\;& C_0^m(\Phi, \Phi')\int_{K^1\backslash K^1_{\mathbf{A}}}(\chi'/\chi)^1(t)\,d^\times t \qquad \text{(by Lemma 2.12 (ii))}\\
=\;& \delta_{\chi,\chi'}\,C_0^m(\Phi, \Phi')\,,
\end{aligned}$$

which proves the second part of (iii). The fourth assertion follows from (ii) and (iii). To prove (v), we fix a $\chi_0 \in \mathcal{X}$. Observe that

$$\sum_{\chi \in \mathcal{X}} \int_{K^1 \backslash K^1_{\mathbf{A}}} (\chi_0/\chi)^1(t)\, f(t)\, d^\times t = f(1)$$

holds for any continuous function f on $K^1 \backslash K^1_{\mathbf{A}}$. It follows that

$$\begin{aligned}
&\sum_{\chi \in \mathcal{X}} (\Theta, \mathcal{T}^m_\chi \Phi) \\
= &\int_{K \backslash K_{\mathbf{A}}} dw \sum_{\chi \in \mathcal{X}} \int_{K^1 \backslash K^1_{\mathbf{A}}} \Theta((w,0)t)\, (\chi_0/\chi)^1(t)\, \overline{\mathcal{T}^m_{\chi_0} \Phi((w,0)t)} d^\times t \\
= &\int_{K \backslash K_{\mathbf{A}}} \Theta((w,0))\, \overline{\mathcal{T}^m_{\chi_0} \Phi((w,0))}\, dw \\
= &\int_{K_{\mathbf{A}}} \Theta((w,0))\, \overline{\Phi(w)}\, dw \,,
\end{aligned}$$

which proves (v). □

2.19 For $\Theta \in \mathbf{T}^m$, we set

$$(\mathcal{F}^m_\chi \Theta)(z) = (\rho'(z,0)\, \Theta, \mathcal{T}^m_\chi \Phi_0) \qquad (z \in K_{\mathbf{A}}). \tag{2.9}$$

2.20 PROPOSITION.

(i) *For* $\Theta \in \mathbf{T}^m$, *we have* $\mathcal{F}^m_\chi \Theta \in V^m$.

(ii) $\mathcal{F}^m_\chi$ *defines an* $R^*_{\mathbf{A}}$*-homomorphism of* $\mathbf{T}^m$ *to* V^m (*with the action* $\mathcal{M}_\chi$).

(iii) *For* $\Theta \in \mathbf{T}^m$, *we have* $\mathcal{F}^m_\chi \Theta = 0$ *except for finitely many* $\chi \in \mathcal{X}$.

(iv) *For* $\Phi \in V^m$ *and* $\chi, \chi' \in \mathcal{X}$, *we have*

$$\mathcal{F}^m_\chi\, \mathcal{T}^m_{\chi'} \Phi = \delta_{\chi,\chi'}\, C^m_0\, \Phi \,.$$

(v) *For* $\Theta \in \mathbf{T}^m$, *we have*

$$\sum_{\chi \in \mathcal{X}} \mathcal{T}^m_\chi\, \mathcal{F}^m_\chi\, \Theta = C^m_0\, \Theta \,.$$

PROOF. (i) Put $\Phi = \mathcal{F}_\chi^m \Theta$. By (2.3), we have

$$\Phi(z + l_f) = \psi_m\left(\frac{1}{2}\langle z, l_f\rangle + \frac{1}{4}\langle l_f, l_f^\sigma\rangle\right)\Phi(z) \tag{2.10}$$

for $l_f \in \mathcal{L}_f$. We next show that $z_f \mapsto \Phi(z_\infty z_f)$ belongs to $\mathcal{S}(K_{\mathbf{A},f})$ for a fixed $z_\infty \in K_\infty$. Take a sufficiently small neighbourhood $\mathcal{U}_f$ of 0 in $K_{\mathbf{A},f}$ such that $\mathcal{U}_f \subset \mathcal{L}_f, \psi_m\left(\frac{1}{4}\langle w_f, w_f^\sigma\rangle\right) = 1$ and $\rho'(w_f, 0)\,\Theta = \Theta$ for $w_f \in \mathcal{U}_f$. Then we have $\Phi(z + w_f) = \psi_m\left(\frac{1}{2}\langle z, w_f\rangle\right)\Phi(z)$ $(z \in K_\mathbf{A}, w_f \in \mathcal{U}_f)$ in view of (2.10). On the other hand, we have

$$\begin{aligned}\Phi(z + w_f) &= \psi_m\left(\frac{1}{2}\langle w_f, z\rangle\right)(\rho'(z,0)\,\rho'(w_f,0)\,\Theta, \mathcal{T}_\chi^m \Phi_0)\\ &= \psi_m\left(\frac{1}{2}\langle w_f, z\rangle\right)\Phi(z) \quad (z \in K_\mathbf{A}, w_f \in \mathcal{U}_f)\,.\end{aligned}$$

These show that, for a fixed $z_\infty \in K_\infty$, $z_f \mapsto \Phi(z_\infty z_f)$ is in $\mathcal{S}(K_{\mathbf{A},f})$ and hence in V_f^m by (2.10). To prove (i), it now remains to show that $z_\infty \mapsto \Phi(z_\infty z_f)$ belongs to V_∞^m for a fixed $z_f \in K_{\mathbf{A},f}$. Put $\Theta' = \rho'(z_f, 0)\,\Theta \in \mathbf{T}^m$ and take $t_1, \ldots, t_s \in K_{\mathbf{A},f}^1$ so that $K_\mathbf{A}^1 = \bigcup_{1 \le j \le s} K^1\, t_j\, K_\infty^1 \mathcal{O}_{K,f}^1$ (disjoint union). Then we have

$$\begin{aligned}\Phi(z_\infty z_f) &= (\Theta', \mathcal{T}_\chi^m(\rho(-z_\infty, 0)\,\Phi_0))\\ &= \int_{K_\mathbf{A}} dw \int_{K^1\backslash K_\mathbf{A}^1} d^\times t\ \Theta'((w,0)t)\,\overline{\mathcal{M}_\chi(t(-z_\infty,0))\,\Phi_0(w)}\\ &= \frac{1}{c}\sum_{j=1}^{s}\int_{K_\mathbf{A}} dw \int_{\mathcal{O}_{K,f}^1} d^\times t_f \int_{K_\infty^1} d^\times t_\infty\, \Theta'((w,0)t_j t_f t_\infty)\\ &\qquad \overline{\mathcal{M}_\chi(t_j t_f t_\infty(-z_\infty,0))\,\Phi_0(w)}\,,\end{aligned}$$

where the measures $d^\times t_f$ and $d^\times t_\infty$ are normalized by $\mathrm{vol}(\mathcal{O}_{K,f}^1) = \mathrm{vol}(K_\infty^1) = 1$ and $c = s \cdot \#(K^1 \cap K_\infty^1 \mathcal{O}_{K,f}^1)$. Lemma 2.14 implies

$$\begin{aligned}&\mathcal{M}_\chi(t_j t_f t_\infty(-z_\infty,0))\,\Phi_0(w_\infty w_f)\\ &\quad = \eta(t_\infty)\,\mathbf{e}_m\left[\frac{\kappa}{2}\left(z_\infty z_\infty^\sigma + w_\infty w_\infty^\sigma - 2t_\infty z_\infty w_\infty^\sigma\right)\right]\mathcal{M}_\chi(t_j t_f)\,\Phi_{0,f}(w_f)\end{aligned}$$

with

$$\eta(t_\infty) = \prod_{j=1}^{n} (t_\infty^{(j)})^{(a_j(\chi)+1)/2}.$$

Hence we have

$$\Phi(z_\infty z_f) = \mathbf{e}_m\left[\frac{\kappa}{2}\, z_\infty z_\infty^\sigma\right]\,\Psi(z_\infty)\,,$$

where

$$\Psi(z_\infty) = \int_{K_\infty} dw_\infty \int_{K_\infty^1} d^\times t_\infty\; \eta(t_\infty)^{-1}\, \varphi(w_\infty, t_\infty)\, \mathbf{e}_m[-\kappa\, w_\infty (t_\infty z_\infty)^\sigma]\,,$$

$$\begin{aligned}\varphi(w_\infty, t_\infty) \;&=\; \frac{1}{c}\,\mathbf{e}_m\left[\frac{\kappa}{2}\, w_\infty w_\infty^\sigma\right] \sum_{j=1}^{s} \int_{K_{\mathbf{A},f}} dw_f \int_{\mathcal{O}_{K,f}^1} d^\times t_f \\ &\qquad \Theta'((w_f,0)t_j t_f \cdot (w_\infty,0)t_\infty)\,\overline{(\mathcal{M}_\chi(t_j t_f)\Phi_{0,f})(w_f)}\,.\end{aligned}$$

Since Θ is right K_∞^1-finite, $\varphi(w_\infty, t_\infty)$ can be written as a finite sum of unitary characters of K_∞^1 as a function of t_∞. It follows that $\Psi(z_\infty)$ is a polynomial in z_∞^σ and hence $z_\infty \mapsto \Phi(z_\infty z_f)$ belongs to V_∞^m. The proof of (i) has been completed.

(ii) It is easily verified that $\mathcal{F}_\chi^m$ is an $N_{\mathbf{A},f}$-homomorphism. To prove (ii), it suffices to show that, for any prime v of F, $\mathcal{M}_\chi(t)\mathcal{F}_\chi^m\Theta = \mathcal{F}_\chi^m\rho'(t)\Theta$ holds for $t \in K_v^1 - \{1\}$. For $w \in K_v$, set

$$f_t(w) = \lambda_{K_v}(\psi_{m,v})^{-1}\,\chi_v\left(\frac{1-t}{\kappa}\right) |N_{K/F}(1-t)|_v^{-1/2}\,\psi_{m,v}\left(\frac{1}{2}\langle w, \frac{w}{1-t}\rangle\right).$$

Since $\overline{f_t(w)} = f_{t^{-1}}(w) = f_{t^{-1}}(-w)$, we have

$$\begin{aligned} &\mathcal{M}_\chi(t)\,\mathcal{F}_\chi^m\Theta(z) \\ =\; &\int_{K_v} f_t(w)\,\psi_{m,v}\left(\frac{1}{2}\langle z, w\rangle\right)\,\mathcal{F}_\chi^m\Theta(z+w)\,dw \\ =\; &\left(\Theta, \mathcal{T}_\chi^m\left(\int_{K_v} f_{t^{-1}}(-w)\,\rho((-w,0)(-z,0))\,\Phi_0\,dw\right)\right) \\ =\; &\left(\Theta, \mathcal{T}_\chi^m(\mathcal{M}_\chi(t^{-1}(-z,0))\,\Phi_0)\right) \\ =\; &\left(\Theta, \rho'(t^{-1}(-z,0))\,\mathcal{T}_\chi^m\,\Phi_0\right) \\ =\; &\mathcal{F}_\chi^m(\rho'(t)\Theta)(z)\end{aligned}$$

for $z \in K_{\mathbf{A}}$. This completes the proof of (ii).

(iii) Fix a $\chi_0 \in \mathcal{X}$ and put $\Theta_0 = \mathcal{T}^m_{\chi_0}\Phi_0$. By Proposition 2.18 (iii), we have

$$\mathcal{F}^m_\chi \Theta(z) = \int_{K^1\backslash K^1_{\mathbf{A}}} I(t,z)\,(\chi/\chi_0)^1(t)\,d^\times t\,,$$

where

$$I(t,z) = \int_{K\backslash K_{\mathbf{A}}} \Theta((w,0)t(z,0))\overline{\Theta_0((w,0)t)}\,dw\,.$$

Since Θ and Θ_0 are right invariant under some open compact subgroup of $N_{\mathbf{A},f}$, there exists an ideal $\mathfrak{a}$ of K such that the equalities

$$I(t, z + w_f) = \psi_m\left(-\frac{1}{2}\langle z_f, w_f\rangle\right) I(t,z)$$

and

$$I(t, z + w_f) = \psi_m\left(\frac{1}{2}\langle z_f, w_f\rangle\right) I(t,z)$$

hold for any $w_f \in \mathfrak{a}_f, t \in K^1_{\mathbf{A}}$ and $z \in K_{\mathbf{A}}$. This implies that $z_f \mapsto I(t, z_\infty z_f)$ has a compact support independent of t and z_∞ and hence that there exists an open compact subgroup U^1_f of $K^1_{\mathbf{A},f}$ satisfying

$$I(t't,z) = I(t,z) \qquad (t' \in U^1_f, t \in K^1_{\mathbf{A}}, z \in K_{\mathbf{A}})\,.$$

Thus there exists an open compact subgroup U_f of $K^\times_{\mathbf{A},f}$ such that

$$\mathcal{F}^m_\chi\Theta \neq 0 \Longrightarrow \chi|_{U_f} = 1\,.$$

On the other hand, in view of the argument of the proof of (i), we see that there exist only a finite number of infinity types of χ such that $\mathcal{F}^m_\chi\Theta \neq 0$. The assertion (iii) now follows.

(iv) For $z \in K_{\mathbf{A}}$, we have

$$\begin{aligned}
\mathcal{F}^m_\chi \mathcal{T}^m_{\chi'}\Phi(z) &= (\rho'(z,0)\mathcal{T}^m_{\chi'}\Phi, \mathcal{T}^m_\chi\Phi_0) \\
&= \delta_{\chi,\chi'}\,C^m_0\,(\rho(z,0)\Phi, \Phi_0) \qquad \text{(by Proposition 2.18 (iv), (iii))} \\
&= \delta_{\chi,\chi'}\,C^m_0\,\Phi(z) \qquad \text{(by Proposition 2.7 (i))}\,.
\end{aligned}$$

(v) In view of (ii), we have

$$(\mathcal{M}_\chi(r)\,\mathcal{F}^m_\chi\,\Theta)(z) = (\rho'((z,0)r)\Theta, \mathcal{T}^m_\chi\Phi_0) \quad (r \in R_{\mathbf{A}}, z \in K_{\mathbf{A}})\,.$$

Thus, for $r \in R_{\mathbf{A}}$, we have

$$\begin{aligned}
&\sum_{\chi\in\mathcal{X}} \mathcal{T}^m_\chi \mathcal{F}^m_\chi \Theta(r) \\
= \;& \sum_{\chi\in\mathcal{X}} \sum_{\xi\in K} (\rho'((\xi,0)r)\Theta, \mathcal{T}^m_\chi\Phi_0) \\
= \;& \sum_{\xi\in K} \int_{K_{\mathbf{A}}} \Theta((w,0)(\xi,0)r)\,\overline{\Phi_0(w)}\,dw \qquad \text{(by Proposition 2.18 (v))} \\
= \;& \sum_{\xi\in K} \int_{K_{\mathbf{A}}} \psi_m(\langle w,\xi\rangle)\Theta((w,0)r)\,\overline{\Phi_0(w)}\,dw \\
= \;& \sum_{\xi\in K} \Theta((\xi,0)r)\,\overline{\Phi_0(\xi)} \qquad \text{(by Poisson summation formula)} \\
= \;& C^m_0\,\Theta(r)\,,
\end{aligned}$$

which proves (v). □

2.21 Put

$$\mathbf{T}^m_\chi = \mathcal{T}^m_\chi(V^m) \subset \mathbf{T}^m\,. \tag{2.11}$$

2.22 THEOREM.

(i) *We have an algebraic direct sum*

$$\mathbf{T}^m = \bigoplus_{\chi\in\mathcal{X}} \mathbf{T}^m_\chi\,.$$

(ii) *For* $(\Phi_\chi) \in \bigoplus_\chi V^m$ *and* $\Theta \in \mathbf{T}^m$, *we have*

$$\left(\sum_{\chi\in\mathcal{X}} \mathcal{T}^m_\chi\,\Phi_\chi, \Theta\right) = \sum_{\chi\in\mathcal{X}} \left(\Phi_\chi, \mathcal{F}^m_\chi\,\Theta\right)\,.$$

(iii) *For* $\Theta, \Theta' \in \mathbf{T}^m$, *we have*

$$\sum_{\chi \in \mathcal{X}} \left(\mathcal{F}_\chi^m \Theta, \mathcal{F}_\chi^m \Theta' \right) = C_0^m \left(\Theta, \Theta' \right) .$$

Proof. The first assertion follows from Proposition 2.20 (v) and (iii). By Proposition 2.20 (v) and Proposition 2.18 (iii), we have

$$\begin{aligned} \left(\sum_\chi \mathcal{T}_\chi^m \Phi_\chi, \Theta \right) &= \left(\sum_\chi \mathcal{T}_\chi^m \Phi_\chi, (C_0^m)^{-1} \sum_\chi \mathcal{T}_\chi^m \mathcal{F}_\chi^m \Theta \right) \\ &= \sum_\chi \left(\Phi_\chi, \mathcal{F}_\chi^m \Theta \right), \end{aligned}$$

which proves (ii). The third assertion is an immediate consequence of (ii) and Proposition 2.20 (v). □

2.23 Corollary. *The mapping* $\bigoplus_\chi \mathcal{T}_\chi^m$ *gives rise to an* $R_{\mathbf{A}}^*$*-isomorphism of* $\bigoplus_\chi V^m$ *onto* $\mathbf{T}^m$, *and extends to an* $R_{\mathbf{A}}$*-isomorphism of* $\widehat{\bigoplus}_\chi \widetilde{V}^m$ *(a direct sum of Hilbert spaces) to* $\widetilde{\mathbf{T}}^m$. *Its inverse is given by* $(C_0^m)^{-1} \sum_\chi \mathcal{F}_\chi^m$.

2.24 Remark. For $\chi \in \mathcal{X}$, let $\mathcal{M}'_\chi$ be the representation of $K_{\mathbf{A}}^1$ on $\widetilde{\mathbf{T}}^m$ constructed from the right translation ρ' of $N_{\mathbf{A}}$ as in §2.9 (see also §0.2). Then $\mathbf{T}^m$ is $K_{\mathbf{A}}^1$-stable under $\mathcal{M}'_\chi$ and $\mathbf{T}_\chi^m$ is characterized as follows:

$$\mathbf{T}_\chi^m = \{\Theta \in \mathbf{T}^m \mid \rho'(t)\Theta = \mathcal{M}'_\chi(t)\Theta \quad (t \in K_{\mathbf{A}}^1)\} .$$

2.25 Let

$$V_{hol,\infty}^m = \mathbf{C} \cdot \Phi_{0,\infty} , \quad V_{hol}^m = V_{hol,\infty}^m \otimes V_f^m \tag{2.12}$$

and

$$\mathcal{X}_0 = \{\chi \in \mathcal{X} \mid a_i(\chi) = -1 \ \ (1 \leq i \leq n)\} . \tag{2.13}$$

For $\chi \in \mathcal{X}_0$, we put

$$\mathbf{T}^m_{hol,\chi} = \mathcal{T}^m_\chi(V^m_{hol}) \subset \mathbf{T}^m_\chi\,. \tag{2.14}$$

2.26 PROPOSITION.

(i) *We have* $\mathbf{T}^m_{hol} = \bigoplus_{\chi\in\mathcal{X}_0} \mathbf{T}^m_{hol,\chi}$.

(ii) *For* $\chi \in \mathcal{X}_0$, *we have* $\mathbf{T}^m_{hol,\chi} = \{\Theta \in \mathbf{T}^m_\chi \mid \rho'(t_\infty)\Theta = \Theta \quad (t_\infty \in K^1_\infty)\}$.

PROOF. Let $\chi \in \mathcal{X}$, $\Phi = \sum_{j=1}^N \Phi_{k^{(j)},\infty} \otimes \Phi_f^{(j)}$ $(k^{(j)} \in (\mathbf{Z}_{\geq 0})^n, \Phi_f^{(j)} \in V_f^m)$, and put $\Theta_\chi = \mathcal{T}^m_\chi\,\Phi$. We show that $\Theta_\chi \in \mathbf{T}^m_{hol}$ if and only if $\chi \in \mathcal{X}_0$ and $k^{(j)} = (0,\dots,0)$ for any j. For $r_f \in R_{\mathbf{A},f}$ and $w_\infty \in K_\infty$, we have

$$\begin{aligned} & \mathbf{e}_m\left[-\frac{\kappa}{2}\,w_\infty w_\infty^\sigma\right]\,\Theta_\chi((w_\infty,0)\,r_f) \\ = \;& \sum_{j=1}^N c_{k^{(j)}} \sum_{\xi\in K} \left(\mathcal{M}_\chi(r_f)\,\Phi_f^{(j)}\right)(\xi) \\ & \cdot\mathbf{e}_m\left[\frac{\kappa}{2}\,\xi\xi^\sigma + \kappa\xi^\sigma w_\infty\right] \prod_{i=1}^n \overline{(w_\infty^{(i)} + \xi^{(i)})}^{k_i^{(j)}}\,, \end{aligned}$$

where $k^{(j)} = (k_i^{(j)})_{1\leq i\leq n}$ (for the definition of $c_{k^{(j)}}$, see §2.13). This implies that Θ_χ satisfies the holomorphy condition (1.10) if and only if $k^{(j)} = (0,\dots,0)$ for any j. Hence we may (and do) assume that $\Phi = \Phi_{0,\infty}\otimes\Phi_f$ with $\Phi_f \in V_f^m$. Since

$$\rho'(t_\infty)\,\Theta_\chi = \mathcal{T}^m_\chi\,(\mathcal{M}_\chi(t_\infty)\Phi) = \prod_{i=1}^n t_i^{(a_i(\chi)+1)/2}\,\Theta_\chi \quad (t_\infty = (t_i)_{1\leq i\leq n} \in K^1_\infty)$$

by Lemma 2.14, we see that Θ_χ is K^1_∞-invariant if and only if $\chi \in \mathcal{X}_0$. Thus we are done. □

2.27 For $\chi \in \mathcal{X}_0$ and an ideal $\mathfrak{a}$ of K, let

$$V^m_{hol}(\mathfrak{a},\chi) = \{\Phi \in V^m_{hol} \mid \mathcal{M}_\chi(r_f)\,\Phi = \Phi \quad (r_f \in R(\mathfrak{a})_f)\}\,. \tag{2.15}$$

and

$$\mathbf{T}^m_{hol}(\mathfrak{a},\chi) = \{\Theta \in \mathbf{T}^m_{hol,\chi} \mid \rho'(r_f)\Theta = \Theta \quad (r_f \in R(\mathfrak{a})_f)\}\,. \tag{2.16}$$

The following is easily verified.

2.28 PROPOSITION.

(i) $V^m_{hol}(\mathfrak{a},\chi)$ *is isomorphic to* $\mathbf{T}^m_{hol}(\mathfrak{a},\chi)$ *via* $\mathcal{T}^m_\chi$.

(ii) *We have* $\mathbf{T}^m_{hol}(\mathfrak{a}) = \bigoplus_{\chi\in\mathcal{X}_0} \mathbf{T}^m_{hol}(\mathfrak{a},\chi)$.

§3. Primitive Theta Functions

3.1 In this section, we study primitive theta functions introduced by Shintani ([Shin]; see also [GlRo] and [MS]).

For an ideal $\mathfrak{a}$ of K, define an endomorphism $\mathcal{P}'_\mathfrak{a}$ of $\mathbf{T}^m$ by

$$\mathcal{P}'_\mathfrak{a}\Theta = \int_{N(\mathfrak{a})_f} \rho'(n)\Theta\, d_\mathfrak{a} n \qquad (\Theta \in \mathbf{T}^m)\,, \tag{3.1}$$

where $d_\mathfrak{a} n$ is the Haar measure on $N(\mathfrak{a})_f$ normalized by $\mathrm{vol}(N(\mathfrak{a})_f) = 1$. Then $\mathcal{P}'_\mathfrak{a} \circ \mathcal{P}'_\mathfrak{a} = \mathcal{P}'_\mathfrak{a}$ and $\mathcal{P}'_\mathfrak{a}(\mathbf{T}^m_{hol}) = \mathbf{T}^m_{hol}(\mathfrak{a})$. Note that $\mathcal{P}'_\mathfrak{a} = 0$ unless $\mathfrak{f}(\mathfrak{a},m)$ is integral (for the definition of $\mathfrak{f}(\mathfrak{a},m)$, see (1.17)).

3.2 From now on, we always assume that

$$\mathfrak{f}(\mathfrak{a},m) \text{ is integral.} \tag{3.2}$$

For $\chi \in \mathcal{X}_0$, put

$$\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi) = \{\Theta \in \mathbf{T}^m_{hol}(\mathfrak{a},\chi) \mid \mathcal{P}'_\mathfrak{b}\Theta = 0 \text{ for any ideal } \mathfrak{b} \supset \mathfrak{a}, \mathfrak{b} \neq \mathfrak{a}\}\,. \tag{3.3}$$

We call $\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$ the space of *primitive theta functions* attached to $(\mathfrak{a},\chi)$.

REMARK. The above definition of primitivity is slightly different from the one in [MS], where we imposed an additional condition that $\mathcal{Q}'_\mathfrak{p}\,\Theta = 0$ for any finite prime $\mathfrak{p}$ of F ramified in K/F with $\mathrm{ord}_\mathfrak{p}\,\mathfrak{f}(\mathfrak{a},m) = 0$. Here

$$\mathcal{Q}'_\mathfrak{p}\,\Theta = \int_{\mathfrak{a}_\mathfrak{p}} \rho'(\Pi_\mathfrak{p}^{-1}w, (\Pi_\mathfrak{p}\Pi_\mathfrak{p}^\sigma)^{-1}x_w)\,\Theta\, d_{\mathfrak{a}_\mathfrak{p}}w \qquad (\Theta \in \mathbf{T}^m)\,. \tag{3.4}$$

($d_{\mathfrak{a}_\mathfrak{p}} w$ is the Haar measure on $\mathfrak{a}_\mathfrak{p}$ normalized by $\mathrm{vol}(\mathfrak{a}_\mathfrak{p}) = 1$ and $\Pi_\mathfrak{p}$ is a prime element of $K_\mathfrak{p}$.)

3.3 For an ideal $\mathfrak{a}$ of K, define $\mathcal{P}_\mathfrak{a} \in \mathrm{End}(V^m)$ by

$$\mathcal{P}_\mathfrak{a}\Phi = \int_{N(\mathfrak{a})_f} \rho(n)\Phi \, d_\mathfrak{a} n \qquad (\Phi \in V^m). \tag{3.5}$$

For $\chi \in \mathcal{X}_0$, we set

$$V^m_{hol,prim}(\mathfrak{a},\chi) = \{\Phi \in V^m_{hol}(\mathfrak{a},\chi) \mid \mathcal{P}_\mathfrak{b}\Phi = 0 \quad \text{for any ideal } \mathfrak{b} \supset \mathfrak{a}, \mathfrak{b} \neq \mathfrak{a}\}. \tag{3.6}$$

Then $V^m_{hol,prim}(\mathfrak{a},\chi)$ is isomorphic to $\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$ via $\mathcal{T}^m_\chi$. On the other hand, we have

$$V^m_{hol,prim}(\mathfrak{a},\chi) = \mathbf{C} \cdot \Phi_{0,\infty} \otimes \bigotimes_{\mathfrak{p}<\infty} V^m_{\mathfrak{p},prim}(\mathfrak{a},\chi),$$

where $V^m_{\mathfrak{p},prim}(\mathfrak{a},\chi)$ is the *primitive* part of $V^m_\mathfrak{p}(\mathfrak{a},\chi) = \{\Phi \in V^m_\mathfrak{p} \mid \mathcal{M}_\chi(r)\Phi = \Phi \ (r \in R(\mathfrak{a}_\mathfrak{p}))\}$ defined similarly as (3.6). By Corollary 6.5 in [MS], $V^m_{\mathfrak{p},prim}(\mathfrak{a},\chi)$ is at most one-dimensional. Thus we have

3.4 THEOREM.

$$\dim_\mathbf{C} \mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi) = \dim_\mathbf{C} V^m_{hol,prim}(\mathfrak{a},\chi) \leq 1.$$

REMARK. This result has been proved, in a classical setting, by Shintani [Shin] in the case where $F = \mathbf{Q}$ and $K = \mathbf{Q}(\sqrt{-1})$, and by Glaubermann and Rogawski [GlRo] in the general CM case.

3.5 We now recall a criterion for the existence of primitive theta functions after [MS]. Let $\mathfrak{p}$ be a finite prime of F. We put

$$\delta_{K_\mathfrak{p}/F_\mathfrak{p}} = \mathrm{ord}_\mathfrak{p}\ N_{K/F}(\mathcal{D}_{K/F}) \tag{3.7}$$

and

$$\mu_\mathfrak{p}(\mathfrak{a},m) = \mathrm{ord}_\mathfrak{p}\ \mathfrak{f}(\mathfrak{a},m). \tag{3.8}$$

Recall that we have assumed $\mu_{\mathfrak{p}}(\mathfrak{a}, m) \geq 0$ for any $\mathfrak{p}$. Let $\chi_{\mathfrak{p}}$ be a character of $K_{\mathfrak{p}}^{\times}$. Define $a(\chi_{\mathfrak{p}})$ to be the smallest nonnegative integer a such that $\chi_{\mathfrak{p}}$ is trivial on $(1+\mathfrak{P}_{\mathfrak{p}}^{a}) \cap \mathcal{O}_{K,\mathfrak{p}}^{\times}$, where

$$\mathfrak{P}_{\mathfrak{p}} = \begin{cases} \pi_{\mathfrak{p}} \mathcal{O}_{K,\mathfrak{p}} & \cdots \quad K_{\mathfrak{p}} = F_{\mathfrak{p}} \oplus F_{\mathfrak{p}} \\ \text{the maximal ideal of } \mathcal{O}_{K,\mathfrak{p}} & \cdots \quad K_{\mathfrak{p}} \text{ is a field}. \end{cases}$$

Let $\epsilon(s, \chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}})$ be Tate's epsilon factor (cf. [Ta]), where $\psi_{m,K_{\mathfrak{p}}} = \psi_{m,\mathfrak{p}} \circ Tr_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} \in (K_{\mathfrak{p}})^{\wedge}$. In what follows, we write $\epsilon(\chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}})$ for $\epsilon(1/2, \chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}})$ to simplify the notation. Recall that, if $K_{\mathfrak{p}}$ is a field, we have

$$\epsilon(\chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}}) = \chi_{\mathfrak{p}}(c) \frac{S}{|S|},$$

where c is an element of $K_{\mathfrak{p}}^{\times}$ satisfying $c\mathcal{O}_{K,\mathfrak{p}} = \pi_{\mathfrak{p}}^{n(\psi_{m,\mathfrak{p}})} \mathfrak{P}_{\mathfrak{p}}^{a(\chi_{\mathfrak{p}})+\delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}}}$ and

$$S = \int_{\mathcal{O}_{K,\mathfrak{p}}^{\times}} \chi_{\mathfrak{p}}^{-1}(u) \, \psi_{m,K_{\mathfrak{p}}}\left(\frac{u}{c}\right) d^{\times}u$$

($d^{\times}u$ is a Haar measure on $\mathcal{O}_{K,\mathfrak{p}}^{\times}$). The following fact is well-known (for example, see [MS, Proposition 3.7]).

3.6 Lemma. *Let $\chi_{\mathfrak{p}} \in \mathcal{X}_{\mathfrak{p}}$ (for the definition of $\mathcal{X}_{\mathfrak{p}}$, see* (2.5)*).*

(i) $\epsilon(\chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}}) = \pm\chi_{\mathfrak{p}}(\kappa^{-1})$.

(ii) *If $K_{\mathfrak{p}} = F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$, we have $\epsilon(\chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}}) = \chi_{\mathfrak{p}}(\kappa^{-1})$.*

(iii) *If $K_{\mathfrak{p}}$ is an unramified quadratic extension of $F_{\mathfrak{p}}$, we have*

$$\epsilon(\chi_{\mathfrak{p}}, \psi_{m,K_{\mathfrak{p}}}) = (-1)^{a(\chi_{\mathfrak{p}})+\mu_{\mathfrak{p}}(\mathfrak{a},m)} \chi_{\mathfrak{p}}(\kappa^{-1}).$$

3.7 Let $\mathcal{X}_{prim}(\mathfrak{a}, m)$ be the set of $\chi = \prod_v \chi_v \in \mathcal{X}$ satisfying, for each finite prime $\mathfrak{p}$,

$$a(\chi_{\mathfrak{p}}) = \begin{cases} \mu_{\mathfrak{p}}(\mathfrak{a}, m) & \cdots \quad \delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} = 0 \\ 2(\mu_{\mathfrak{p}}(\mathfrak{a}, m) + \delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}}) & \cdots \quad \delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} > 0 \text{ and } \mu_{\mathfrak{p}}(\mathfrak{a}, m) > 0 \\ 2\delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} \text{ or } 2\delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} - 1 & \cdots \quad \delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}} > 0 \text{ and } \mu_{\mathfrak{p}}(\mathfrak{a}, m) = 0. \end{cases}$$

Set

$$\mathcal{X}^+_{prim}(\mathfrak{a}, m) = \{\chi \in \mathcal{X}_{prim}(\mathfrak{a}, m) \mid \epsilon(\chi_\mathfrak{p}, \psi_{m,K_\mathfrak{p}}) = \chi_\mathfrak{p}(\kappa^{-1}) \text{ for each finite prime } \mathfrak{p}\} \tag{3.9}$$

and

$$\mathcal{X}^+_{0,prim}(\mathfrak{a}, m) = \mathcal{X}_0 \cap \mathcal{X}^+_{prim}(\mathfrak{a}, m) \tag{3.10}$$

(recall that $\mathcal{X}_0$ is defined in §2.25). The following criterion for the existence of primitive theta functions is a direct consequence of the corresponding fact for $V^m_{\mathfrak{p},prim}(\mathfrak{a}, \chi)$ proved in our previous paper (see [MS, Theorem 6.4]).

3.8 THEOREM. *Let $\chi \in \mathcal{X}$ and $\mathfrak{a}$ be an ideal of K such that $\mathfrak{f}(\mathfrak{a}, m)$ is integral.*

(i) *We have $\mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi) \neq \{0\}$ if and only if $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$. In this case, we have $\mathbf{T}^m_{hol}(\mathfrak{a}, \chi) = \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi)$ and $\dim_\mathbf{C} \mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi) = 1$.*

(ii) *Let $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$ and Θ be a nonzero element of $\mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi)$. Let $\mathfrak{p}$ be a finite prime ramified in K/F with $\mu_\mathfrak{p}(\mathfrak{a}, m) = 0$. Then*

$$\begin{aligned} a(\chi_\mathfrak{p}) = 2\delta_{K_\mathfrak{p}/F_\mathfrak{p}} \quad &\Longleftrightarrow \quad \mathcal{Q}'_\mathfrak{p}\,\Theta = 0 \\ a(\chi_\mathfrak{p}) = 2\delta_{K_\mathfrak{p}/F_\mathfrak{p}} - 1 \quad &\Longleftrightarrow \quad \mathcal{Q}'_\mathfrak{p}\,\Theta = \Theta\,, \end{aligned}$$

where $\mathcal{Q}'_\mathfrak{p}$ is defined by (3.4)*.*

REMARK. Theorem 3.8 is an refinement of the *epsilon dichotomy* for $(U(1), U(1))$, which asserts that $\chi_\mathfrak{p} \in \mathcal{X}_\mathfrak{p}$ appears in the metaplectic representation of $K^1_\mathfrak{p}$ if and only if $\epsilon(\chi_\mathfrak{p}, \psi_{m,K_\mathfrak{p}}) = \chi_\mathfrak{p}(\kappa^{-1})$. The epsilon dichotomy for $(U(1), U(1))$ was proved by Moen ([Mo]) in the odd residual characteristic case and by Rogawski ([Ro]) in the general case by using a global method (see also [HKS] for a purely local proof, which is different from the one in [MS]).

The following result is due to Shintani ([Shin]; see also [MS, §10]).

3.9 THEOREM. *Let $\mathfrak{a}$ be an ideal of K such that $\mathfrak{f}(\mathfrak{a}, m)$ is integral. Then we have a direct sum decomposition*

$$\mathbf{T}^m_{hol}(\mathfrak{a}) = \sum_{\mathfrak{b}} \sum_{\chi} \mathbf{T}^m_{hol,prim}(\mathfrak{b}, \chi)\,,$$

where $\mathfrak{b}$ runs over the ideals of K such that $\mathfrak{b} \supset \mathfrak{a}$ and $\mathfrak{f}(\mathfrak{b}, m)$ is integral, and $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{b}, m)$.

§4. Main Result

4.1 Recall that m is a totally positive element of F. Let $\mathfrak{a}$ be an ideal of K such that $\mathfrak{f}(\mathfrak{a}, m)$ is integral and let $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a}, m)$. Let Θ be a nonzero element of $\mathbf{T}^m_{hol,prim}(\mathfrak{a}, \chi)$, which is uniquely determined by $(\mathfrak{a}, \chi)$ up to constant multiples.

For $f \in \mathcal{A}_l(\mathcal{K}_f)$, we call the inner product

$$(f^m_{\mathfrak{a}}, \Theta) = \int_{R_F \backslash R_{\mathbf{A}}} f^m_{\mathfrak{a}}(r)\, \overline{\Theta(r)}\, dr \tag{4.1}$$

the *primitive* $(\mathfrak{a}, \chi)$*-component* of f (with respect to Θ).

4.2 Let l be an even positive integer and Ω a Hecke character of K satisfying

$$\Omega|_{\mathcal{O}^\times_{K,f}} = 1 \text{ and } \Omega(z_\infty) = \prod_{i=1}^{n} \left(\frac{z^{(i)}_\infty}{|z^{(i)}_\infty|} \right)^l \quad (z_\infty \in K^\times_\infty)\,. \tag{4.2}$$

Let P be a minimal parabolic subgroup of G given by $P_F = \{ r\,\mathbf{d}(z) \mid r \in R_F, z \in K^\times \}$. Recall that the Iwasawa decomposition $G_{\mathbf{A}} = P_{\mathbf{A}} \mathcal{K}_f \mathcal{K}_\infty$ holds. Define a function ϕ_Ω on $G_{\mathbf{A}}$ by

$$\phi_\Omega(r\,\mathbf{d}(z)\,k_f\,k_\infty) = \Omega(z)\,|N_{K/F}(z)|^{l/2}_{\mathbf{A}}\, j(k_\infty, Z_0)^{-l} \tag{4.3}$$

for $r \in R_{\mathbf{A}}, z \in K^\times_{\mathbf{A}}, k_f \in \mathcal{K}_f, k_\infty \in \mathcal{K}_\infty$. Then ϕ_Ω is left P_F-invariant. We set

$$E_\Omega(g) = \sum_{\gamma \in P_F \backslash G_F} \phi_\Omega(\gamma g) \qquad (g \in G_{\mathbf{A}})\,. \tag{4.4}$$

If $l>4$, the series (4.4) is absolutely convergent and defines an element of $\mathcal{A}_l(\mathcal{K}_f)$. We call E_Ω the *holomorphic Eisenstein series* attached to Ω. From now on, we fix an even integer l with $l>4$.

4.3 To state the main result of the paper, we let $m,\mathfrak{a},\chi,\Theta$ be as in §4.1. Let Ω be a Hecke character of K satisfying (4.2). Let $\alpha_f=(\alpha_\mathfrak{p})_{\mathfrak{p}<\infty}$ be an element of $K_{\mathbf{A},f}^\times$ corresponding to $\mathfrak{a}$ as in §1.10. Let $A(\chi)$ be the set of finite primes $\mathfrak{p}$ of F such that

$$\text{(4.5)} \qquad \mathfrak{p} \text{ is ramified in } K/F,\ \mu_\mathfrak{p}(\mathfrak{a},m)=0 \text{ and } a(\chi_\mathfrak{p})=2\delta_{K_\mathfrak{p}/F_\mathfrak{p}}-1.$$

For each finite prime $\mathfrak{p}$, we choose an element $b_\mathfrak{p}(\kappa)$ of $F_\mathfrak{p}^\times$ such that $(\kappa\,\mathcal{D}_{K/F}^{-1})_\mathfrak{p}=b_\mathfrak{p}(\kappa)\,\mathcal{O}_{K,\mathfrak{p}}$. Let $I(\Theta)$ be the *period* of Θ given by

$$I(\Theta)=\int_{K^1\backslash K_\mathbf{A}^1}\Theta(t)\,d^\times t\,, \tag{4.6}$$

where $d^\times t$ is normalized by $\mathrm{vol}(K^1\backslash K_\mathbf{A}^1)=1$. In the next section, we state a criterion for the non-vanishing of $I(\Theta)$ (cf. Corollary 5.4).

4.4 THEOREM. *For* $\Theta\in\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$, *we have*

$$\begin{aligned}((E_\Omega)^m_\mathfrak{a},\Theta) &= c(\Omega)\cdot\prod_{\mathfrak{p}\in A(\chi)}(1+\Omega(\Pi_\mathfrak{p})\,q_\mathfrak{p}^{1-l/2})\\ &\quad\times\frac{L(\chi\Omega;(l-1)/2)}{L(\omega\,\Omega_F;l-1)\,L(\Omega;l/2)}\cdot\overline{I(\Theta)}\,.\end{aligned}$$

Here $\Omega_F=\Omega|_{F_\mathbf{A}^\times}$ *and*

$$\begin{aligned}c(\Omega) &= \left\{\frac{(2\pi\sqrt{-1})^l}{(l-1)!}\right\}^n d_F^{-3/2}\,N_{F/\mathbf{Q}}(m^{l-2})\,N_{K/\mathbf{Q}}(\mathfrak{a}^{l/2-2}\,\kappa^{l/2-1}\,\mathcal{D}_{K/F}^{-l/2})\\ &\quad\times\mathbf{e}_m\left[\frac{\kappa}{2}\right]\prod_{\mathfrak{p}<\infty}\Omega_\mathfrak{p}^{-1}\left(b_\mathfrak{p}(\kappa)\,\alpha_\mathfrak{p}^\sigma\right)\,.\end{aligned}$$

§5. Algebraicity of Primitive Components of Eisenstein Series

5.1 Let $m,\mathfrak{a},\chi,\Omega$ be as in §4 and let $\Theta\in\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$. The object of this section is to give a criterion for the nonvanishing of the primitive component $((E_\Omega)^m_{\mathfrak{a}}, \Theta)$ and to prove its algebraicity under a suitable normalization of Θ. We begin with a formula for $|I(\Theta)|^2$, which is essentially due to T. Yang.

5.2 Theorem. *We have*

$$|I(\Theta)|^2 = B(\chi)\, L\left(\chi; \frac{1}{2}\right) (\Theta, \Theta)\,,$$

where

$$\begin{aligned} B(\chi) &= \frac{1}{2\,L(\omega;1)} \prod_{\mathfrak{p}\in A_1} (1+q_{\mathfrak{p}}^{-1})^{-1} \prod_{\mathfrak{p}\in A_2} (1-q_{\mathfrak{p}}^{-1})^{-1} \\ A_1 &= \{\mathfrak{p}\colon \text{unramified in}\, K/F,\ a(\chi_{\mathfrak{p}})>0\} \\ A_2 &= \{\mathfrak{p}\colon \text{split in}\, K/F,\ a(\chi_{\mathfrak{p}})>0\}\,. \end{aligned}$$

The theorem is proved by using Yang's arguments in [Yan] with a slight modification. In the course of the proof, we need the following formula for matrix coefficients of $\mathcal{M}_\chi$.

5.3 Lemma.

(i) *Let $\mathfrak{p}$ be a finite prime of F and $\Phi \in V^m_{\mathfrak{p},prim}(\mathfrak{a},\chi)$. For $t \in K^1_{\mathfrak{p}}$, we have*

$$(\mathcal{M}_\chi(t)\,\Phi, \Phi) = (\Phi,\Phi) \times \begin{cases} \mathrm{ch}_{\mathcal{O}^1_{K,\mathfrak{p}}}(t) & \cdots\ \ \mathfrak{p} \text{ split in } K/F,\ a(\chi_{\mathfrak{p}})>0 \\ q_{\mathfrak{p}}^{-|\operatorname{ord}_{\mathfrak{p}} z_t|/2}\,\chi_{\mathfrak{p}}(t) & \cdots\ \ \mathfrak{p} \text{ split in } K/F,\ a(\chi_{\mathfrak{p}})=0 \\ 1 & \cdots\ \ \text{otherwise}\,. \end{cases}$$

Here we write $t=(z_t, z_t^{-1})$ when $K_{\mathfrak{p}} = F_{\mathfrak{p}} \oplus F_{\mathfrak{p}}$.

(ii) *For $t_\infty \in K^1_\infty$, we have*

$$(\mathcal{M}_\chi(t_\infty)\,\Phi_{0,\infty}, \Phi_{0,\infty}) = (\Phi_{0,\infty}, \Phi_{0,\infty})\,.$$

We omit the proofs of Theorem 5.2 and Lemma 5.3. In view of Theorem 4.4 and Theorem 5.2, we have obtained the following criterion for the primitive components of E_Ω.

5.4 COROLLARY. *For* $\Theta \in \mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi) - \{0\}$*, we have*

$$((E_\Omega)^m_{\mathfrak{a}},\Theta) \neq 0 \iff I(\Theta) \neq 0 \iff L\left(\chi;\frac{1}{2}\right) \neq 0\,.$$

5.5 Let $m,\mathfrak{a}$ be as above and $\chi \in \mathcal{X}^+_{0,prim}(\mathfrak{a},m)$. If $L\left(\chi;\frac{1}{2}\right) \neq 0$, let $\Theta_{\mathfrak{a},\chi}$ be the element of $\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$ such that

$$I(\Theta_{\mathfrak{a},\chi}) = \pi^{-n}\mathbf{e}_m\left[-\frac{\kappa}{2}\right]\cdot\overline{L\left(\chi;\frac{1}{2}\right)}\,.$$

If $L\left(\chi;\frac{1}{2}\right) = 0$, we take any nonzero element $\Theta_{\mathfrak{a},\chi}$ of $\mathbf{T}^m_{hol,prim}(\mathfrak{a},\chi)$.

5.6 THEOREM. $((E_\Omega)^m_{\mathfrak{a}},\Theta_{\mathfrak{a},\chi})$ *is an algebraic number.*

PROOF. For $a,b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and $a/b \in \overline{\mathbf{Q}}$. Set

$$p(\chi,\Omega) = \frac{L(\chi\Omega;(l-1)/2)\,L(\chi;1/2)}{L(\Omega;l/2)}\,.$$

Since $L(\omega\Omega_F;l-1) \sim \pi^{(l-1)n}$, we see that $((E_\Omega)^m_{\mathfrak{a}},\Theta_{\mathfrak{a},\chi}) \sim p(\chi,\Omega)$ by Theorem 4.4. On the other hand, we have $p(\chi,\Omega) \sim 1$ by Shimura's results ([Shim1, Theorem 2], [Shim3, Theorem 1.1]; see also [Yo]). Thus we are done. □

REMARK. E_Ω is arithmetic in the sense of [Shim2] (see [Shim2, Theorem 5.3]).

§6. Fourier-Jacobi Expansion of Eisenstein Series

6.1 Let the notation be as in §4. Define a function $J^{\alpha_f}_\Omega$ on $K_{\mathbf{A}}$ by

$$J^{\alpha_f}_\Omega(w) = \int_{F_{\mathbf{A}}} \phi_\Omega(\Upsilon_0\,(\alpha_f^{-1}w,x))\,\psi_m(-N_{K/F}(\alpha_f)x)\,dx \qquad (w \in K_{\mathbf{A}})\,, \tag{6.1}$$

where dx is normalized as in §1.1 and

$$\Upsilon_0 = \begin{bmatrix} & & 1 \\ & 1 & \\ -1 & & \end{bmatrix} \in G_F \,.$$

The next lemma follows from the definition (4.4) of E_Ω and the Bruhat decomposition $G_F = P_F \cup P_F\, \Upsilon_0\, N_F$.

6.2 Lemma. *For $r = (w, 0)t \in R_{\mathbf{A}}$, we have*

$$(E_\Omega)^m_{\mathfrak{a}}(r) = \Omega(\alpha_f^\sigma)^{-1}\, |N_{K/F}(\alpha_f)|_{\mathbf{A}}^{1-l/2} \sum_{\xi\in K} \psi_m\left(\frac{1}{2}\langle \xi, w\rangle\right) J_\Omega^{\alpha_f}(t^{-1}(w+\xi))\,.$$

6.3 We have

$$J_\Omega^{\alpha_f}(w) = J_{\Omega_\infty}(w_\infty) \prod_{\mathfrak{p}} J_{\Omega_{\mathfrak{p}}}^{\alpha_{\mathfrak{p}}}(w_{\mathfrak{p}}) \qquad (w \in K_{\mathbf{A}})\,, \tag{6.2}$$

where

$$\begin{aligned} J_{\Omega_\infty}(w_\infty) &= \int_{F_\infty} \phi_{\Omega_\infty}(\Upsilon_0(w_\infty, x_\infty))\, \mathbf{e}_m[-x_\infty]\, dx_\infty \\ J_{\Omega_{\mathfrak{p}}}^{\alpha_{\mathfrak{p}}}(w_{\mathfrak{p}}) &= \int_{F_{\mathfrak{p}}} \phi_{\Omega_{\mathfrak{p}}}(\Upsilon_0(\alpha_{\mathfrak{p}}^{-1} w_{\mathfrak{p}}, x_{\mathfrak{p}}))\, \psi_{m,\mathfrak{p}}(-N_{K/F}(\alpha_{\mathfrak{p}}) x_{\mathfrak{p}})\, dx_{\mathfrak{p}}\,. \end{aligned}$$

Here we put $\Omega_\infty = \Omega|_{K_\infty^\times}$, $\phi_{\Omega_\infty} = \phi_\Omega|_{G_\infty}$, $\Omega_{\mathfrak{p}} = \Omega|_{K_{\mathfrak{p}}^\times}$ and $\phi_{\Omega_{\mathfrak{p}}} = \phi_\Omega|_{G_{\mathfrak{p}}}$. Set

$$c_\infty = \left\{ \frac{(2\pi\sqrt{-1})^l}{(l-1)\,!} \right\}^n N_{F/\mathbf{Q}}(m)^{l-1}\, |d_F|^{-1/2}\, \mathbf{e}_m[\kappa/2]\,. \tag{6.3}$$

6.4 Lemma. *We have*

$$J_{\Omega_\infty}(w_\infty) = c_\infty\, \Phi_{0,\infty}(w_\infty) \qquad (w_\infty \in K_\infty)\,.$$

PROOF. Since $\phi_{\Omega_\infty}(g) = j(g, Z_0)^{-l}$ for $g \in G_\infty$, we have

$$\begin{aligned} J_{\Omega_\infty}(w_\infty) &= \int_{F_\infty} j(\Upsilon_0(w_\infty, x_\infty), Z_0)^{-l}\, \mathbf{e}_m[-x_\infty]\, dx_\infty \\ &= |d_F|^{-1/2} \prod_{i=1}^{n} \int_{\mathbf{R}} \left\{ x_\infty^{(i)} + \frac{\kappa^{(i)}}{2}\,(w_\infty^{(i)}\overline{w_\infty^{(i)}} + 1) \right\}^{-l} \\ &\qquad\qquad \times \mathbf{e}[-m^{(i)} x_\infty^{(i)}]\, d' x_\infty^{(i)} \\ &= c_\infty\, \Phi_{0,\infty}(w_\infty) \qquad (w_\infty = (w_\infty^{(i)})_{1 \le i \le n} \in K_\infty)\,, \end{aligned}$$

where $d'x_\infty^{(i)}$ is the usual Lebesgue measure on $\mathbf{R}$. □

6.5 Let Φ be the element of $V^m_{hol,prim}(\mathfrak{a}, \chi)$ such that $\Theta = \mathcal{T}^m_\chi(\Phi)$. Then

$$\Phi(w) = \Phi_{0,\infty}(w_\infty) \prod_{\mathfrak{p}<\infty} \Phi_{\mathfrak{p}}(w_{\mathfrak{p}}) \qquad (w \in K_{\mathbf{A}})\,, \tag{6.4}$$

where $\Phi_{0,\infty}$ is defined in §2.6 and $\Phi_{\mathfrak{p}} \in V^m_{\mathfrak{p},prim}(\mathfrak{a}, \chi)$. We put

$$W^{\alpha_f}_{\Omega,\Phi}(z) = \int_{K_{\mathbf{A}}} \overline{J^{\alpha_f}_{\Omega}(w)}\, \rho(w,0)\, \Phi(z)\, dw \qquad (z \in K_{\mathbf{A}})\,. \tag{6.5}$$

Later we will see that $W^{\alpha_f}_{\Omega,\Phi}$ is not in V^m, but in $\widetilde{V}^m$ (cf. Proposition 10.2).

6.6 LEMMA. *We have*

$$(\Theta, (E_\Omega)^m_{\mathfrak{a}}) = \Omega(\alpha_f^\sigma)\, |N_{K/F}(\alpha_f)|_{\mathbf{A}}^{1-l/2} \int_{K^1\backslash K^1_{\mathbf{A}}} \mathcal{T}^m_\chi(W^{\alpha_f}_{\Omega,\Phi})(t)\, d^\times t\,.$$

PROOF. By Lemma 6.2, we have

$$\begin{aligned} &\Omega(\alpha_f^\sigma)^{-1} |N_{K/F}(\alpha_f)|_{\mathbf{A}}^{l/2-1} \cdot (\Theta, (E_\Omega)^m_{\mathfrak{a}}) \\ &\quad = \int_{K\backslash K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t\, \Theta((w,0)t) \\ &\qquad \times \sum_{\xi \in K} \psi_m\left(-\frac{1}{2}\langle \xi, w\rangle\right) \overline{J^{\alpha_f}_{\Omega}(t^{-1}(w+\xi))} \end{aligned}$$

$$
\begin{aligned}
&= \int_{K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t\, \Theta((w,0)t)\, \overline{J^{\alpha_f}_{\Omega}(t^{-1}w)} \\
&= \int_{K_{\mathbf{A}}} dw \int_{K^1\backslash K^1_{\mathbf{A}}} d^\times t\, \Theta(t(w,0))\, \overline{J^{\alpha_f}_{\Omega}(w)} \\
&= \int_{K^1\backslash K^1_{\mathbf{A}}} \mathcal{T}^m_\chi(W^{\alpha_f}_{\Omega,\Phi})(t)\, d^\times t\,,
\end{aligned}
$$

which proves the lemma. □

6.7 In view of (6.2) and (6.4), $W^{\alpha_f}_{\Omega,\Phi}$ is decomposed as

$$
W^{\alpha_f}_{\Omega,\Phi}(z) = W_\infty(z_\infty) \prod_{\mathfrak{p}<\infty} W^{\alpha_\mathfrak{p}}_{\Omega_\mathfrak{p},\Phi_\mathfrak{p}}(z_\mathfrak{p}) \qquad (z \in K_{\mathbf{A}})\,,
$$

where

$$
W_\infty(z_\infty) = \int_{K_\infty} \overline{J_{\Omega_\infty}(w_\infty)}\, \rho(w_\infty,0)\, \Phi_{0,\infty}(z_\infty)\, dw_\infty \quad (z_\infty \in K_\infty) \tag{6.6}
$$

$$
W^{\alpha_\mathfrak{p}}_{\Omega_\mathfrak{p},\Phi_\mathfrak{p}}(z_\mathfrak{p}) = \int_{K_\mathfrak{p}} \overline{J^{\alpha_\mathfrak{p}}_{\Omega_\mathfrak{p}}(w_\mathfrak{p})}\, \rho(w_\mathfrak{p},0)\, \Phi_\mathfrak{p}(z_\mathfrak{p})\, dw_\mathfrak{p} \quad (z_\mathfrak{p} \in K_\mathfrak{p})\,. \tag{6.7}
$$

6.8 Lemma. *We have*

$$
W_\infty(z_\infty) = c_\infty \cdot \Phi_{0,\infty}(z_\infty) \qquad (z_\infty \in K_\infty)\,.
$$

Proof. This follows from Lemma 6.4 and Lemma 2.3. □

§7. Local Calculation: (I) Preparation

7.1 In §§7–9, we fix a finite prime $\mathfrak{p}$ of F and calculate the local factor $W^{\alpha_\mathfrak{p}}_{\Omega_\mathfrak{p},\Phi_\mathfrak{p}}$ defined by (6.7). To simplify the notation, we often omit the subscript $\mathfrak{p}$ in these sections. For example, we write $F, K, N,$ $|\cdot|_F, \mathrm{ord}_F, \pi, \psi_m, \omega, \delta_{K/F}$ and V^m for $F_\mathfrak{p}, K_\mathfrak{p}, N_\mathfrak{p}, |\cdot|_\mathfrak{p}, \mathrm{ord}_\mathfrak{p}, \pi_\mathfrak{p}, \psi_{m,\mathfrak{p}},$ $\omega_\mathfrak{p}, \delta_{K_\mathfrak{p}/F_\mathfrak{p}}$ and $V^m_\mathfrak{p}$. For $z \in K$, we write $N(z)$ for $N_{K/F}(z)$ if there is no fear of confusion. We put $q = \#(\mathcal{O}_F/\pi\mathcal{O}_F)$. Denote by τ_0 (resp. τ) the characteristic function of $\mathcal{O}_F$ (resp. $\mathcal{O}_K$). From now on, we fix an element θ of K satisfying

(i) $\{1,\theta\}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_K$.

(ii) θ is a prime element of K when K/F is ramified.

We have

$$\delta_{K/F} = \mathrm{ord}_F\, N(\theta - \theta^\sigma)\,. \tag{7.1}$$

Set

$$\nu = -\frac{\theta^\sigma}{\theta - \theta^\sigma}\,. \tag{7.2}$$

Then ν satisfies the conditions in §1.15.

7.2 Let $\mathfrak{a} = \alpha\mathcal{O}_K$ be an ideal of K. Note that

$$\mu(\mathfrak{a}, m) := \mathrm{ord}_F\ \mathfrak{f}(\mathfrak{a}, m) = \mathrm{ord}_F(b(\kappa)N(\alpha)) + n(\psi_m)\,, \tag{7.3}$$

where we put

$$b(\kappa) = \frac{\kappa}{\theta - \theta^\sigma} \in F^\times\,. \tag{7.4}$$

We also note that

$$d_{\mathfrak{a}} z = q^{\mu(\mathfrak{a},m)+\delta_{K/F}}\, dz \tag{7.5}$$

(recall that $d_{\mathfrak{a}} z$ is normalized by $\mathrm{vol}(\mathfrak{a}) = 1$).

7.3 Let ξ (resp. Ξ) be a character of $F^\times$ (resp. $K^\times$). As usual, we define the local L-factors attached to ξ and Ξ by

$$L(\xi; s) = \begin{cases} (1 - \xi(\pi)q^{-s})^{-1} & \cdots \quad \xi|_{\mathcal{O}_F^\times} = 1 \\ 1 & \cdots \quad \text{otherwise} \end{cases}$$

and

$$L(\Xi; s) = \begin{cases} (1 - \Xi(\Pi)|N(\Pi)|_F^s)^{-1} & \cdots \quad K \text{ is a field and } \Xi|_{\mathcal{O}_K^\times} = 1 \\ \prod_{i=1,2} (1 - \Xi(\Pi_i)q^{-s})^{-1} & \cdots \quad K = F \oplus F \text{ and } \Xi|_{\mathcal{O}_K^\times} = 1 \\ 1 & \cdots \quad \Xi|_{\mathcal{O}_K^\times} \neq 1\,. \end{cases}$$

Here Π is a prime element of K when K is a field, and $\Pi_1 = (\pi, 1), \Pi_2 = (1, \pi)$ when $K = F \oplus F$.

7.4 Let $C^\infty(K)$ be the space of locally constant functions on K. Define a smooth representation ρ of N on $C^\infty(K)$ by

$$\rho(w,x)\, f(z) = \psi_m\left(\frac{1}{2}\,\langle z, w\rangle + x\right) f(z+w) \quad (f \in C^\infty(K), z \in K). \tag{7.6}$$

Recall that $\rho(N)V^m \subset V^m$ and that ρ defines a smooth irreducible representation of N on V^m.

For an ideal $\mathfrak{a}$ of K, we define an endomorphism $\mathcal{P}_{\mathfrak{a}}$ of $C^\infty(K)$ by

$$\mathcal{P}_{\mathfrak{a}}\, f = \int_{N(\mathfrak{a})} \rho(n)\, f\, d_{\mathfrak{a}} n \quad (f \in C^\infty(K)),$$

where $d_{\mathfrak{a}}n$ is normalized by $\mathrm{vol}(N(\mathfrak{a})) = 1$. If $\mu(\mathfrak{a}, m) \geq 0$, we have

$$\mathcal{P}_{\mathfrak{a}}\, f = \int_{\mathfrak{a}} \rho(w, x_w)\, f\, d_{\mathfrak{a}} w,$$

where

$$x_w = \frac{\kappa}{2}\,(\nu - \nu^\sigma) w w^\sigma. \tag{7.7}$$

If $\mu(\mathfrak{a}, m) < 0$, we have $\mathcal{P}_{\mathfrak{a}}\, f = 0$.

7.5 For $f \in C^\infty(K)$ and $\Phi \in V^m$, we put

$$\Lambda(f, \Phi)(z) = \int_K \overline{f(w)}\, (\rho(w,0)\,\Phi)\,(z)\, dw \qquad (z \in K). \tag{7.8}$$

The integral (7.8) is absolutely convergent for every $z \in K$ and defines an element of $C^\infty(K)$. The following is easily verified.

7.6 Lemma.

(i) *For* $f \in C^\infty(K), \Phi \in V^m$ *and* $w \in K$, *we have*

$$\Lambda(\overline{\rho(w,0) f}, \Phi) = \rho(-w, 0)\, \Lambda(\overline{f}, \Phi).$$

(ii) *For an ideal* $\mathfrak{a}$ *of* K, *we have*

$$\Lambda(f, \mathcal{P}_{\mathfrak{a}}\Phi) = \Lambda(\mathcal{P}_{\mathfrak{a}} f, \Phi) \qquad (f \in C^\infty(K), \Phi \in V^m).$$

7.7 Until the end of §9, we fix an ideal $\mathfrak{a} = \alpha\,\mathcal{O}_K$ with $\mu(\mathfrak{a}, m) \geq 0$, $\Omega \in (K^\times)^\wedge$ with $\Omega|_{\mathcal{O}_K^\times} = 1$, $\chi \in \mathcal{X}^+_{prim}(\mathfrak{a}, m)$ and $\Phi \in V^m(\mathfrak{a}, \chi) = V^m_{prim}(\mathfrak{a}, \chi)$. We write μ for $\mu(\mathfrak{a}, m)$ if there is no fear of confusion. Set

$$W^*(z) = \overline{\Omega}(b(\kappa))\,|b(\kappa)|_F^l\, L_{\mathfrak{p}}\Big(\overline{\Omega}\,; \frac{l}{2}\Big) \cdot W^\alpha_{\Omega,\Phi}(z) \qquad (z \in K) \tag{7.9}$$

and

$$\begin{aligned} J^*(w) &= \Omega(b(\kappa))\,|b(\kappa)|_F^l\, L_{\mathfrak{p}}\Big(\Omega; \frac{l}{2}\Big) \cdot J^\alpha_\Omega(w) \\ &= \int_F \phi^*_\Omega(\Upsilon_0(\alpha^{-1}w, x))\psi_m(-N(\alpha)x)\,dx \qquad (w \in K)\,, \end{aligned} \tag{7.10}$$

where

$$\phi^*_\Omega(g) = \Omega(b(\kappa))\,|b(\kappa)|_F^l\, L_{\mathfrak{p}}\Big(\Omega; \frac{l}{2}\Big) \cdot \phi_\Omega(g) \quad (g \in G)\,.$$

In view of (6.5), we have

$$W^*(z) = \Lambda(J^*, \Phi)(z)\,. \tag{7.11}$$

To calculate $W^*(z)$, we need the following integral expression of J^*.

7.8 LEMMA. *For* $w \in K$, *we have*

$$\begin{aligned} J^*(w) &= |N(\alpha)|_F^{-1}\,\psi_m(-x_w) \int_{K^\times} d^\times z \int_F dx\; \psi_m(-x)\,\Omega(z)\,|N(z)|_F^{l/2} \\ &\qquad \tau(z)\,\tau(\alpha^{-1}wz^\sigma)\,\tau\Big(b(\kappa)^{-1}N(\alpha)^{-1}(x - b(\kappa)ww^\sigma\theta^\sigma)z\Big)\,, \end{aligned}$$

where $d^\times z$ *is normalized by* $\mathrm{vol}(\mathcal{O}_K^\times) = 1$.

PROOF. Put $L' = {}^tL \cdot (\kappa S) = \mathcal{O}_K \oplus \kappa\mathcal{O}_K \oplus b(\kappa)\mathcal{O}_K$. Then L' is $\mathcal{K}$-invariant under right multiplication. By a standard argument, we have

$$\phi^*_\Omega(g) = \int_{K^\times} \mathrm{ch}_{L'}\Big((0,0,z)\,g\Big)\,\Omega(z)\,|N(z)|_F^{l/2}\,d^\times z \qquad (g \in G)$$

and hence

$$J^*(w) = \int_{K^\times} d^\times z \int_F dx\, \psi_m(-N(\alpha)x)\, \Omega(z)\, |N(z)|_F^{l/2}$$
$$\tau(z)\tau(\alpha^{-1}wz^\sigma)\tau\left(b(\kappa)^{-1}(x+\frac{\kappa}{2}\, N(\alpha)^{-1}ww^\sigma)z\right).$$

Changing the variable x into $N(\alpha)^{-1}(x+x_w)$, we obtain the assertion of the lemma. □

§8. Local Calculation: (II) The Case Where K is a Field

8.1 In this section, we calculate W^* when K is a field. We first consider the unramified case.

8.2 Lemma. *Suppose that K/F is an unramified quadratic extension. Then we have*

$$J^*(w) = |b(\kappa)|_F\, \psi_m(-x_w) \sum_{k=0}^{\mu} (\Omega(\pi)q^{1-l})^k\, \tau(\pi^{[k/2]}\alpha^{-1}w) \qquad (w \in K).$$

Proof. By Lemma 7.8, we have

$$J^*(w) = |N(\alpha)|_F^{-1}\, \psi_m(-x_w) \sum_{k=0}^{\infty} (\Omega(\pi)q^{-l})^k\, I_k(w),$$

where

$$I_k(w) = \tau(\pi^k\alpha^{-1}w) \int_F \tau\Big(b(\kappa)^{-1}N(\alpha)^{-1}\pi^k\,(x - b(\kappa)ww^\sigma\theta^\sigma)\Big)\, \psi_m(-x)\, dx.$$

Since $\{1,\theta^\sigma\}$ is an $\mathcal{O}_F$-basis of $\mathcal{O}_K$, we have

$$I_k(w) = \tau(\pi^k\alpha^{-1}w)\, \tau_0(\pi^k N(\alpha^{-1}w)) \int_F \tau_0(\pi^{k-\mu+n(\psi_m)}x)\, \psi_m(-x)\, dx$$
$$= \tau(\pi^{[k/2]}\alpha^{-1}w) \times \begin{cases} q^{k-\mu+n(\psi_m)} & \cdots \quad 0 \le k \le \mu \\ 0 & \cdots \quad k > \mu. \end{cases}$$

This proves the lemma, since $q^{-\mu+n(\psi_m)} = |b(\kappa)N(\alpha)|_F$. □

8.3 PROPOSITION. *Suppose that K/F is an unramified quadratic extension. Then we have*

$$W^* = q^{-\mu-\delta_{K/F}}\,|b(\kappa)|_F\,\frac{L\left(\overline{\chi\Omega}\,;(l-1)/2\right)}{L\left(\overline{\omega\Omega_F}\,;l-1\right)}\,\Phi\,.$$

PROOF. In view of (7.11) and Lemma 8.2, we have

$$\begin{aligned} W^* &= |b(\kappa)|_F \sum_{k=0}^{\mu}\left(\overline{\Omega}(\pi)q^{1-l}\right)^k \int_K \tau(\pi^{[k/2]}\alpha^{-1}w)\,\rho(w,x_w)\,\Phi\,dw \\ &= |b(\kappa)|_F \sum_{k=0}^{\mu}\left(\overline{\Omega}(\pi)q^{1-l}\right)^k q^{2[k/2]-\mu-\delta_{K/F}}\,\mathcal{P}_{\pi^{-[k/2]}\mathfrak{a}}\,\Phi\,. \end{aligned}$$

The primitivity of Φ implies that

$$W^* = q^{-\mu-\delta_{K/F}}\,|b(\kappa)|_F\cdot\Phi\times\begin{cases} 1 & \cdots \quad \mu=0 \\ 1+\overline{\Omega}(\pi)\,q^{1-l} & \cdots \quad \mu>0\,. \end{cases}$$

Observe that $\chi(\pi)=\omega(\pi)=-1$ and that χ is trivial on $\mathcal{O}_K^\times$ if and only if $\mu=0$. It follows that $L(\overline{\omega\Omega_F};l-1)=(1+\overline{\Omega}(\pi)\,q^{1-l})^{-1}$ and

$$L\left(\overline{\chi\Omega};\frac{l-1}{2}\right)=\begin{cases} (1+\overline{\Omega}(\pi)\,q^{1-l})^{-1} & \cdots \quad \mu=0 \\ 1 & \cdots \quad \mu>0\,. \end{cases}$$

These prove the proposition. □

8.4 Next suppose that K/F is a ramified quadratic extension and let Π be a prime element of K. Define an endomorphism $\mathcal{Q}$ of V^m by

$$\mathcal{Q}\,\Psi = \int_{\mathfrak{a}'} \rho(w,x_w)\,\Psi\,d_{\mathfrak{a}'}w \quad (\Psi\in V^m)\,,$$

where $\mathfrak{a}'=\Pi^{-\mu-1}\mathfrak{a}$.

8.5 LEMMA. *Let $\Phi\in V^m_{prim}(\mathfrak{a},\chi)$.*

(i) *If $\mu>0$, we have $\mathcal{Q}\,\Phi=0$.*

(ii) *If* $\mu = 0$, *we have*

$$\mathcal{Q}\,\Phi = \begin{cases} 0 & \cdots \quad a(\chi) = 2\delta_{K/F} \\ \Phi & \cdots \quad a(\chi) = 2\delta_{K/F} - 1\,. \end{cases}$$

PROOF. First suppose that $\mu > 0$. A straightforward calculation shows that $\mathcal{Q} = \mathcal{Q}\,\mathcal{P}_{\Pi^{-1}\mathfrak{a}}$ and hence $\mathcal{Q}\,\Phi = 0$ by the primitivity of Φ. The second assertion follows from Theorem 3.8 (ii). $\square$

8.6 LEMMA. *Suppose that* K/F *is a ramified quadratic extension. Then we have*

$$J^*(w) = |b(\kappa)|_F\,\psi_m(-x_w)\sum_{k=0}^{2\mu+1} q^{[k/2]}\,(\Omega(\Pi)\,q^{-l/2})^k\,\tau(\Pi^{[(k+1)/2]}\alpha^{-1}w)$$

for $w \in K$.

PROOF. By Lemma 7.8, we have

$$J^*(w) = |N(\alpha)|_F^{-1}\,\psi_m(-x_w)\sum_{k=0}^{\infty}(\Omega(\Pi)\,q^{-l/2})^k\,I_k(w) \quad (w \in K)\,,$$

where

$$I_k(w) = \tau(\Pi^k\alpha^{-1}w)\int_F \tau\left(b(\kappa)^{-1}N(\alpha)^{-1}\Pi^k(x - b(\kappa)ww^\sigma\theta^\sigma)\right)\,\psi_m(-x)\,dx\,.$$

First suppose that $k = 2k'$ is even. Since $\Pi^k \in \pi^{k'}\mathcal{O}_K^\times$, we have

$$\begin{aligned} I_k(w) &= \tau(\pi^{k'}\alpha^{-1}w)\,\tau_0(\pi^{k'}N(\alpha^{-1}w))\int_F \tau_0(\pi^{k'-\mu+n(\psi_m)}\,x)\,\psi_m(-x)\,dx \\ &= \tau(\Pi^{k'}\alpha^{-1}w) \times \begin{cases} q^{k'-\mu+n(\psi_m)} & \cdots \quad 0 \le k' \le \mu \\ 0 & \cdots \quad k' > \mu\,. \end{cases} \end{aligned}$$

We next consider the case where $k = 2k' + 1$ is odd. Recall that we have assumed that θ is a prime element of K. It follows that $\Pi^k \in \pi^{k'}\theta\,\mathcal{O}_K^\times$ and hence

$$\begin{aligned} I_k(w) &= \tau(\Pi^{2k'+1}\alpha^{-1}w)\,\tau_0(\pi^{k'+1}N(\alpha^{-1}w)) \\ &\quad \times \int_F \tau_0(\pi^{k'-\mu+n(\psi_m)}\,x)\,\psi_m(-x)\,dx \\ &= \tau(\Pi^{k'+1}\alpha^{-1}w) \times \begin{cases} q^{k'-\mu+n(\psi_m)} & \cdots \quad 0 \le k' \le \mu \\ 0 & \cdots \quad k' > \mu\,. \end{cases} \end{aligned}$$

These yield the required result. □

8.7 Proposition. *Suppose that K/F is a ramified quadratic extension. Then we have*

$$W^* = q^{-\mu-\delta_{K/F}} |b(\kappa)|_F \frac{L\left(\overline{\chi\Omega}\,;(l-1)/2\right)}{L\left(\overline{\omega\Omega_F}\,;l-1\right)} \times \left\{ \begin{array}{lll} 1+\overline{\Omega}(\Pi)q^{1-l/2} & \cdots & \mu=0, a(\chi)=2\delta_{K/F}-1 \\ 1 & \cdots & \text{otherwise} \end{array} \right\} \times \Phi\,.$$

Proof. By an argument similar to the proof of Proposition 8.3, we have

$$\begin{aligned} W^* &= |b(\kappa)|_F \sum_{k=0}^{2\mu+1} q^{[k/2]} \left(\overline{\Omega}(\Pi)q^{-l/2}\right)^k \int_{\Pi^{-[(k+1)/2]}\mathfrak{a}} \rho(w,x_w)\,\Phi\,dw \\ &= |b(\kappa)|_F \Big\{ \sum_{k=0}^{2\mu} q^{[k/2]+[(k+1)/2]-\mu-\delta_{K/F}} \left(\overline{\Omega}(\Pi)q^{-l/2}\right)^k \mathcal{P}_{\Pi^{-[(k+1)/2]}\mathfrak{a}}\,\Phi \\ &\quad + q^{\mu+1-\delta_{K/F}} (\overline{\Omega}(\Pi)\,q^{-l/2})^{2\mu+1}\,\mathcal{Q}\,\Phi \Big\}\,. \end{aligned}$$

If $\mu>0$, the primitivity of Φ and Lemma 8.5 (i) imply that

$$W^* = q^{-\mu-\delta_{K/F}} |b(\kappa)|_F\,\Phi\,.$$

If $\mu=0$, Lemma 8.5 (ii) implies

$$W^* = q^{-\mu-\delta_{K/F}} |b(\kappa)|_F \times \left\{ \begin{array}{lll} 1 & \cdots & a(\chi)=2\delta_{K/F} \\ 1+\overline{\Omega}(\Pi)\,q^{1-l/2} & \cdots & a(\chi)=2\delta_{K/F}-1 \end{array} \right\} \times \Phi\,.$$

On the other hand, we have $L(\overline{\chi\Omega};s) = L(\omega\Omega_F;s) = 1$, since $\chi\Omega$ (resp. $\omega\Omega_F$) is nontrivial on $\mathcal{O}_K^\times$ (resp. $\mathcal{O}_F^\times$). This completes the proof of the proposition. □

§9. Local Calculation: (III) The Case Where $K = F\oplus F$

9.1 In this section, we calculate W^* when $K=F\oplus F$. We take $\theta=(1,0)$. Then we have $x_w = -\dfrac{1}{2}\,b(\kappa)\,N(w)$ and $\langle z,w\rangle = b(\kappa)(z_2w_1 - z_1w_2)$

for $z=(z_1,z_2), w=(w_1,w_2)\in K$. Set $\xi_i=\Omega(\Pi_i)\,q^{-l/2}$ $(i=1,2)$, where $\Pi_1=(\pi,1)$ and $\Pi_2=(1,\pi)$. We let $\alpha=(\alpha_1,\alpha_2)\in K^\times$.

9.2 Lemma. *For $w\in K$, we have*

$$J^*(w)=|b(\kappa)|_F\sum_{k_1,k_2=0}^{\infty}\xi_1^{k_1}\xi_2^{k_2}\,I_{k_1,k_2}(w)\,,$$

where

$$\begin{aligned}&I_{k_1,k_2}(w)\\&\quad=\tau\left(\Pi_1^{k_2}\Pi_2^{k_1}\alpha^{-1}w\right)\\&\qquad\times\begin{cases}q^{k_1}\,\tau_0(\pi^{k_2}N(\alpha^{-1}w))\,\psi_m(-x_w) & \cdots\quad k_1\le k_2 \text{ and } k_1\le\mu\\ q^{k_2}\,\tau_0(\pi^{k_1}N(\alpha^{-1}w))\,\psi_m(x_w) & \cdots\quad k_1\ge k_2 \text{ and } k_2\le\mu\\ 0 & \cdots\quad \text{otherwise}\,.\end{cases}\end{aligned}$$

Proof. By Lemma 7.8, we obtain

$$J^*(w)=|N(\alpha)|_F^{-1}\,\psi_m(-x_w)\sum_{k_1,k_2=0}^{\infty}\xi_1^{k_1}\xi_2^{k_2}\,\tau\left(\Pi_1^{k_2}\Pi_2^{k_1}\alpha^{-1}w\right)\,I'_{k_1,k_2}(w)\,,$$

where

$$I'_{k_1,k_2}(w)=\int_F\tau_0(\pi^{k_1-\mu+n(\psi_m)}x)\,\tau_0(\pi^{k_2-\mu+n(\psi_m)}(x+2x_w))\,\psi_m(-x)\,dx\,.$$

First suppose that $k_1\le k_2$. Then

$$\begin{aligned}I'_{k_1,k_2}(w)&=\tau_0(\pi^{k_2-\mu+n(\psi_m)}\cdot 2x_w)\int_F\tau_0(\pi^{k_1-\mu+n(\psi_m)}x)\,\psi_m(-x)\,dx\\&=\tau_0(\pi^{k_2}N(\alpha^{-1}w))\times\begin{cases}q^{k_1}|b(\kappa)N(\alpha)|_F & \cdots\quad 0\le k_1\le\mu\\ 0 & \cdots\quad k_1>\mu\,.\end{cases}\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}I'_{k_1,k_2}(w)&=\psi_m(2x_w)\,\tau_0(\pi^{k_1}N(\alpha^{-1}w))\\&\qquad\times\begin{cases}q^{k_2}|b(\kappa)N(\alpha)|_F & \cdots\quad 0\le k_2\le\mu\\ 0 & \cdots\quad k_2>\mu\end{cases}\end{aligned}$$

if $k_1 \geq k_2$. Hence the lemma has been proved. □

9.3 For $(k, k') \in (\mathbf{Z}_{\geq 0})^2$, put

$$\eta^{\pm}_{k,k'}(w) = \tau\left(\Pi_1^k \Pi_2^{k'} \alpha^{-1} w\right) \psi_m(\pm x_w) \quad (w \in K).$$

Note that $\eta^{+}_{k,k'} = \eta^{-}_{k,k'}$ if $k + k' \leq \mu$, in which case we write simply $\eta_{k,k'}$ for $\eta^{\pm}_{k,k'}$. Let $\mathcal{N}$ be the subspace of $\mathcal{S}(K)$ spanned by $\eta^{\pm}_{k,k'}$ ($(k, k') \in (\mathbf{Z}_{\geq 0})^2 - \{(0,0)\}$).

9.4 LEMMA.

(i) $I_{0,0} = \eta_{0,0}$.

(ii) *If* $\mu \geq 1$, *then* $I_{1,1} = q\,(-\eta_{0,0} + \eta_{1,0} + \eta_{0,1}) \in -q\,\eta_{0,0} + \mathcal{N}$.

(iii) *If* $(k_1, k_2) \neq (0,0), (1,1)$, *then* $I_{k_1,k_2} \in \mathcal{N}$.

PROOF. The first and second assertions are immediate from the definition of I_{k_1,k_2}. We show (iii) in the case $k_2 \geq k_1$ by induction on k_1. Let $\alpha^{-1}w = (w_1', w_2')$. First suppose that $k_2 > k_1 = 0$. Then $I_{k_1,k_2}(w) = \tau_0(\pi^{k_2} w_1')\,\tau_0(w_2')\,\psi_m(-x_w) = \eta^{-}_{k_2,0}(w) \in \mathcal{N}$. Next suppose that $k_2 \geq k_1 > 0$ and $(k_1, k_2) \neq (1,1)$. If $k_1 > \mu$, we have $I_{k_1,k_2} = 0$. Assume $k_1 \leq \mu$. Then

$$\begin{aligned} & q^{-k_1}\, I_{k_1,k_2}(w) \\ = \;& \tau_0(\pi^{k_2} w_1')\,\tau_0(\pi^{k_1} w_2')\,\tau_0(\pi^{k_2} w_1' w_2')\,\psi_m(-x_w) \\ = \;& \tau_0(\pi^{k_2} w_1') \left\{\tau_0(\pi^{k_1} w_2') - \tau_0(\pi^{k_1-1} w_2')\right\} \tau_0(\pi^{k_2} w_1' w_2')\,\psi_m(-x_w) \\ & + \tau_0(\pi^{k_2} w_1')\,\tau_0(\pi^{k_1-1} w_2')\,\tau_0(\pi^{k_2} w_1' w_2')\,\psi_m(-x_w)\,. \end{aligned}$$

Since $\tau_0(\pi^{k_1} w_2') - \tau_0(\pi^{k_1-1} w_2') = \delta(\pi^{k_1} w_2' \in \mathcal{O}_F^{\times})$, we have

$$\begin{aligned} & q^{-k_1}\, I_{k_1,k_2}(w) \\ = \;& \tau_0(\pi^{k_2-k_1} w_1') \left\{\tau_0(\pi^{k_1} w_2') - \tau_0(\pi^{k_1-1} w_2')\right\} \psi_m(-x_w) \\ & + q^{1-k_1}\, I_{k_1-1,k_2}(w) \\ = \;& \eta^{-}_{k_2-k_1,k_1}(w) - \eta^{-}_{k_2-k_1,k_1-1}(w) + q^{1-k_1}\, I_{k_1-1,k_2}(w)\,. \end{aligned}$$

In view of the assumption $(k_1, k_2) \neq (1,1)$, we have $\eta^-_{k_2-k_1,k_1} - \eta^-_{k_2-k_1,k_1-1} \in \mathcal{N}$ and hence $I_{k_1,k_2} \in \mathcal{N}$ by the induction hypothesis. We can show (iii) in the case $k_2 < k_1$ in a similar manner. □

The following fact is easily verified and we omit its proof.

9.5 LEMMA. *Let $k, k' \in \mathbf{Z}_{\geq 0}$ and $\epsilon = \pm$.*

(i) *We have*

$$\mathcal{P}_{\mathfrak{a}}\, \eta^\epsilon_{k,k'}(w) = \eta^\epsilon_{k,k'}(w) \times \begin{cases} \tau_0(\pi^\mu \alpha_1^{-1} w_1) & \cdots \quad \epsilon = + \\ \tau_0(\pi^\mu \alpha_2^{-1} w_2) & \cdots \quad \epsilon = - . \end{cases}$$

(ii) *If $\mu > 0$ and $k > 0$, we have*

$$\mathcal{P}_{\Pi_1^{-1}\mathfrak{a}}\, \eta^\epsilon_{k,k'}(w) = \eta^\epsilon_{k,k'}(w) \times \begin{cases} \tau_0(\pi^\mu \alpha_1^{-1} w_1) & \cdots \quad \epsilon = + \\ \tau_0(\pi^{\mu-1} \alpha_2^{-1} w_2) & \cdots \quad \epsilon = - . \end{cases}$$

(iii) *If $\mu > 0$ and $k' > 0$, we have*

$$\mathcal{P}_{\Pi_2^{-1}\mathfrak{a}}\, \eta^\epsilon_{k,k'}(w) = \eta^\epsilon_{k,k'}(w) \times \begin{cases} \tau_0(\pi^{\mu-1} \alpha_1^{-1} w_1) & \cdots \quad \epsilon = + \\ \tau_0(\pi^\mu \alpha_2^{-1} w_2) & \cdots \quad \epsilon = - . \end{cases}$$

9.6 LEMMA. *Let $\Phi \in V^m_{prim}(\mathfrak{a}, \chi)$.*

(i) *We have $\Lambda(\eta_{0,0}, \Phi) = q^{-\mu}\, \Phi$.*

(ii) *If $\mu \geq 1$, we have $\Lambda(\eta, \Phi) = 0$ for $\eta \in \mathcal{N}$.*

PROOF. The first assertion is immediate from the fact that $\Lambda(\eta_{0,0}, \Phi) = q^{-\mu}\, \mathcal{P}_{\mathfrak{a}}\, \Phi$. Suppose that $\mu \geq 1$. The primitivity of Φ implies that $\Phi = (\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_1^{-1}\mathfrak{a}})\, \Phi = (\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_2^{-1}\mathfrak{a}})\, \Phi$. If $k > 0$, we have

$$\begin{aligned} & \left(\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_1^{-1}\mathfrak{a}}\right) \eta^\epsilon_{k,0}(w) \\ = \;& \eta^\epsilon_{k,0}(w) \times \begin{cases} 0 & \cdots \quad \epsilon = + \\ \tau_0(\pi^\mu \alpha_2^{-1} w_2) - \tau_0(\pi^{\mu-1} \alpha_2^{-1} w_2) & \cdots \quad \epsilon = - \end{cases} \\ = \;& 0 \end{aligned}$$

by Lemma 9.5 and hence $\Lambda(\eta^{\epsilon}_{k,0},\Phi) = \Lambda(\eta^{\epsilon}_{k,0},(\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_1^{-1}\mathfrak{a}})\Phi) = \Lambda((\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_1^{-1}\mathfrak{a}})\eta^{\epsilon}_{k,0},\Phi) = 0$. A simialr argument shows that $\Lambda(\eta^{\epsilon}_{0,k'},\Phi) = 0$ for $k' > 0$. Finally let $k, k' > 0$. By Lemma 9.5, we have $(\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_1^{-1}\mathfrak{a}})\,\eta^{+}_{k,k'} = 0$ and $(\mathcal{P}_{\mathfrak{a}} - \mathcal{P}_{\Pi_2^{-1}\mathfrak{a}})\,\eta^{-}_{k,k'} = 0$, which imply $\Lambda(\eta^{\pm}_{k,k'},\Phi) = 0$. □

9.7 PROPOSITION. *Suppose that* $K = F \oplus F$ *and* $\mu \geq 1$. *Then*

$$W^* = q^{-\mu-\delta_{K/F}}\,|b(\kappa)|_F\,\frac{L\left(\overline{\chi\Omega};(l-1)/2\right)}{L\left(\overline{\omega\Omega_F};l-1\right)}\,\Phi\,.$$

PROOF. By (7.11), Lemma 9.2, Lemma 9.4 and Lemma 9.6, we obtain

$$\begin{aligned} W^* &= |b(\kappa)|_F\,\Lambda((1-q\xi_1\xi_2)\eta_{0,0},\Phi) \\ &= |b(\kappa)|_F\,q^{-\mu}\left(1-q\overline{\xi_1\xi_2}\right)\Phi \\ &= |b(\kappa)|_F\,q^{-\mu}\,L(\overline{\omega\Omega_F}\,;l-1)^{-1}\Phi \end{aligned}$$

(note that ω is trivial when $K = F \oplus F$). Since $\mu \geq 1$, $\chi\Omega$ is nontrivial on $\mathcal{O}_K^{\times}$ and hence $L\left(\overline{\chi\Omega}\,;s\right) = 1$, which proves the proposition. □

9.8 Until the end of this section, we assume that $\mu = 0$. In this case, by Lemma 9.2, we have

$$I_{k_1,k_2} = \begin{cases} 0 & \cdots \quad k_1,k_2 > 0 \\ \eta^{+}_{0,k_1} & \cdots \quad k_2 = 0 \\ \eta^{-}_{k_2,0} & \cdots \quad k_1 = 0 \end{cases}$$

and hence

$$J^* = |b(\kappa)|_F\left\{\eta_{0,0} + \sum_{k=1}^{\infty}\left(\xi_1^k\,\eta^{+}_{0,k} + \xi_2^k\,\eta^{-}_{k,0}\right)\right\}. \tag{9.1}$$

9.9 For $t \in K^1$, we define $M(t) \in \mathrm{End}(\mathcal{S}(K))$ as follows. If $t = 1$, we put $M(t) = \mathrm{Id}_{\mathcal{S}(K)}$. If $t \neq 1$, we put

$$\begin{aligned} & M(t)\varphi(z) \\ = \;& |N_{K/F}(1-t)|_F^{1/2}\int_K \psi_m\left(\frac{1}{2}\,\langle w,tw\rangle\right)\,\rho((1-t)w,0)\,\varphi(z)\,dw \\ = \;& |N_{K/F}(1-t)|_F^{-1/2}\int_K \psi_m\left(a_t w w^{\sigma} + \frac{1}{2}\,\langle z,w\rangle\right)\,\varphi(z+w)\,dw\,, \end{aligned}$$

where $\varphi \in \mathcal{S}(K)$ and $a_t = \dfrac{\kappa}{2}\dfrac{1+t}{1-t} \in F$. Note that

$$\mathcal{M}_\chi(t)\Phi = \chi\left(\frac{1-t}{\kappa}\right) M(t)\Phi \qquad (t \in K^1 - \{1\}) \tag{9.2}$$

for $\Phi \in V^m$ and $\chi \in \mathcal{X}$ (cf. §2.9).

9.10 LEMMA. *For $\eta \in \mathcal{S}(K), \Phi \in V^m$ and $t \in K^1$, we have*

$$\Lambda(\overline{M(t)\eta}, \Phi) = M(t)\Lambda(\overline{\eta}, \Phi)\,.$$

PROOF. This follows from Lemma 7.6 and the definition of $M(t)$. □

9.11 LEMMA. *For $k \geq 0$, we have*

$$\begin{aligned} q^{k/2}\, M(t_k)\eta_{0,0} &= \eta^-_{0,k} = \overline{\eta^+_{0,k}} \\ q^{k/2}\, M(t_k^{-1})\eta_{0,0} &= \eta^+_{k,0} = \overline{\eta^-_{k,0}} \end{aligned}$$

where $t_k = (\Pi_1/\Pi_2)^k = (\pi^k, \pi^{-k})$.

PROOF. Since the assertion for $k = 0$ is trivial, we assume that $k > 0$. For $z \in K$, we have

$$\begin{aligned} & q^{k/2}\, M(t_k)\eta_{0,0}(z) \\ = \;& q^{k/2}\, |N_{K/F}(1-t_k)|_F^{-1/2} \\ & \times \int_K \psi_m\left(a_{t_k}(w-z)(w-z)^\sigma + \frac{1}{2}\langle z, w\rangle\right) \eta_{0,0}(w)\, dw \\ = \;& \int_{\mathfrak{a}} \psi_m\Bigg(\frac{1}{2}\frac{1+\pi^k}{1-\pi^k}\, b(\kappa)\, (w_1 - z_1)(w_2 - z_2) \\ & \qquad\qquad + \frac{1}{2}\, b(\kappa)\, (z_2 w_1 - z_1 w_2) + \frac{1}{2}\, b(\kappa)\, w_1 w_2\Bigg)\, d_{\mathfrak{a}} w \\ = \;& \psi_m\left(-x_z + \frac{\pi^k}{1-\pi^k}\, b(\kappa)\, z_1 z_2\right) \\ & \times \int_{\mathfrak{a}} \psi_m\left(-\frac{\pi^k}{1-\pi^k}\, b(\kappa)\, z_2 w_1 - \frac{1}{1-\pi^k}\, b(\kappa)\, z_1 w_2\right) d_{\mathfrak{a}} w \end{aligned}$$

$$
\begin{aligned}
&= \psi_m\left(-x_z + \frac{\pi^k}{1-\pi^k}\, b(\kappa)\, z_1 z_2\right) \tau_0(\alpha_1^{-1} z_1)\, \tau_0(\pi^k \alpha_2^{-1} z_2) \\
&= \tau(\Pi_2^k \alpha^{-1} z)\, \psi_m(-x_z) \\
&= \eta_{0,k}^-(z)\,.
\end{aligned}
$$

We can prove the second formula in a similar manner. □

9.12 PROPOSITION. *Suppose that $K = F \oplus F$ and $\mu = 0$. Then we have*

$$
\begin{aligned}
W^* \;=\; & q^{-\mu-\delta_{K/F}}\, |b(\kappa)|_F \cdot \left\{ \Phi + \sum_{k=1}^{\infty} \left(q^{-(l-1)/2}\, \overline{\Omega}(\Pi_1)\right)^k M((\Pi_1/\Pi_2)^k)\, \Phi \right. \\
& \left. + \sum_{k=1}^{\infty} \left(q^{-(l-1)/2}\, \overline{\Omega}(\Pi_2)\right)^k M((\Pi_1/\Pi_2)^{-k})\, \Phi \right\}.
\end{aligned}
$$

PROOF. It follows from (7.11) and (9.1) that

$$
W^* = |b(\kappa)|_F \left\{ \Lambda(\eta_{0,0}, \Phi) + \sum_{k=1}^{\infty} \overline{\xi_1}^k\, \Lambda(\eta_{0,k}^+, \Phi) + \sum_{k=1}^{\infty} \overline{\xi_2}^k\, \Lambda(\eta_{k,0}^-, \Phi) \right\}.
$$

For $k \geq 0$, we have

$$
\begin{aligned}
\Lambda(\eta_{0,k}^+, \Phi) &= \Lambda(\overline{\eta_{0,k}^-}, \Phi) = q^{k/2}\, \Lambda(\overline{M(t_k)\eta_{0,0}}, \Phi) \\
&= q^{k/2} M(t_k) \Lambda(\overline{\eta_{0,0}}, \Phi) = q^{k/2} M(t_k) \Phi
\end{aligned}
$$

by Lemma 9.11 and Lemma 9.6 (i). Similarly we have

$$
\Lambda(\eta_{k,0}^-, \Phi) = q^{k/2} M(t_k^{-1}) \Phi\,.
$$

These prove the proposition. □

§10. Proof of Theorem 4.4

10.1 Let the notation and the assumptions be the same as in §4. Let $\Phi \in V_{hol,prim}^m(\mathfrak{a}, \chi)$ be as in §6.5. Recall that

$$
(10.1) \qquad (\Theta, (E_\Omega)_{\mathfrak{a}}^m) = \Omega(\alpha_f^\sigma)\, |N_{K/F}(\alpha_f)|_{\mathbf{A}}^{1-l/2} \int_{K^1 \backslash K_{\mathbf{A}}^1} \mathcal{T}_\chi^m(W_{\Omega,\Phi}^{\alpha_f})(t)\, d^\times t
$$

and

$$W^{\alpha_f}_{\Omega,\Phi}(z) = c_\infty\, \Phi_{0,\infty}(z_\infty) \prod_{\mathfrak{p}<\infty} W_\mathfrak{p}(z_\mathfrak{p}) \qquad (z \in K_\mathbf{A})\,, \tag{10.2}$$

where c_∞ is defined by (6.3) and $W_\mathfrak{p} = W^{\alpha_\mathfrak{p}}_{\Omega_\mathfrak{p},\Phi_\mathfrak{p}}$ (cf. Lemma 6.6 and Lemma 6.8). We now summarize an explicit formula for $W_\mathfrak{p}$ calculated in §§8–9. Let S be the set of finite primes $\mathfrak{p}$ of F such that $K_\mathfrak{p} = F_\mathfrak{p} \oplus F_\mathfrak{p}$ and $\mu_\mathfrak{p}(\mathfrak{a}, m) = 0$. Note that S is an infinite set. If $\mathfrak{p} \in S$, we set

$$\begin{aligned}
\mathcal{R}_\mathfrak{p}\Phi_\mathfrak{p} \;&=\; \Phi_\mathfrak{p} + \sum_{k=1}^{\infty} \left(q_\mathfrak{p}^{-(l-1)/2}\,\overline{\Omega}(\Pi_{\mathfrak{p},1})\right)^k M((\Pi_{\mathfrak{p},1}/\Pi_{\mathfrak{p},2})^k)\,\Phi_\mathfrak{p} \\
&\quad + \sum_{k=1}^{\infty} \left(q_\mathfrak{p}^{-(l-1)/2}\,\overline{\Omega}(\Pi_{\mathfrak{p},2})\right)^k M((\Pi_{\mathfrak{p},1}/\Pi_{\mathfrak{p},2})^{-k})\,\Phi_\mathfrak{p}
\end{aligned}$$

for $\Phi_\mathfrak{p} \in V^m_\mathfrak{p}$, where $\Pi_{\mathfrak{p},1} = (\pi_\mathfrak{p}, 1)$ and $\Pi_{\mathfrak{p},2} = (1, \pi_\mathfrak{p})$ (for the definition of M, see §9.9).

10.2 PROPOSITION.

(i) *Let $\mathfrak{p}$ be a finite prime of F. Then we have*

$$\begin{aligned}
W_\mathfrak{p} \;&=\; q_\mathfrak{p}^{-\mu_\mathfrak{p}-\delta_{K_\mathfrak{p}/F_\mathfrak{p}}}\, |b_\mathfrak{p}(\kappa)|_\mathfrak{p}^{1-l}\, \Omega_\mathfrak{p}(b_\mathfrak{p}(\kappa))\, L_\mathfrak{p}\left(\overline{\Omega}\,;\frac{l}{2}\right)^{-1} \\
&\times \begin{cases}
\mathcal{R}_\mathfrak{p}\Phi_\mathfrak{p} & \cdots\quad \mathfrak{p} \in S \\
L_\mathfrak{p}\left(\overline{\chi\Omega}\,;\dfrac{l-1}{2}\right) L_\mathfrak{p}\left(\overline{\omega\Omega_F}\,; l-1\right)^{-1} & \\
\quad \left(1 + \overline{\Omega}(\Pi_\mathfrak{p}) q_\mathfrak{p}^{1-l/2}\right) \Phi_\mathfrak{p} & \cdots\quad \mathfrak{p} \in A(\chi) \\
L_\mathfrak{p}\left(\overline{\chi\Omega}\,;\dfrac{l-1}{2}\right) L_\mathfrak{p}\left(\overline{\omega\Omega_F}\,; l-1\right)^{-1} \Phi_\mathfrak{p} & \cdots\quad \text{otherwise}\,.
\end{cases}
\end{aligned}$$

(ii) *We have $W_\mathfrak{p} \in \widetilde{V}^m_\mathfrak{p}$ for any $\mathfrak{p}$, and $W_\mathfrak{p} \in V^m_\mathfrak{p}$ if $\mathfrak{p} \notin S$.*

PROOF. The first assertion follows from Proposition 8.3, Proposition 8.7, Proposition 9.7, Proposition 9.12 and (7.9). The second is a direct consequence of (i). □

10.3 LEMMA. *If $\mathfrak{p} \in S$, we have*

$$\int_{K^1\backslash K^1_\mathbf{A}} \mathcal{T}^m_\chi \left(M((\Pi_{\mathfrak{p},1}/\Pi_{\mathfrak{p},2})^k)\Phi\right)(t)\, d^\times t = \overline{\chi_\mathfrak{p}}(\Pi_{\mathfrak{p},1})^k \cdot I(\Theta)$$

for $k \in \mathbf{Z}$.

PROOF. The assertion is trivial for $k = 0$. Suppose that $k \neq 0$ and put $t_k = (\Pi_{\mathfrak{p},1}/\Pi_{\mathfrak{p},2})^k$. In view of (9.2), we have

$$\mathcal{T}_\chi^m\left(M(t_k)\Phi\right)(t) = \chi_{\mathfrak{p}}^{-1}\left(\frac{1-t_k}{\kappa}\right)\mathcal{T}_\chi^m(\mathcal{M}_\chi(t_k)\,\Phi)(t) = \overline{\chi_{\mathfrak{p}}}(\Pi_{\mathfrak{p},1})^k\,\Theta(tt_k)$$

for $t \in K_{\mathbf{A}}^1$ (note that $\chi_{\mathfrak{p}}$ is trivial on $F_{\mathfrak{p}}^{\times}$). This immediately implies the lemma. □

10.4 We are now ready to prove Theorem 4.4. In view of (10.1), (10.2), Proposition 10.2 and Lemma 10.3, $(\Theta, (E_\Omega)_{\mathfrak{a}}^m)$ is equal to

$$c'(\Omega)\prod_{\mathfrak{p}<\infty,\mathfrak{p}\notin S}\frac{L_{\mathfrak{p}}\left(\overline{\chi\Omega}\,;\dfrac{l-1}{2}\right)}{L_{\mathfrak{p}}\left(\overline{\omega\Omega_F}\,;l-1\right)L_{\mathfrak{p}}\left(\overline{\Omega};l/2\right)}$$
$$\times\prod_{\mathfrak{p}\in A(\chi)}\left(1+\overline{\Omega}(\Pi_{\mathfrak{p}})q_{\mathfrak{p}}^{1-l/2}\right)\prod_{\mathfrak{p}\in S}L_{\mathfrak{p}}\left(\overline{\Omega}\,;\frac{l}{2}\right)^{-1}Z_{\mathfrak{p}}\left(\frac{l-1}{2}\right)\cdot I(\Theta)\,,$$

where

$$c'(\Omega) = c_\infty\cdot\Omega(\alpha_f^\sigma)\,|N_{K/F}(\alpha_f)|_{\mathbf{A}}^{1-l/2}\prod_{\mathfrak{p}<\infty}\left(q_{\mathfrak{p}}^{-\mu_{\mathfrak{p}}-\delta_{K_{\mathfrak{p}}/F_{\mathfrak{p}}}}\,|b_{\mathfrak{p}}(\kappa)|_{\mathfrak{p}}^{1-l}\,\Omega_{\mathfrak{p}}(b_{\mathfrak{p}}(\kappa))\right)$$

and

$$Z_{\mathfrak{p}}(s) = 1+\sum_{k=1}^{\infty}\left(q_{\mathfrak{p}}^{-s}\,\overline{\chi_{\mathfrak{p}}\Omega_{\mathfrak{p}}}(\Pi_{\mathfrak{p},1})\right)^k+\sum_{k=1}^{\infty}\left(q_{\mathfrak{p}}^{-s}\,\overline{\chi_{\mathfrak{p}}\Omega_{\mathfrak{p}}}(\Pi_{\mathfrak{p},2})\right)^k.$$

By a straightforward calculation, we have $c'(\Omega) = \overline{c(\Omega)}$ and

$$Z_{\mathfrak{p}}(s) = \frac{1-q_{\mathfrak{p}}^{-2s}\,\overline{\chi_{\mathfrak{p}}\Omega_{\mathfrak{p}}}(\pi_{\mathfrak{p}})}{\left(1-q_{\mathfrak{p}}^{-s}\,\overline{\chi_{\mathfrak{p}}\Omega_{\mathfrak{p}}}(\Pi_{\mathfrak{p},1})\right)\left(1-q_{\mathfrak{p}}^{-s}\,\overline{\chi_{\mathfrak{p}}\Omega_{\mathfrak{p}}}(\Pi_{\mathfrak{p},2})\right)} = \frac{L_{\mathfrak{p}}\left(\overline{\chi\Omega}\,;s\right)}{L_{\mathfrak{p}}\left(\overline{\omega\Omega_F}\,;2s\right)}.$$

Thus Theorem 4.4 has been established.

References

[GeRo] Gelbart, S. S. and J. D. Rogawski, L-functions and Fourier-Jacobi coefficients for the unitary group $U(3)$, Invent. math. **105** (1991), 445–472.

[GlRo] Glauberman, G. I. and J. D. Rogawski, On theta functions with complex multiplication, J. reine angew. Math. **395** (1989), 68–101.

[Hi] Hickey, T. J., On the Fourier-Jacobi coefficients of certain Eisenstein series on a unitary group, Thesis, University of Chicago (1986).

[HKS] Harris, M., Kudla, S. S. and W. J. Sweet, Theta dichotomy for unitary groups, J. Amer. Math. Soc. **9** (1996), 941–1004.

[Ig] Igusa, J., Theta functions, Springer–Verlag, Berlin Heidelberg New York (1972).

[Is] Ishikawa, Y., The generalized Whittaker functions for $\mathrm{SU}(2,1)$ and the Fourier expansion of automorphic forms, J. Math. Sci. Univ. Tokyo **6** (1999), 477–526.

[Mo] Moen, C., The dual pair $(U(1), U(1))$ over a p-adic field, Pacific J. Math. **158** (1991), 365–386.

[MVW] Mœglin, C., Vignéras, M.-F. and J.-L. Waldspurger, Correspondences de Howe sur un corps p-adique, Lecture Notes in Math. **1291** (1987), Springer–Verlag, New York.

[MS] Murase, A. and T. Sugano, Local theory of primitive theta functions, Compositio Math. **123** (2000), 273–302.

[PS] Piatetski-Shapiro, I. I., Tate theory for reductive groups and distinguished representations, Proc. Int. Cong. Math., Helsinki 1978, 585–590.

[Ro] Rogawski, J. D., The multiplicity formula for A-packets, in *The Zeta Functions of Picard Modular Surfaces*, ed. by R. P. Langlands and D. Ramakrishnan (1992), 395–419.

[Shim1] Shimura, G., On some arithmetical properties of modualr forms of one and several variables, Ann. Math. **102** (1975), 491–515.

[Shim2] Shimura, G., The arithmetic of automorphic forms with respect to a unitary group, Ann. Math. **107** (1978), 569–605.

[Shim3] Shimura, G., The arithmetic of certain zeta functions and automorphic forms on orthogonal groups, Ann. Math. **111** (1980), 313–375.

[Shin] Shintani, T., On automorphic forms on unitary groups of order 3, unpublished manuscript (1979).

[Ta] Tate, J., Number theoretic background, Proc. Symp. in Pure Math. **33**, part 2 (1979), 3–26.

[Yam] Yamauchi, A., On a certain extended Galois action on the space of arithmetic modular forms with respect to a unitary group, J. Math. Kyoto Univ. **41** (2001), 183–231.

[Yan] Yang, T., Theta liftings and Hecke L-functions, J. reine angew. Math. **485** (1997), 25–53.

[Yo] Yoshida, H., On absolute CM-periods II, Amer. J. Math. **120** (1998), 1199–1236.

(Received May 28, 2001)

Atsushi Murase
Department of Mathematics
Faculty of Science
Kyoto Sangyo University
Kamigamo-Motoyama, Kita-ku
Kyoto 603-8555, Japan
E-mail: murase@cc.kyoto-su.ac.jp

Takashi Sugano
Department of Mathematics
Faculty of Science
Kanazawa University
Kakuma-machi
Kanazawa 920-1192, Japan
E-mail: sugano@kappa.s.kanazawa-u.ac.jp